# POSITIVE OPERATOR-VALUED KERNELS AND NON-COMMUTATIVE PROBABILITY 

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#### Abstract

We prove new factorization and dilation results for general positive operator-valued kernels, and we present their implications for associated Hilbert space-valued Gaussian processes, and their covariance structure. Further applications are to non-commutative probability theory, including a non-commutative Radon-Nikodym theorem for systems of completely positive maps.


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## 1. Introduction

In this paper we present a result which offers a canonical link between a general setting for operator valued completely positive maps on the one hand, with an induced scalar valued counterpart. We further offer applications to a variety of neighboring areas, including the following six closely interrelated areas: (i) operator valued Gaussian processes, and their associated covariance structure; (ii) universal factorizations; (iii) non-commutative operator valued Radon-Nikodym derivatives and their applications to quantum gates and to quantum states; (iv) partial orders on operator valued completely positive maps via their associated reproducing kernel Hilbert spaces; (v) intertwining operators for representations induced from completely positive maps; and (vi) applications of (iii) to completely positive maps and associated quantum gates.

The paper is organized as follows: In Section 2 we present the general framework, and the main theorems, for operator valued completely positive maps, as well as their associated structures, including operator valued Gaussian processes. The applications to non-commutative operator valued Radon-Nikodym derivatives are given in Section 3, while the focus in Section 4 is that of completely positive maps.

[^0]For the reader's benefit, we include the following citations: For the theory of positive definite kernels [Gue22b, AA22, Dav21], Hilbert space valued Gaussian processes [AJ22, Kre19, LM17], completely positive maps [Tho24, Kod23, ZD22, SCC21], quantum gates [ZHL24, PY24, GLL24, LCH24], and operator valued RadonNikodym derivatives [MPR20, Ahm13].

The literature on the theory of reproducing kernels is vast, encompassing both theoretical foundations and recent applications. For a comprehensive overview, including the latest developments, we refer to the following resources: [AS56, Aro48, AJ21, PR16, AFMP94, LYA23, CY17, Alp91, AD93, AL95, AB97, AD06, AJ15].

Notation. Throughout the paper, we use the physics convention that inner products are linear in the second variable. Let $|a\rangle\langle b|$ denote Dirac's rank-1 operator, $c \mapsto a\langle b, c\rangle . \mathcal{L}(H)$ denotes the algebra of all bounded linear operators in a Hilbert space $H$.

For a positive definite (p.d.) kernel $K: S \times S \rightarrow \mathbb{C}$, let $H_{K}$ be the corresponding reproducing kernel Hilbert space (RKHS). $H_{K}$ is the Hilbert completion of

$$
\begin{equation*}
\operatorname{span}_{\mathbb{C}}\left\{K_{y}(\cdot):=K(\cdot, y) \mid y \in S\right\} \tag{1.1}
\end{equation*}
$$

with respect to the inner product

$$
\begin{equation*}
\left\langle\sum_{i} c_{i} K\left(\cdot, x_{i}\right), \sum_{i} d_{j} K\left(\cdot, y_{j}\right)\right\rangle_{H_{K}}:=\sum_{i} \sum_{j} \bar{c}_{i} d_{j} K\left(x_{i}, x_{j}\right) \tag{1.2}
\end{equation*}
$$

The following reproducing property holds:

$$
\begin{equation*}
\varphi(x)=\langle K(\cdot, x), \varphi\rangle_{H_{K}}, \forall x \in S, \forall \varphi \in H_{K} \tag{1.3}
\end{equation*}
$$

Any scalar-valued kernel $K$ as above is associated with a zero-mean Gaussian process, where $K$ is the covariance:

$$
\begin{equation*}
K(s, t)=\mathbb{E}\left[\overline{W_{s}} W_{t}\right] \tag{1.4}
\end{equation*}
$$

and $W_{s} \sim N(0, K(s, s))$.
An $\mathcal{L}(H)$-valued kernel $K: S \times S \rightarrow \mathcal{L}(H)$ is p.d. if

$$
\begin{equation*}
\sum_{i=1}^{n}\left\langle a_{i}, K\left(s_{i}, s_{j}\right) a_{j}\right\rangle_{H} \geq 0 \tag{1.5}
\end{equation*}
$$

for all $\left(a_{i}\right)_{1}^{n}$ in $H$, and all $n \in \mathbb{N}$.

## 2. $\mathcal{L}(H)$-valued kernels and $H$-valued Gaussian processes

In this section, we begin by revisiting a universal construction for operator-valued positive definite kernels. This construction will be adapted for various applications, which will be explored in the following sections.

Theorem 2.1 (Universal Factorization). Let $K: S \times S \rightarrow \mathcal{L}(H)$ be a p.d. kernel. Then there exists a RKHS $H_{\tilde{K}}$, and a family of operators $V(s): H \rightarrow H_{\tilde{K}}, s \in S$, such that

$$
\begin{equation*}
H_{\tilde{K}}=\overline{\operatorname{span}}\{V(s) a: a \in H, s \in S\} \tag{2.1}
\end{equation*}
$$

and

$$
\begin{equation*}
K(s, t)=V(s)^{*} V(t) \tag{2.2}
\end{equation*}
$$

Conversely, if there is a Hilbert space $L$ and operators $V(s): H \rightarrow L, s \in S$, such that

$$
\begin{equation*}
L=\overline{\operatorname{span}}\{V(s) a: a \in H, s \in S\} \tag{2.3}
\end{equation*}
$$

and (2.2) holds, then $L \simeq H_{\tilde{K}}$.
Proof. For a detailed proof, we refer the reader to [JT24b, JT24a]. We provide a brief outline of the main steps below:

Given a p.d. kernel $K: S \times S \rightarrow \mathcal{L}(H)$, define $\tilde{K}:(S \times H) \times(S \times H) \rightarrow \mathbb{C}$ as

$$
\begin{equation*}
\tilde{K}((s, a),(t, b)):=\langle a, K(s, t) b\rangle_{H} . \tag{2.4}
\end{equation*}
$$

Then $\tilde{K}$ is a scalar-valued p.d. kernel. Let $H_{\tilde{K}}$ be the associated RKHS.
Set $V(s): H \rightarrow H_{\tilde{K}}$ by

$$
\begin{equation*}
V(s) a=\tilde{K}_{(s, a)}=\tilde{K}(\cdot,(s, a)):(S \times H) \rightarrow \mathbb{C}, \quad \forall a \in H \tag{2.5}
\end{equation*}
$$

One verifies that

$$
V(s)^{*} \tilde{K}(\cdot,(t, b))=K(s, t) b
$$

and the factorization $K(s, t)=V(s)^{*} V(t)$ holds.
Remark 2.2. The setting in Theorem 2.1 includes and extends well known constructions in classical dilation theory. Notably, transitioning from an operator-valued kernel $K$ to the scalar-valued positive definite kernel $\tilde{K}$ enables a function-based approach to dilation theory, contrasting with the traditional abstract spaces of equivalence classes. The literature is vast, and here we call attention to [Arv10], and the papers cited there.

Building upon Ito's seminal work, Hilbert spaces and their corresponding operators have become essential tools in the stochastic analysis of Gaussian processes. For more detailed insights, see e.g., [Sch52, AJ15, AJ21], as well as the earlier works cited therein. We explore two principal types of Gaussian processes in this field: (i) scalar-valued processes that are indexed by a Hilbert space, typically through an Ito-isometry; and (ii) processes where the Gaussian values are embedded directly within a Hilbert space (see e.g., [Lop23, Gue22a, KM20, FS20, Kat19, JT23]). This paper concentrates on the second type, which provides enhanced adaptability in modeling covariance structures-a key element in processing large datasets. We now proceed to discuss the details of this approach.

Theorem 2.3. Every operator-valued p.d. kernel $K: S \times S \rightarrow \mathcal{L}(H)$ is associated with an $H$-valued Gaussian process $\{W(s)\}_{s \in S}$ with zero-mean, realized on some probability space $(\Omega, \mathbb{P})$, such that

$$
\begin{equation*}
K(s, t)=\int_{\Omega}|W(s)\rangle\langle W(t)| d \mathbb{P} \tag{2.6}
\end{equation*}
$$

Conversely, every $H$-valued Gaussian process is obtained from such a $\mathcal{L}(H)$-valued kernel.

Remark 2.4. More precisely, the identity (2.6) holds in the sense that

$$
\begin{equation*}
\mathbb{E}\left[\langle a, W(s)\rangle_{H}\langle W(t), b\rangle_{H}\right]=\langle a, K(s, t) b\rangle_{H} \tag{2.7}
\end{equation*}
$$

for all $s, t \in S$, and all $a, b \in H$. For a detailed proof, see [JT24a].

Definition 2.5. Consider the $H$-valued Gaussian process $W_{s}: \Omega \rightarrow H$,

$$
\begin{equation*}
W(t)=\sum_{i}\left(V(t)^{*} \varphi_{i}\right) Z_{i}, \tag{2.8}
\end{equation*}
$$

where $\left(\varphi_{i}\right)$ is an ONB in $H_{\tilde{K}}$. Following standard conventions, here $\left\{Z_{i}\right\}$ refers to a choice of an independent identically distributed (i.i.d.) system of standard scalar Gaussians $N(0,1)$ random variables, and with an index matching the choice of ONB.

Theorem 2.6. We have

$$
\begin{equation*}
\mathbb{E}\left[\langle a, W(s)\rangle_{H}\langle W(t), b\rangle_{H}\right]=\langle a, K(s, t) b\rangle_{H} \tag{2.9}
\end{equation*}
$$

Proof. Note that $\mathbb{E}\left[Z_{i} Z_{j}\right]=\delta_{i, j}$. From this, we get

$$
\begin{aligned}
\operatorname{LHS}_{(2.9)} & =\mathbb{E}\left[\langle a, W(s)\rangle_{H}\langle W(t), b\rangle_{H}\right] \\
& =\sum_{i} \sum_{j}\left\langle a, V(s)^{*} \varphi_{i}\right\rangle\left\langle V(t)^{*} \varphi_{j}, b\right\rangle \mathbb{E}\left[Z_{i} Z_{j}\right] \\
& =\sum_{i}\left\langle V(s) a, \varphi_{i}\right\rangle_{H_{\tilde{K}}}\left\langle\varphi_{i}, V(t) b\right\rangle_{H_{\tilde{K}}} \\
& =\langle V(s) a, V(t) b\rangle_{H_{\tilde{K}}}=\left\langle a, V(s)^{*} V(t) b\right\rangle_{H}=\langle a, K(s, t) b\rangle_{H} .
\end{aligned}
$$

Remark 2.7. The Gaussian process $\{W(t)\}_{t \in S}$ in (2.8) is well defined and possesses the stated properties. This is an application of the central limit theorem to the choice $\left\{Z_{i}\right\}$ of i.i.d. $N(0,1)$ Gaussians on the right-side of (2.8).

Varying the choices of ONBs $\left(\varphi_{i}\right)$ and i.i.d. $N(0,1)$ Gaussian random variables $\left(Z_{i}\right)$ will result in different Gaussian processes $\{W(t)\}_{t \in S}$, but all will adhere to the covariance condition specified in (2.9). Importantly, once the ONB $\left(\varphi_{i}\right)$ and the random variables $\left(Z_{i}\right)$ are fixed, the resulting Gaussian process $\{W(t)\}_{t \in S}$ is uniquely determined by its first two moments (mean and covariance).

## 3. A non-commutative Radon-Nikodym theorem

As is known, the theory of von Neumann algebras offers a framework for noncommutative measure theory, see e.g., [Sak65]. In this interpretation, the projections in the algebra are the characteristic functions of the (non-commuting) "measurable sets." From the elements of the Hilbert space $H$ we then build the bounded measurable functions; and the (normal) states are the probability measures on the underlying (non-commutative) measure space. Here we shall supplement this theory with Aronszajn's notion of systems of ordered p.d. kernels and contractive containment of the corresponding reproducing kernel Hilbert spaces (RKHSs.) From this we then build a natural (but different) notion of non-commutative Radon-Nikodym derivatives.

For the sake of completeness, we include a proof of Aronszajn's inclusion theorem below. It states that, for two (scalar-valued) p.d. kernels $K$ and $L$ on $S \times S, K \leq L$ (i.e., $L-K$ is p.d.) if and only if $H_{K}$ is contractively contained in $H_{L}$ (see e.g., [Aro50]).
Theorem 3.1 (Aronszajn). If $K \leq L$, then $H_{K} \subset H_{L}$ and $\|f\|_{H_{L}} \leq\|f\|_{H_{K}}$ for all $f \in H_{K}$.

Proof. Let $f \in H_{K}$. To verify that $f \in H_{L}$, it is equivalent to show that

$$
\left|\sum c_{i} f\left(s_{i}\right)\right|^{2} \leq \mathrm{const} \cdot\left\|\sum c_{i} L_{s_{i}}\right\|_{H_{L}}^{2}
$$

for all $\left(c_{i}\right)_{i=1}^{n}$ in $\mathbb{C},\left(s_{i}\right)_{i=1}^{n}$ in $S$, and $n \in \mathbb{N}$. This follows from the reproducing property, and the assumption $K \leq L$ :

$$
\begin{aligned}
\left|\sum c_{i} f\left(s_{i}\right)\right|^{2} & =\left|\left\langle\sum \overline{c_{i}} K_{s_{i}}, f\right\rangle_{H_{K}}\right|^{2} \leq\|f\|_{H_{K}}^{2}\left\|\sum \overline{c_{i}} K_{s_{i}}\right\|_{H_{K}}^{2} \\
& \leq\|f\|_{H_{K}}^{2} \sum \overline{c_{i}} c_{j} K\left(s_{i}, s_{j}\right) \leq\|f\|_{H_{K}}^{2} \sum \overline{c_{i}} c_{j} L\left(s_{i}, s_{j}\right) \\
& =\|f\|_{H_{K}}^{2}\left\|\sum c_{i} L_{s_{i}}\right\|_{H_{L}}^{2}
\end{aligned}
$$

Proposition 3.2. Suppose $K, L$ are p.d. on $S \times S$, and $K \leq L$. There exists $a$ unique positive selfadjoint operator $T$ on $H_{L}$, such that $0 \leq T \leq I_{H_{K}}$, and

$$
\begin{equation*}
K(s, t)=\left\langle T^{1 / 2} L_{s}, T^{1 / 2} L_{t}\right\rangle_{H_{L}}, \quad s, t \in S \tag{3.1}
\end{equation*}
$$

Proof. Define a map $\Phi$ as

$$
\Phi\left(L_{s}, L_{t}\right)=K(s, t)
$$

and extend it by linearity:

$$
\Phi\left(\sum_{i=1}^{m} c_{i} L_{s_{i}}, \sum_{j=1}^{n} d_{j} L_{t_{j}}\right)=\sum_{i=1}^{m} \sum_{j=1}^{n} \overline{c_{i}} d_{j} K\left(s_{i}, t_{j}\right)
$$

Then $\Phi$ extends by density to a unique bounded sesquilinear form on $H_{L}$.
The remaining assertions follow from the general theory of quadratic forms. In particular, if $j: H_{K} \rightarrow H_{L}$ is the inclusion map, then $T=j j^{*}: H_{L} \rightarrow H_{L}$ is the desired operator.

We now consider $\mathcal{L}(H)$-valued p.d. kernels.
Definition 3.3. Suppose $K, L$ are $\mathcal{L}(H)$-valued p.d. kernels defined on $S \times S$. We say that $K \leq L$ if

$$
\begin{equation*}
\sum\left\langle a_{i}, K\left(s_{i}, s_{j}\right) a_{j}\right\rangle_{H} \leq \sum\left\langle a_{i}, L\left(s_{i}, s_{j}\right) a_{j}\right\rangle_{H} \tag{3.2}
\end{equation*}
$$

for all $\left(s_{i}\right)_{i=1}^{n}$ in $S,\left(a_{i}\right)_{i=1}^{n}$ in $H$, and $n \in \mathbb{N}$.
Let $\tilde{K}, \tilde{L}$ be the associated scalar-valued p.d. kernels on $S \times S$, and $H_{\tilde{K}}, H_{\tilde{L}}$ be the respective RKHSs (see (2.4)). Definition 3.3 means that

$$
\begin{equation*}
K \leq L \Longleftrightarrow \tilde{K} \leq \tilde{L} \tag{3.3}
\end{equation*}
$$

Theorem 3.4. Let $K, L: S \times S \rightarrow \mathcal{L}(H)$ be operator-valued p.d. kernels on $S \times S$. Let $\tilde{K}, \tilde{L}$ be the associated scalar-valued p.d. kernels, and $H_{\tilde{K}}, H_{\tilde{L}}$ be the respective RKHSs.

The following are equivalent:
(1) $K \leq L$ in the sense of (3.2)-(3.3).
(2) There exists a positive selfadjoint operator $T: H_{\tilde{L}} \rightarrow H_{\tilde{L}}, 0 \leq T \leq I_{H_{\tilde{L}}}$, such that

$$
\begin{equation*}
K(s, t)=V_{L}(s)^{*} T V_{L}(t) \tag{3.4}
\end{equation*}
$$

Equivalently,

$$
\begin{equation*}
V_{K}(t)=T^{1 / 2} V_{L}(t) \tag{3.5}
\end{equation*}
$$

Here, $L(s, t)=V_{L}(s)^{*} V_{L}(t)$ and $K(s, t)=V_{K}(s)^{*} V_{K}(t)$ are the respective canonical factorizations of $L$ and $K$ (Theorem 2.1).

Proof. $(1) \Rightarrow$ (2) Assume $K \leq L$. By Theorem 3.1, $H_{\tilde{K}} \subset H_{\tilde{L}}$ and $\|f\|_{H_{\tilde{L}}} \leq\|f\|_{H_{\tilde{K}}}$ for all $f \in H_{\tilde{K}}$.

Recall that $L$ factors as

$$
L(s, t)=V_{L}(s)^{*} V_{L}(t)
$$

where $V_{L}(t): H \rightarrow H_{\tilde{L}}$ is given by $V_{L}(t) a=\tilde{L}_{(t, a)}$, for all $t \in S$ and $a \in H$ (see (2.5)).

Apply Proposition 3.2 to the scalar-valued kernels $\tilde{K}$ and $\tilde{L}$ : There exists a unique selfadjoint operator $T$ in $H_{\tilde{L}}$, with $0 \leq T \leq I_{H_{\tilde{L}}}$, so that

$$
\begin{aligned}
\langle a, K(s, t) b\rangle_{H} & =\left\langle\tilde{K}_{(s, a)}, \tilde{K}_{(t, b)}\right\rangle_{H_{\tilde{K}}} \\
& =\left\langle T^{1 / 2} \tilde{L}_{(s, a)}, T^{1 / 2} \tilde{L}_{(t, b)}\right\rangle_{H_{\tilde{L}}} \\
& =\left\langle T^{1 / 2} V_{L}(s) a, T^{1 / 2} V_{L}(t) b\right\rangle_{H_{\tilde{L}}} \\
& =\left\langle a, V_{L}(s)^{*} T V_{L}(t) b\right\rangle_{H}
\end{aligned}
$$

for all $a, b \in H$, and all $s, t \in S$. Therefore, (3.4) holds. The identity (3.5) follows from the above argument and the proof of Proposition 3.2.
$(2) \Rightarrow(1)$ Conversely, from (3.4) and the fact that $0 \leq T \leq I_{H_{\tilde{L}}}$, we have

$$
\begin{aligned}
\sum_{i, j}\left\langle a_{i}, K\left(s_{i}, s_{j}\right) a_{j}\right\rangle_{H} & =\sum_{i, j}\left\langle a_{i}, V_{L}\left(s_{i}\right)^{*} T V_{L}\left(s_{j}\right) a_{j}\right\rangle_{H} \\
& =\left\|T^{1 / 2} \sum_{i} V_{L}\left(s_{i}\right) a_{i}\right\|_{H_{\tilde{L}}}^{2} \leq\left\|\sum_{i} V_{L}\left(s_{i}\right) a_{i}\right\|_{H_{\tilde{L}}}^{2} \\
& =\sum_{i, j}\left\langle a_{i}, V_{L}\left(s_{i}\right)^{*} V_{L}\left(s_{j}\right) a_{j}\right\rangle_{H} \\
& =\sum_{i, j}\left\langle a_{i}, L\left(s_{i}, s_{j}\right) a_{j}\right\rangle_{H}
\end{aligned}
$$

and so $K \leq L$.
Corollary 3.5. Suppose $K, L: S \times S \rightarrow \mathcal{L}(H)$ p.d., and $K \leq L$. Let $T=d K / d L$ be the Radon-Nikodym derivative from Theorem 3.4.

Let $W_{L}(t)$ be the $H$-valued Gaussian process from (2.8), i.e.,

$$
\begin{equation*}
W_{L}(t)=\sum_{i}\left(V_{L}(t)^{*} \varphi_{i}\right) Z_{i} \tag{3.6}
\end{equation*}
$$

Set

$$
\begin{equation*}
W_{K}(t):=\sum_{i}\left(V_{L}(t)^{*} T^{1 / 2} \varphi_{i}\right) Z_{i} \tag{3.7}
\end{equation*}
$$

Then $K$ admits the following decomposition

$$
\begin{equation*}
K(s, t)=\int_{\Omega}\left|W_{K}(s)\right\rangle\left\langle W_{K}(t)\right| d \mathbb{P} \tag{3.8}
\end{equation*}
$$

Proof. Recall that (3.8) is equivalent to

$$
\begin{equation*}
\mathbb{E}\left[\left\langle a, W_{K}(s)\right\rangle_{H}\left\langle W_{K}(t), b\right\rangle_{H}\right]=\langle a, K(s, t) b\rangle_{H} \tag{3.9}
\end{equation*}
$$

for all $a, b \in H$ and $s, t \in S$.
Given $L: S \times S \rightarrow \mathcal{L}(H)$ p.d., recall that

$$
L(s, t)=V_{L}(s)^{*} V_{L}(t)
$$

where $V_{L}(t): H \rightarrow H_{\tilde{L}}=$ the RKHS of $\tilde{L}$.
Let $\left(\varphi_{i}\right)$ be an ONB in $H_{\tilde{L}}$, and apply the identity

$$
I_{H_{\tilde{L}}}=\sum_{i}\left|\varphi_{i}\right\rangle\left\langle\varphi_{i}\right|
$$

we then get

$$
\begin{aligned}
L(s, t) & =V_{L}(s)^{*} I_{H_{\tilde{L}}} V_{L}(t) \\
& =\sum_{i}\left|V_{L}(s)^{*} \varphi_{i}\right\rangle\left\langle V_{L}(t)^{*} \varphi_{i}\right| .
\end{aligned}
$$

Similarly, using $V_{K}(t)=T^{1 / 2} V_{L}(t)$ from (3.5),

$$
\begin{equation*}
K(s, t)=\sum_{i}\left|V_{L}(s)^{*} T^{1 / 2} \varphi_{i}\right\rangle\left\langle V_{L}(t)^{*} T^{1 / 2} \varphi_{i}\right| \tag{3.10}
\end{equation*}
$$

Thus,
l.h.s.(3.9)

$$
\begin{align*}
& =\sum_{i, j}\left\langle a, V_{L}(s)^{*} T^{1 / 2} \varphi_{i}\right\rangle_{H}\left\langle V_{L}(t)^{*} T^{1 / 2} \varphi_{j}, b\right\rangle_{H} \underbrace{\mathbb{E}\left[Z_{i} Z_{j}\right]}_{=\delta_{i, j}} \\
& =\sum_{i}\left\langle a, V_{L}(s)^{*} T^{1 / 2} \varphi_{i}\right\rangle_{H}\left\langle V_{L}(t)^{*} T^{1 / 2} \varphi_{i}, b\right\rangle_{H} \\
& =\left\langle a,\left(\sum_{i}\left|V_{L}(s)^{*} T^{1 / 2} \varphi_{i}\right\rangle\left\langle V_{L}(t)^{*} T^{1 / 2} \varphi_{i}\right|\right) b\right\rangle_{H} \tag{1}
\end{align*}
$$

## 4. Completely positive maps

The role in physics of completely positive maps includes the following: It is the kind of transformation resulting, for example, from passing a beam in a certain mixed state through some device thereby producing another beam in a different mixed state, hence allowing for dissipative effects. Such transformations must map states into states and hence be positive. But complete positivity is stronger. Under standard assumptions of unitary dynamics as a whole, completely positive maps arise as restrictions of representations realized in the bigger system, e.g., beam
plus transformer. In mathematical terms, this corresponds to the form given in Stinespring's theorem. Recall that Stinespring's theorem (see (4.1)) states that each completely positive map can be realized as a compression of a unitary dynamics, hence is experimentally realizable.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital $C^{*}$-algebras. A map $\psi: \mathfrak{A} \rightarrow \mathfrak{B}$ is said to be completely positive (CP) if $\psi \otimes I_{n}: \mathfrak{A} \otimes M_{n} \rightarrow \mathfrak{B} \otimes M_{n}$ is positive for all $n \in \mathbb{N}$.

Let $\mathfrak{A}$ be a unital $C^{*}$-algebra. Suppose $\psi: \mathfrak{A} \rightarrow \mathcal{L}(H)$ is completely positive and $\psi(I)=I_{H}$. Stinespring's dilation theorem states that, there exists a Hilbert space $\mathscr{K}$, a representation $\pi: \mathfrak{A} \rightarrow \mathcal{L}(\mathscr{K})$, and an isometric embedding $V: H \rightarrow \mathscr{K}$ such that

$$
\begin{equation*}
\psi(A)=V^{*} \pi(A) V \tag{4.1}
\end{equation*}
$$

Further, $(\pi, V, \mathscr{K})$ may be chosen to be minimal, i.e., $\mathscr{K}=\overline{\pi(\mathfrak{A}) V H}$. In that case, the dilation is unique up to unitary equivalence.

We sketch a proof of (4.1) as an application of results in Sections 2-3.
Corollary 4.1. Let $\psi: \mathfrak{A} \rightarrow \mathcal{L}(H)$ be as above, i.e., completely positive, and $\psi(I)=I$. Let $K=K_{\varphi}: \mathfrak{A} \times \mathfrak{A} \rightarrow \mathcal{L}(H)$ be given by

$$
K(A, B):=\psi\left(A^{*} B\right), \quad A, B \in \mathfrak{A}
$$

as a $\mathcal{L}(H)$-valued p.d. kernel on $\mathfrak{A} \times \mathfrak{A}$. Let $\mathscr{K}=H_{\tilde{K}}=$ the RKHS of the associated scalar-valued kernel $\tilde{K}$. Define $V=V(I): H \rightarrow \mathscr{K}$ by

$$
V h=\tilde{K}_{(I, h)}: \mathfrak{A} \times H \rightarrow \mathbb{C}
$$

set $\pi: \mathfrak{A} \rightarrow \mathcal{L}(\mathscr{K})$ by

$$
\pi(A) \tilde{K}_{(B, h)}=\tilde{K}_{(A B, h)}
$$

for all $(B, h) \in \mathfrak{A} \times H$. Then (4.1) holds, and $(\pi, V, \mathscr{K})$ is minimal.
Proof. This is a direct "translation" of Theorem 2.1 to the setting of CP maps.
Specifically, when $K(A, B)=\psi\left(A^{*} B\right)$, it factors into

$$
K(A, B)=V^{*}(A) V(B)
$$

as in (2.2). That is,

$$
K(A, B) h=V^{*}(A) V(B) h=V^{*}(A) \tilde{K}_{(B, h)}=\psi\left(A^{*} B\right) h
$$

By setting $A=I$ and $V:=V(I)$, this reduces to

$$
\begin{aligned}
K(I, B) h & =V^{*} V(B) h=V^{*} \tilde{K}_{(B, h)} \\
& =V^{*} \pi(B) \tilde{K}_{(I, h)} \\
& =V^{*} \pi(B) V h=\psi(B) h
\end{aligned}
$$

which is (4.1).
Moreover, since

$$
\begin{aligned}
\mathscr{K}=H_{\tilde{K}} & =\overline{\operatorname{span}}\left\{\tilde{K}_{(A, h)}: A \in \mathfrak{A}, h \in H\right\} \\
& =\overline{\operatorname{span}}\{\pi(A) V h: A \in \mathfrak{A}, h \in H\}=\overline{\pi(\mathfrak{A}) V H}
\end{aligned}
$$

the dilation is minimal. The assertion of uniqueness (up to unitary equivalence) is immediate.

In the above setting, there is also a Radon-Nikodym type theorem that characterizes all CP maps $\varphi, \psi$ for which $\varphi \leq \psi$, in the sense that $\psi-\varphi$ is CP. In term of the operator-valued kernels, say $K_{\varphi}$ and $K_{\psi}$, we have $\varphi \leq \psi \Longleftrightarrow K_{\varphi} \leq K_{\psi}$, and the latter is in the sense of Definition 3.3.

We sketch below that this result can be derived as an application of Theorem 3.4.
Corollary 4.2. Let $\varphi, \psi: \mathfrak{A} \rightarrow \mathcal{L}(H)$ be CP maps. Let $\left(\pi_{\psi}, V_{\psi}, \mathscr{K}_{\psi}\right)$ be the minimal Stinespring dilation from Corollary 4.1, i.e., $\mathscr{K}_{\psi}=H_{\tilde{K}_{\psi}}=$ the $R K H S$ of $\tilde{K}_{\psi}$, and

$$
\tilde{K}((A, a),(B, b))=\left\langle a, \psi\left(A^{*} B\right) b\right\rangle_{H}
$$

for all $A, B \in \mathfrak{A}$, and $a, b \in H$.
Then $\varphi \leq \psi$ if and only if there exists a unique positive selfadjoint operator $T$ in the commutant $\pi_{\psi}(\mathfrak{A})^{\prime}, 0 \leq T \leq I_{\mathscr{K}_{\psi}}$, such that

$$
\begin{equation*}
\varphi(A)=V_{\psi}^{*} T^{1 / 2} \pi_{\psi}(A) T^{1 / 2} V_{\psi}, \quad A \in \mathfrak{A} \tag{4.2}
\end{equation*}
$$

Proof. Suppose $\varphi \leq \psi$ (i.e., $K_{\varphi} \leq K_{\psi}$ ).
Set $K_{\psi}(A, B)=\psi\left(A^{*} B\right)$ as before, so it factors into

$$
K_{\psi}(A, B)=V_{\psi}(A)^{*} V_{\psi}(B)
$$

By Theorem 3.4, there exists a unique $T, 0 \leq T \leq I_{H_{\tilde{K}_{\psi}}}$, such that (see (3.4))

$$
\begin{equation*}
K_{\varphi}(A, B)=V_{\psi}(A)^{*} T V_{\psi}(B) \tag{4.3}
\end{equation*}
$$

Set $A=I$ in (4.3), then

$$
\begin{equation*}
\varphi(B)=K_{\varphi}(I, B)=V_{\psi}(I)^{*} T V_{\psi}(B)=V_{\psi} T \pi_{\psi}(B) V_{\psi} \tag{4.4}
\end{equation*}
$$

where $V_{\psi}:=V_{\psi}(I)$, and $\pi_{\psi}: \mathfrak{A} \rightarrow \mathcal{L}\left(\mathscr{K}_{\psi}\right)$ is as in Corollary 4.1. In particular,

$$
\begin{equation*}
V_{\psi}(A)=\pi_{\psi}(A) V_{\psi}, \quad A \in \mathfrak{A} \tag{4.5}
\end{equation*}
$$

It remains to show that $T \in \pi_{\psi}(\mathfrak{A})^{\prime}$. For this, one checks that

$$
\begin{align*}
\varphi(B) & =\varphi\left(\left(B^{*}\right)^{*} I\right) \\
& =K_{\varphi}\left(B^{*}, I\right) \\
& =  \tag{4.3}\\
(4.3) & V_{\psi}\left(B^{*}\right)^{*} T V_{\psi}(I) \\
= & \left(\pi_{\psi}\left(B^{*}\right) V_{\psi}\right)^{*} T V_{\psi}  \tag{4.6}\\
(4.5) & \\
& =V_{\psi}^{*} \pi_{\psi}(B) T V_{\psi}
\end{align*}
$$

Combining (4.4) and (4.6), we conclude that $\pi_{\psi}(B) T=T \pi_{\psi}(B)$, for all $B \in \mathfrak{A}$. Thus, $T \in \pi_{\psi}(\mathfrak{A})^{\prime}$ and (4.2) follows.

Conversely, given (4.2), it is clear that $\varphi$ is CP. See e.g., the proof of " $(2) \Rightarrow(1)$ " in Theorem 3.4.

In view of the correspondence between operator-valued p.d. kernels and Hilbert space-valued Gaussian processes, we have:
Corollary 4.3. Every $C P$ map $\psi: \mathfrak{A} \rightarrow \mathcal{L}(H)$ admits a direct integral decomposition

$$
\begin{equation*}
\psi\left(A^{*} B\right)=\int_{\Omega}\left|W_{\psi}(A)\right\rangle\left\langle W_{\psi}(B)\right| d \mathbb{P} \tag{4.7}
\end{equation*}
$$

where $\left\{W_{\psi}(A)\right\}_{A \in \mathfrak{A}}$ is a mean-zero, H-valued Gaussian process, realized in some probability space $(\Omega, \mathbb{P})$.

Proof. Let $\left(V_{\psi}, \pi_{\psi}, \mathscr{K}_{\psi}\right)$ be the minimal Stinespring dilation in Corollary 4.1, i.e., $\mathscr{K}_{\psi}=H_{\tilde{K}_{\psi}}=$ the RKHS of $\tilde{K}_{\psi}$. Then,

$$
\begin{aligned}
\psi\left(A^{*} B\right) & =V_{\psi}^{*} \pi_{\psi}\left(A^{*} B\right) V_{\psi} \\
& =\left(\pi_{\psi}(A) V_{\psi}\right)^{*}\left(\pi_{\psi}(B) V_{\psi}\right) \\
& =\sum_{i}\left|\left(\pi_{\psi}(A) V_{\psi}\right)^{*} \varphi_{i}\right\rangle\left\langle\left(\pi_{\psi}(B) V_{\psi}\right)^{*} \varphi_{i}\right|
\end{aligned}
$$

where $\left(\varphi_{i}\right)$ is an ONB in $\mathscr{K}_{\psi}$. Setting

$$
W_{\psi}(A):=\sum_{i}\left(\left(\pi_{\psi}(A) V_{\psi}\right)^{*} \varphi_{i}\right) Z_{i}
$$

with $\left\{Z_{i}\right\}$ i.i.d. $N(0,1)$ as above, we get

$$
\left\langle a, \psi\left(A^{*} B\right) b\right\rangle_{H}=\mathbb{E}\left[\left\langle a, W_{\psi}(A)\right\rangle_{H}\left\langle W_{\psi}(B), b\right\rangle_{H}\right]
$$

for all $a, b \in H$, and $A, B \in \mathfrak{A}$, which is (4.7).

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