Generalizing Quantum Tanner Codes

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***Abstract*—In this work, we present a generalization of the recently proposed quantum Tanner codes by Leverrier and Zémor, which contains a construction of asymptotically good quantum LDPC codes. Quantum Tanner codes have so far been constructed equivalently from groups, Cayley graphs, or square complexes constructed from groups. We show how to enlarge this to group actions on finite sets, Schreier graphs, and a family of square complexes which is the largest possible in a certain sense. Furthermore, we discuss how the proposed generalization opens up the possibility of finding other families of asymptotically good quantum codes.**

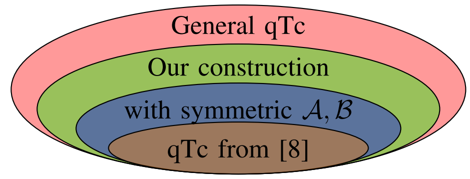

Fig. 1. Our construction (green) generalizes the quantum Tanner codes (qTc) from [8] (brown). We also consider a slightly less general construction (blue), and in Section III-D we show that there are codes not from our construction that also reasonably may be called general quantum Tanner codes (red).

I. Introduction

Quantum computers, which are based on the peculiarities of quantum mechanics, have been predicted to revolutionize several computing tasks for a long time, e.g., solving challenging problems that arise in chemistry and finance. A quantum computer works by taking advantage of the quantum behavior of particles, which makes it possible to have superpositions of states. However, quantum computers are prone to errors due to the fragile nature of quantum states, in particular when the number of states grows. The use of quantum error-correction codes can mitigate the effect of such errors and hence make it possible to build large-scale quantum computers.

The existence of quantum error-correcting codes was first established independently by Shor and Steane in the mid-nineties [1], [2]. CSS codes allowing to build a quantum code from two classical codes with the requirement that the dual of one should be contained in the other [3], [4] were introduced shortly after and then followed by quantum stabilizer codes [5], [6] which are in many ways analogous to classical linear codes. Since then, protecting quantum information has received considerable interest, and it was a long-standing open problem if asymptotically good quantum error-correcting codes, i.e., codes with a minimum distance growing linearly with the block length, could exist. This was largely due to the CSS restriction that made it difficult to directly extend classical asymptotically good code constructions. This was settled in a 2022 paper by Panteleev and Kalachev [7]. Subsequently, the construction in [7] was modified and improved in [8], resulting in a construction with an improved estimate of the minimum distance growth rate. Independently, very similar constructions have also answered a long-standing open question about locally testable classical codes [7], [9]. The recent renewed interest in quantum error correction has come due to recent progress in building intermediate-scale quantum computers with 300-1000 qubits, enough to make them close to performing some tasks faster than state-of-the-art classical computers [10].

In this work, we propose a construction of quantum Tanner codes that can loosely be described as follows. Take two regular[1] graphs on the same vertex set that commute, which can be thought of as their union having many four cycles. From the graphs, create a two-dimensional space of squares by filling in certain of these cycles. This type of space is fittingly called a square complex. The two diagonals in a square give rise to two new graphs, and by putting bits on the edges and parity-check constraints at the vertices of these graphs in a clever way, we obtain quantum codes. When applied to bipartite double covers of Cayley graphs, our construction gives the quantum Tanner codes from [8] (see Proposition 3), illustrated by the brown region of Fig. 1.

Our main technical result (Theorem 1) gives a necessary and sufficient condition such that if one starts with a graph where this clever assignment is possible, then the graph can always be viewed as the graph of diagonals of a square complex, and this complex can always be made from commuting graphs. This condition puts us in the green region of Fig. 1. Such an assignment may also be possible without this condition, in which case the resulting codes still could reasonably be called general quantum Tanner codes. Lemma 2 gives a condition for this, putting us in the red region of Fig. 1. There is potential to find other families of asymptotically good codes that do not come from Cayley graphs. The methods in [11] can likely be adapted if one, e.g., has a non-Cayley Ramanujan Schreier graph commuting with a Cayley graph of two Ramanujan components, as discussed in Section IV. The codes we look at there are part of the blue region of Fig. 1, a subset of our codes that are easier to work with (see Section III-B2).

All proofs are omitted due to lack of space.

A. Notation

Vectors are denoted by bold letters, matrices by sans serif uppercase letters, and sets (and groups) by calligraphic uppercase letters, e.g., $\boldsymbol{a}$, A, and $\mathcal{A}$, respectively. The neutral element of a group will be denoted by 1, while e is reserved for an edge in a graph. Linear codes, graphs, and square complexes are denoted by script uppercase letters, e.g., $\mathscr{C}$.

[1]A graph is called *regular* if all its vertices have the same degree.

A graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$ is denoted by $\mathscr{G} = (\mathcal{V}, \mathcal{E})$, and may have parallel edges and self-loops unless stated otherwise. The edges incident to a vertex v is called the local view of v and denoted $\mathcal{E}(v)$. We work with undirected graphs with an ordering on the edges, and view the graphs as directed graphs (digraphs) with twice as many edges as we see fit. The disjoint union of sets $\mathcal{A}, \mathcal{B}$ is denoted by $\mathcal{A} \sqcup \mathcal{B} \triangleq \{(a, 0), (b, 1) : a \in \mathcal{A}, b \in \mathcal{B}\}$. A linear code $\mathscr{C}$ of length n, dimension k, and minimum distance d is sometimes referred to by $[n, k, d]$, and its dual code is denoted $\mathscr{C}^{\perp}$. The binary field is denoted by $\mathbb{F}_2$, the identity matrix of size a by I_a, the all-zero matrix (of arbitrary size) by $\mathsf{0}$, and the transpose of a matrix by $(\cdot)^{\mathsf{T}}$. Standard order notation $\Theta(\cdot)$ is used for asymptotic results.

II. Preliminaries

We recall some background on particular types of graphs, their (spectral) expansion, definitions of classical and quantum error-correcting codes, and the notion of a square complex.

A. Graphs

Definition 1. *A labeling η on a digraph $(\mathcal{V}, \mathcal{E})$ by elements of $\mathcal{A}$ is a function $\eta : \mathcal{E} \to \mathcal{A}$. A digraph with a labeling is called a labeled digraph, and we say it is well labeled if for every vertex $v \in \mathcal{V}$ and label $a \in \mathcal{A}$ there is exactly one edge starting at v labeled by a and exactly one edge ending in v labeled by a.*

A labeling on the local views of an undirected graph is equivalent to a labeling on the corresponding digraph. An edge $v \overset{e}{\text{---}} w$ corresponds to a pair of directed edges $v \xrightarrow{\vec{e}} w$, $v \xleftarrow{\overleftarrow{e}} w$, and we use the convention that e has the label of $\vec{e}$ in the local view of v and the label of $\overleftarrow{e}$ in the local view of w. We write $s(\vec{e}) = v = t(\overleftarrow{e})$ and $t(\vec{e}) = w = s(\overleftarrow{e})$, where "$s$" and "$t$" indicate the source and target vertices of a directed edge, respectively. For bipartite graphs with vertex set $\mathcal{V} = \mathcal{V}_0 \sqcup \mathcal{V}_1$, we let $\vec{e}$ go from $\mathcal{V}_0$ to $\mathcal{V}_1$. Given a (undirected) graph $(\mathcal{V}, \mathcal{E})$, we write $\mathcal{E}^{\text{dir}}$ for the edges of the corresponding digraph.

Given a group $\mathcal{G}$, we will call a subset $\mathcal{A} \subseteq \mathcal{G}$ *symmetric* if $a^{-1} \in \mathcal{A}$ for all $a \in \mathcal{A}$.

Definition 2. *Given a group $\mathcal{G}$ and a symmetric subset $\mathcal{A} \subseteq \mathcal{G}$, the left Cayley graph $\mathrm{Cay}_{\mathrm{l}}(\mathcal{G}, \mathcal{A})$ is the regular graph with vertex set $\mathcal{G}$ and an edge (g, g') if $g' = ag$ for an $a \in \mathcal{A}$, in which case we label the edge by a and a^{-1} in the local view of g and g', respectively.*[2]

Right Cayley graphs $\mathrm{Cay}_{\mathrm{r}}(\mathcal{G}, \mathcal{A})$ are defined similarly.

Definition 3. *A group action of $\mathcal{G}$ on $\mathcal{V}$ is a function*

$$\varphi : \mathcal{G} \times \mathcal{V} \to \mathcal{V}$$

such that $\varphi(1, v) = v$ and $\varphi(g, \varphi(h, v)) = \varphi(\varphi(g, h), v)$ for all $v \in \mathcal{V}$ and $g, h \in \mathcal{G}$, where 1 is the neutral element of $\mathcal{G}$.

Definition 4. *Given a group $\mathcal{G}$ acting on a set $\mathcal{V}$ and a subset $\mathcal{A} \subseteq \mathcal{G}$, the Schreier digraph $\mathrm{Sch}(\mathcal{G}, \mathcal{V}, \mathcal{A})$ is the digraph with vertices $\mathcal{V}$ and an edge (v, v') labeled a whenever there is an $a \in \mathcal{A}$ mapping v to v' by the group action. A Schreier graph is a Schreier digraph with a choice of pairs $e : v \rightleftarrows w : e'$ such that every directed edge is part of exactly one pair. These pairs are the edges of the graph.*[2]

For symmetric $\mathcal{A}$, we will pair edges with inverse labels, as we do for Cayley graphs. Note that a directed edge can be paired with itself if it is a self-loop. It is known that all graphs can be given the structure of a Schreier graph where $\mathcal{A}$ is not necessarily symmetric.

Remark 1. *Schreier graphs are regular graphs where the local views are labeled such that the corresponding digraph is well-labeled. Cayley graphs are the Schreier graphs where the vertex set is the group $\mathcal{G}$. Both are labeled by the group elements $\mathcal{A} \subseteq \mathcal{G}$.*

By Cayley's theorem [12], the elements of any group can be viewed as permutations of a set, turning multiplication of elements in the group into composition of functions. Going the other way, a set of permutations on a set $\mathcal{V}$ will generate a group and define a directed Schreier graph of that group with vertex set $\mathcal{V}$. Concretely, we get a directed edge $v \to w$ labeled π if $\pi(v) = w$.

With a stricter definition of Schreier graphs, most regular graphs are still Schreier.

Proposition 1 ([13])**.** *All regular graphs of even degree can be given the structure of a Schreier graph with a symmetric labeling set. The same is true for graphs of odd degree precisely when they have a perfect matching.*[3]

We will look at group actions of products of groups, i.e., commuting group actions (see Remark 3), and the following will be useful.

Lemma 1. *Two permutations $\pi_1, \pi_2 : \mathcal{V} \to \mathcal{V}$ commute if and only if π_2 is a digraph homomorphism on the digraph $\mathscr{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ where $\mathcal{E}_1 = \{(v, \pi_1(v)) : v \in \mathcal{V}\}$, i.e., $(\pi_2(v), \pi_2(w)) \in \mathcal{E}_1$ whenever $(v, w) \in \mathcal{E}_1$.*

Given two Schreier graphs $\mathscr{G}_{\mathrm{A}} = (\mathcal{V}, \mathcal{E}_{\mathrm{A}})$ and $\mathscr{G}_{\mathrm{B}} = (\mathcal{V}, \mathcal{E}_{\mathrm{B}})$ on the same vertex set, we will say they *commute* if their defining permutations commute pairwise. That is, if $\mathscr{G}_{\mathrm{A}}$ and $\mathscr{G}_{\mathrm{B}}$ are labeled by η_{A} and η_{B} and we are given edges $v_0 \xleftarrow{\overleftarrow{e}_1} v_1 \xleftarrow{\overleftarrow{e}_2} v_2 \xrightarrow{\vec{e}_3} v_3 \xrightarrow{\vec{e}_4} v_4$ such that $e_1, e_3 \in \mathcal{E}_{\mathrm{A}}$, $e_2, e_4 \in \mathcal{E}_{\mathrm{B}}$, $\eta_{\mathrm{A}}(\overleftarrow{e}_1) = \eta_{\mathrm{A}}(\vec{e}_3)$, and $\eta_{\mathrm{B}}(\overleftarrow{e}_2) = \eta_{\mathrm{B}}(\vec{e}_4)$, then $v_0 = v_4$. We will say they have *overlapping edges* if there is a pair of vertices v, w such that both graphs have at least one edge between v and w.

B. Graph Expansion

By picking an order on the vertices of a graph $\mathscr{G} = (\mathcal{V}, \mathcal{E})$, we get an adjacency matrix $\mathsf{M}^{\mathscr{G}}$ where $\mathsf{M}^{\mathscr{G}}_{ij}$ is the number of edges from the j-th vertex to the i-th vertex.

Since $\mathsf{M}^{\mathscr{G}}$ is symmetric, it will have real eigenvalues $\lambda_1 \geq \cdots \geq \lambda_{|\mathcal{V}|}$, where $\lambda_1 = \Delta$ when $\mathscr{G}$ is Δ-regular, and $\lambda_{|\mathcal{V}|} = -\Delta$ if and only if it also is bipartite [14]. For $\mathscr{G}$ connected

[2] Cayley and Schreier graphs are often defined to have symmetric labeling sets, and so that they have no self-loops or parallel edges.

[3] A perfect matching is a set of edges where no edges share endpoints and all vertices are endpoints of edges in the set. We allow for self-loops in this set of edges.

and $|\mathcal{V}| > 2$, define $\lambda(\mathscr{G}) \triangleq \max\{|\lambda_i|: \lambda_i \neq \pm\Delta\}$. This is a measure of the (spectral) expansion of the graph, and the graph is called *Ramanujan* when $\lambda(\mathscr{G}) \leq 2\sqrt{\Delta - 1}$ [14].

C. Tanner Codes and Quantum CSS Codes

Tanner codes were introduced by Tanner in [15] and famously give asymptotically good families of classical codes. Loosely speaking, the construction takes a graph, puts bits on the edges of the graph, and assigns a code to each vertex. We will be using a regular graph with the same code on every vertex. A choice of bits is then in the Tanner code if, for any vertex, the bits on the edges connected to the vertex are in the code assigned to it. Formally, we use the following definition, where the restriction of a vector $\boldsymbol{c} \in \mathbb{F}_2^{|\mathcal{E}|}$ defined on the edges $\mathcal{E}$ of a graph to the local view of a vertex v is denoted $\boldsymbol{c}_v$. Note that we assume an ordering on $\mathcal{E}$ so that we may use $\mathbb{F}_2^{|\mathcal{E}|}$ instead of $\{\mathcal{E} \to \mathbb{F}_2\}$ as our vector space.

Definition 5. *Let $\mathscr{C}$ be a linear code of length Δ and $\mathscr{G} = (\mathcal{V}, \mathcal{E})$ be a Δ-regular graph, possibly with parallel edges but without self-loops. We define the Tanner code on $\mathscr{G}$ and $\mathscr{C}$ as* $\mathrm{Tan}(\mathscr{G}, \mathscr{C}) \triangleq \{\boldsymbol{c} \in \mathbb{F}_2^{|\mathcal{E}|} : \boldsymbol{c}_v \in \mathscr{C}$ *for all* $v \in \mathcal{V}\}$.

The definition assumes a well-labeling on $\mathscr{G}$. One may think of this as an order on each local view, where each order is independent of the other orderings.

For our main construction, the local code $\mathscr{C}$ will be the dual of a tensor product code.

Definition 6. *Given linear codes $\mathscr{C}_\mathrm{A}, \mathscr{C}_\mathrm{B}$ of length n_A and n_B, respectively, their tensor code $\mathscr{C}_\mathrm{A} \otimes \mathscr{C}_\mathrm{B}$ is defined as the set of $n_\mathrm{A} \times n_\mathrm{B}$ matrices with columns in $\mathscr{C}_\mathrm{A}$ and rows in $\mathscr{C}_\mathrm{B}$.*

If $\mathscr{C}_\mathrm{A}$ and $\mathscr{C}_\mathrm{B}$ have parameters $[n_\mathrm{A}, k_\mathrm{A}, d_\mathrm{A}]$ and $[n_\mathrm{B}, k_\mathrm{B}, d_\mathrm{B}]$, then $\mathscr{C}_\mathrm{A} \otimes \mathscr{C}_\mathrm{B}$ has parameters $[n_\mathrm{A}n_\mathrm{B}, k_\mathrm{A}k_\mathrm{B}, d_\mathrm{A}d_\mathrm{B}]$. The dual code $(\mathscr{C}_\mathrm{A} \otimes \mathscr{C}_\mathrm{B})^\perp$ is equal to $\mathscr{C}_\mathrm{A}^\perp \otimes \mathbb{F}_2^{n_\mathrm{B}} + \mathbb{F}_2^{n_\mathrm{A}} \otimes \mathscr{C}_\mathrm{B}^\perp$ and has minimum distance $\min(d_\mathrm{A}, d_\mathrm{B})$.

Definition 7. *We say the classical codes $\mathscr{C}_0$ and $\mathscr{C}_1$ form a CSS code when $\mathscr{C}_0^\perp \subseteq \mathscr{C}_1$.*

If the classical codes $\mathscr{C}_0$ and $\mathscr{C}_1$ have parity-check matrices H_0 and H_1, respectively, Definition 7 is equivalent to $\mathsf{H}_0\mathsf{H}_1^\intercal = 0$. CSS codes were introduced in [3], where they show that the classical codes $\mathscr{C}_0$ and $\mathscr{C}_1$ can be used to construct good quantum error-correcting codes.

The dimension k of a CSS code where the classical codes are of length n is $k = \dim(\mathscr{C}_0 \setminus \mathscr{C}_1^\perp) = \dim \mathscr{C}_0 + \dim \mathscr{C}_1 - n$, and the minimum distance d of the quantum CSS code can be given as the minimum of $d_\mathrm{X} = \min_{\boldsymbol{c} \in \mathscr{C}_0 \setminus \mathscr{C}_1^\perp} |\boldsymbol{c}|$ and $d_\mathrm{Z} = \min_{\boldsymbol{c} \in \mathscr{C}_1 \setminus \mathscr{C}_0^\perp} |\boldsymbol{c}|$.

A CSS code $(\mathscr{C}_0, \mathscr{C}_1)$ is called a *quantum LDPC* code when both codes $\mathscr{C}_0$ and $\mathscr{C}_1$ are defined by sparse parity-check matrices. For families of codes, we require that the columns and rows of the parity-check matrices have weight at most Δ, for some constant Δ independent of the code length n. A code family is called *asymptotically good* if it has parameters $[n, k = \Theta(n), d = \Theta(n)]$.

D. Square Complexes

We will need the notion of square complexes, normally defined as two-dimensional cube complexes, a particular type of a CW complex. We will use the following definition, which is equivalent for our purposes.

Definition 8. *A square complex $\mathcal{X} = (\mathcal{V}, \mathcal{E}, \mathcal{Q})$ is a triple of sets such that $(\mathcal{V}, \mathcal{E})$ is a graph and the elements of $\mathcal{Q}$ are of the form $((v_1, v_2), (v_1, v_3), (v_2, v_4), (v_3, v_4)) \in \mathcal{E}^{\times 4}$.*

III. Proposed Generalized Construction

We start by giving an example of commuting non-Cayley Schreier graphs. Then, we give a construction of quantum LDPC codes that generalizes the quantum Tanner codes of [8] and can take these Schreier graphs as input. We compare the two constructions and characterize the new one in three different ways.

A. Example

The Petersen graph, pictured to the left in Fig. 2, is known to be a non-Cayley graph, and can be considered as two 5-cycles joined in a certain way. Let

$$\mathsf{C}_5 = \begin{bmatrix} 0&1&0&0&1 \\ 1&0&1&0&0 \\ 0&1&0&1&0 \\ 0&0&1&0&1 \\ 1&0&0&1&0 \end{bmatrix} \quad \text{and} \quad \mathsf{C}_5' = \begin{bmatrix} 0&0&1&1&0 \\ 0&0&0&1&1 \\ 1&0&0&0&1 \\ 1&1&0&0&0 \\ 0&1&1&0&0 \end{bmatrix}$$

denote different adjacency matrices for a 5-cycle (vertices labeled by $1, 2, \ldots, 5$ for C_5 and vertices labeled by $1', \ldots, 5'$ for C_5', as in the left graph in Fig. 2). In a certain basis, the Petersen graph has the adjacency matrix

$$\mathsf{M}_\mathrm{A} = \left[\begin{array}{c|c} \mathsf{C}_5 & \mathsf{I}_5 \\ \hline \mathsf{I}_5 & \mathsf{C}_5' \end{array}\right], \text{ commuting with } \mathsf{M}_\mathrm{B} = \left[\begin{array}{c|c} \mathsf{C}_5 & 0 \\ \hline 0 & \mathsf{C}_5 \end{array}\right],$$

which is the adjacency matrix of the second graph depicted in Fig. 2, so the pair of graphs gives an example of a non-Cayley graph commuting with a 2-component graph when labeled as in the figure. By reordering the vertices, one can also write the adjacency matrices as

$$\mathsf{M}_\mathrm{A} = \left[\begin{array}{c|c} \mathsf{C}_5 & \mathsf{P} \\ \hline \mathsf{P}^\intercal & \mathsf{C}_5 \end{array}\right] \quad \text{and} \quad \mathsf{M}_\mathrm{B} = \left[\begin{array}{c|c} \mathsf{C}_5 & 0 \\ \hline 0 & \mathsf{C}_5' \end{array}\right],$$

for a certain permutation matrix P.

The two graphs have overlapping edges and different degrees. This is unwanted for our applications and may be remedied, for example, in the following way. First, add self-loops to all vertices of the second graph to make their degrees equal. Then, take two copies of the resulting graph, and use the bipartite double cover of the Petersen graph (the Desargues graph), as explained in Section III-C. If one in the end also wants both graphs to be bipartite on the same partition of vertices, one may take the bipartite double cover of both resulting graphs.

B. New Construction

1) General Case: Let $\mathscr{G}_\mathrm{A} = (\mathcal{V}, \mathcal{E}_\mathrm{A})$ and $\mathscr{G}_\mathrm{B} = (\mathcal{V}, \mathcal{E}_\mathrm{B})$ be (non-directed) commuting Δ-regular Schreier graphs with no overlapping edges and a chosen partition $\mathcal{V} = \mathcal{V}_0 \sqcup \mathcal{V}_1$ for which both graphs are bipartite. We treat the graphs as digraphs and call the labelings they have in virtue of being Schreier graphs $\eta_\mathrm{A} : \mathcal{E}_\mathrm{A}^\mathrm{dir} \to \mathcal{A}$ and $\eta_\mathrm{B} : \mathcal{E}_\mathrm{B}^\mathrm{dir} \to \mathcal{B}$. Furthermore, assume that if two vertices are connected by an edge in $\mathscr{G}_\mathrm{A}$, then the pairs $\{(\eta_\mathrm{B}(\vec{e}_i), \eta_\mathrm{B}(\overleftarrow{e}_i)) : 0 < i \leq \Delta\}$ are the same for all edges e_i, $0 < i \leq \Delta$, in the local views of

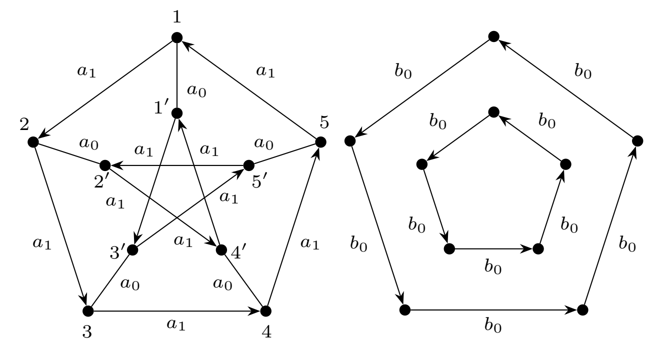

Fig. 2. The Petersen graph is shown to the left. On the right is a Schreier graph commuting with the Petersen graph. They are labeled by $\mathcal{A} = \{a_0, a_1, a_2\}$ and $\mathcal{B} = \{b_0, b_1\}$, respectively, where $a_0^{-1} = a_0$, $a_1^{-1} = a_2$, and $b_0^{-1} = b_1$.

the two vertices, and vice versa. This can loosely be thought of as the labels being locally invertible.

From the commuting graphs $\mathscr{G}_\mathrm{A}, \mathscr{G}_\mathrm{B}$, we may construct a square complex $\mathcal{X}$ with vertices $\mathcal{V}$, edges $\mathcal{E}_\mathrm{A} \cup \mathcal{E}_\mathrm{B}$, and for each $v \in \mathcal{V}_0$, $a \in \mathcal{A}$, and $b \in \mathcal{B}$, a square $(e_1, e_2, e_3, e_4) \in \mathcal{E}_\mathrm{A} \times \mathcal{E}_\mathrm{A} \times \mathcal{E}_\mathrm{B} \times \mathcal{E}_\mathrm{B}$ given by (1) below. We illustrate it by the square on the left when $t(\vec{e}_1) = (w, 1)$, $t(\vec{e}_3) = (w', 1)$, $s(\vec{e}_2) = (v', 0)$, $\eta_\mathrm{A}(\vec{e}_1) = a$, and $\eta_\mathrm{B}(\vec{e}_3) = b$.

$$\begin{array}{ccc} (w',1) & \xrightarrow[e_2]{a} & (v',0) \\ b\big\uparrow e_3 & & b\big\uparrow e_4 \\ (v,0) & \xrightarrow[e_1]{a} & (w,1) \end{array} \qquad \begin{aligned} &s(\vec{e}_1) = s(\vec{e}_3) = (v,0), \\ &\eta_\mathrm{A}(\vec{e}_1) = \eta_\mathrm{A}(\overleftarrow{e}_2), \quad s(\overleftarrow{e}_2) = t(\vec{e}_3), \\ &\eta_\mathrm{B}(\vec{e}_3) = \eta_\mathrm{B}(\overleftarrow{e}_4), \quad s(\overleftarrow{e}_4) = t(\vec{e}_1). \end{aligned} \tag{1}$$

The squares (e_1, e_2, e_3, e_4) and (e_2, e_1, e_4, e_3) are identified. We refer to $\mathcal{X}$ as the Schreier complex on $\mathscr{G}_\mathrm{A}$ and $\mathscr{G}_\mathrm{B}$, and denote its set of squares by $\mathcal{Q}$.

We define $\mathscr{G}_0^\square = (\mathcal{V}_0, \mathcal{E}_0^\mathcal{Q})$ as the Δ^2-regular graph with vertices $\mathcal{V}_0$ and an edge $(v, v') \in \mathcal{E}_0^\mathcal{Q}$ labeled by $(\eta_\mathrm{A}(\vec{e}_1), \eta_\mathrm{B}(\vec{e}_3)) = (a, b) \in \mathcal{A} \times \mathcal{B}$ in the local view of v and $(\eta_\mathrm{A}(\vec{e}_2), \eta_\mathrm{B}(\vec{e}_4))$ in the local view of v', for each square on the form (1). Similarly, we let $\mathscr{G}_1^\square = (\mathcal{V}_1, \mathcal{E}_1^\mathcal{Q})$ be the Δ^2-regular graph with vertices $\mathcal{V}_1$ and an edge (w, w') labeled $(\eta_\mathrm{A}(\overleftarrow{e}_1), \eta_\mathrm{B}(\overleftarrow{e}_4))$ in the local view of w and $(\eta_\mathrm{A}(\overleftarrow{e}_2), \eta_\mathrm{B}(\overleftarrow{e}_3))$ in the local view of w' for each square on the form (1).

Remark 2. *The graphs $\mathscr{G}_0^\square$ and $\mathscr{G}_1^\square$ are the subgraphs of the two bipartite halves of $\mathscr{G}_\mathrm{A} \cup \mathscr{G}_\mathrm{B}$ where all edges contain an edge from each graph $\mathscr{G}_\mathrm{A}, \mathscr{G}_\mathrm{B}$.*

Definition 9. *Given graphs as above and classical codes $\mathscr{C}_\mathrm{A}$, $\mathscr{C}_\mathrm{B}$ of length Δ, define $\mathscr{C}_0$ and $\mathscr{C}_1$ as the Tanner codes*

$$\mathscr{C}_0 = \mathrm{Tan}(\mathscr{G}_0^\square, (\mathscr{C}_\mathrm{A} \otimes \mathscr{C}_\mathrm{B})^\perp),\ \mathscr{C}_1 = \mathrm{Tan}(\mathscr{G}_1^\square, (\mathscr{C}_\mathrm{A}^\perp \otimes \mathscr{C}_\mathrm{B}^\perp)^\perp).$$

Proposition 2 below is proved similarly to the corresponding statement in [8].

Proposition 2. *The codes $\mathscr{C}_0$ and $\mathscr{C}_1$ form a CSS code which is also a quantum LDPC code.*

Remark 3. *Two Schreier graphs commute precisely when the group actions $\mathcal{G}_\mathrm{A} \times \mathcal{V} \to \mathcal{V}$ and $\mathcal{G}_\mathrm{B} \times \mathcal{V} \to \mathcal{V}$ defining them form a group action $\mathcal{G}_\mathrm{A} \times \mathcal{G}_\mathrm{B} \times \mathcal{V} \to \mathcal{V}$. Hence, without loss of generality, we can define our construction using a group action $\mathcal{G}_\mathrm{A} \times \mathcal{G}_\mathrm{B} \times \mathcal{V} \to \mathcal{V}$ and subsets $\mathcal{A} \subseteq \mathcal{G}_\mathrm{A}, \mathcal{B} \subseteq \mathcal{G}_\mathrm{B}$ instead of the commuting Schreier graphs* $\mathrm{Sch}(\mathcal{G}_\mathrm{A}, \mathcal{V}, \mathcal{A})$ *and* $\mathrm{Sch}(\mathcal{G}_\mathrm{B}, \mathcal{V}, \mathcal{B})$.

2) Symmetric Labeling Set: The construction used in Definition 9 can be somewhat simplified when the labeling sets are symmetric. In this case, the inverse of each label is well defined. When the graphs involved are not already bipartite (with respect to the same partition of vertices), we can make them so by using the bipartite double cover of the graphs, simplifying it further. In this case, $\mathscr{G}_0^\square = \mathscr{G}_1^\square$.

C. Connection With Previous Constructions

To create commuting graphs $\mathscr{G}_\mathrm{A}, \mathscr{G}_\mathrm{B}$, one may start with a group $\mathcal{G}$ and two symmetric subsets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{G}$. Then the Cayley graphs $\mathscr{G}_\mathrm{A} = \mathrm{Cay}_\mathrm{l}(\mathcal{G}, \mathcal{A})$ and $\mathscr{G}_\mathrm{B} = \mathrm{Cay}_\mathrm{r}(\mathcal{G}, \mathcal{B})$ will commute because group multiplication is associative. Our construction on the bipartite double covers of these graphs is equivalent to the approach used to create quantum Tanner codes so far [8].

Our assumption that the graphs $\mathscr{G}_\mathrm{A}, \mathscr{G}_\mathrm{B}$ have no overlapping edges plays the same role as the total no-conjugacy (TNC) condition for the quantum Tanner codes defined on groups, which states that $ag \neq gb$ for all $g \in \mathcal{G}, a \in \mathcal{A}, b \in \mathcal{B}$. It ensures that v and v' in (1) are different so that there are no self-loops in $\mathscr{G}_0^\square, \mathscr{G}_1^\square$. Many authors use what is often called "the quadripartite construction" to avoid dealing with the TNC condition.

In our setup, the quadripartite construction corresponds to the regular construction on two copies of one of the graphs and the bipartite double cover of the other. In other words, for graphs with adjacency matrices M_A and M_B, use the graphs with adjacency matrices

$$\left[\begin{array}{c|c} 0 & \mathsf{M}_\mathrm{A} \\ \hline \mathsf{M}_\mathrm{A} & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} \mathsf{M}_\mathrm{B} & 0 \\ \hline 0 & \mathsf{M}_\mathrm{B} \end{array}\right].$$

It can easily be seen that the two graphs still commute after this step when the obvious labeling is chosen. In the case of Cayley graphs, one may equivalently swap the group $\mathcal{G}$ for $\mathcal{G} \times \mathbb{F}_2$, and use $\mathcal{A}' = \{(a, 1) : a \in \mathcal{A}\}$ and $\mathcal{B}' = \{(b, 0) : b \in \mathcal{B}\}$. This means that using the quadripartite construction is quite restrictive when looking for concrete finite-length examples.

The following proposition should be clear when comparing our construction with the one from [8].

Proposition 3. *Let $\mathcal{G}$ be a group with generating symmetric subsets $\mathcal{A}$ and $\mathcal{B}$ of size Δ satisfying the TNC condition and not containing the neutral element, and let $\mathscr{C}_\mathrm{A}, \mathscr{C}_\mathrm{B}$ be codes of length Δ. Then, our construction applied to the bipartite double covers of the graphs* $\mathrm{Cay}_\mathrm{l}(\mathcal{G}, \mathcal{A}), \mathrm{Cay}_\mathrm{r}(\mathcal{G}, \mathcal{B})$ *and the codes $\mathscr{C}_\mathrm{A}, \mathscr{C}_\mathrm{B}$ gives the same CSS code as the construction from [8] applied to $\mathcal{G}, \mathcal{A}, \mathcal{B}, \mathscr{C}_\mathrm{A}, \mathscr{C}_\mathrm{B}$.*

From Proposition 1, most regular graphs can be used to construct quantum Tanner codes. However, to have freedom when choosing the other graph, the automorphism group of the graph should be large. Moving away from Cayley graphs means getting a smaller automorphism group, see Section IV.

D. Equivalent Characterizations

It is natural to ask when a square complex can give CSS codes the way left-right Cayley complexes and our square complexes described in Section III-C do, namely, by changing which diagonal of the squares that determine their endpoints when viewed as edges. We now turn to prove that these are precisely the square complexes that can be made from two commuting Schreier graphs (see Corollary 1). Along the way, we present another view of quantum Tanner codes (Lemma 2), and show how our construction fits in (Theorem 1).

In Lemma 2, we consider Δ^2-regular Schreier graphs. The local views are viewed as $\Delta \times \Delta$-matrices so that the Tanner codes and the rows and columns are well defined.

Lemma 2. *Let $\mathscr{G}_0 = (\mathcal{V}_0, \mathcal{E}_0)$ and $\mathscr{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ be Δ^2-regular Schreier graphs such that $|\mathcal{V}_0| = |\mathcal{V}_1|$, and let $\psi : \mathcal{E}_0 \to \mathcal{E}_1$ be the bijection given by the order on the edges. Then, (i) and (ii) are equivalent.*

(i) The Tanner codes $\mathscr{C}_0 = \mathrm{Tan}(\mathscr{G}_0, (\mathscr{C}_\mathrm{A} \otimes \mathscr{C}_\mathrm{B})^\perp)$ and $\mathscr{C}_1 = \mathrm{Tan}(\mathscr{G}_1, (\mathscr{C}_\mathrm{A}^\perp \otimes \mathscr{C}_\mathrm{B}^\perp)^\perp)$ form a CSS code for all classical codes $\mathscr{C}_\mathrm{A}, \mathscr{C}_\mathrm{B}$ of length Δ.

(ii) For any vertices $v \in \mathcal{V}_0, w \in \mathcal{V}_1$, either $\psi(\mathcal{E}_0(v)) \cap \mathcal{E}_1(w) = \emptyset$, or the intersection forms one or more rows or columns in $\mathcal{E}_0(v)$ and $\mathcal{E}_1(w)$ such that each row (column) is mapped onto a row (column) by ψ, preserving the order inside the row (column).

We find it reasonable to call any codes $\mathscr{C}_0, \mathscr{C}_1$ constructed as in (i) *general quantum Tanner codes*, so these codes fit in the red region of Fig. 1. Note that Δ^m-regular graphs with m local codes $\mathscr{C}_{\mathrm{A}_1}, \ldots, \mathscr{C}_{\mathrm{A}_m}$ also are of interest and could share this name. However, we restrict ourselves to the case $m = 2$.

Theorem 1 gives a condition that restricts the red region of Fig. 1 to the green region.

Theorem 1. *Let $\mathscr{G}_0$, $\mathscr{G}_1$, and ψ be as in Lemma 2. Then, (i) and (ii) of Lemma 2 are equivalent to (iii) below if and only if for each of the two labels of an edge ψ fixes, the labels share one index in the labeling set $\Delta \times \Delta$.*

(iii) There exist Schreier graphs $\mathscr{G}_\mathrm{A}, \mathscr{G}_\mathrm{B}$ such that $\mathscr{G}_i^\square = \mathscr{G}_i$, $i = 0, 1$.

Corollary 1. *All square complexes that give CSS codes using the construction from Section III-B, can be constructed from a pair of commuting Schreier graphs as in Section III-B.*

IV. Asymptotically Good Quantum Codes

In this section, we discuss how our proposed construction might be used to create new asymptotically good codes. The discussion assumes that the graphs are not already bipartite, and will be made so by taking their bipartite double cover. We start by stating Proposition 4 below, which gives an obstruction for when a pair of commuting graphs can be non-Cayley.

Proposition 4. *If $\mathscr{G}_\mathrm{A}$ and $\mathscr{G}_\mathrm{B}$ commute, then they are either Cayley graphs, or one of them has more than one component.*

At first glance, it might seem like Proposition 4 tells us there is no hope of finding asymptotically good quantum codes using the methods from [11]. After all, $\lambda(\mathscr{G}) = \Delta$ for a Δ-regular graph $\mathscr{G}$ with more than one component, which is as large as it can get. However, as already seen in Section III-C, we can get good codes even in this case, as one of the graphs will have two components when using the quadripartite construction. This stems from the fact that $\mathscr{G}_0^\square, \mathscr{G}_1^\square$ and the components of $\mathscr{G}_\mathrm{A}$ and $\mathscr{G}_\mathrm{B}$ may have a small λ.

Since the adjacency matrices M_A and M_B are symmetric and commute, they are simultaneously diagonalizable. Therefore, $\mathsf{M}_\mathrm{A} + \mathsf{M}_\mathrm{B}$ and $\mathsf{M}_\mathrm{A}\mathsf{M}_\mathrm{B}$, which are the adjacency matrices of respectively $(\mathcal{V}, \mathcal{E}_\mathrm{A} \cup \mathcal{E}_\mathrm{B})$ and both $\mathscr{G}_0^\square$ and $\mathscr{G}_1^\square$, have eigenvalues the sums and products, respectively, of the eigenvalues of M_A and M_B.

We know that the all-ones vector $\boldsymbol{u}$ will correspond to $\lambda_1 = \Delta$ for any regular graph, and for a graph with two components commuting with a connected graph, the other eigenvector corresponding to this eigenvalue that is also an eigenvector for the other graph, will have to be $(\boldsymbol{u}, -\boldsymbol{u})$.

Let $\mathsf{M}_\mathrm{A} = \left[\begin{array}{c|c} \mathsf{A}_1 & \mathsf{A}_2 \\ \hline \mathsf{A}_2^\intercal & \mathsf{A}_3 \end{array}\right]$ and $\mathsf{M}_\mathrm{B} = \left[\begin{array}{c|c} \mathsf{B}_1 & 0 \\ \hline 0 & \mathsf{B}_2 \end{array}\right]$ be the adjacency matrices of $\mathscr{G}_\mathrm{A}$ and $\mathscr{G}_\mathrm{B}$, respectively. We demand that $\mathsf{M}_\mathrm{A}\mathsf{M}_\mathrm{B} = \mathsf{M}_\mathrm{B}\mathsf{M}_\mathrm{A}$, which means that the product is a symmetric matrix. These two products are

$$\left[\begin{array}{c|c} \mathsf{A}_1\mathsf{B}_1 & \mathsf{A}_2\mathsf{B}_2 \\ \hline \mathsf{A}_2^\intercal\mathsf{B}_1 & \mathsf{A}_3\mathsf{B}_2 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} \mathsf{B}_1\mathsf{A}_1 & \mathsf{B}_1\mathsf{A}_2 \\ \hline \mathsf{B}_2\mathsf{A}_2^\intercal & \mathsf{B}_2\mathsf{A}_3 \end{array}\right],$$

which are equal if and only if we have the relations $\mathsf{A}_2\mathsf{B}_2 = \mathsf{B}_1\mathsf{A}_2$, $\mathsf{A}_1\mathsf{B}_1 = \mathsf{B}_1\mathsf{A}_1$, and $\mathsf{A}_3\mathsf{B}_2 = \mathsf{B}_2\mathsf{A}_3$. Assuming that B_1, B_2, A_1, and A_3 correspond to connected graphs, Proposition 4 now tells us they all have to be Cayley graphs. Lemma 1 tells us that A_2 either is the 0 matrix or consists of graph isomorphisms between the two components of $\mathscr{G}_\mathrm{B}$, so the two components of $\mathscr{G}_\mathrm{B}$ are equal up to a rearrangement of the vertices since we assume $\mathscr{G}_\mathrm{A}$ is connected. All this fits what we saw in the example of Section III-A.

If we assume that the blocks of M_A come from distinct generators and let a be the regularity of the graph corresponding to A_1, then the eigenvalue of M_A corresponding to $(\boldsymbol{u}, -\boldsymbol{u})$ is $2a - \Delta$. So, when the weights of the rows in A_1 and A_2 are equal, then $\lambda(\mathscr{G}_i^\square) = 2\sqrt{\Delta - 1}$ for $i = 0, 1$ when $\mathscr{G}_\mathrm{A}$ and the two components of $\mathscr{G}_\mathrm{B}$ are Ramananujan graphs, because $\lambda_2 = \Delta$ for $\mathscr{G}_\mathrm{B}$ is multiplied with 0.

We end with pointing at a possible way to create a connected Schreier graph $\mathscr{G}_\mathrm{A}$ that commutes with a Cayley graph $\mathscr{G}_\mathrm{B}$ with two components. Let $\mathscr{G}_\mathrm{A}$ be a Cayley graph on the above form, and let P be a permutation matrix of the same size as A_3. If $\mathsf{P}\mathsf{A}_3\mathsf{P}^\intercal$ commutes with B_2, then the matrix

$$\left[\begin{array}{c|c} \mathsf{A}_1 & \mathsf{A}_2 \\ \hline \mathsf{A}_2^\intercal & \mathsf{P}\mathsf{A}_3\mathsf{P}^\intercal \end{array}\right] \tag{2}$$

will still commute with M_B. We can ensure this by letting P be given by an isomorphism between two isomorphic Cayley graphs on different generating sets. For example, given a Cayley graph $\mathrm{Cay}(\mathscr{G}, \mathcal{B})$ and an automorphism σ on $\mathscr{G}$, then σ also is an isomorphism between $\mathrm{Cay}(\mathcal{G}, \mathcal{B})$ and $\mathrm{Cay}(\mathcal{G}, \mathcal{B}')$, where $\mathcal{B}' = \{\sigma b \sigma^{-1} : b \in \mathcal{B}\}$. A clever choice of σ should make (2) the adjacency matrix of a non-Cayley Schreier graph.

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