On the Smooth Curve of Entire Vector Fields that Solves the Navier-Stokes Equation

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May 17, 2024

Abstract

In this paper we prove the existence and smoothness of the Navier-Stokes Equation for viscosity large enough which after rescaling implies a solution for any positive viscosity, additionally, we show the existence of a curve of entire vector fields of order 2 that extends the solution to the complex domain for positive time.

Key words: Navier-Stokes Equation, Riezs convolution, spaces of functions that decrease fast, spaces of functions dominated by Fourier Caloric functions, global solution.

Introduction

Fluid dynamics, the study of how fluids move and interact with their surroundings, encompasses a vast array of phenomena observed in nature and engineering applications. From the graceful flow of a river to the turbulent swirls in a cup of coffee, understanding the behavior of fluids is essential in fields ranging from aerospace engineering to climate science.

At the heart of this discipline lies the Navier-Stokes Equation, a set of partial differential equations that govern the motion of fluids. Named after Claude-Louis Navier and George Gabriel Stokes, who independently contributed to its formulation in the 19th century, this equation encapsulates the fundamental principles underlying fluid motion, including conservation of mass and momentum.

Despite its seemingly straightforward appearance, the Navier-Stokes Equation conceals a wealth of complexity. Its solutions exhibit a rich variety of behaviors, from laminar flow patterns characterized by smooth, orderly motion, to turbulent regimes marked by chaotic fluctuations and eddies. Understanding and predicting these phenomena have been among the central challenges in fluid dynamics, with profound implications for fields as diverse as weather forecasting, aircraft design, and biomedical engineering.

In this paper we stick to solve the Navier-Stokes Equation for dimension $d \geq 3$ stated in [8] since the results for dimension $d = 2$ are well known (See [24]). A fundamental difference between the Navier-Stokes Equation and Euler Equation (the case $\nu = 0$) is that in the latter the existence of solutions with finite blow up time $T > 0$ implies that the norm $L^{\infty,1}(\mathbb{R}^3 \times [0, T))$ of the vorticity $\omega(x, t) = curl_x u(x, t)$ is infinite ([19]). However, in the case $\nu > 0$ we have a solution extended without blow up time, in other words $T = \infty$.

In [15], Leray showed the existence of weak solutions of the Navier-Stokes Equation. The partial regularity Theory of the Navier-Stokes Equation started with Scheffer [23] and also for suitable solutions in the work of Caffarelli-Kohn-Nirenberg [4] . A direct and simplified proof of the main result of [4] can be found in the work of Lin [18].

The plan of this article is as follows: In Section 1 we fix ideas about convenient notation and useful results. In Section 2 we considered a well-known transformation in order to reduce the proof of the existence of a smooth solution of the Navier-Stokes Equation to the case of a special viscosity $\nu > 0$. In Section 3 we study spaces of functions decreasing fast which are fundamental in order to construct the smooth solution for large viscosity $\nu > 0$. In Section 4 we study a kind of generalization of the convolution that we call Riezs convolution and its properties with respect to functions decreasing fast and Lebesgue spaces. In Section 5 we study a remarkable class of spaces of functions decreasing fast involving time that are dominated by Fourier Caloric functions and their properties with respect to operations such as convolution and the product associated to the Navier-Stokes Equation that we denote by $\odot$. In Section 6 we construct the solution for viscosity large enough using the results for spaces dominated by Fourier Caloric functions decreasing fast. In Section 7 we show that for positive time we can extend the solution to the complex domain obtaining a curve of entire vector fields of order 2.

1 Notation and Preliminary Results

In this Section we clarify the notation that we use throughout the paper.

1.1 Notation

For a field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ we denote:

$$\mathbb{K}_{+}^{d+1} = \mathbb{K}^{d} \times [0, \infty) \text{ and } \mathbb{K}_{>0}^{d+1} = \mathbb{K}^{d} \times (0, \infty).$$

Let X be a vector space and $\|\cdot\|_1, \|\cdot\|_2$ be two norms over X. We define the norm $\|\cdot\|_{1\oplus 2}$,

$$\|x\|_{1\oplus 2} = \|x\|_1 + \|x\|_2, \text{ for } x \in X.$$

Note that we can take Lebesgue spaces of two degrees p_0 and p_1 and $X = L^{p_0}(\mathbb{R}^d) \cap L^{p_1}(\mathbb{R}^d)$ we can consider the norm $\|\cdot\|_{p_0 \oplus p_1}$ on X, we denote this space by $L^{p_0 \oplus p_1}(\mathbb{R}^d)$.

For $1 \leq p \leq \infty$ we denote the conjugate exponent by $p^{'}$, i.e., $p^{'} = \frac{p}{p-1}$. We call the map $p \longmapsto p'$ the conjugate function.

For $\xi, \eta \in \mathbb{R}^d$ we define the tensor product $\xi \otimes \eta = \xi \eta^T$.

Let $\alpha > 0$, we say that $f : \mathbb{R}_{+}^{d+1} \to \mathbb{C}$ is a Fourier Caloric function if:

$$f(\xi, t) = e^{-\lambda t |\xi|^{\alpha}} f^0(\xi), \tag{1}$$

for $(\xi, t) \in \mathbb{R}_{+}^{d+1}$, for some $\lambda > 0$ and $f^0 : \mathbb{R}_{+}^{d} \to \mathbb{C}$. This name is motivated because the Fourier transform $\widehat{f}(\cdot, t)$ of a Fourier Caloric function f is a solution of the fractional heat equation

$$\frac{\partial u}{\partial t} = \lambda \Delta^{\frac{\alpha}{2}} u.$$

with initial condition $\widehat{f^0}$.

Let $(A, \cdot)$ be a nonassociative algebra. For elements $a_1, \cdots a_k \in A$ we denote by $w^{\cdot}(a_1, \cdots, a_k)$ an arbitrary monomial of degree k. If $a_1 = \cdots = a_k$ we write $w_k^{\cdot}(a) = w^{\cdot}(a_1, \cdots, a_k)$.

Remark 1. Note that we emphasize the product $\cdot$ in the definition of the monomials since we can have more than one product acting on the same set and even at the same time.

Furthermore, $w^{\cdot}(a_1, \cdots, a_k)$ denote the order in which each variable appears. For example, the notation for the monomial $a \cdot b$ is of the form $w^{\cdot}(a, b)$. We denote by $M(a_1, \cdots, a_k)$ the set of nonassociative monomials of degree k in $a_1, \cdots, a_k$.

1.2 Preliminary Results

In this Subsection we propose some useful results that will be used later.

Proposition 1. *Let $\alpha > 0$, for every $\xi, \eta \in \mathbb{R}^d$ we have:*

$$-|\xi - \eta|^\alpha - |\eta|^\alpha \leq -r_\alpha |\xi|^\alpha ,$$

for some $r_\alpha \leq 1$.

Proof. By Theorem 15 we have that

$$(s+t)^\alpha \leq \max\left\{2^{\alpha-1}, 1\right\}(s^\alpha + t^\alpha),$$

for every $s \geq 0$, $t \geq 0$. Therefore, for every $\xi, \eta \in \mathbb{R}^d$:

$$|\xi|^\alpha \leq (|\xi - \eta| + |\eta|)^\alpha \leq \max\left\{2^{\alpha-1}, 1\right\}(|\xi - \eta|^\alpha + |\eta^\alpha|).$$

Thus, we have the contention by defining $r_\alpha = \frac{1}{\max\left\{2^{\alpha-1},1\right\}}$. □

Proposition 2. *Let X be an inner product space. For every $z \in X$ we have:*

$$\max_{x+y=z, x,y\in X} \langle x, y\rangle = \frac{\|z\|^2}{4}.$$

Proof. Let $x, y \in X$, $z = x + y$, then the polarization identity implies that:

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) \leq \frac{\|x+y\|^2}{4} = \frac{\|z\|^2}{4}.$$

Therefore, $\max_{x+y=z,x,y\in X}\langle x, y\rangle \leq \frac{\|z\|^2}{4}$. However, taking $x = y = \frac{z}{2}$ we obtain:

$$\max_{x+y=z, x,y\in X} \langle x, y\rangle \geq \left\langle \frac{z}{2}, \frac{z}{2}\right\rangle = \frac{\|z\|^2}{4}.$$

□

Additionally, we state a useful result about the iteration of a linear operator.

Lemma 1. *Let X be a vector space, $L : X \to X$ be a linear operator, $x, y \in X$, $\alpha \in \mathbb{C}$ such that:*

$$Lx = \alpha x + y.$$

Then, for every $n \in \mathbb{N}$ we have:

$$L^n x = \alpha^n x + \sum_{j=0}^{n-1} \alpha^j L^{n-1-j} y.$$

Proof. The proof is by induction over n.
For $n = 1$ we have

$$Lx = \alpha x + \sum_{j=0}^{0} \alpha^j L^{1-1-j} y = \alpha x + y,$$

so it is valid for $n = 1$.

Assume for n and note that:

$$
\begin{aligned}
L^{n+1}x &= L(L^n x) = L\left(\alpha^n x + \sum_{j=0}^{n-1} \alpha^j L^{n-1-j} y\right) \\
&= \alpha^n Lx + \sum_{j=0}^{n-1} \alpha^j L^{n-j} y \\
&= \alpha^n (\alpha x + y) + \sum_{j=0}^{n-1} \alpha^j L^{n-j} y \\
&= \alpha^{n+1} x + \alpha^n y + \sum_{j=0}^{n-1} \alpha^j L^{n-j} y \\
&= \alpha^{n+1} x + \sum_{j=0}^{n} \alpha^j L^{n-j} y.
\end{aligned}
$$

Therefore, it is true for $n+1$. The result follows by induction. □

2 Reduction of the Problem to Special Viscosities

Let us remind that $(u,p) : \mathbb{R}_+^{d+1} \to \mathbb{R}^{d+1}$ is a solution of the Navier-Stokes equation if

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} u = \nu \Delta u - \nabla p + f, \\
&div(u) = 0, \\
&u(x,0) = u^0(x),
\end{aligned}
$$

for $(u,p) \in C^\infty(\mathbb{R}_+^{d+1}, \mathbb{R}^{d+1})$ such that:

$$
\sup_{t \geq 0} \int_{\mathbb{R}^d} |u(x,t)|^2 \, dx < \infty.
$$

Here, $\nu > 0$, $f : \mathbb{R}_+^{d+1} \to \mathbb{R}^d$, $u^0 : \mathbb{R}^d \to \mathbb{R}^d$ are smooth functions that decrease fast, i.e., $f \in S(\mathbb{R}_+^{d+1})^d$ and $u^0 \in S(\mathbb{R}^d)^d$. For simplicity we will consider the homogeneous case in which $f = 0$.

Now we show that there is a solution for $\nu > 0$ if and only if there is a solution for $\alpha\nu > 0$ for every $\alpha > 0$.

Let us consider the function $(v,q) : \mathbb{R}_+^{d+1} \to \mathbb{R}^{d+1}$ given by

$$
v(x,t) = u\left(\frac{x}{\alpha}, \frac{t}{\alpha}\right), q(x,t) = p\left(\frac{x}{\alpha}, \frac{t}{\alpha}\right),
$$

for some $\alpha > 0$ arbitrary.

Then, $\frac{\partial v}{\partial t}(x,t) = \frac{1}{\alpha}\frac{\partial u}{\partial t}\left(\frac{x}{\alpha}, \frac{t}{\alpha}\right)$, $\frac{\partial v}{\partial x}(x,t) = \frac{1}{\alpha}\frac{\partial u}{\partial x}(\frac{x}{\alpha}, \frac{t}{\alpha})$, $\Delta V(x,t) = \frac{1}{\alpha^2}\Delta u(\frac{x}{\alpha}, \frac{t}{\alpha})$, $\nabla q(x,t) = \frac{1}{\alpha}\nabla p(\frac{x}{\alpha}, \frac{t}{\alpha})$.

Therefore,

$$\begin{aligned}
&\frac{\partial v}{\partial t}(x,t)+\frac{\partial v}{\partial x}(x,t)v(x,t)=\frac{1}{\alpha}\frac{\partial u}{\partial t}\left(\frac{x}{\alpha},\frac{t}{\alpha}\right)+\frac{1}{\alpha}\frac{\partial u}{\partial x}\left(\frac{x}{\alpha},\frac{t}{\alpha}\right)u\left(\frac{x}{\alpha},\frac{t}{\alpha}\right)\\
&=\frac{1}{\alpha}\left(\frac{\partial u}{\partial t}\left(\frac{x}{\alpha},\frac{t}{\alpha}\right)+\frac{\partial u}{\partial x}\left(\frac{x}{\alpha},\frac{t}{\alpha}\right)u\left(\frac{x}{\alpha},\frac{t}{\alpha}\right)\right)\\
&=\frac{1}{\alpha}\left(\nu\Delta u\left(\frac{x}{\alpha},\frac{t}{\alpha}\right)-\nabla p\left(\frac{x}{\alpha},\frac{t}{\alpha}\right)\right)\\
&=\alpha\nu\left(\frac{1}{\alpha^2}\Delta u\left(\frac{x}{\alpha},\frac{t}{\alpha}\right)\right)-\frac{1}{\alpha}\nabla p\left(\frac{x}{\alpha},\frac{t}{\alpha}\right)\\
&=\alpha\nu\Delta v(x,t)-\nabla q(x,t).
\end{aligned}$$

Furthermore,

$$div(v(x,t))=Tr\left(\frac{\partial v}{\partial x}(x,t)\right)=\frac{\nu}{\alpha}Tr\left(\frac{\partial u}{\partial x}\left(\frac{x}{\alpha},\frac{t}{\alpha}\right)\right)=\frac{\nu}{\alpha}div(u)\left(\frac{x}{\alpha},\frac{t}{\alpha}\right)=0,$$

for $(x,t)\in\mathbb{R}_+^{d+1}$.

On the other hand, we have bounded energy,

$$\int_{\mathbb{R}^d}|v(x,t)|^2\,dx=\int_{\mathbb{R}^d}\left|u\left(\frac{x}{\alpha},\frac{t}{\alpha}\right)\right|^2dx=\alpha^{2d}\int_{\mathbb{R}^d}\left|u\left(y,\frac{t}{\alpha}\right)\right|^2dy.$$

In particular,

$$\sup_{t\geq 0}\int_{\mathbb{R}^d}|v(x,t)|^2\,dx=\sup_{t\geq 0}\alpha^{2d}\int_{\mathbb{R}^d}\left|u\left(x,\frac{t}{\alpha}\right)\right|^2dx=\alpha^{2d}\sup_{t\geq 0}\int_{\mathbb{R}^d}|u(x,t)|^2\,dx<\infty.$$

Additionally, the initial condition is:

$$v^0(x)=v(x,0)=u\left(\frac{x}{\alpha},0\right)=u^0\left(\frac{x}{\alpha}\right).$$

Since $v^0\in S(\mathbb{R}^d)^d$ if $u\in S(\mathbb{R}^d)^d$ it is enough to solve the Navier-Stokes Equation for an arbitrary or sufficiently large viscosity $\nu>0$.

In conclusion, if we have solved the Navier-Stokes Equation:

$$\begin{aligned}
&\frac{\partial v}{\partial t}+\frac{\partial v}{\partial x}v=\alpha\nu\Delta v-\nabla q,\\
&div(v)=0,\\
&v(x,0)=v^0(x),\\
&\sup_{t\geq 0}\int_{\mathbb{R}^d}|v(x,t)|^2\,dx<\infty,
\end{aligned}$$

and we set the initial condition to be $u^0(x)=v^0(\alpha x)$ then the solution of

$$
\begin{aligned}
&\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}u=\nu\Delta u-\nabla p,\\
&div(u)=0,\\
&u(x,0)=u^0(x),\\
&\sup_{t\geq 0}\int_{\mathbb{R}^d}|u(x,t)|^2\,dx<\infty
\end{aligned}
$$

is given by $u(x,t)=v(\alpha x,\alpha t), p(x,t)=q(\alpha x,\alpha t)$.

3 Space of Functions Decreasing Fast

In this Section we study spaces of functions that have good behaviour at infinity. For a function $\phi:\mathbb{R}^d\to\mathbb{R}$ we consider the multiplication map $(M\phi)(\xi)=(1+|\xi|^2)\phi(\xi)$, with this we can state the following:

Definition 1. Let $(\mathcal{B},\|\cdot\|)$ a Banach space of functions $\phi:\mathbb{R}^d\to\mathbb{R}$. We define the space of functions decreasing fast associated to $\mathcal{B}$ as:

$$\mathcal{E}_{\mathcal{B}}=\left\{\phi\in\mathcal{B}\mid M^n(\phi)\in\mathcal{B},\forall n\in\mathbb{N}\right\}.$$

We provide the space $\mathcal{E}_{\mathcal{B}}$ with the topology given by the family of norms:

$$p_n(\phi)=\|M^n(\phi)\|,$$

for all $n\in\mathbb{N}$. With this topology we have that $\mathcal{E}_{\mathcal{B}}$ is a Frechet space. Furthermore, we have the multiplication map:

$$M^n:\mathcal{E}\to\mathcal{E},(M\phi)(\xi)=(1+|\xi|^2)^n\phi(\xi).$$

It is continuous, since $p_j(M^n\phi)=p_{n+j}(\phi)$ for every $\phi\in\mathcal{E}$, $j\in\mathbb{N}$.

We denote this Frechet space simply by $\mathcal{E}$ when there is no way to confusion and by $\mathcal{E}^+=\{\phi\in\mathcal{E}\mid\phi\geq 0\}$.

Remark 2. We can consider more generally Banach spaces of vector fields $\phi:\mathbb{R}^d\to\mathbb{R}^e$ and the definition of $\mathcal{E}_{\mathcal{B}}$ applies. However, we explore only scalar fields $\phi:\mathbb{R}^d\to\mathbb{R}$ in this Section.

We have an interesting algebraic results.

Lemma 2. *If $(\mathcal{B},+,\cdot)$ is a Banach algebra with the usual pointwise sum and product of functions:*

$$(\phi+\psi)(\xi)=\phi(\xi)+\psi(\xi),\ (\phi\psi)(\xi)=\phi(\xi)\psi(\xi).$$

then $\mathcal{E}$ is an ideal of $\mathcal{B}$.

Proof. Let $\phi,\psi\in\mathcal{E}$, since $M(\phi+\psi)=M\phi+M\psi$ and $M(\phi\psi)=(M\phi)\psi=\phi(M\psi)$ we have that

$$
\begin{aligned}
&p_n(\phi+\psi)=\|M^n(\phi+\psi)\|\leq\|M^n(\phi)+M^n(\psi)\|\\
&\leq\|M^n(\phi)\|+\|M^n(\psi)\|=p_n(\phi)+p_n(\psi).
\end{aligned}
$$

On the other hand, if $\phi\in\mathcal{E}$, $\psi\in\mathcal{B}$:

$$
\begin{aligned}
&p_n(\phi\psi)=\|M^n(\phi\psi)\|\leq\|M^n(\phi)\psi\|\\
&\leq\|M^n(\phi)\|\,\|\psi\|=p_n(\phi)\,\|\psi\|.
\end{aligned}
$$

Therefore, $\phi\psi\in\mathcal{E}$. We conclude that $\mathcal{E}\triangleleft\mathcal{B}$. $\square$

We are interested in the convolution operation:

$$(\phi * \psi)(\xi) = \int_{\mathbb{R}^d} \phi(\xi - \eta)\psi(\eta) d\eta,$$

for $\phi, \psi \in \mathcal{B}$.

We study the continuity of the convolution operation, so we consider the following result:

Lemma 3. *For every, $\xi, \eta \in \mathbb{R}^d$ we have:*

$$1 + |\xi|^2 \leq 2(1 + |\xi - \eta|^2)(1 + |\eta|^2),$$

and

$$|\xi|^2 \leq 2(|\xi - \eta|^2 + |\eta|^2).$$

Proof. It is enough to make $\alpha = 2$ in Corollaries 27 and 28 to obtain this result. □

With this we can state the following result:

Theorem 1. *Assume that $(\mathcal{B}, +, *)$ is a Banach algebra and with a monotone norm, i.e., $|\phi| \leq |\psi|$ implies $\|\phi\| \leq \|\psi\|$ and $\|\phi\| = \||\phi|\|$ then $\phi * \psi \in \mathcal{E}$ if $\phi, \psi \in \mathcal{E}$.*

Proof. Let $\phi, \psi \in \mathcal{E}$, note that Lemma 3 we have $M^n(|\phi * \psi|) \leq 2^n(|M^n(\phi)| * |M^n(\psi)|)$, since $\|\cdot\|$ is monotone:

$$p_n(\phi * \psi) = \|M^n(|\phi * \psi|)\| \leq 2^n \||M^n(\phi)| * |M^n(\psi)|\| \leq 2^n \||M^n(\phi)|\| \||M^n(\psi)|\| = 2^n p_n(\phi) p_n(\psi),$$

for every $n \in \mathbb{N}$. Therefore, $\phi * \psi \in \mathcal{E}$. □

Corollary 1. *Assume that $(\mathcal{B}, +, *)$ is a Banach algebra and with a monotone norm, i.e., $|\phi| \leq |\psi|$ implies $\|\phi\| \leq \|\psi\|$ then $(\mathcal{E}, +, *)$ is a subalgebra de Banach of $(\mathcal{B}, +, *)$.*

In the next Subsections we study the case in which $\mathcal{B} = L^\infty(\mathbb{R}^d)$ and spaces related to it with singularities at the origin.

3.1 Space of Functions Decreasing Fast Associated to $L^\infty(\mathbb{R}^d)$

Let us consider the space of functions that decrease fast associated to $\mathcal{B} = L^\infty(\mathbb{R}^d)$. We denote it by $\mathcal{D} = \mathcal{E}_{\mathcal{B}}$. We remind that in this case:

$$\mathcal{D} = \left\{ \phi \in L^\infty(\mathbb{R}^d) \mid \sup_{\xi \in \mathbb{R}^d} (1 + |\xi|^2)^n |\phi(\xi)| < \infty, \forall n \in \mathbb{N} \right\}.$$

We provide the space $\mathcal{D}$ with the topology given by the family of norms:

$$p_n(\phi) = \sup_{\xi \in \mathbb{R}^d} (1 + |\xi|^2)^n |\phi(\xi)|.$$

Remark 3. Notice that we do not require that every element $\phi \in \mathcal{D}$ to be smooth. However, we have that $S(\mathbb{R}^d) \subset \mathcal{D}$.

Observe that $p_n \leq p_{n+1}$, for all $n \in \mathbb{N}$. Note that $p_0 = \|\cdot\|_{L^\infty(\mathbb{R}^d)}$ and we can write $p_n(\phi) = \|M^n(\phi)\|_{L^\infty(\mathbb{R}^d)}$ for every $\phi \in \mathcal{D}$.

Now, we study the relationship of the space $\mathcal{D}$ with other Lebesgue spaces.

Proposition 3. *We have $\mathcal{D} \subset \bigcap_{1 \leq p \leq \infty} L^p(\mathbb{R}^d)$ with continuous inclusions $\mathcal{D} \subset L^p(\mathbb{R}^d)$ for every $1 \leq p \leq \infty$.*

Proof. Since $p_0(\phi) = \|\phi\|_{L^\infty(\mathbb{R}^d)}$ we have the claim for $p = \infty$. Let $1 \leq p < \infty$ and note that $|\phi|^p \in \mathcal{D}$ if $\phi \in \mathcal{D}$.

In fact,

$$\begin{aligned}
p_n(|\phi|^p) &= \|M^n(|\phi|^p)\|_{L^\infty(\mathbb{R}^d)} = \left\|(1+|\cdot|^2)^n |\phi|^p\right\|_{L^\infty(\mathbb{R}^d)} \\
&\leq \left\|(1+|\cdot|^2)^{np} |\phi|^p\right\|_{L^\infty(\mathbb{R}^d)} = \left\|(1+|\cdot|^2)^n |\phi|\right\|^p_{L^\infty(\mathbb{R}^d)} = \|M^n(|\phi|)\|^p_{L^\infty(\mathbb{R}^d)} = p_n(\phi)^p.
\end{aligned}$$

With this it is enough to check that $\phi \in L^1(\mathbb{R}^d)$ for $\phi \in \mathcal{D}$ since $|\phi|^p \in L^1(\mathbb{R}^d)$ if and only if $|\phi| \in L^p(\mathbb{R}^d)$.

Note that,

$$\int_{\mathbb{R}^d} |\phi(\xi)|\, d\xi = \int_{\mathbb{R}^d} (1+|\xi|^2)^{\frac{d+1}{2}} \frac{|\phi(\xi)|\, d\xi}{(1+|\xi|^2)^{\frac{d+1}{2}}} \leq C_d p_{[\frac{d}{2}]+1}(\xi),$$

with

$$C_d = \int_{\mathbb{R}^d} \frac{d\xi}{(1+|\xi|^2)^{\frac{d+1}{2}}}.$$

□

Remark 4. From here on C will denote a constant depending of the order of the involved Lebesgue spaces and the dimension. In particular, if we fixed the Lebesgue spaces we will have that C is a dimensionality constant $C = C(d)$.

3.2 Space of Functions Decreasing Fast with Singularities

Now we will try to look for more general spaces with singularities at the origin.

Definition 2. Let us consider $0 < \alpha < d$ and the operator:

$$S_\alpha(\phi)(\xi) = \frac{\phi(\xi)}{|\xi|^\alpha},$$

for $\xi \neq 0$.

In the next result we give some integrability properties of $S_\alpha(\phi)$ for $\phi \in \mathcal{D}$.

Proposition 4. *If $d \geq 2$ and $\phi \in L^{1\oplus p}(\mathbb{R}^d)$ for some $p \in \left(\frac{d}{d-\alpha}, \infty\right]$ then*

$$\|S_\alpha(\phi)\|_{L^1(\mathbb{R}^d)} \leq C\, \|\phi\|_{1\oplus p}.$$

Proof. In fact, if $p > \frac{d}{d-\alpha}$ then $p' < \frac{d}{\alpha}$ and $\alpha p' < d$. Therefore,

$$\begin{aligned}
\|S_\alpha(\phi)\|_{L^1(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |S_\alpha(\phi)(\xi)|\, d\xi = \int_{\mathbb{R}^d} \frac{\phi(\xi)}{|\xi|^\alpha} d\xi = \int_{|\xi|\leq 1} \frac{\phi(\xi)}{|\xi|^\alpha} d\xi + \int_{|\xi|\geq 1} \frac{\phi(\xi)}{|\xi|^\alpha} d\xi \\
&\leq \left(\int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{\alpha p'}}\right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^d} |\phi(\xi)^p|\, d\xi\right)^p + \int_{\mathbb{R}^d} |\phi(\xi)|\, d\xi \leq C\, \|\phi\|_{1\oplus p}.
\end{aligned}$$

□

Corollary 2. *If $d \geq 2$, $1 \leq q < \frac{d}{\alpha}$ and $\phi \in L^{p\oplus q}(\mathbb{R}^d)$ for some $p \in \left(\frac{dq}{d-\alpha q}, \infty\right]$ then*

$$\|S_\alpha(\phi)\|_{L^q(\mathbb{R}^d)} \leq C \|\phi\|_{p\oplus q}.$$

Proof. In fact, applying Proposition 4 to $|\phi|^p$ we have:

$$\begin{aligned}\|S_\alpha(\phi)\|^q_{L^q(\mathbb{R}^d)} &= \|S_\alpha(|\phi|^q)\|_{L^1(\mathbb{R}^d)} \leq C\left(\||\phi|^q\|_{L^{\frac{p}{q}}(\mathbb{R}^d)} + \||\phi|^q\|_{L^1(\mathbb{R}^d)}\right)\\ &\leq C\left(\|\phi\|^q_{L^p(\mathbb{R}^d)} + \||\phi|\|^q_{L^q(\mathbb{R}^d)}\right) \leq C\|\phi\|_{p\oplus q}.\end{aligned}$$

□

Corollary 3. *The map $S_\alpha : L^{1\oplus p}(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is continuous for $p \in \left(\frac{d}{d-\alpha}, \infty\right]$.*

Corollary 4. *The map $S_\alpha : L^{p\oplus q}(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is continuous for $1 \leq q < \frac{d}{\alpha}$, $p \in \left(\frac{dq}{d-\alpha q}, \infty\right]$.*

Since $\mathcal{D} \subset \bigcap_{1\leq p\leq\infty} L^p(\mathbb{R}^d)$ it is useful to consider the subspace:

$$\mathcal{D}_\alpha = S_\alpha(\mathcal{D}) = \{S_\alpha(\phi) \mid \phi \in \mathcal{D}\} = \left\{\phi \in L^1(\mathbb{R}^d) \mid |\cdot|^\alpha \phi \in \mathcal{D}\right\},$$

with the topology that makes the map $S_\alpha : \mathcal{D} \to \mathcal{D}_\alpha$ continuous, i.e.,

$$\phi_j \to_{j\to\infty} \phi \text{ in } \mathcal{D}_\alpha \iff |\cdot|^\alpha \phi_j \to_{j\to\infty} |\cdot|^\alpha \phi \text{ in } \mathcal{D}.$$

We conclude this section with a continuity result of the convolution operation on $\mathcal{D}_\alpha$.

Theorem 2. *For $\phi, \psi \in \mathcal{D}_\alpha$ we have*

$$p_n(|\cdot|^\alpha (\phi * \psi)) \leq C\left(p_n(|\cdot|^\alpha \phi)p_{n+[\frac{d}{2}]+1}(|\cdot|^\alpha \psi) + p_{n+[\frac{d}{2}]+1}(|\cdot|^\alpha \phi)p_n(|\cdot|^\alpha \psi)\right),$$

for every $n \in \mathbb{N}$.

Proof. Since $\phi, \psi \in \mathcal{D}_\alpha$ we have that $|\cdot|^\alpha \phi, |\cdot|^\alpha \psi \in \mathcal{D}$ and

$$\begin{aligned}&|\xi|^\alpha |(\phi * \psi)(\xi)| \leq 2^\alpha \left[\int_{\mathbb{R}^d} \left(|\xi-\eta|^\alpha |\phi(\xi-\eta)|\right)|\psi(\eta)|\,d\eta + \int_{\mathbb{R}^d} |\phi(\xi-\eta)| \left(|\eta|^\alpha |\psi(\eta)|\right) d\eta\right]\\ &= 2^\alpha \left[\int_{\mathbb{R}^d} \left(|\xi-\eta|^\alpha |\phi(\xi-\eta)|\right)\left(|\eta|^\alpha |\psi(\eta)|\right)\frac{d\eta}{|\eta|^\alpha} + \int_{\mathbb{R}^d} \left(|\xi-\eta|^\alpha |\phi(\xi-\eta)|\right)\left(|\eta|^\alpha |\psi(\eta)|\right)\frac{d\eta}{|\xi-\eta|^\alpha}\right]\\ &= 2^\alpha \left[\int_{\mathbb{R}^d} \left(|\xi-\eta|^\alpha |\phi(\xi-\eta)|\right)\left(|\eta|^\alpha |\psi(\eta)|\right)\frac{d\eta}{|\eta|^\alpha} + \int_{\mathbb{R}^d} \left(|\eta|^\alpha |\phi(\eta)|\right)\left(|\xi-\eta|^\alpha |\psi(\xi-\eta)|\right)\frac{d\eta}{|\eta|^\alpha}\right].\end{aligned}$$

Therefore,

$$\begin{aligned}&(1+|\xi|^2)^n|\xi|^\alpha |(\phi * \psi)(\xi)| \leq 2^{n+\alpha}\left[\int_{\mathbb{R}^d}\left((1+|\xi-\eta|^2)^n|\xi-\eta|^\alpha|\phi(\xi-\eta)|\right)\left((1+|\eta|^2)^n|\eta|^\alpha|\psi(\eta)|\right)\frac{d\eta}{|\eta|^\alpha}\right.\\ &\left.+\int_{\mathbb{R}^d}\left((1+|\eta|^2)^n|\eta|^\alpha|\phi(\eta)|\right)\left((1+|\xi-\eta|^2)^n|\xi-\eta|^\alpha|\psi(\xi-\eta)|\right)\frac{d\eta}{|\eta|^\alpha}\right]\\ &\leq 2^{n+\alpha}\left[p_n(|\cdot|^\alpha\phi)\left(\int_{\mathbb{R}^d} S_\alpha\left((1+|\cdot|^2)^n|\cdot|^\alpha|\psi|\right)(\eta)d\eta\right) + p_n(|\cdot|^\alpha\psi)\left(\int_{\mathbb{R}^d} S_\alpha\left((1+|\cdot|^2)^n|\cdot|^\alpha|\phi|\right)(\eta)d\eta\right)\right]\\ &= 2^{n+\alpha}\left[p_n(|\cdot|^\alpha\phi)\left\|S_\alpha\left((1+|\cdot|^2)^n|\cdot|^\alpha|\psi|\right)\right\|_{L^1(\mathbb{R}^d)} + p_n(|\cdot|^\alpha\psi)\left\|S_\alpha\left((1+|\cdot|^2)^n|\cdot|^\alpha|\phi|\right)\right\|_{L^1(\mathbb{R}^d)}\right].\end{aligned}$$

By Proposition 4 we have

$$\|S_\alpha(\phi)\|_{L^1(\mathbb{R}^d)} \leq C\,\|\phi\|_{1\oplus p}\,,$$

for $\phi \in L^{1\oplus p}(\mathbb{R}^d)$ and $p \in \left(\frac{d}{d-\alpha}, \infty\right]$.

However, $\|\phi\|_{L^1(\mathbb{R}^d)} \leq C p_{[\frac{d}{2}]+1}(\phi)$ and

$$\|\phi\|_{L^p(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |\phi(\xi)|^p\,d\xi\right)^{\frac{1}{p}} \leq C p_{[\frac{d}{2}]+1}(|\phi|^p)^{\frac{1}{p}} \leq C p_{[\frac{d}{2}]+1}(\phi),$$

for $p \in \left(\frac{d}{d-\alpha}, \infty\right]$.

Therefore,

$$\|S_\alpha(\phi)\|_{L^1(\mathbb{R}^d)} \leq C p_{[\frac{d}{2}]+1}(\phi). \tag{2}$$

Consequently,

$$(1+|\xi|^2)^n\,|\xi|^\alpha\,|(\phi*\psi)(\xi)| \leq \left[p_n(|\cdot|^\alpha\,\phi)p_{n+[\frac{d}{2}]+1}(|\cdot|^\alpha\,\psi) + p_{n+[\frac{d}{2}]+1}(|\cdot|^\alpha\,\phi)p_n(|\cdot|^\alpha\,\psi)\right],$$

for every $\xi \in \mathbb{R}^d$.

Thus,

$$p_n(|\cdot|^\alpha\,(\phi*\psi)) \leq C\left(p_n(|\cdot|^\alpha\,\phi)p_{n+[\frac{d}{2}]+1}(|\cdot|^\alpha\,\psi) + p_{n+[\frac{d}{2}]+1}(|\cdot|^\alpha\,\phi)p_n(|\cdot|^\alpha\,\psi)\right).$$

□

Corollary 5. *The convolution product $*: \mathcal{D}_\alpha \times \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is continuous.*

Remark 5. Note that Theorem 2 implies that $(\phi*\psi) \in \mathcal{D}_\alpha$ for $\phi, \psi \in \mathcal{D}_\alpha$ however for some cases we can have even that $(\phi*\psi) \in \mathcal{D}$, this important case will be treated now.

Theorem 3. *Let $0 \leq \alpha, \beta < d$ such that $\alpha+\beta < d$ then*

$$p_n(\phi*\psi) \leq C\left(p_n(|\cdot|^\alpha\,\phi)p_{n+[\frac{d}{2}]+1}(|\cdot|^\beta\,\psi) + p_{n+[\frac{d}{2}]+1}(|\cdot|^\alpha\,\phi)p_n(|\cdot|^\beta\,\psi)\right),$$

for every $\phi \in \mathcal{D}_\alpha$, $\psi \in \mathcal{D}_\beta$ and $n \in \mathbb{N}$.

Proof. Since $\phi \in \mathcal{D}_\alpha$, $\psi \in \mathcal{D}_\beta$ we have that $|\cdot|^\alpha$, $|\cdot|^\beta \in \mathcal{D}$. For every $n \in \mathbb{N}$ we have:

$$\begin{aligned}
(1+|\xi|^2)^n\,|(\phi*\psi)(\xi)| &\leq 2^n\left[\int_{\mathbb{R}^d}\Big((1+|\xi-\eta|^2)^n\,|\phi(\xi-\eta)|\Big)\Big((1+|\eta|^2)^n\,|\psi(\eta)|\Big)\,d\eta\right] \\
&\leq 2^n\left[\int_{\mathbb{R}^d}\Big((1+|\xi-\eta|^2)^n\,|\xi-\eta|^\alpha\,|\phi(\xi-\eta)|\Big)\Big((1+|\eta|^2)^n\,|\eta|^\beta\,|\psi(\eta)|\Big)\frac{d\eta}{|\xi-\eta|^\alpha\,|\eta|^\beta}\right] \\
&\leq 2^n\left[\int_{|\xi-\eta|\geq|\eta|}\Big((1+|\xi-\eta|^2)^n\,|\xi-\eta|^\alpha\,|\phi(\xi-\eta)|\Big)\Big((1+|\eta|^2)^n\,|\eta|^\beta\,|\psi(\eta)|\Big)\frac{d\eta}{|\xi-\eta|^\alpha\,|\eta|^\beta}\right. \\
&\quad\left.+\int_{|\xi-\eta|\geq|\eta|}\Big((1+|\eta|^2)^n\,|\eta|^\alpha\,|\phi(\eta)|\Big)\Big((1+|\xi-\eta|^2)^n\,|\xi-\eta|^\beta\,|\psi(\xi-\eta)|\Big)\frac{d\eta}{|\eta|^\alpha\,|\xi-\eta|^\beta}\right]
\end{aligned}$$

$$\leq 2^n \left[\int_{|\xi-\eta|\geq|\eta|} \left((1+|\xi-\eta|^2)^n |\xi-\eta|^\alpha |\phi(\xi-\eta)| \right) \left((1+|\eta|^2)^n |\eta|^\beta |\psi(\eta)| \right) \frac{d\eta}{|\eta|^{\alpha+\beta}} \right.$$
$$\left. + \int_{|\xi-\eta|\geq|\eta|} \left((1+|\eta|^2)^n |\eta|^\alpha |\phi(\eta)| \right) \left((1+|\xi-\eta|^2)^n |\xi-\eta|^\beta |\psi(\xi-\eta)| \right) \frac{d\eta}{|\eta|^{\alpha+\beta}} \right]$$
$$\leq 2^n \left[p_n(|\cdot|^\alpha \phi) \left(\int_{\mathbb{R}^d} S_{\alpha+\beta} \left((1+|\cdot|^2)^n |\cdot|^\beta |\psi| \right)(\eta) d\eta \right) + p_n(|\cdot|^\beta \psi) \left(\int_{\mathbb{R}^d} S_{\alpha+\beta} \left((1+|\cdot|^2)^n |\cdot|^\alpha |\phi| \right)(\eta) d\eta \right) \right]$$
$$= 2^n \left[p_n(|\cdot|^\alpha \phi) \left\| S_\alpha \left((1+|\cdot|^2)^n |\cdot|^\beta |\psi| \right) \right\|_{L^1(\mathbb{R}^d)} + p_n(|\cdot|^\beta \psi) \left\| S_\alpha \left((1+|\cdot|^2)^n |\cdot|^\alpha |\phi| \right) \right\|_{L^1(\mathbb{R}^d)} \right].$$

In consequence, if we apply Equation (2) we obtain that:

$$(1+|\xi|^2)^n |(\phi*\psi)(\xi)| \leq C \left(p_n(|\cdot|^\alpha \phi) p_{n+[\frac{d}{2}]+1}(|\cdot|^\beta \psi) + p_{n+[\frac{d}{2}]+1}(|\cdot|^\alpha \phi) p_n(|\cdot|^\beta \psi) \right),$$

for every $\xi \in \mathbb{R}^d$. Thus,

$$p_n(\phi*\psi) \leq C \left(p_n(|\cdot|^\alpha \phi) p_{n+[\frac{d}{2}]+1}(|\cdot|^\beta \psi) + p_{n+[\frac{d}{2}]+1}(|\cdot|^\alpha \phi) p_n(|\cdot|^\beta \psi) \right).$$

□

Corollary 6. *We have the bilinear continuous operator* $*: \mathcal{D}_\alpha \times \mathcal{D}_\beta \to \mathcal{D}$, *if* $0 \leq \alpha+\beta < d$.

Remark 6. The condition $\alpha+\beta < d$ is necessary. If we define

$$\phi(\xi) = \frac{e^{-|\xi|^2}}{|\xi|^\alpha},$$

and

$$\psi(\xi) = \frac{e^{-|\xi|^2}}{|\xi|^\beta},$$

we have that $\phi \in \mathcal{D}_\alpha$, $\psi \in \mathcal{D}_\beta$. However,

$$(\phi*\psi)(0) = \int_{\mathbb{R}^d} \phi(-\eta)\psi(\eta) d\eta = \int_{\mathbb{R}^d} \frac{e^{-2|\eta|^2}}{|\eta|^{\alpha+\beta}} d\eta \geq e^{-2} \int_{|\eta|\leq 1} \frac{d\eta}{|\eta|^{\alpha+\beta}} = \infty,$$

if $\alpha+\beta \geq d$. Thus, $\phi*\psi \notin \mathcal{D}$.

Corollary 7. *Let* $0 \leq \alpha < \frac{d}{2}$ *then*

$$p_n(\phi*\psi) \leq C \left(p_n(|\cdot|^\alpha \phi) p_{n+[\frac{d}{2}]+1}(|\cdot|^\alpha \psi) + p_{n+[\frac{d}{2}]+1}(|\cdot|^\alpha \phi) p_n(|\cdot|^\alpha \psi) \right),$$

for every $\phi, \psi \in \mathcal{D}_\alpha$ *and* $n \in \mathbb{N}$.

Corollary 8. *We have the bilinear continuous operator* $*: \mathcal{D}_\alpha \times \mathcal{D}_\alpha \to \mathcal{D}$, *if* $0 \leq \alpha < \frac{d}{2}$.

4 Riesz Convolution

In this Section we consider a generalization of the convolution operation.

Definition 3. Let $f, g : \mathbb{R}^d \to \mathbb{R}$ such that $f*g : \mathbb{R}^d \to \mathbb{R}$ is well defined. Let $0 < \alpha < d$, we define the Riesz convolution between f and g to be $f *_\alpha g : \mathbb{R}^d \to \mathbb{R}$,

$$(f *_\alpha g)(\xi) = S_\alpha(f*g)(\xi) = \frac{(f*g)(\xi)}{|\xi|^\alpha}.$$

Note that $*_\alpha$ is commutative. In fact, since

$$(f *_\alpha g)(\xi) = \frac{(f * g)(\xi)}{|\xi|^\alpha} = \frac{(g * f)(\xi)}{|\xi|^\alpha} = (g *_\alpha f)(\xi),$$

for $\xi \in \mathbb{R}^d - \{0\}$, we obtain $f *_\alpha g = g *_\alpha f$.

Additionally, $*_\alpha$ is distributive but not associative.

Remark 7. A remarkable case is when we consider $f, g \in \mathcal{D}$ since $f * g \in \mathcal{D}$ we have that $f *_\alpha g \in \mathcal{D}_\alpha$ and the operation $*_\alpha : \mathcal{D} \times \mathcal{D} \to \mathcal{D}_\alpha$. If $0 \leq \alpha < \frac{d}{2}$ we obtain by Corollary 8 the operation $*_\alpha : \mathcal{D}_\alpha \times \mathcal{D}_\alpha \to \mathcal{D}_\alpha$.

In the following result we study the behaviour of $*_\alpha$ with respect to L^p-spaces.

Proposition 5. *Let $1 \leq q < \frac{d}{\alpha}$ and $1 \leq p \leq \infty$, $d \geq 2$. If $f \in L^{p\oplus q}(\mathbb{R}^d)$ and $g \in L^{1\oplus p'}(\mathbb{R}^d)$ then*

$$\|f *_\alpha g\|_{L^q(\mathbb{R}^d)} \leq C \left(\|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^{p'}(\mathbb{R}^d)} + \|f\|_{L^q(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)} \right).$$

Proof. By Corollary 3 we have that

$$\|S_\alpha(\phi)\|_{L^q(\mathbb{R}^d)} \leq C \|\phi\|_{r\oplus q},$$

for $1 \leq q < \frac{d}{\alpha}$, $r \in \left(\frac{dq}{d-\alpha q}, \infty \right]$, $d \geq 2$.

If $f \in L^{p\oplus q}(\mathbb{R}^d)$, $g \in L^{1\oplus p'}(\mathbb{R}^d)$ we take $\phi = f * g \in L^\infty(\mathbb{R}^d)$ we have:

$$\begin{aligned}
&\|f *_\alpha g\|_{L^q(\mathbb{R}^d)} = \|S_\alpha(\phi)\|_{L^q(\mathbb{R}^d)} \leq C \|\phi\|_{\infty\oplus q} \\
&= C \left(\|f * g\|_{L^\infty(\mathbb{R}^d)} + \|f * g\|_{L^q(\mathbb{R}^d)} \right) = C \left(\|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^{p'}(\mathbb{R}^d)} + \|f\|_{L^q(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)} \right).
\end{aligned}$$

□

Corollary 9. *Let $1 \leq q < \frac{d}{\alpha}$ and $1 \leq p \leq \infty$, $d \geq 2$. If $f \in L^{p\oplus q}(\mathbb{R}^d)$ and $g \in L^{1\oplus p'}(\mathbb{R}^d)$ then:*

$$\|f *_\alpha g\|_{L^q(\mathbb{R}^d)} \leq C \|f\|_{p\oplus q} \|g\|_{p'\oplus 1}.$$

Corollary 10. *Let $1 \leq p \leq \infty$, $d \geq 2$. If $f \in L^{p\oplus 1}(\mathbb{R}^d)$ and $g \in L^{1\oplus p'}(\mathbb{R}^d)$ then:*

$$\|f *_\alpha g\|_{L^1(\mathbb{R}^d)} \leq C \|f\|_{p\oplus 1} \|g\|_{p'\oplus 1}.$$

Theorem 4. *Let $\frac{d}{d-\alpha} < p < \frac{d}{\alpha}$, $d \geq 2$. If $f \in L^{p\oplus 1}(\mathbb{R}^d)$ and $g \in L^{1\oplus p'}(\mathbb{R}^d)$ then:*

$$\|f *_\alpha g\|_{1\oplus p\oplus p'} \leq C \|f\|_{p\oplus 1} \|g\|_{p'\oplus 1}.$$

Proof. By Corollary 10 we have:

$$\|f *_\alpha g\|_{L^1(\mathbb{R}^d)} \leq C \|f\|_{p\oplus 1} \|g\|_{p'\oplus 1}.$$

By Proposition 5 with $p = q$ we have:

$$\|f *_\alpha g\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{p'\oplus 1} \leq C \|f\|_{p\oplus 1} \|g\|_{p'\oplus 1}.$$

Since $*_\alpha$ is commutative and $\frac{d}{d-\alpha} < p$ we have $1 \leq p' < \frac{d}{\alpha}$ and we can exchange p and p' to obtain:

$$\begin{aligned}
&\|f *_\alpha g\|_{L^{p'}(\mathbb{R}^d)} = \|g *_\alpha f\|_{L^{p'}(\mathbb{R}^d)} \\
&\leq C \|g\|_{p'\oplus 1} \|g\|_{p\oplus 1} \leq C \|f\|_{p\oplus 1} \|g\|_{p'\oplus 1},
\end{aligned}$$

since $p'' = p$.

In consequence,

$$\|f *_\alpha g\|_{1\oplus p\oplus p'} \leq C \|f\|_{p\oplus 1} \|g\|_{p'\oplus 1}.$$

□

Remark 8. Observe that the condition $\frac{d}{d-\alpha} < \frac{d}{\alpha}$ implies that $\alpha < \frac{d}{2}$. In particular, when $d = 2$ and $1 < \alpha < 2$ we have that $\alpha < 1$. If $d \geq 3$ then $\alpha < \frac{3}{2}$. Since we are interested in the case $\alpha \geq 1$ it is convenient to take $d \geq 3$.

From now on we will consider $d \geq 3$.

Remark 9. Note that if $A_p = L^{1\oplus p\oplus p'}(\mathbb{R}^d)$, $\frac{d}{d-\alpha} < p < \frac{d}{\alpha}$ then $*_\alpha : A_p \times A_p \to A_p$ is a closed operation and is continuous since:

$$\|f *_\alpha g\|_{1\oplus p\oplus p'} \leq C \|f\|_{1\oplus p\oplus p'} \|g\|_{1\oplus p\oplus p'}. \tag{3}$$

Since $L^2(\mathbb{R}^d) \subset L^{p\oplus p'}(\mathbb{R}^d)$ for every $1 \leq p \leq \infty$ it is interesting to take $p = 2$. In this case, $A_2 = L^{1\oplus 2}(\mathbb{R}^d)$ and we have:

$$\|f *_\alpha g\|_{1\oplus 2} \leq C \|f\|_{1\oplus 2} \|g\|_{1\oplus 2}.$$

It is useful to generalize Equation (3) to monomials of degree greater than 2.

Theorem 5. *If $f_1, \cdots f_k \in A_p$, for $\frac{d}{d-\alpha} < p < \frac{d}{\alpha}$ then*

$$\sup_{w\in M(x_1,\cdots,x_k)} \|w^{*_\alpha}(f_1,\cdots,f_k)\|_{1\oplus p\oplus p'} \leq C^{k-1} \prod_{j=1}^{k} \|f_j\|_{1\oplus p\oplus p'}.$$

Proof. By induction over k. For $k = 1$ is obvious. For $k = 2$ it is given by Equation (3).

Assume the statement of this Theorem for $1 \leq j < k$. Let $w \in M(x_1, \cdots, x_k)$ then we can write $w = w_{(1)}w_{(2)}$ for $w_{(1)} \in M(x_1, \cdots, x_j)$ and $w_{(2)} \in M(x_{j+1}, \cdots, x_k)$.

Applying the case $k = 2$ and the induction hypothesis for $w_{(1)}$ and $w_{(2)}$ we have:

$$\begin{aligned}
&\|w^{*_\alpha}(f_1,\cdots,f_k)\|_{1\oplus p\oplus p'} = \left\|w^{*_\alpha}_{(1)}(f_1,\cdots,f_j) *_\alpha w^{*_\alpha}_{(2)}(f_{j+1},\cdots,f_k)\right\|_{1\oplus p\oplus p'} \\
&\leq C \left\|w^{*_\alpha}_{(1)}(f_1,\cdots,f_j)\right\|_{1\oplus p\oplus p'} \left\|w^{*_\alpha}_{(2)}(f_{j+1},\cdots,f_k)\right\|_{1\oplus p\oplus p'} \\
&\leq C \left(C^{j-1} \prod_{l=1}^{j} \|f_l\|_{1\oplus p\oplus p'}\right) \left(C^{k-j-1} \prod_{l=j+1}^{k} \|f_l\|_{1\oplus p\oplus p'}\right) \\
&= C^{k-1} \prod_{j=1}^{k} \|f_j\|_{1\oplus p\oplus p'}.
\end{aligned}$$

□

So far we have some estimates of $\|f *_\alpha g\|_{L^p(\mathbb{R}^d)}$ for $\frac{d}{d-\alpha} < p < \frac{d}{\alpha}$ however we need some estimates of the L^∞-norm of the Riesz convolution. Since in general $f *_\alpha g \notin L^\infty(\mathbb{R}^d)$ it is a good idea to consider this norm on the complement of a ball centered at the origin. For simplicity we can consider the unit ball.

Lemma 4. *If $f, g \in A_p$, for $\frac{d}{d-\alpha} < p < \frac{d}{\alpha}$ then:*

$$\sup_{|\xi|\geq 1} |(f *_\alpha g)(\xi)| \leq \|f\|_{1\oplus p\oplus p'} \|g\|_{1\oplus p\oplus p'}. \tag{4}$$

Proof. In fact, if $|\xi| \geq 1$ then by Holder Inequality:

$$|(f *_\alpha g)(\xi)| \leq |(f * g)(\xi)| \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^{p'}(\mathbb{R}^d)} \leq \|f\|_{1\oplus p\oplus p'} \|g\|_{1\oplus p\oplus p'}.$$

□

Now we generalize Equation (4) to monomials of degree greater than 2.

Theorem 6. *If $f_1, \cdots f_k \in A_p$, for $\frac{d}{d-\alpha} < p < \frac{d}{\alpha}$ then*

$$\sup_{w\in M(x_1,\cdots,x_k)} \sup_{|\xi|\geq 1} |w^{*_\alpha}(f_1,\cdots,f_k)(\xi)| \leq C^{k-2} \prod_{j=1}^{k} \|f_j\|_{1\oplus p\oplus p'}.$$

Proof. By induction over $k \geq 2$. For $k = 2$ we use Equation (4) . Assume the statement for $1 \leq j < k$. Let $w \in M(x_1, \cdots, x_k)$ then we can write $w = w_{(1)}w_{(2)}$ for $w_{(1)} \in M(x_1, \cdots, x_j)$ and $w_{(2)} \in M(x_{j+1}, \cdots, x_k)$.

Applying the case $k = 2$ and Theorem 5 for $w_{(1)}$ and $w_{(2)}$ we have for $|\xi| \geq 1$:

$$\begin{aligned}
&|w^{*_\alpha}(f_1,\cdots,f_k)(\xi)| = \left|(w^{*_\alpha}_{(1)}(f_1,\cdots,f_j) *_\alpha w^{*_\alpha}_{(2)}(f_{j+1},\cdots,f_k))(\xi)\right| \\
&\leq \left\|w^{*_\alpha}_{(1)}(f_1,\cdots,f_j)\right\|_{1\oplus p\oplus p'} \left\|w^{*_\alpha}_{(2)}(f_{j+1},\cdots,f_k)\right\|_{1\oplus p\oplus p'} \\
&\leq \left(C^{j-1}\prod_{l=1}^{j} \|f_l\|_{1\oplus p\oplus p'}\right)\left(C^{k-j-1}\prod_{l=j+1}^{k} \|f_l\|_{1\oplus p\oplus p'}\right) \\
&= C^{k-2}\prod_{j=1}^{k} \|f_j\|_{1\oplus p\oplus p'}.
\end{aligned}$$

□

Corollary 11. *If $f \in A_p$, for $\frac{d}{d-\alpha} < p < \frac{d}{\alpha}$ then*

$$\sup_{w\in M(x_1,\cdots,x_k)} \sup_{|\xi|\geq 1} \left|w^{*_\alpha}_k(f)(\xi)\right| \leq C^{k-2} \|f\|^k_{1\oplus p\oplus p'}.$$

Now we consider a variation of the estimates in the unit ball.

Corollary 12. *If $f_1, \cdots f_k \in A_p$, for $\frac{d}{d-\alpha} < p < \frac{d}{\alpha}$ then*

$$\sup_{w\in M(x_1,\cdots,x_k)} \sup_{|\xi|\leq 1} |\xi|^\alpha |w^{*_\alpha}(f_1,\cdots,f_k)(\xi)| \leq C^{k-2} \prod_{j=1}^{k} \|f_j\|_{1\oplus p\oplus p'}.$$

Proof. We make induction over $k \geq 2$. For $k = 2$ we have that $M(x_1, x_2) = \{x_1x_2, x_2x_1\}$ however since $*_\alpha$ is commutative is enough to consider $w(x_1, x_2) = x_1x_2$.

Note that for $|\xi| \leq 1$ we have:

$$\begin{aligned}
&|\xi|^\alpha |w^{*_\alpha}(f_1, f_2)(\xi)| = |\xi|^\alpha |(f_1 *_\alpha f_2)(\xi)| \leq |(f_1 * f_2)(\xi)| \\
&\leq \|f_1 * f_2\|_{L^\infty(\mathbb{R}^d)} \leq \|f_1\|_{L^p(\mathbb{R}^d)} \|f_2\|_{L^{p'}(\mathbb{R}^d)} \leq \prod_{j=1}^{2} \|f_j\|_{1\oplus p\oplus p'}.
\end{aligned}$$

Assume the statement for $1 \leq j < k$. Let $w \in M(x_1, \cdots, x_k)$ then we can write $w = w_{(1)}w_{(2)}$ for $w_{(1)} \in M(x_1, \cdots, x_j)$ and $w_{(2)} \in M(x_{j+1}, \cdots, x_k)$.

Applying Theorem 5 for $g_1 = w^{*_\alpha}_{(1)}(f_1, \cdots, f_j)$ and $g_2 = w^{*_\alpha}_{(2)}(f_{j+1}, \cdots, f_k)$ we have for $|\xi| \le 1$:

$$\begin{aligned}
&|\xi|^\alpha \left| w^{*_\alpha}(f_1, \cdots, f_k)(\xi) \right| = |\xi|^\alpha \left| (g_1 *_\alpha g_2)(\xi) \right| \\
&\le \|g_1\|_{L^p(\mathbb{R}^d)} \|g_2\|_{L^{p'}(\mathbb{R}^d)} \le \left(C^{j-1} \prod_{l=1}^{j} \|f_l\|_{1\oplus p\oplus p'} \right) \left(C^{k-j-1} \prod_{l=j+1}^{k} \|f_l\|_{1\oplus p\oplus p'} \right) \\
&= C^{k-2} \prod_{j=1}^{k} \|f_j\|_{1\oplus p\oplus p'}.
\end{aligned}$$

□

5 Spaces of Functions Dominated by Fourier Caloric Functions

In this Section we consider spaces of functions in which time is involved. It is motivated by the solution of the Navier-Stokes Equation since this is an evolution equation.

Definition 4. Let $\mathcal{E}$ be a space of functions decreasing fast such that $\mathcal{E}^+$ is closed by pointwise addition, convolution and maximum.

Let $p, q, n \in \mathbb{N}$, $\alpha > 0$ we define $\mathcal{C}_\alpha(\mathcal{E})^{p\times q}_n$ be the complex space generated by functions $f : \mathbb{R}^{d+1}_+ \to \mathbb{C}^{p\times q}$ such that:

$$|f(\xi, t)| \le t^n e^{-\lambda t |\xi|^\alpha} f^0(\xi), \tag{5}$$

for $(\xi, t) \in \mathbb{R}^{d+1}_+$, for some $\lambda > 0$, $f^0 \in \mathcal{E}^+$.

We denote $\mathcal{C}_\alpha(\mathcal{E})^{p\times q} = \oplus_{n=0}^{\infty} \mathcal{C}_\alpha(\mathcal{E})^{p\times q}_n$. For $f \in \mathcal{C}_\alpha(\mathcal{E})^{p\times q}_n$ we say that λ in Equation (5) is an exponent of f and we will use the notation $\exp(f)$.

We simplify the notation to $\mathcal{C}^{p\times q}_\alpha$ when there is no place to confusion.

Remark 10.
- Observe that if $f_1, \cdots f_k \in \mathcal{C}^{p\times q}_\alpha$ and $\lambda_j = \exp(f_j)$ for $1 \le j \le k$ then we can take a common exponent $\lambda = \min \{\lambda_j \mid 1 \le j \le k\}$.

  On the other hand, since $\mathcal{E}^+$ is closed by maximum we can take $f^0 = \max \left\{ f^0_j \mid 1 \le j \le k \right\}$.

  Thus, we have that every element $f \in \mathcal{C}^{p\times q}_\alpha$ satisfies:

  $$|f(\xi, t)| \le p(t) e^{-\lambda t |\xi|^\alpha} f^0(\xi),$$

  for a polynomial $p \in \mathbb{C}[t]$ such that $p([0, \infty)) \subset [0, \infty)$.

With Remark 10 we can obtain the following result.

Proposition 6. *For every $p, q \in \mathbb{N}$, $\mathcal{C}^{p\times q}_\alpha$ is a graded $\mathcal{E}$-module and a $\mathbb{C}[t]$-module with pointwise operations.*

We can consider the product of two elements $f \in \mathcal{C}^{p\times q}_\alpha$, $g \in \mathcal{C}^{r\times q}_\alpha$

$$(f \cdot g)(\xi, t) = f(\xi, t) g(\xi, t)^*, \; (\xi, t) \in \mathbb{R}^{d+1}_+.$$

Note that $f \cdot g : \mathbb{R}^{d+1}_+ \to \mathbb{C}^{p\times r}$. In the following result we verify that this product is well behaved.

Proposition 7. *If $\mathcal{E}^+$ is closed by pointwise product then the product $\cdot : \mathcal{C}^{p\times q}_\alpha \times \mathcal{C}^{r\times q}_\alpha \to \mathcal{C}^{p\times r}_\alpha$ is a bilinear operator. The set $\mathcal{C}^{p\times p}_\alpha$ is a $\mathcal{E}$-algebra with the product $\cdot$.*

Proof. In fact, since $f \in \mathcal{C}_\alpha^{p\times q}$, $g \in \mathcal{C}_\alpha^{r\times q}$ they are compatible. We can assume without loss of generality that they are generators, so we can write:

$$|f(\xi,t)| \leq t^{n_1} e^{-\lambda_1 t|\xi|^\alpha} f^0(\xi),\ |g(\xi,t)| \leq t^{n_2} e^{-\lambda_2 t|\xi|^\alpha} g^0(\xi),$$

for $(\xi,t) \in \mathbb{R}_+^{d+1}$.

Therefore,

$$|(f\cdot g)(\xi,t)| \leq t^{n_1+n_2} e^{-(\lambda_1+\lambda_2)t|\xi|^\alpha} f^0 g^0(\xi),$$

for $(\xi,t) \in \mathbb{R}_+^{d+1}$.

Since $f^0 g^0 \in \mathcal{E}^+$ we have that $f\cdot g \in \mathcal{C}_\alpha^{p\times r}$. In particular $\mathcal{C}_\alpha^{p\times p}$ is a graded $\mathcal{E}$-algebra. Clearly $\cdot$ is bilinear. □

Corollary 13. *If $\mathcal{E}^+$ is closed by pointwise product and $p \in \mathbb{N}$, then $\mathcal{C}_\alpha^{p\times p}$ is a $\mathbb{C}$-algebra.*

Now, we consider an interesting operation in the spaces $\mathcal{C}_\alpha^{p\times q}$.

Definition 5. For $f \in \mathcal{C}_\alpha^{p\times q}$, $g \in \mathcal{C}_\alpha^{r\times q}$ we define the *tensor convolution* of f and g to be:

$$(f * g)(\xi,t) = \int_{\mathbb{R}^d} f(\xi-\eta,t)\cdot g(\eta,t)d\eta.$$

Similarly to the pointwise product we have the following result for tensor convolution.

Theorem 7. *The tensor convolution $* : \mathcal{C}_\alpha^{p\times q} \times \mathcal{C}_\alpha^{r\times q} \to \mathcal{C}_\alpha^{p\times r}$ is a bilinear operator. The set $\mathcal{C}_\alpha^{p\times p}$ is a $\mathcal{E}$-algebra with the product $*$.*

Proof. By Proposition 7 we have that $\cdot$ is bilinear so $*$ is bilinear. Let $f \in \mathcal{C}_\alpha^{p\times q}$ and $g \in \mathcal{C}_\alpha^{r\times q}$. We can assume without loss of generality that they are generators, so we can write:

$$|f(\xi,t)| \leq t^{n_1} e^{-\lambda t|\xi|^\alpha} f^0(\xi),\ |g(\xi,t)| \leq t^{n_2} e^{-\lambda t|\xi|^\alpha} g^0(\xi),$$

for $(\xi,t) \in \mathbb{R}_+^{d+1}$.

By Proposition 1 we have that for every $\xi,\eta \in \mathbb{R}^d$:

$$-|\xi-\eta|^\alpha - |\eta|^\alpha \leq -r_\alpha |\xi|^\alpha,$$

for some $r_\alpha \leq 1$. Consequently,

$$\begin{aligned} &|f(\xi-\eta,t)\cdot g(\eta,t)| \leq |f(\xi-\eta,t)|\,|g(\eta,t)| \\ &\leq t^{n_1+n_2} e^{-\lambda t(|\xi-\eta|^\alpha+|\eta|^\alpha)} f^0(\xi-\eta) g^0(\eta) \leq t^{n_1+n_2} e^{-r_\alpha \lambda t|\xi|^\alpha} f^0(\xi-\eta) g^0(\eta). \end{aligned}$$

Thus,

$$|(f*g)(\xi,t)| \leq t^{n_1+n_2} e^{-r_\alpha \lambda t|\xi|^\alpha} (f^0 * g^0)(\xi).$$

Since $f^0 * g^0 \in \mathcal{E}^+$ for $f^0, g^0 \in \mathcal{E}^+$ we have that $f * g \in \mathcal{C}_\alpha^{p\times r}$. □

For $p,q \in \mathbb{N}$ and $\lambda > 0$ we denote $\mathcal{C}_\alpha^{p\times q}(\lambda) = \left\{ f \in \mathcal{C}_\alpha^{p\times q} \mid exp(f) \in (0,\lambda) \right\}$. Additionally, a remarkable case is when $\alpha = 2$ that we denote simply by $\mathcal{C}^{p\times q}(\lambda)$.

In order to solve the Navier-Stokes Equation we need to consider a product associated with the gaussian distribution.

Definition 6. Let $K : \mathbb{R}^d \to \mathbb{C}^{d\times d}$ such that $\sup_{\xi\in\mathbb{R}^d} \|K(\xi)\|_{\mathcal{L}(\mathbb{C}^d)} \le 1$, we define the product:

$$(f \odot g)(\xi,t) = 2\pi i K(\xi)\left[\int_0^t e^{-4\pi^2\nu(t-s)|\xi|^2}(f*g)(\xi,s)ds\right]\xi,$$

for $f,g \in \mathcal{C}_{d,1}(4\pi^2\nu)$.

Theorem 8. *Assume that $\mathcal{E}^+$ is closed by Riezs convolution $*_1$, i.e., $f^0 *_1 g^0 \in \mathcal{E}^+$ for $f^0, g^0 \in \mathcal{E}^+$ or it is closed by convolution after multiplication by $|\cdot|$, i.e., $|\cdot|\,(f^0 * g^0) \in \mathcal{E}^+$ for $f^0, g^0 \in \mathcal{E}^+$. Then the product $\odot$: $\mathcal{C}^{d\times1}(4\pi^2\nu) \times \mathcal{C}^{d\times1}(4\pi^2\nu) \to \mathcal{C}^{d\times1}(4\pi^2\nu)$ is a bilinear operator.*

Proof. By Theorem 7 we have that $*$ is bilinear so $\odot$ is bilinear. However, we need to check that $f\odot g \in \mathcal{C}^{d\times1}(4\pi^2\nu)$ for $f,g \in \mathcal{C}^{d\times1}(4\pi^2\nu)$.

Let $f \in \mathcal{C}^{d\times1}(4\pi^2\nu)$ and $g \in \mathcal{C}^{d\times1}(4\pi^2\nu)$. We can assume without loss of generality that they are generators, so we can write:

$$|f(\xi,t)| \le t^{n_1}e^{-\lambda t|\xi|^2}f^0(\xi),\ |g(\xi,t)| \le t^{n_2}e^{-\lambda t|\xi|^2}g^0(\xi),$$

for $(\xi,t)\in\mathbb{R}^{d+1}_+$.

By Theorem 7 implies that:

$$|(f*g)(\xi,t)| \le t^{n_1+n_2}e^{-\frac{\lambda t}{2}|\xi|^2}(f^0*g^0)(\xi),$$

for $(\xi,t)\in\mathbb{R}^{d+1}_+$.

Therefore,

$$\begin{aligned}
|(f\odot g)(\xi,t)| &\le 2\pi\left(\int_0^t e^{-4\pi^2\nu(t-s)|\xi|^2}s^{n_1+n_2}e^{-\frac{\lambda s}{2}|\xi|^2}(f^0*g^0)(\xi)ds\right)|\xi| \\
&= 2\pi t^{n_1+n_2}e^{-4\pi^2\nu t|\xi|^2}\left(\int_0^t e^{\left(4\pi^2\nu-\frac{\lambda}{2}\right)s|\xi|^2}ds\right)(f^0*g^0)(\xi)\,|\xi| \\
&= 2\pi t^{n_1+n_2}\,|\xi|\,e^{-4\pi^2\nu t|\xi|^2}\left[\frac{e^{\left(4\pi^2\nu-\frac{\lambda}{2}\right)s|\xi|^2}}{\left(4\pi^2\nu-\frac{\lambda}{2}\right)|\xi|^2}\right]_0^t(f^0*g^0)(\xi) \\
&= 2\pi t^{n_1+n_2}\,|\xi|\,e^{-4\pi^2\nu t|\xi|^2}\left(\frac{e^{\left(4\pi^2\nu-\frac{\lambda}{2}\right)t|\xi|^2}-1}{\left(4\pi^2\nu-\frac{\lambda}{2}\right)|\xi|^2}\right)(f^0*g^0)(\xi) \\
&= 2\pi t^{n_1+n_2}\,|\xi|\left(\frac{e^{-\frac{\lambda}{2}t|\xi|^2}-e^{-4\pi^2\nu t|\xi|^2}}{\left(4\pi^2\nu-\frac{\lambda}{2}\right)|\xi|^2}\right)(f^0*g^0)(\xi) \\
&= 2\pi t^{n_1+n_2}\,|\xi|\,e^{-\frac{\lambda}{2}t|\xi|^2}\left(\frac{1-e^{-\left(4\pi^2\nu-\frac{\lambda}{2}\right)t|\xi|^2}}{\left(4\pi^2\nu-\frac{\lambda}{2}\right)|\xi|^2}\right)(f^0*g^0)(\xi).
\end{aligned}$$

- If $\mathcal{E}^+$ is closed by Riezs convolution $*_1$ then we have that

  $$|(f\odot g)(\xi,t)| \le \left(\frac{2\pi}{4\pi^2\nu-\frac{\lambda}{2}}\right)t^{n_1+n_2}e^{-\frac{\lambda t}{2}|\xi|^2}(f^0 *_1 g^0)(\xi),$$

  for $(\xi,t)\in\mathbb{R}^{d+1}_+$. Since $f^0 *_1 g^0 \in \mathcal{E}^+$ we have that $f\odot g \in \mathcal{C}^{d\times1}(4\pi^2\nu)$.

- If $\mathcal{E}$ it is closed by convolution after multiplication by $|\cdot|$, we use that

$$\max_{x>0}\left(\frac{1-e^{-x}}{x}\right)=1,$$

to conclude that

$$|(f\odot g)(\xi,t)|\leq 2\pi t^{n_1+n_2+1}e^{-\frac{\lambda t}{2}|\xi|^2}(f^0*g^0)(\xi)=t^{n_1+n_2+1}e^{-\frac{\lambda t}{2}|\xi|^2}h^0(\xi),$$

for $(\xi,t)\in\mathbb{R}_+^{d+1}$ with $h^0(\xi)=2\pi|\xi|(f^0*g^0)(\xi)$. Therefore $h^0\in\mathcal{E}^+$ and we have that $f\odot g\in\mathcal{C}^{d\times 1}(4\pi^2\nu)$.

□

We assume the notation of Theorem 8 to state the following:

Corollary 14. *If $\mathcal{E}^+$ is closed by Riezs convolution $*_1$ and $f,g\in\mathcal{C}^{d\times 1}(4\pi^2\nu)$ then:*

$$|(f\odot g)(\xi,t)|\leq\frac{t^{n_1+n_2}}{\pi\nu}e^{-\frac{\lambda t}{2}|\xi|^2}(f^0*_1g^0)(\xi).$$

Proof. Since $\lambda\leq 4\pi^2\nu$ we have that $4\pi^2\nu-\frac{\lambda}{2}\geq 2\pi^2\nu$ then

$$\frac{2\pi}{4\pi^2\nu-\frac{\lambda}{2}}\leq\frac{2\pi}{2\pi^2\nu}\leq\frac{1}{\pi\nu}.$$

□

Since we are looking for solutions of the Navier-Stokes Equation is important to consider derivatives with respect to time. This motivates the following definition.

Definition 7. Let $\mathcal{E}$ be a space of functions decreasing fast closed by pointwise addition, convolution and maximum.

Let $d\geq 3$, we define $\mathcal{V}(\mathcal{E})$ be the complex space generated by functions $f:\mathbb{R}_+^{d+1}\to\mathbb{C}^{d\times 1}$ such that $f\in\mathcal{C}(\mathcal{E})_0$, $f(\xi,\cdot)\in C^\infty([0,\infty),\mathbb{C}^d)$ for a.e $\xi\in\mathbb{R}^d$ and we have the automorphisms $\frac{\partial}{\partial t}:\mathcal{V}(\mathcal{E})\to\mathcal{V}(\mathcal{E})$ and $|\cdot|:\mathcal{V}(\mathcal{E})\to\mathcal{V}(\mathcal{E})$ satisfying that for every $m,n\in\mathbb{N}$:

$$\left|(\lambda^{\frac{1}{2}}|\xi|)^m\frac{\partial^n f}{\partial t^n}(\xi,t)\right|\leq(\lambda a)^{\frac{m}{2}}(\lambda b)^n e^{-\lambda t|\xi|^2}f^0_{m,n}(\xi),$$

for every $(\xi,t)\in\mathbb{R}_+^{d+1}$, for some $\lambda>0$, $f^0_{m,n}\in\mathcal{E}^+$, $a,b\in\mathbb{R}_{>0}$.

Remark 11. Note that we have a uniform exponent $\lambda=\exp\left(|\cdot|^m\frac{\partial^n f}{\partial t^n}\right)$ for every $m,n\in\mathbb{N}$.

In the next result we study the behaviour of the restriction of tensor convolution to the space $\mathcal{V}(\mathcal{E})$. For simplicity we will denote it by $\mathcal{V}$.

Theorem 9. *The tensor convolution $*:\mathcal{V}\times\mathcal{V}\to\mathcal{V}$ is a bilinear operator. The set $\mathcal{V}$ is a $\mathcal{E}$-algebra with the product $*$.*

Proof. Let us take $f,g\in\mathcal{V}$ and write:

$$\left|(\beta^{\frac{1}{2}}|\xi|)^{m_1}\frac{\partial^{n_1}f}{\partial t^{n_1}}(\xi,t)\right|\leq(\beta a)^{\frac{m_1}{2}}(\beta b)^{n_1}e^{-\beta t|\xi|^2}f^0_{m_1,n_1}(\xi),\tag{6}$$

and

$$\left|(\beta^{\frac{1}{2}}|\xi|)^{m_2}\frac{\partial^{n_2}g}{\partial t^{n_2}}(\xi,t)\right|\leq(\beta c)^{\frac{m_2}{2}}(\beta d)^{n_2}e^{-\beta t|\xi|^2}g^0_{m_2,n_2}(\xi),\tag{7}$$

for every $(\xi,t)\in\mathbb{R}_+^{d+1}$, for some $\beta>0$, $f^0_{m_1,n_1},g^0_{m_2,n_2}\in\mathcal{E}^+$, $a,b,c,d\in\mathbb{R}_{>0}$.

By Newton binomial and triangle inequality we have for every $\xi, \eta \in \mathbb{R}^d$:

$$|\xi|^m \leq (|\xi - \eta| + |\eta|)^m = \sum_{j=0}^{m} \binom{m}{j} |\xi - \eta|^j |\eta|^{n-j}.$$

Therefore, applying the Leibnitz rule we have:

$$\begin{aligned}
&\left| (\beta^{\frac{1}{2}} |\xi|)^m \frac{\partial^n (f * g)}{\partial t^n}(\xi, t) \right| \\
&= \left| \sum_{r=0}^{n} \binom{n}{r} (\beta^{\frac{1}{2}} |\xi|)^m \left(\frac{\partial^r f}{\partial t^r} * \frac{\partial^{n-r} f}{\partial t^{n-r}} \right)(\xi, t) \right| \\
&\leq \sum_{r=0}^{n} \sum_{j=0}^{m} \binom{m}{j} \binom{n}{r} \left(\left((\beta^{\frac{1}{2}} |\cdot|)^j \left| \frac{\partial^r f}{\partial t^r} \right| \right) * \left((\beta^{\frac{1}{2}} |\cdot|)^{m-j} \left| \frac{\partial^{n-r} g}{\partial t^{n-r}} \right| \right) \right)(\xi, t).
\end{aligned}$$

By Theorem 7 we have that:

$$\left(\left((\beta^{\frac{1}{2}} |\cdot|)^j \left| \frac{\partial^r f}{\partial t^r} \right| \right) * \left((\beta^{\frac{1}{2}} |\cdot|)^{m-j} \left| \frac{\partial^{n-r} g}{\partial t^{n-r}} \right| \right) \right)(\xi, t) \leq (\beta a)^{\frac{j}{2}} (\beta b)^r (\beta c)^{\frac{m-j}{2}} (\beta d)^{n-r} e^{-\frac{\beta t}{2}|\xi|^2} (f^0_{j,r} * g^0_{m-j,n-r})(\xi).$$

Thus,

$$\begin{aligned}
&\left| (\beta^{\frac{1}{2}} |\xi|)^m \frac{\partial^n (f * g)}{\partial t^n}(\xi, t) \right| \\
&\leq \sum_{r=0}^{n} \sum_{j=0}^{m} \binom{m}{j} \binom{n}{r} (\beta a)^{\frac{j}{2}} (\beta b)^r (\beta c)^{\frac{m-j}{2}} (\beta d)^{n-r} e^{-\frac{\beta t}{2}|\xi|^2} (f^0_{j,r} * g^0_{m-j,n-r})(\xi) \\
&= (\beta^{\frac{1}{2}} (a^{\frac{1}{2}} + c^{\frac{1}{2}}))^m (\beta(b+d))^n e^{-\frac{\beta t}{2}|\xi|^2} (f * g)^0_{m,n}(\xi),
\end{aligned}$$

with $(f * g)^0_{m,n}(\xi) = \max_{0 \leq j \leq m, 0 \leq r \leq n} (f^0_{j,r} * g^0_{m-j,n-r})(\xi)$. Since $(f * g)^0_{m,n} \in \mathcal{E}^+$ for every $m, n \in \mathbb{N}$, we obtain that $f * g \in \mathcal{V}$. $\square$

Remark 12. Note that, after a simple inspection we see that for every $\alpha \geq 0$, $(f *_\alpha g)^0_{m,n} \leq (f *_\alpha g)^0_{m+1,n}$ and $(f *_\alpha g)^0_{m,n} \leq (f *_\alpha g)^0_{m,n+1}$ for every $m, n \in \mathbb{N}$ if $f^0_{m,n} \leq f^0_{m+1,n}$, $f^0_{m,n} \leq f^0_{m,n+1}$ and $g^0_{m,n} \leq g^0_{m+1,n}$, $g^0_{m,n} \leq g^0_{m,n+1}$, for every $m, n \in \mathbb{N}$.

In other words, if $f^0_{m,n} \leq f^0_{m',n'}$ and $g^0_{m,n} \leq g^0_{m',n'}$ for every $m, m', n, n' \in \mathbb{N}$ such that $m \leq m'$ and $n \leq n'$ then $(f *_\alpha g)^0_{m,n} \leq (f *_\alpha g)^0_{m',n'}$ for every $m, m', n, n' \in \mathbb{N}$ such that $m \leq m'$ and $n \leq n'$.

Remark 13. If $a \geq 1$ and $c \geq 1$ in Equations (6) and (7) respectively then we obtain the simpler inequality:

$$\left| (\beta^{\frac{1}{2}} |\xi|)^m \frac{\partial^n (f * g)}{\partial t^n}(\xi, t) \right| \leq (\beta^{\frac{1}{2}} (a + c))^m (\beta(b+d))^n e^{-\frac{\beta t}{2}|\xi|^2} (f * g)^0_{m,n}(\xi).$$

Furthermore, we can write the original inequality using the Riesz convolution $*_1$:

$$\left| (\beta^{\frac{1}{2}} |\xi|)^m \frac{\partial^n (f *_1 g)}{\partial t^n}(\xi, t) \right| \leq (\beta^{\frac{1}{2}} (a^{\frac{1}{2}} + c^{\frac{1}{2}}))^m (\beta(b+d))^n e^{-\frac{\beta t}{2}|\xi|^2} (f *_1 g)^0_{m,n}(\xi).$$

Definition 8. For $\beta > 0$ we denote $\mathcal{V}(\beta) = \left\{ f \in \mathcal{V} \mid \exp\left(|\cdot|^m \frac{\partial^n f}{\partial t^n} \right) \leq \beta, \forall m, n \in \mathbb{N} \right\}$. Additionally we define $\lambda = 4\pi^2 \nu$.

With this definition we can state the following result.

Lemma 5. *For every $f, g \in \mathcal{V}(\lambda)$ and $n \in \mathbb{N}$ we have:*

$$\frac{\partial^n (f \odot g)}{\partial t^n}(\xi, t) = \left(-\lambda |\xi|^2\right)^n (f \odot g)(\xi, t) + 2\pi i K(\xi) \left(\sum_{j=0}^{n-1} \left(-\lambda |\xi|^2\right)^j \frac{\partial^{n-1-j}(f *_1 g)}{\partial t^{n-1-j}}(\xi, t) |\xi| \right) \xi,$$

for every $(\xi, t) \in \mathbb{R}^{d+1}_+$.

Proof. The proof is a consequence of Lemma 1 with:
$L = \frac{\partial}{\partial t}$, $x(\xi, t) = (f \odot g)(\xi, t)$, $y(\xi, t) = 2\pi i K(\xi)(f *_1 g)(\xi, t) |\xi| \xi$, $\alpha(\xi) = -\lambda |\xi|^2$ and:

$$\frac{\partial (f \odot g)}{\partial t}(\xi, t) = \left(-\lambda |\xi|^2\right) (f \odot g)(\xi, t) + 2\pi i K(\xi)(f *_1 g)(\xi, t) |\xi| \xi.$$

□

Remark 14. Note that for every $f \in \mathcal{V}(\lambda)$ and $m, n \in \mathbb{N}$ we can write:

$$\left| (\lambda^{\frac{1}{2}} |\xi|)^m \frac{\partial^n f}{\partial t^n}(\xi, t) \right| \le (\lambda a)^{\frac{m}{2}} (\lambda b)^n e^{-\beta t |\xi|^2} f^0_{m,n}(\xi), \tag{8}$$

for every $(\xi, t) \in \mathbb{R}^{d+1}_+$, for $\beta = \exp(f)$. Furthermore, by taking $a \vee b = \max\{a, b\}$ we can simplify the inequality to:

$$\left| (\lambda^{\frac{1}{2}} |\xi|)^m \frac{\partial^n f}{\partial t^n}(\xi, t) \right| \le (\lambda (a \vee b))^{\frac{m}{2}+n} e^{-\beta t |\xi|^2} f^0_{m,n}(\xi), \tag{9}$$

for every $(\xi, t) \in \mathbb{R}^{d+1}_+$.

We use the notation considered so far to state the following result.

Theorem 10. *For every $m, n \in \mathbb{N}$, $f, g \in \mathcal{V}(\lambda)$ we have*

$$\begin{aligned} &\left| (\lambda^{\frac{1}{2}} |\xi|)^m \frac{\partial^n (f \odot g)}{\partial t^n}(\xi, t) \right| \\ &\le \frac{(\lambda (a^{\frac{1}{2}} + c^{\frac{1}{2}})^2)^{\frac{m}{2}+1} (\lambda (b+d))^{n-1}}{\pi \nu} \left(\frac{\left(a^{\frac{1}{2}} + c^{\frac{1}{2}}\right)^{2n} - (b+d)^n}{(b+d)^{n-1} \left[\left(a^{\frac{1}{2}} + c^{\frac{1}{2}}\right)^2 - (b+d) \right]} \right) e^{-\frac{\beta t}{2} |\xi|^2} (f \odot g)^0_{m,n}(\xi), \end{aligned}$$

for some $(f \odot g)^0_{m,n} \in \mathcal{E}^+$.

Proof. Let us take $f, g \in \mathcal{V}(\lambda)$ and use Remark 14 to write:

$$\left| (\lambda^{\frac{1}{2}} |\xi|)^{m_1} \frac{\partial^{n_1} f}{\partial t^{n_1}}(\xi, t) \right| \le (\lambda a)^{\frac{m_1}{2}} (\lambda b)^{n_1} e^{-\beta t |\xi|^2} f^0_{m_1,n_1}(\xi), \tag{10}$$

and

$$\left| (\lambda^{\frac{1}{2}} |\xi|)^{m_2} \frac{\partial^{n_2} g}{\partial t^{n_2}}(\xi, t) \right| \le (\lambda c)^{\frac{m_2}{2}} (\lambda d)^{n_2} e^{-\beta t |\xi|^2} g^0_{m_2,n_2}(\xi), \tag{11}$$

for every $(\xi, t) \in \mathbb{R}^{d+1}_+$, for some $0 < \beta \le \lambda$, $f^0_{m_1,n_1}, g^0_{m_2,n_2} \in \mathcal{E}^+$, $a, b, c, d \in \mathbb{R}_{>0}$.

By Theorem 9 and Remark 14 we have:

$$\left| (\lambda^{\frac{1}{2}} |\xi|)^m \frac{\partial^n (f *_1 g)}{\partial t^n}(\xi, t) \right| \le (\lambda^{\frac{1}{2}} (a^{\frac{1}{2}} + c^{\frac{1}{2}}))^m (\lambda (b+d))^n e^{-\frac{\beta t}{2} |\xi|^2} (f *_1 g)^0_{m,n}(\xi).$$

Using the case $n = 0$ and Corollary 14 we obtain:

$$\left|(\lambda^{\frac{1}{2}}|\xi|)^m (f \odot g)(\xi, t)\right| \leq \frac{(\lambda^{\frac{1}{2}}(a^{\frac{1}{2}} + c^{\frac{1}{2}}))^m}{\pi\nu} e^{-\frac{\beta t}{2}|\xi|^2} (f *_1 g)^0_{m,0}(\xi).$$

Therefore,

$$\begin{aligned}
&\left|(\lambda^{\frac{1}{2}}|\xi|)^m \frac{\partial^n (f \odot g)}{\partial t^n}(\xi, t)\right| \leq (\lambda^{\frac{1}{2}}|\xi|)^{m+2n} |(f \odot g)(\xi, t)| + \frac{2\pi}{\lambda} \sum_{j=0}^{n-1} (\lambda^{\frac{1}{2}}|\xi|)^{m+2j+2} \left|\frac{\partial^n (f *_1 g)}{\partial t^n}(\xi, t)\right| \\
&\leq \frac{(\lambda^{\frac{1}{2}}(a^{\frac{1}{2}} + c^{\frac{1}{2}}))^{m+2n}}{\pi\nu} e^{-\frac{\beta t}{2}|\xi|^2} (f *_1 g)^0_{m+2n,0}(\xi) \\
&+ \frac{2\pi}{\lambda} \sum_{j=0}^{n-1} (\lambda^{\frac{1}{2}}(a^{\frac{1}{2}} + c^{\frac{1}{2}}))^{m+2j+2} (\lambda(b+d))^{n-1-j} e^{-\frac{\beta t}{2}|\xi|^2} (f *_1 g)^0_{m+2j+2,n-1-j}(\xi) \\
&\leq \frac{(\lambda^{\frac{1}{2}}(a^{\frac{1}{2}} + c^{\frac{1}{2}}))^{m+2n}}{\pi\nu} e^{-\frac{\beta t}{2}|\xi|^2} (f *_1 g)^0_{m+2n,0}(\xi) \\
&+ \frac{2\pi}{\lambda} (\lambda^{\frac{1}{2}}(a^{\frac{1}{2}} + c^{\frac{1}{2}}))^{m+2} (\lambda(b+d))^{n-1} \left(\sum_{j=0}^{n-1} \left[\frac{(a^{\frac{1}{2}} + c^{\frac{1}{2}})^2}{b+d}\right]^j \right) e^{-\frac{\beta t}{2}|\xi|^2} (f \odot g)^0_{m,n}(\xi),
\end{aligned}$$

with $(f \odot g)^0_{m,n}(\xi) = \max_{0 \leq j \leq n-1} (f *_1 g)^0_{m+2j+2,n-1-j}(\xi)$.

However,

$$(\lambda^{\frac{1}{2}}(a^{\frac{1}{2}} + c^{\frac{1}{2}}))^{m+2n} \leq (\lambda^{\frac{1}{2}}(a^{\frac{1}{2}} + c^{\frac{1}{2}}))^{m+2} (\lambda(b+d))^{n-1} \left(\sum_{j=0}^{n-1} \left[\frac{(a^{\frac{1}{2}} + c^{\frac{1}{2}})^2}{b+d}\right]^j \right).$$

In fact, the term on the left hand side is the term on the right hand side when $j = n-1$. Additionally, $(f *_1 g)^0_{m+2n,0}(\xi) \leq (f \odot g)^0_{m,n}(\xi)$ and

$$\sum_{j=0}^{n-1} \left[\frac{\left(a^{\frac{1}{2}} + c^{\frac{1}{2}}\right)^2}{b+d}\right]^j = \frac{\left(\frac{(a^{\frac{1}{2}}+c^{\frac{1}{2}})^2}{b+d}\right)^n - 1}{\frac{\left(a^{\frac{1}{2}}+c^{\frac{1}{2}}\right)^2}{b+d} - 1} = \frac{\left(a^{\frac{1}{2}} + c^{\frac{1}{2}}\right)^{2n} - (b+d)^n}{(b+d)^{n-1}\left[\left(a^{\frac{1}{2}} + c^{\frac{1}{2}}\right)^2 - (b+d)\right]}.$$

Thus,

$$\begin{aligned}
&\left|(\lambda^{\frac{1}{2}}|\xi|)^m \frac{\partial^n (f \odot g)}{\partial t^n}(\xi, t)\right| \\
&\leq \frac{(\lambda(a^{\frac{1}{2}} + c^{\frac{1}{2}})^2)^{\frac{m}{2}+1} (\lambda(b+d))^{n-1}}{\pi\nu} \left(\frac{\left(a^{\frac{1}{2}} + c^{\frac{1}{2}}\right)^{2n} - (b+d)^n}{(b+d)^{n-1}\left[\left(a^{\frac{1}{2}} + c^{\frac{1}{2}}\right)^2 - (b+d)\right]} \right) e^{-\frac{\beta t}{2}|\xi|^2} (f \odot g)^0_{m,n}(\xi),
\end{aligned}$$

□

Corollary 15. *If* $(a^{\frac{1}{2}} + c^{\frac{1}{2}})^2 \leq 2(b+d)$ *then:*

$$\left|(\lambda^{\frac{1}{2}}|\xi|)^m \frac{\partial^n (f \odot g)}{\partial t^n}(\xi, t)\right| \leq \frac{2^{n+1}(\lambda(a^{\frac{1}{2}} + c^{\frac{1}{2}})^2)^{\frac{m}{2}} (\lambda(b+d))^n}{\pi\nu} e^{-\frac{\beta t}{2}|\xi|^2} (f \odot g)^0_{m,n}(\xi),$$

for every $(\xi, t) \in \mathbb{R}^{d+1}_+$.

Remark 15. Note that in Corollary 15 the condition is satisfied if $a \le b$ and $c \le d$.

Corollary 16. *If $a \ge b$ and $c \ge d$ then:*

$$\left|(\lambda^{\frac{1}{2}}|\xi|)^m \frac{\partial^n (f \odot g)}{\partial t^n}(\xi, t)\right| \le \frac{(\lambda(a^{\frac{1}{2}} + c^{\frac{1}{2}})^2)^{\frac{m}{2}+n}}{2a^{\frac{1}{2}}c^{\frac{1}{2}}\pi\nu} e^{-\frac{\beta t}{2}|\xi|^2}(f \odot g)^0_{m,n}(\xi),$$

for every $(\xi, t) \in \mathbb{R}^{d+1}_+$.

Proof. In fact, if $a \ge b$ and $c \ge d$ then

$$(a^{\frac{1}{2}} + c^{\frac{1}{2}})^2 - (b + d) = a + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + c - b - d \ge 2a^{\frac{1}{2}}b^{\frac{1}{2}}.$$

□

Remark 16. Observe that $f \in \mathcal{V}(\lambda)$ and $m, n \in \mathbb{N}$ we can write:

$$\left|(\lambda^{\frac{1}{2}}|\xi|)^m \frac{\partial^n f}{\partial t^n}(\xi, t)\right| \le (\lambda a)^{\frac{m}{2}}(\lambda b)^n e^{-\beta t|\xi|^2} f^0_{m,n}(\xi), \tag{12}$$

for every $(\xi, t) \in \mathbb{R}^{d+1}_+$, for $\beta = \exp(f)$. Furthermore, by taking $a \vee b = \max\{a, b\}$ we can simplify the inequality to:

$$\left|(\lambda^{\frac{1}{2}}|\xi|)^m \frac{\partial^n f}{\partial t^n}(\xi, t)\right| \le (\lambda(a \vee b))^{\frac{m}{2}+n} e^{-\beta t|\xi|^2} f^0_{m,n}(\xi), \tag{13}$$

for every $(\xi, t) \in \mathbb{R}^{d+1}_+$. It means that it is enough to consider the case in which $a = b$.

Corollary 17. *If $f, g \in \mathcal{V}(\lambda)$ such that:*

$$\left|(\lambda^{\frac{1}{2}}|\xi|)^{m_1} \frac{\partial^{n_1} f}{\partial t^{n_1}}(\xi, t)\right| \le (\lambda a)^{\frac{m_1}{2}+n_1} e^{-\beta t|\xi|^2} f^0_{m_1,n_1}(\xi), \tag{14}$$

and

$$\left|(\lambda^{\frac{1}{2}}|\xi|)^{m_2} \frac{\partial^{n_2} g}{\partial t^{n_2}}(\xi, t)\right| \le (\lambda c)^{\frac{m_2}{2}+n_2} e^{-\beta t|\xi|^2} g^0_{m_2,n_2}(\xi), \tag{15}$$

for every $(\xi, t) \in \mathbb{R}^{d+1}_+$, for some $0 < \beta \le \lambda$, $f^0_{m_1,n_1}, g^0_{m_2,n_2} \in \mathcal{E}^+$, $a, c \in \mathbb{R}_{>0}$. Then,

$$\left|(\lambda^{\frac{1}{2}}|\xi|)^m \frac{\partial^n (f \odot g)}{\partial t^n}(\xi, t)\right| \le \frac{(\lambda(a^{\frac{1}{2}} + c^{\frac{1}{2}})^2)^{\frac{m}{2}+n}}{2a^{\frac{1}{2}}c^{\frac{1}{2}}\pi\nu} e^{-\frac{\beta t}{2}|\xi|^2}(f \odot g)^0_{m,n}(\xi), \tag{16}$$

for every $(\xi, t) \in \mathbb{R}^{d+1}_+$.

Proof. It is enough to apply Corollary 16 since in this case $a = b$ and $c = d$. □

Corollary 18. *Assume that $\mathcal{E}^+$ is closed by Riezs convolution $*_1$, i.e., $f^0 *_1 g^0 \in \mathcal{E}^+$ for $f^0, g^0 \in \mathcal{E}^+$. Then the product $\odot : \mathcal{V}(\lambda) \times \mathcal{V}(\lambda) \to \mathcal{V}(\lambda)$ is a bilinear operator.*

Proof. This is a direct consequence of Inequality (16) and the properties defining $\mathcal{E}$.

A remarkable case of Corollary 17 is when $a, c \in \mathbb{Z}_+$ and we state now because is fundamental to construct the solution of the Navier-Stokes Equation.

Corollary 19. *Let $f, g \in \mathcal{V}$ such that $a, c \in \mathbb{Z}_+$ then:*

$$\left|(\lambda^{\frac{1}{2}}|\xi|)^m \frac{\partial^n (f \odot g)}{\partial t^n}(\xi, t)\right| \le \frac{(\lambda(a^{\frac{1}{2}} + c^{\frac{1}{2}})^2)^{\frac{m}{2}+n}}{2\pi\nu} e^{-\frac{\beta t}{2}|\xi|^2}(f \odot g)^0_{m,n}(\xi), \tag{17}$$

for every $(\xi, t) \in \mathbb{R}^{d+1}_+$.

Remark 17. Note that we have that:

$$\begin{aligned}(f \odot g)^0_{m,n}(\xi) &= \max_{0\leq j\leq n-1} (f *_1 g)^0_{m+2j+2,n-1-j}(\xi) \\ &= \max_{0\leq j\leq n-1} \max_{\substack{0\leq j\leq m+2j+2 \\ 0\leq r\leq n-1-j}} (f^0_{l,r} *_1 g^0_{m+2j+2-l,n-1-j-r})(\xi) \\ &= \max_{\substack{0\leq r_1+r_2\leq n-1 \\ l_1+l_2+2(r_1+r_2)=m+2n}} (f^0_{l_1,r_1} *_1 g^0_{l_2,r_2})(\xi).\end{aligned}$$

In particular, after a simple inspection we see that $(f \odot g)^0_{m,n} \leq (f \odot g)^0_{m+1,n}$ and $(f \odot g)^0_{m,n} \leq (f \odot g)^0_{m,n+1}$ for every $m, n \in \mathbb{N}$ if $f^0_{m,n} \leq f^0_{m+1,n}$, $f^0_{m,n} \leq f^0_{m,n+1}$ and $g^0_{m,n} \leq g^0_{m+1,n}$, $g^0_{m,n} \leq g^0_{m,n+1}$, for every $m, n \in \mathbb{N}$.

In other words, if $f^0_{m,n} \leq f^0_{m',n'}$ and $g^0_{m,n} \leq g^0_{m',n'}$ for every $m, m', n, n' \in \mathbb{N}$ such that $m \leq m'$ and $n \leq n'$ then $(f \odot g)^0_{m,n} \leq (f \odot g)^0_{m',n'}$ for every $m, m', n, n' \in \mathbb{N}$ such that $m \leq m'$ and $n \leq n'$.

6 Existence and Smoothness of the Navier-Stokes Equation

In this Section we will construct a smooth solution $(u, p) : \mathbb{R}^{d+1}_+ \to \mathbb{R}^{d+1}$ of the Navier-Stokes Equation. We start by defining the following recurrence relation:

$$\begin{aligned} v_0(\xi, t) &= e^{-\lambda t|\xi|^2}\widehat{u^0}(\xi), \\ v_k &= \sum_{j=0}^{k-1} v_j \odot v_{k-1-j},\ k \geq 1. \end{aligned}$$

With our convention $\lambda = 4\pi^2\nu$. We denote by $\{c_k\}_{k\in\mathbb{N}}$ the sequence of Catalan numbers, i.e.,

$$\begin{aligned} c_0 &= 1, \\ c_k &= \frac{1}{k}\binom{2(k-1)}{k-1},\ k \geq 1. \end{aligned}$$

Note that we have the alternative expression

$$c_k = \frac{4^k \left(\frac{1}{2}\right)_k}{k!},\ k \geq 1.$$

Moreover its generating function is given by $c(t) = \frac{1-\sqrt{1-4t}}{2t}$. Additionally, it satisfies the recurrence relation:

$$\begin{aligned} c_0 &= 1, \\ c_k &= \sum_{j=0}^{k-1} c_j c_{k-1-j}. \end{aligned}$$

The sequence $\{v_k\}_{k=0}^{\infty}$ satisfies a remarkable family of inequalities that we state in the incoming result.

Proposition 8. *For every $m, n, k \in \mathbb{N}$ we obtain:*

$$\left|(\lambda^{\frac{1}{2}}\,|\xi|)^m \frac{\partial^n v_k}{\partial t^n}(\xi, t)\right| \leq \frac{c_k(\lambda(k+1)^2)^{\frac{m}{2}+n}}{(2\pi\nu)^k} e^{-\frac{\lambda t}{2^k}|\xi|^2} v^0_{k,m,n}(\xi),$$

for some $v^0_{k,m,n} \in \mathcal{D}_1$, $m, n, k \in \mathbb{N}$. Furthermore, for every $k \in \mathbb{N}$, $v^0_{k,m,n} \leq v^0_{k,m',n'}$ for $m, m', n, n' \in \mathbb{N}$ such that $m \leq m'$ and $n \leq n'$.

Proof. By induction over k.

- For $k=0$, we have that
$$\left|(\lambda^{\frac{1}{2}}\,|\xi|)^m \frac{\partial^n v_0}{\partial t^n}(\xi,t)\right| \leq (\lambda\,|\xi|^2)^{\frac{m}{2}+n} e^{-\lambda t|\xi|^2}\left|\widehat{u^0}(\xi)\right|,$$
for all $m,n\in\mathbb{N}$.

  Hence,
$$\left|(\lambda^{\frac{1}{2}}\,|\xi|)^m \frac{\partial^n v_0}{\partial t^n}(\xi,t)\right| \leq \lambda^{\frac{m}{2}+n} e^{-\lambda t|\xi|^2} v^0_{0,m,n}(\xi),$$
with $v^0_{0,m,n}(\xi)=\max\{1,|\xi|\}^{m+2n}\left|\widehat{u^0}(\xi)\right|$ for all $m,n\in\mathbb{N}$.

  Note that $v^0_{0,m,n}=v^0_{0,m+2n,0}$ and $v^0_{0,m,n}\leq v^0_{0,m',n'}$ for $m,m',n,n'\in\mathbb{N}$ such that $m\leq m'$ and $n\leq n'$. Additionally, we define $v^0_{0,\alpha}(\xi)=\max\{1,|\xi|\}^{\alpha}\left|\widehat{u^0}(\xi)\right|$ for $\alpha\in\mathbb{R}$, in particular $v^0_{0,m,n}=v^0_{0,m+2n}$.

- For $k=1$, note that $v_1=v_0\odot v_0$ and by Corollary 19 we have that:
$$\left|(\lambda^{\frac{1}{2}}\,|\xi|)^m \frac{\partial^n v_1}{\partial t^n}(\xi,t)\right| \leq \frac{c_1(4\lambda)^{\frac{m}{2}+n}}{2\pi\nu} e^{-\frac{\lambda t}{2}|\xi|^2}(v_0\odot v_0)^0_{m,n}(\xi).$$

  Therefore, it is enough to take $v^0_{1,m,n}(\xi)=(v_0\odot v_0)^0_{m,n}(\xi)$. Note that by Remark 17, $v^0_{1,m,n}\leq v^0_{1,m',n'}$ for $m,m',n,n'\in\mathbb{N}$ such that $m\leq m'$ and $n\leq n'$.

  Assume the result for $0\leq j<k$, i.e.,
$$\left|(\lambda^{\frac{1}{2}}\,|\xi|)^m \frac{\partial^n v_j}{\partial t^n}(\xi,t)\right| \leq \frac{c_j(\lambda(j+1)^2)^{\frac{m}{2}+n}}{(2\pi\nu)^j} e^{-\frac{\lambda t}{2^j}|\xi|^2} v^0_{j,m,n}(\xi).$$

  Using the recursion relation we have:
$$\begin{aligned}\left|(\lambda^{\frac{1}{2}}\,|\xi|)^m \frac{\partial^n v_k}{\partial t^n}(\xi,t)\right| &\leq \sum_{j=0}^{k-1}\left|(\lambda^{\frac{1}{2}}\,|\xi|)^m \frac{\partial^n (v_j\odot v_{k-1-j})}{\partial t^n}(\xi,t)\right|\\ &\leq \sum_{j=0}^{k-1}\frac{c_jc_{k-1-j}(\lambda([(j+1)^2]^{\frac{1}{2}}+[(k-j)^2]^{\frac{1}{2}})^2)^{\frac{m}{2}+n}}{(2\pi\nu)(2\pi\nu)^j(2\pi\nu)^{k-1-j}} e^{-\frac{1}{2}\min\left\{\frac{\lambda t}{2^j},\frac{\lambda t}{2^{k-1-j}}\right\}|\xi|^2}(v_j\odot v_{k-1-j})^0_{m,n}(\xi)\\ &\leq \frac{c_k(\lambda(k+1)^2)^{\frac{m}{2}+n}}{(2\pi\nu)^k} e^{-\frac{\lambda t}{2^k}|\xi|^2} v^0_{k,m,n}(\xi),\end{aligned}$$
with $v^0_{k,m,n}(\xi)=\max_{0\leq j\leq k-1}(v_j\odot v_{k-1-j})^0_{m,n}(\xi)$.

  Note that by Remark 17, $v^0_{k,m,n}\leq v^0_{k,m',n'}$ for $m,m',n,n'\in\mathbb{N}$ such that $m\leq m'$ and $n\leq n'$.

  Additionally, if we expand using the definition of $(f\odot g)^0$ for $f,g\in\mathcal{V}(\lambda)$ we have:
$$\begin{aligned}v^0_{k,m,n}(\xi) &= \max_{0\leq j\leq k-1}\max_{0\leq q\leq n-1}\max_{\substack{0\leq l\leq m+2q+2\\ 0\leq r\leq n-1-q}}(v^0_{j,l,r}\odot v^0_{k-1-j,m+2q+2-l,n-1-q-r})(\xi)\\ &= \max_{0\leq j\leq k-1}\max_{\substack{0\leq n_1+n_2\leq n-1\\ m_1+m_2+2(n_1+n_2)=m+2n}}(v^0_{j,m_1,n_1}*_1 v^0_{k-1-j,m_2,n_2})(\xi).\end{aligned}$$

□

Note that $v^0_{k,m,n} \in \mathcal{D}_1$ for $m,n,k \in \mathbb{N}$, in the following result we see that we can bound every of such element by a monimial in the nonassociative algebra $(\mathcal{D}_1, *_1)$, since $d \geq 3$.

Theorem 11. *For $m,n,k \in \mathbb{N}$ we have that:*

$$v^0_{k,m,n}(\xi) \leq \max_{\substack{0 \leq l_1+\cdots+l_{k+1} \leq m+2n \\ w \in M(x_1,\cdots,x_{k+1})}} w^{*1}(v^0_{0,l_1}, v^0_{0,l_2}, \cdots, v^0_{0,l_{k+1}})(\xi).$$

Proof. By induction over k.

- For $k=0$ it is obvious since

$$v^0_{0,m,n}(\xi) = v^0_{0,m+2n}(\xi) = \max_{0 \leq l \leq m+2n} v^0_{0,l}(\xi).$$

- Assume that it is true for every $0 \leq j < k$. For $0 \leq j \leq k-1$ we write:

$$v^0_{j,m_1,n_1}(\xi) \leq \max_{\substack{0 \leq r_1+\cdots+r_{j+1} \leq m_1+2n_1 \\ w \in M(y_1,\cdots,y_{j+1})}} w^{*1}(v^0_{0,r_1}, v^0_{0,r_2}, \cdots, v^0_{0,r_{k+1}})(\xi).$$

  Therefore,

$$\begin{aligned}
v^0_{k,m,n}(\xi) &= \max_{0 \leq j \leq k-1} \max_{\substack{0 \leq n_1+n_2 \leq n-1 \\ m_1+m_2+2(n_1+n_2)=m+2n}} (v^0_{j,m_1,n_1} *_1 v^0_{k-1-j,m_2,n_2})(\xi) \\
&\leq \max_{0 \leq j \leq k-1} \max_{\substack{0 \leq q \leq n-1 \\ m_1+m_2=m+2q+2 \\ n_1+n_2=n-1-q}} \left(\max_{\substack{0 \leq r_1+\cdots+r_{j+1} \leq m_1+2n_1 \\ w \in M(y_1,\cdots,y_{j+1})}} w^{*1}(v^0_{0,r_1}, v^0_{0,r_2}, \cdots, v^0_{0,r_{j+1}}) \right) \\
&*_1 \left(\max_{\substack{0 \leq s_1+\cdots+s_{k-j} \leq m_2+2n_2 \\ w \in M(z_1,\cdots,z_{k-j})}} w^{*1}(v^0_{0,s_1}, v^0_{0,s_2}, \cdots, v^0_{0,s_{k-j}}) \right)(\xi) \\
&\leq \max_{\substack{0 \leq l_1+\cdots+l_{k+1} \leq m+2n \\ w \in M(x_1,\cdots,x_{k+1})}} w^{*1}(v^0_{0,l_1}, v^0_{0,l_2}, \cdots, v^0_{0,l_{k+1}})(\xi).
\end{aligned}$$

□

Note that by Theorem 6, Corollary 12 and Theorem 11 we obtain:

$$\begin{aligned}
\sup_{|\xi| \leq 1} |\xi|\, v^0_{k,m,n}(\xi) &\leq \max_{\substack{0 \leq l_1+\cdots+l_{k+1} \leq m+2n \\ w \in M(x_1,\cdots,x_{k+1})}} \sup_{|\xi| \leq 1} |\xi|\, w^{*1}(v^0_{0,l_1}, v^0_{0,l_2}, \cdots, v^0_{0,l_{k+1}})(\xi) \\
&\leq C^{k-1} \max_{0 \leq l_1+\cdots+l_{k+1} \leq m+2n} \prod_{j=1}^{k+1} \left\| v^0_{0,l_j} \right\|_{1 \oplus p \oplus p'},
\end{aligned}$$

and for every $\beta \geq 0$:

$$\begin{aligned}
\sup_{|\xi| \geq 1} |\xi|^\beta\, v^0_{k,m,n}(\xi) &\leq \max_{\substack{0 \leq l_1+\cdots+l_{k+1} \leq m+2n \\ w \in M(x_1,\cdots,x_{k+1})}} \sup_{|\xi| \geq 1} |\xi|^\beta\, w^{*1}(v^0_{0,l_1}, v^0_{0,l_2}, \cdots, v^0_{0,l_{k+1}})(\xi) \\
&\leq 2^\beta \max_{\substack{0 \leq l_1+\cdots+l_{k+1} \leq m+2n \\ w \in M(x_1,\cdots,x_{k+1})}} \sup_{|\xi| \geq 1} w^{*1}(v^0_{0,l_1+\beta}, v^0_{0,l_2+\beta}, \cdots, v^0_{0,l_{k+1}+\beta})(\xi) \\
&\leq 2^\beta C^{k-1} \max_{0 \leq l_1+\cdots+l_{k+1} \leq m+2n} \prod_{j=1}^{k+1} \left\| v^0_{0,l_j+\beta} \right\|_{1 \oplus p \oplus p'}.
\end{aligned}$$

Remark 18. Note that we can extend the definition of $v^0_{k,m,n}$ for $m \in \mathbb{R}$ by defining $v^0_{k,\beta,n} = |\cdot|^\beta \, v^0_{k,0,n}$.

Corollary 20. *For every $m, n, k \in \mathbb{N}$ and $1 \leq q < d$ we have that:*

$$\left\| v^0_{k,m,n} \right\|_{1 \oplus q} \leq C^{k+\frac{d+1}{2}} \max_{0 \leq l_1 + \cdots + l_{k+1} \leq m+2n} \prod_{j=1}^{k+1} \left\| v^0_{0,l_j+\frac{d+1}{2}} \right\|_{1 \oplus p \oplus p'}.$$

Proof. Note that for every $1 \leq q < d$ we have:

$$\begin{aligned}
&\left\| v^0_{k,m,n} \right\|^q_{L^q(\mathbb{R}^q)} = \int_{\mathbb{R}^d} v^0_{k,m,n}(\xi)^q d\xi \\
&= \int_{|\xi| \leq 1} \frac{|\xi|^q \, v^0_{k,m,n}(\xi)^q}{|\xi|^q} d\xi + \int_{|\xi| \geq 1} \frac{|\xi|^{\left(\frac{d+1}{2}\right)q} \, v^0_{k,m,n}(\xi)^q}{|\xi|^{\left(\frac{d+1}{2}\right)q}} d\xi \\
&\leq C^{kq} \max_{0 \leq l_1 + \cdots + l_{k+1} \leq m+2n} \prod_{j=1}^{k+1} \left\| v^0_{0,l_j} \right\|^q_{1 \oplus p \oplus p'} + C^{\left(k+\frac{d+1}{2}\right)q} \max_{0 \leq l_1 + \cdots + l_{k+1} \leq m+2n} \prod_{j=1}^{k+1} \left\| v^0_{0,l_j+\frac{d+1}{2}} \right\|^q_{1 \oplus p \oplus p'} \\
&\leq C^{\left(k+\frac{d+1}{2}\right)q} \max_{0 \leq l_1 + \cdots + l_{k+1} \leq m+2n} \prod_{j=1}^{k+1} \left\| v^0_{0,l_j+\frac{d+1}{2}} \right\|^q_{1 \oplus p \oplus p'}.
\end{aligned}$$

Therefore,

$$\left\| v^0_{k,m,n} \right\|_{L^q(\mathbb{R}^q)} \leq C^{k+\frac{d+1}{2}} \max_{0 \leq l_1 + \cdots + l_{k+1} \leq m+2n} \prod_{j=1}^{k+1} \left\| v^0_{0,l_j+\frac{d+1}{2}} \right\|_{1 \oplus p \oplus p'}. \tag{18}$$

If we apply Equation (18) and the special case for $q = 1 < d$ and sum we obtain:

$$\left\| v^0_{k,m,n} \right\|_{1 \oplus q} \leq C^{k+\frac{d+1}{2}} \max_{0 \leq l_1 + \cdots + l_{k+1} \leq m+2n} \prod_{j=1}^{k+1} \left\| v^0_{0,l_j+\frac{d+1}{2}} \right\|_{1 \oplus p \oplus p'}.$$

□

Corollary 21. *For every $m, n, k \in \mathbb{N}$, $1 \leq q < d$ we have that :*

$$\left\| v^0_{k,m,n} \right\|_{1 \oplus q} \leq C^{k+\frac{d+1}{2}} \left\| v_{0,m+2n+\frac{d+1}{2}} \right\|^{k+1}_{1 \oplus p \oplus p'},$$

if $0 \leq k \leq m + 2n$,

$$\left\| v^0_{k,m,n} \right\|_{1 \oplus q} \leq C^{k+\frac{d+1}{2}} \left\| v_{0,m+2n+\frac{d+1}{2}} \right\|^{m+2n}_{1 \oplus p \oplus p'} \left\| v_{0,\frac{d+1}{2}} \right\|^{k+1-m-2n}_{1 \oplus p \oplus p'},$$

if $k \geq m + 2n + 1$.

Proof. We consider the inequalities of the form:

$$0 \leq l_1 + \cdots + l_{k+1} \leq m + 2n, \text{ for } l_1, \cdots, l_{k+1} \geq 0.$$

Note that $0 \leq l_j \leq m + 2n$ implies:

$$v^0_{0,l_j+\frac{d+1}{2}} \leq v^0_{0,m+2n+\frac{d+1}{2}}.$$

- If $0 \leq k \leq m + 2n$ then

$$\left\| v^0_{k,m,n} \right\|_{1 \oplus q} \leq C^{k+\frac{d+1}{2}} \left\| v^0_{0,m+2n+\frac{d+1}{2}} \right\|^{k+1}_{1 \oplus p \oplus p'},$$

- If $k \geq m+2n+1$, since $l_1 + \cdots + l_{k+1} \leq m+2n$ we can assume that without loss of generality $l_{m+2n+1} = \cdots = l_{k+1} = 0$, therefore

$$\begin{aligned}\left\|v^0_{k,m,n}\right\|_{1\oplus q} &\leq C^{k+\frac{d+1}{2}} \prod_{j=1}^{m+2n} \left\|v^0_{0,l_j+\frac{d+1}{2}}\right\|_{1\oplus p\oplus p'} \prod_{j=m+2n+1}^{k+1} \left\|v^0_{0,l_j+\frac{d+1}{2}}\right\|_{1\oplus p\oplus p'} \\ &\leq C^{k+\frac{d+1}{2}} \left\|v^0_{0,m+2n+\frac{d+1}{2}}\right\|^{m+2n}_{1\oplus p\oplus p'} \left\|v^0_{0,\frac{d+1}{2}}\right\|^{k+1-m-2n}_{1\oplus p\oplus p'}.\end{aligned}$$

□

Corollary 22. *For every $m, n, k \in \mathbb{N}$ we have that:*

$$\left\|(\lambda^{\frac{1}{2}}\,|\cdot|)^m \frac{\partial^n v_k}{\partial t^n}\right\|_{1\oplus 2,\infty} \leq \frac{c_k(\lambda(k+1)^2)^{\frac{m}{2}+n}}{(2\pi\nu)^k} C^{k+\frac{d+1}{2}} \left\|v^0_{0,m+2n+\frac{d+1}{2}}\right\|^{k+1}_{1\oplus 2},$$

if $0 \leq k \leq m+2n$,

$$\left\|(\lambda^{\frac{1}{2}}\,|\cdot|)^m \frac{\partial^n v_k}{\partial t^n}\right\|_{1\oplus 2,\infty} \leq \frac{c_k(\lambda(k+1)^2)^{\frac{m}{2}+n}}{(2\pi\nu)^k} C^{k+\frac{d+1}{2}} \left\|v^0_{0,m+2n+\frac{d+1}{2}}\right\|^{m+2n}_{1\oplus 2} \left\|v^0_{0,\frac{d+1}{2}}\right\|^{k+1-m-2n}_{1\oplus 2},$$

if $k \geq m+2n+1$.

Proof. We use the family of inequalities of Proposition 8 and Corollary 21 to obtain this result. □

Now we consider the Banach space $\mathcal{B} = L^{1\oplus 2}(\mathbb{R}^d, \mathbb{C}^d)$ and the associated space of functions that decrease fast

$$\mathcal{E}_{\mathcal{B}} = \left\{\phi : \mathbb{R}^d \to \mathbb{C}^d \mid M^n(\phi) \in \mathcal{B}, \forall n \in \mathbb{N}\right\}.$$

Corollary 23. *There exists $v \in C^\infty([0,\infty), \mathcal{E}_{\mathcal{B}})$ such that*

$$v = v_0 + v^{\odot 2},$$

for ν large enough.

Proof. Let us consider $v = \sum_{k=0}^\infty v_k$. By Corollary 22 we have that $v \in C^\infty([0,\infty), \mathcal{E}_{\mathcal{B}})$ for ν large enough.

In fact, note that

$$\begin{aligned}&\sum_{k=0}^{m+2n} \frac{c_k(\lambda(k+1)^2)^{\frac{m}{2}+n}}{(2\pi\nu)^k} C^{k+\frac{d+1}{2}} \left\|v_{0,m+2n+\frac{d+1}{2}}\right\|^{k+1}_{1\oplus 2} \\ &+ \sum_{k=m+2n+1}^{\infty} \frac{c_k(\lambda(k+1)^2)^{\frac{m}{2}+n}}{(2\pi\nu)^k} C^{k+\frac{d+1}{2}} \left\|v_{0,m+2n+\frac{d+1}{2}}\right\|^{m+2n}_{1\oplus 2} \left\|v_{0,\frac{d+1}{2}}\right\|^{k+1-m-2n}_{1\oplus 2},\end{aligned}$$

converges after applying the Ratio test:

$$\lim_{k\to\infty} \left(\frac{c_{k+1}}{c_k}\right)\left(\frac{\lambda(k+2)^2}{\lambda(k+1)^2}\right)\left(\frac{1}{2\pi\nu}\right) C \left\|v^0_{0,\frac{d+1}{2}}\right\|_{1\oplus 2} = \frac{4C}{2\pi\nu}\left\|v^0_{0,\frac{d+1}{2}}\right\|_{1\oplus 2} = \frac{2C}{\pi\nu}\left\|v^0_{0,\frac{d+1}{2}}\right\|_{1\oplus 2} < 1,$$

if and only if $\nu > \frac{2C}{\pi}\left\|v^0_{0,\frac{d+1}{2}}\right\|_{1\oplus 2}$ independently on m and n.

Furthermore, we have

$$\begin{aligned}v &= \sum_{k=0}^\infty v_k = v_0 + \sum_{k=1}^\infty v_k = v_0 + \sum_{k=1}^\infty \sum_{j=0}^{k-1} v_j \odot v_{k-1-j} \\ &= v_0 + \sum_{j=0}^\infty \sum_{k=j+1}^\infty v_j \odot v_{k-1-j} = v_0 + \left(\sum_{j=0}^\infty v_j\right)^{\odot 2} = v_0 + v^{\odot 2}.\end{aligned}$$

$\square$

Theorem 12. *Let $q : \mathbb{R}_+^{d+1} \to \mathbb{C}$ defined by*

$$q(\xi, t) = 2\pi i \left(\left(\frac{\xi \otimes \xi}{|\xi \otimes \xi|} \right) (v * v)(\xi, t)\xi \right),$$

then,

$$\frac{\partial v}{\partial t}(\xi, t) = -4\pi^2 \nu v + 2\pi i (v * v)(\xi, t)\xi - q(\xi, t).$$

Proof. In fact, since $v = v_0 + v^{\odot 2}$ we have that:

$$\frac{\partial v}{\partial t} = \frac{\partial v_0}{\partial t} + \frac{\partial v^{\odot 2}}{\partial t} = -4\pi^2 \nu v_0 + \frac{\partial v^{\odot 2}}{\partial t}.$$

However, by Lemma 5 we have for every $f, g \in \mathcal{V}(\lambda)$ and $n \in \mathbb{N}$:

$$\frac{\partial^n (f \odot g)}{\partial t^n}(\xi, t) = \left(-\lambda |\xi|^2\right)^n (f \odot g)(\xi, t) + 2\pi i K(\xi) \left(\sum_{j=0}^{n-1} \left(-\lambda |\xi|^2\right)^j \frac{\partial^{n-1-j}(f *_1 g)}{\partial t^{n-1-j}}(\xi, t) |\xi| \right) \xi.$$

Taking $f = g = v$, $n = 1$ and $K(\xi) = I_d - \frac{\xi \otimes \xi}{|\xi \otimes \xi|}$ with $I_d \in \mathbb{R}^{d \times d}$ the identity matrix we obtain:

$$\frac{\partial v^{\odot 2}}{\partial t} = -4\pi^2 \nu v^{\odot 2} + 2\pi i K(\xi)(v * v)(\xi, t)\xi = -4\pi^2 \nu v^{\odot 2} + 2\pi i (v * v)(\xi, t)\xi - q(\xi, t).$$

Thus,

$$\frac{\partial v}{\partial t} = -4\pi^2 \nu v_0 - 4\pi^2 \nu v^{\odot 2} + 2\pi i (v * v)(\xi, t)\xi - q(\xi, t) = -4\pi^2 \nu v + 2\pi i (v * v)(\xi, t)\xi - q(\xi, t).$$

$\square$

Corollary 24. *Define $(u, p) : \mathbb{R}_+^{d+1} \to \mathbb{R}^d$,*

$$u(x, t) = \int_{\mathbb{R}^d} v(\xi, t) e^{-2\pi i x \cdot \xi} d\xi,$$

$$p(x, t) = -\frac{1}{2\pi i} \int_{\mathbb{R}^d} \frac{\xi^T q(\xi, t)}{|\xi|^2} e^{-2\pi i x \cdot \xi} d\xi,$$

then (u, p) is the solution of the Navier-Stokes Equation for $\nu > 0$ large enough.

Proof. By Corollary 23 we have that:

$$\left| (\lambda^{\frac{1}{2}} |\cdot|)^m \frac{\partial^n v}{\partial t^n} \right| \in L^{1 \oplus 2}(\mathbb{R}_+^{d+1}),$$

for every $m, n \in \mathbb{N}$.

By Theorem 12 we obtain:

$$\frac{\partial v}{\partial t} = -4\pi^2 \nu v_0 - 4\pi^2 \nu v^{\odot 2} + 2\pi i (v * v)(\xi, t)\xi - q(\xi, t) = -4\pi^2 \nu v + 2\pi i (v * v)(\xi, t)\xi - q(\xi, t).$$

Additionally, since:

$$q(\xi, t) = 2\pi i \left(\left(\frac{\xi \otimes \xi}{|\xi \otimes \xi|} \right) (v * v)(\xi, t)\xi \right),$$

we deduce that $\frac{\xi\otimes\xi}{|\xi\otimes\xi|}q(\xi,t) = q(\xi,t)$ and:

$$\nabla p(x,t) = \int_{\mathbb{R}^d} \frac{\xi\otimes\xi}{|\xi\otimes\xi|} q(\xi,t) e^{-2\pi i x\cdot\xi} d\xi = \int_{\mathbb{R}^d} q(\xi,t) e^{-2\pi i x\cdot\xi} d\xi.$$

On the other hand, note that:

$$\begin{aligned}
&2\pi i \int_{\mathbb{R}^d} (v * v)(\xi,t)\xi e^{-2\pi i x\cdot\xi} d\xi \\
&= 2\pi i \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v(\xi-\eta,t)\otimes v(\eta,t)\xi e^{-2\pi i x\cdot\xi} d\eta d\xi \\
&= 2\pi i \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v(\xi-\eta,t)\otimes v(\eta,t)\xi e^{-2\pi i x\cdot\xi} d\xi d\eta \\
&= 2\pi i \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v(\xi,t)\otimes v(\eta,t)(\xi+\eta) e^{-2\pi i x\cdot(\xi+\eta)} d\eta d\xi \\
&= 2\pi i \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v(\xi,t)v(\eta,t)^T\xi e^{-2\pi i x\cdot(\xi+\eta)} d\eta d\xi \\
&= 2\pi i \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v(\xi,t)\xi^T v(\eta,t) e^{-2\pi i x\cdot(\xi+\eta)} d\eta d\xi \\
&= -\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v(\xi,t)(-2\pi i\xi^T) e^{-2\pi i x\cdot\xi} d\xi\right)\left(\int_{\mathbb{R}^d} v(\eta,t) e^{-2\pi i x\cdot\eta} d\eta\right) \\
&= -\frac{\partial u}{\partial x}(x,t)u(x,t).
\end{aligned}$$

Therefore,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}u = \nu\Delta u - \nabla p.$$

Furthermore, $(u,p) \in C^\infty(\mathbb{R}^{d+1}_+, \mathbb{R}^{d+1})$, $u(x,0) = u^0(x)$, $div(u)(x,t) = \int_{\mathbb{R}^d} 2\pi i\xi^T v(\xi,t)e^{2\pi i x\cdot\xi}d\xi = 0$.

Therefore, have bounded energy for all the derivatives of u since by Plancherel identity:

$$\left\|\frac{\partial^n}{\partial t^n}\left(\frac{\partial^\gamma u}{\partial x^\gamma}\right)\right\|_{L^{2,\infty}(\mathbb{R}^{d+1}_+)} \le \left\|(4\pi^2 |\cdot|^2)^{|\gamma|}\frac{\partial^n v}{\partial t^n}\right\|_{L^{2,\infty}(\mathbb{R}^{d+1}_+)} < \infty,$$

for every multi-index $\gamma \in \mathbb{N}^d$ and $n \in \mathbb{N}$. □

Thus, we have the existence of a smooth solution of the Navier-Stokes Equation. In fact, we have a stronger result, the existence of an entire extension $(U(z,t), P(z,t))$ for positive time that we explore in the next section.

7 The existence of the Curve of Entire Vector Fields of order 2

In this Section we show the existence of the curve $(U,P) : \mathbb{C}^{d+1}_{>0} \to \mathbb{C}^{d+1}$ such that $U(,\cdot)$ and $P(,\cdot)$ are entire of order 2 and $U(x,t) = u(x,t)$, $P(x,t) = p(x,t)$ for every $x \in \mathbb{R}^d, t > 0$.

We start with a useful result about uniform convergence.

Lemma 6. *Let $\{g_k\}_{k\in\mathbb{N}} \subset C^0(\mathbb{R}^d - \{0\}, \mathbb{R})$ such that:*

- *There is a sequence $\{r_k\}_{k\in\mathbb{N}}$ such that $r_k \to_{k\to\infty} \infty$ and $\sup_{|x|\ge r_k} g_k(x) \le 0$, for all $k \in \mathbb{N}$.*
- *We have $g_k \to_{k\to\infty} 0$ uniformly in $\mathbb{R}^d - B(0,\delta)$ for some $\delta > 0$. Then, $\sup_{|x|\ge r} g_k(x) \le 0$, for all $k \in \mathbb{N}$, for some $r = r(\delta) \ge 1$.*

Proof. Let $\epsilon > 0$ fixed. Since $r_k \to_{k\to\infty} \infty$ and $g_k \to_{k\to\infty} 0$ uniformly in $\mathbb{R}^d - B(0,\delta)$ we have $r_{k_0} \leq \delta < r_{k_0+1}$ and $\left\|g_k - g_{k_0}\right\|_{L^\infty(\mathbb{R}^d - B(0,\delta))} < \epsilon$ for all $k \geq k_0$. Since $\sup_{|x|\geq r} g_{k_0}(x) \leq \sup_{|x|\geq r_{k_0}} g_{k_0}(x) \leq 0$ we have that $g_{k_0}(x) \leq 0$ for $|x| \geq \delta$.

Therefore,

$$\left|g_k(x) - g_{k_0}(x)\right| \leq \left\|g_k - g_{k_0}\right\|_{L^\infty(\mathbb{R}^d - B(0,\delta))} < \epsilon,$$

for all $k \geq k_0$, $|x| \geq \delta$.

With this we conclude that if $|x| \geq \delta$ and $k \geq k_0$:

$$g_k(x) = (g_k(x) - g_{k_0}(x)) + g_{k_0}(x) \leq \left|g_k(x) - g_{k_0}(x)\right| < \epsilon.$$

Let $r = \max\left\{r_1, \cdots, r_{k_0}, \delta, 1\right\}$ then:

$$\sup_{|x|\geq r} g_j(x) \leq \sup_{|x|\geq r_j} g_j(x) \leq 0 < \epsilon,$$

for $1 \leq j \leq k_0$ and

$$\sup_{|x|\geq r} g_j(x) \leq \sup_{|x|\geq \delta} g_j(x) < \epsilon,$$

for $j \geq k_0$.

In particular, $\sup_{k\in\mathbb{N}} \sup_{|x|\geq r} g_k(x) < \epsilon$ for $\epsilon > 0$ arbitrary. Letting $\epsilon \to 0^+$ we have $\sup_{k\in\mathbb{N}} \sup_{|x|\geq r} g_k(x) \leq 0$. □

Now we have some remarkable spaces.

Definition 9. Let $\mathcal{E}$ be a space of functions that decreasing fast closed by pointwise addition, convolution and maximum .

Let $d \geq 3$ and $\alpha \geq 1$ we define $\mathcal{V}_\alpha(\mathcal{E})$ be the complex space generated by functions $f : \mathbb{R}^{d+1}_+ \to \mathbb{C}^{d\times 1}$ such that $f \in \mathcal{C}(\mathcal{E})_0$, $f(\xi, \cdot) \in C^\infty([0,\infty), \mathbb{C}^d)$ for a.e $\xi \in \mathbb{R}^d$ and we have the automorphisms $\frac{\partial}{\partial t} : \mathcal{V}_\alpha(\mathcal{E}) \to \mathcal{V}_\alpha(\mathcal{E})$ and $|\cdot| : \mathcal{V}_\alpha(\mathcal{E}) \to \mathcal{V}_\alpha(\mathcal{E})$ satisfying that for every $m, n \in \mathbb{N}$:

$$\left|(\lambda^{\frac{1}{2}} |\xi|)^m \frac{\partial^n f}{\partial t^n}(\xi, t)\right| \leq (\lambda a)^{\frac{m}{2}} (\lambda b)^n e^{-\lambda t|\xi|^\alpha} f^0_{m,n}(\xi),$$

for every $(\xi, t) \in \mathbb{R}^{d+1}_+$, for some $\lambda > 0$, $f^0_{m,n} \in \mathcal{E}^+$, $a, b \in \mathbb{R}_{>0}$.

Remark 19. Note that we have a uniform exponent $\lambda = \exp\left(|\cdot|^m \frac{\partial^n f}{\partial t^n}\right)$ for every $m, n \in \mathbb{N}$.

Let us consider the Banach space $\mathcal{B} = L^1(\mathbb{R}^d)$ and $\mathcal{E}$ its associated space of functions decreasing fast. The spaces $\mathcal{V}_\alpha(\mathcal{E})$ for $\alpha > 1$ are interesting because of the following

Theorem 13. *For every $f \in \mathcal{V}_\alpha(\mathcal{E})$ we have that $\hat{f}(\cdot, t)$ has an entire extension to a function $F(\cdot, t) : \mathbb{C}^d \to \mathbb{C}^d$ such that:*

$$|F(z,t)| \leq e^{\left((\alpha-1)\lambda t\left(\frac{\pi}{\lambda t}\right)^{\alpha'} + c\right)|Im(z)|^{\alpha'}},$$

for every $(z, t) \in \mathbb{C}^{d+1}_{>0}$ such that $|Im(z)|$ large enough and some constant $c > 0$.

Proof. Note that by Corollary 29 in Appendix we have that

$$-|\xi|^\alpha + c\xi \cdot \eta \leq (\alpha - 1)\lambda t \left(\frac{\pi}{\lambda t}\right)^{\alpha'} |\eta|^{\alpha'},$$

for $\xi, \eta \in \mathbb{R}^d$.

Furthermore,

$$|f(\xi,t)| \le e^{-\lambda t|\xi|^{\alpha}} f^0(\xi), \text{ for } (\xi,t) \in \mathbb{R}^{d+1}_+.$$

Let us consider the Laplace-Fourier transform:

$$F(z,t) = \int_{\mathbb{R}^d} f(\xi,t) e^{-2\pi i z\cdot\xi} d\xi, \text{ for } (z,t) \in \mathbb{C}^{d+1}_{>0}.$$

If we write $z = x + iy$, $x, y \in \mathbb{R}^d$ then:

$$\begin{aligned} |f(\xi,t)| \left|e^{-2\pi i z\cdot\xi}\right| &= |f(\xi,t)| e^{2\pi y\cdot\xi} \le e^{-\lambda t|\xi|^{\alpha}} e^{2\pi y\cdot\xi} f^0(\xi) \le e^{\lambda t\left(-|\xi|^{\alpha} + \frac{2\pi}{\lambda t} y\cdot\xi\right)} f^0(\xi) \\ &\le e^{(\alpha-1)\lambda t\left(\frac{\pi}{\lambda t}\right)^{\alpha'} |y|^{\alpha'}} f^0(\xi). \end{aligned} \tag{19}$$

Then the integrand belongs to $L^1(\mathbb{R}^d)$ for every $(z,t) \in \mathbb{C}^{d+1}_{>0}$. We can apply the Morera's Theorem to obtain that F is entire. Furthermore, by Equation (19) we obtain

$$|F(z,t)| \le e^{(\alpha-1)\lambda t\left(\frac{\pi}{\lambda t}\right)^{\alpha'} |Im(z)|^{\alpha'}} \left\|f^0\right\|_{L^1(\mathbb{R}^d)},$$

for every $(z,t) \in \mathbb{C}^{d+1}_{>0}$. □

Corollary 25. *For every $f \in \mathcal{V}_\alpha(\mathcal{E})$ there exists $F : \mathbb{C}^{d+1}_{>0} \to \mathbb{C}^d$ such that $F(\cdot,t)$ is entire for every $t > 0$ and $F(z,\cdot)$ is smooth for every $z \in \mathbb{C}^d$ and $F|_{\mathbb{R}^{d+1}_+} = \hat{f}$.*

Proof. Since $f \in \mathcal{V}_\alpha(\mathcal{E})$ we have:

$$\left|\frac{\partial^n f}{\partial t^n}(\xi,t)\right| \le e^{-\lambda t|\xi|^{\alpha}} f^0_n(\xi),$$

for $(\xi,t) \in \mathbb{R}^{d+1}_+$, $n \in \mathbb{N}$.

Therefore, we can exchange $\frac{\partial^n}{\partial t^n}$ with the integral defining F in such a way that:

$$\frac{\partial^n F}{\partial t^n}(z,t) = \int_{\mathbb{R}^d} \frac{\partial^n f}{\partial t^n}(\xi,t) e^{-2\pi i z\cdot\xi} d\xi.$$

By Theorem 13 we have that $\frac{\partial^n F}{\partial t^n}(\cdot,t)$ is entire for every $n \in \mathbb{N}$, $t > 0$ and $F(z,\cdot) \in C^\infty([0,\infty))$ for every $z \in \mathbb{C}^d$.

Additionally, if $z = x \in \mathbb{R}^d$ we have $F(x,t) = \hat{f}(x,t)$, $t \ge 0$.

Finally, note that:

$$\left|\frac{\partial^n F}{\partial t^n}(z,t)\right| \le e^{(\alpha-1)\lambda t\left(\frac{\pi}{\lambda t}\right)^{\alpha'} |Im(z)|^{\alpha'}} \left\|f^0_n\right\|_{L^1(\mathbb{R}^d)},$$

for every $n \in \mathbb{N}$ and $(z,t) \in \mathbb{C}^{d+1}_{>0}$. □

By results of the previous section we have that $v_k \in \mathcal{V}_2(\mathcal{E})$ for every $k \in \mathbb{N}$ with exponent $\exp(v_k) = \frac{4\pi^2\nu}{2^k}$, therefore we can not find a common positive exponent in order to have $v \in \mathcal{V}_2(\mathcal{E})$. However, we can apply the previous results in this section to obtain that $v \in \mathcal{V}_\alpha(\mathcal{E})$ for every $1 < \alpha < 2$. Now we define $\lambda = 4\pi^2\nu$.

Theorem 14. *There exists a smooth function $(U,P) : \mathbb{C}^{d+1}_{>0} \to \mathbb{C}^{d+1}$ extending $(u,p) : \mathbb{R}^{d+1}_{>0} \to \mathbb{R}^{d+1}$ such that $(U(\cdot,t), P(\cdot,t))$ is entire for $t > 0$,*

$$\left|\frac{\partial^n U}{\partial t^n}(z,t)\right| \le e^{\left((\alpha-1)\lambda t\left(\frac{\pi}{\lambda t}\right)^{\alpha'} + c_1(n)\right)|Im(z)|^{\alpha'}}, \left|\frac{\partial^n P}{\partial t^n}(z,t)\right| \le e^{\left((\alpha-1)r_\alpha\lambda t\left(\frac{\pi}{r_\alpha\lambda t}\right)^{\alpha'} + c_2(n)\right)|Im(z)|^{\alpha'}},$$

for every $(z,t) \in \mathbb{C}^{d+1}_{>0}$ *such that* $|Im(z)|$ *is large enough for some constants* $c_1(n) > 0, c_2(n) > 0$*, for every* $n \in \mathbb{N}$.

Proof. Let $1 < \alpha < 2$ and consider $\xi \in \mathbb{R}^d$ such that $|\xi| \geq 2^{\frac{k}{2-\alpha}}$ then $|\xi|^{2-\alpha} \geq 2^k$, in other words $|\xi|^\alpha \leq \frac{|\xi|^2}{2^k}$. Therefore,

$$\left|(\lambda^{\frac{1}{2}}\,|\xi|)^m \frac{\partial^n v_k}{\partial t^n}(\xi,t)\right| \leq \frac{c_k(\lambda(k+1)^2)^{\frac{m}{2}+n}}{(2\pi\nu)^k} e^{-\lambda t|\xi|^\alpha} v^0_{k,m,n}(\xi),$$

for $|\xi| \geq r_k$ with $r_k = 2^{\frac{k}{2-\alpha}}$.

Define $w_{k,m,n} : (\mathbb{R}^d - \{0\}) \times [0,\infty) \to \mathbb{R}$,

$$w_{k,m,n}(\xi,t) = \left|(\lambda^{\frac{1}{2}}\,|\xi|)^m \frac{\partial^n v_k}{\partial t^n}(\xi,t)\right| - \frac{c_k(\lambda(k+1)^2)^{\frac{m}{2}+n}}{(2\pi\nu)^k} e^{-\lambda t|\xi|^\alpha} v^0_{k,m,n}(\xi).$$

Therefore,

$$\sup_{|\xi|\geq r_k} w_{k,m,n}(\xi) \leq 0, \text{ for } m,n,k \in \mathbb{N}.$$

Additionally applying Theorem 6 with $p = 2$ we have:

$$\sup_{|\xi|\geq 1} w_{k,m,n}(\xi) \leq \frac{2C^{k+\frac{d+1}{2}}c_k}{(2\pi\nu)^k}(\lambda(k+1)^2)^{\frac{m}{2}+n} \left\|v^0_{0,m+2n+\frac{d+1}{2}}\right\|^{m+2n}_{1\oplus 2} \left\|v^0_{0,\frac{d+1}{2}}\right\|^{k+1-m-2n}_{1\oplus 2}$$
$$\leq 2c_k C^{\frac{d+1}{2}} \left(\frac{C}{2\pi\nu}\left\|v^0_{0,\frac{d+1}{2}}\right\|_{1\oplus 2}\right)^k \left(\frac{\left\|v^0_{0,m+2n+\frac{d+1}{2}}\right\|_{1\oplus 2}}{\left\|v^0_{0,\frac{d+1}{2}}\right\|_{1\oplus 2}}\right)^{m+2n} \left\|v^0_{0,\frac{d+1}{2}}\right\|_{1\oplus 2} \to_{k\to\infty} 0,$$

if $\nu > \frac{C}{2\pi}\left\|v^0_{0,\frac{d+1}{2}}\right\|_{1\oplus 2}$.

Then Lemma 6 implies that:

$$\sup_{|\xi|\geq r_{m,n}} w_{k,m,n}(\xi) \leq 0, \text{ for all } m,n,k \in \mathbb{N}, \text{ for some } r_{m,n} \geq 1.$$

Let us define $f_{k,m,n} : \mathbb{R}^{d+1}_+ \to \mathbb{R}$,

$$f_{k,m,n}(\xi,t) = \frac{c_k(\lambda(k+1)^2)^{\frac{m}{2}+n}}{(2\pi\nu)^k} v^0_{k,m,n}(\xi)\chi_{\{0<|\xi|\leq r_{m,n}\}}(\xi,t)$$
$$+ \frac{c_k(\lambda(k+1)^2)^{\frac{m}{2}+n}}{(2\pi\nu)^k} e^{-\lambda t|\xi|^\alpha} v^0_{k,m,n}(\xi)\chi_{\{|\xi|\geq r_{m,n}\}}(\xi,t)$$

and $f_{k,m,n}(0,t) = 0$ for $k \geq 1$, $m,n \in \mathbb{N}$, $t \geq 0$.

Hence,

$$\left|(\lambda^{\frac{1}{2}}\,|\xi|)^m \frac{\partial^n v_k}{\partial t^n}(\xi,t)\right| \leq f_{k,m,n}(\xi,t),$$

for $(\xi,t) \in \mathbb{R}^{d+1}_+$.

Furthermore,

$$f_{k,m,n}(\xi,t) \leq \frac{c_k(\lambda(k+1)^2)^{\frac{m}{2}+n}}{(2\pi\nu)^k} v^0_{k,m,n}(\xi).$$

Therefore,

$$\sum_{k=0}^{\infty} \|f_{k,m,n}(\cdot,t)\|_{1\oplus 2} \leq \sum_{k=0}^{\infty} \frac{c_k(\lambda(k+1)^2)^{\frac{m}{2}+n}}{(2\pi\nu)^k} \left\|v^0_{k,m,n}\right\|_{1\oplus 2}$$
$$\leq \sum_{k=0}^{m+2n} \frac{c_k(\lambda(k+1)^2)^{\frac{m}{2}+n}}{(2\pi\nu)^k} C^{k+\frac{d+1}{2}} \left\|v^0_{0,m+2n+\frac{d+1}{2}}\right\|^{k+1}_{1\oplus 2}$$
$$+ \sum_{k=m+2n+1}^{\infty} \frac{c_k(\lambda(k+1)^2)^{\frac{m}{2}+n}}{(2\pi\nu)^k} C^{k+\frac{d+1}{2}} \left\|v^0_{0,m+2n+\frac{d+1}{2}}\right\|^{m+2n}_{1\oplus 2} \left\|v^0_{0,\frac{d+1}{2}}\right\|^{k+1-m-2n}_{1\oplus 2}.$$

Since,

$$\lim_{k\to\infty} \left(\frac{c_{k+1}}{c_k}\right)\left(\frac{\lambda(k+2)^2}{\lambda(k+1)^2}\right)\left(\frac{1}{2\pi\nu}\right) C \left\|v^0_{0,\frac{d+1}{2}}\right\|_{1\oplus 2} = \frac{2C}{\pi\nu}\left\|v^0_{0,\frac{d+1}{2}}\right\|_{1\oplus 2} < 1,$$

if and only if $\nu > \frac{2C}{\pi}\left\|v^0_{0,\frac{d+1}{2}}\right\|_{1\oplus 2}$ independently on $m,n \in \mathbb{N}$, we have that $\sum_{k=0}^{\infty} \|f_{k,m,n}\|_{1\oplus 2,\infty} < \infty$.

Therefore, $f_{(m,n)} = \sum_{k=0}^{\infty} f_{k,m,n} \in L^{1\oplus 2,\infty}(\mathbb{R}^{d+1}_+)$.

Note that we can write:

$$f_{k,m,n}(\xi,t) \leq e^{-\lambda t|\xi|^\alpha} f^0_{k,m,n,\alpha}(\xi),$$

for $(\xi,t) \in \mathbb{R}^{d+1}_+$.

With

$$f^0_{k,m,n,\alpha}(\xi) = \left(e^{\lambda t|\xi|^\alpha}\chi_{\{|\xi|\leq r_{m,n}\}}(\xi) + \chi_{\{|\xi|\geq r_{m,n}\}}(\xi)\right) \frac{c_k(\lambda(k+1)^2)^{\frac{m}{2}+n}}{(2\pi\nu)^k} v^0_{k,m,n}(\xi).$$

Therefore,

$$f_{(m,n)}(\xi,t) \leq e^{-\lambda t|\xi|^\alpha} f^0_{(m,n,\alpha)}(\xi),$$

with $f^0_{(m,n,\alpha)} = \sum_{k=0}^{\infty} f_{k,m,n,\alpha} \in L^{1\oplus 2,\infty}(\mathbb{R}^{d+1}_+)$ for $\nu > \frac{2C}{\pi}\left\|v^0_{0,\frac{d+1}{2}}\right\|_{1\oplus 2}$.

Thus,

$$\left|(\lambda^{\frac{1}{2}}\,|\cdot|)^m \frac{\partial^n v}{\partial t^n}\right| \leq e^{-\lambda t|\xi|^\alpha} f^0_{(m,n,\alpha)}(\xi),$$

and $v \in \mathcal{V}_\alpha(\mathcal{E})$ for $1 < \alpha < 2$.

Note that we have the identity

$$\frac{\xi^T}{|\xi|^2} q(\xi,t) = -\frac{\xi^T}{|\xi|}(v * v)(\xi,t)\frac{\xi}{|\xi|},$$

for every $(\xi,t) \in \mathbb{R}^{d+1}_+$. By Theorem 7 and $\mathcal{V}_\alpha(\mathcal{E}) \subset \mathcal{C}^{d\times 1}_\alpha(\mathcal{E})$ we obtain that $\mathcal{V}_\alpha(\mathcal{E})$ is closed by convolution and therefore $q \in \mathcal{V}_\alpha(\mathcal{E})$ for $1 < \alpha < 2$.

Applying Theorem 14 we have that $u(\cdot,t) = \widehat{v}(\cdot,t)$ and

$$p(x,t) = \int_{\mathbb{R}^d} \frac{\xi^T}{|\xi|^2} q(\xi,t) e^{-2\pi i x\cdot\xi} d\xi$$

have entire extensions that gives rise to an smooth vector field $(U,P) : \mathbb{C}^{d+1}_{>0} \to \mathbb{C}^{d+1}$ such that $(U(\cdot,t), P(\cdot,t))$ is entire for $t > 0$,

$$\left|\frac{\partial^n U}{\partial t^n}(z,t)\right| \le e^{\left((\alpha-1)\lambda t\left(\frac{\pi}{\lambda t}\right)^{\alpha'} + c_1(n)\right)|Im(z)|^{\alpha'}}, \left|\frac{\partial^n P}{\partial t^n}(z,t)\right| \le e^{\left((\alpha-1)r_\alpha \lambda t\left(\frac{\pi}{r_\alpha \lambda t}\right)^{\alpha'} + c_2(n)\right)|Im(z)|^{\alpha'}},$$

for every $1 < \alpha < 2$, $(z,t) \in \mathbb{C}^{d+1}_{>0}$ such that $|Im(z)|$ is large enough, for some constants $c_1(n) > 0$, $c_2(n) > 0$, for every $n \in \mathbb{N}$. □

Corollary 26. *There exists a curve $(U,P) : \mathbb{C}^{d+1}_{>0} \to \mathbb{C}^{d+1}$ of entire vector fields of order 2 that such that $U(x,t) = u(x,t)$, $P(x,t) = p(x,t)$ for every $(x,t) \in \mathbb{R}^{d+1}_{>0}$.*

Proof. Since the conclusion of Theorem 14 is valid for $\alpha' > 2$ arbitrary (it is valid for $\alpha < 2$ arbitrary and the conjugate function is continuous) we have that $(U(\cdot,t), P(\cdot,t))$ is an entire function of order 2 for every $t > 0$. Furthermore,

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} U = \nu \Delta U - \nabla P \text{ and } div(U)(z,t) = \int_{\mathbb{R}^d} 2\pi i \xi^T v(\xi,t) e^{2\pi i z\cdot\xi} d\xi = 0.$$

□

Remark 20. Note that since

$$\lim_{t\to 0} e^{\left((\alpha-1)\lambda t\left(\frac{\pi}{\lambda t}\right)^{\alpha'} + c_1(n)\right)|Im(z)|^{\alpha'}} = \lim_{t\to 0} e^{\left((\alpha-1)r_\alpha \lambda t\left(\frac{\pi}{r_\alpha \lambda t}\right)^{\alpha'} + c_2(n)\right)|Im(z)|^{\alpha'}} = \infty,$$

for every $(z,t) \in \mathbb{C}^{d+1}_{>0}$ with $|Im(z)| > 0$ we can not assure the entire extension until the boundary $\partial\mathbb{R}^{d+1}_{+} = \mathbb{R}^d \times \{0\}$. However, in the boundary we have that (u,p) is smooth. Furthermore, $u(\cdot,0) = u^0 \in S(\mathbb{R}^d)^d$.

Appendix A

In this Appendix we remind basic properties of the power function.

Let $f : [0,\infty) \to \mathbb{R}$,

$$f(t) = \frac{(1+t)^\alpha}{1+t^\alpha},$$

for $\alpha > 0$.

Then $f \in C^\infty((0,\infty),\mathbb{R})$ satisfying $f(0) = 1$, $f(1) = \frac{2^\alpha}{2} = 2^{\alpha-1}$ and $\lim_{t\to\infty} f(t) = 1$.

Note that

$$f'(t) = \frac{\alpha(1+t)^{\alpha-1}(1-t^{\alpha-1})}{(1+t^\alpha)^2}.$$

Therefore, $f'(1) = 0$.

- If $\alpha \ge 1$ then $f(1) = 2^{\alpha-1} > 1$, hence $\max_{t\ge 0} f(t) = f(1)$ and $\min_{t\ge 0} f(t) = f(0) = 1$.

- If $0 < \alpha < 1$ then $f(1) = 2^{\alpha-1} < 1$, hence $\max_{t\geq 0} f(t) = 1$ and $\min_{t\geq 0} f(t) = f(1) = 2^{\alpha-1}$.

Therefore, $1 \leq f(t) \leq 2^{\alpha-1}$ for $t \geq 0$ when $\alpha \geq 1$ and $2^{\alpha-1} \leq f(t) \leq 1$ for $t \geq 0$ when $0 < \alpha < 1$. Multiplying by $1 + t^\alpha$ we obtain:

$$\begin{aligned} &1 + t^\alpha \leq (1+t)^\alpha \leq 2^{\alpha-1}(1+t^\alpha), \text{ for } t \geq 0,\ \alpha \geq 1, \\ &2^{\alpha-1}(1+t^\alpha) \leq (1+t)^\alpha \leq 1 + t^\alpha, \text{ for } t \geq 0,\ 0 < \alpha < 1, \end{aligned}$$

In particular, we have the following result:

Theorem 15. *For every $s \geq 0$, $t \geq 0$,*

$$s^\alpha + t^\alpha \leq (s+t)^\alpha \leq 2^{\alpha-1}(s^\alpha + t^\alpha), \textit{ for } \alpha \geq 1, \tag{20}$$

$$2^{\alpha-1}(s^\alpha + t^\alpha) \leq (s+t)^\alpha \leq s^\alpha + t^\alpha, \textit{ for } 0 < \alpha < 1. \tag{21}$$

As a byproduct we have

Corollary 27. *For every $t \in \mathbb{R}^d$, $t \geq 0$ we have*

$$\|t\|_\alpha \leq \|t\|_1 \leq 2^{\frac{1}{\alpha'}} \|t\|_\alpha ,$$

for $\alpha \geq 1$.

We can use this to get a useful result in normed spaces.

Corollary 28. *Let X be a normed space and $x_1, \cdots x_n \in X$ then*

$$1 + \left\| \sum_{j=1}^n x_j \right\|^\alpha \leq 2^{\alpha-1} \prod_{j=1}^n (1 + \|x_j\|^\alpha).$$

We conclude with an important inequality.

Proposition 9. *For $\alpha > 1$ and $a > 0$ consider $g : [0, \infty) \to \mathbb{R}$,*

$$g(t) = -t^\alpha + at,$$

then

$$g(t) \leq (\alpha - 1)\left(\frac{a}{\alpha}\right)^{\alpha'}, \textit{ for } t \geq 0.$$

Proof. Note that $g'(t) = -\alpha t^{\alpha-1} + a$, then the only critical point is

$$t_0 = \left(\frac{a}{\alpha}\right)^{\alpha'}.$$

Since $g''(t) = -\alpha(\alpha-1)t^{\alpha-2}$ satisfies $g''(t_0) < 0$ we have that:

$$\begin{aligned} \max_{t\geq 0} g(t) = g(t_0) &= -\left(\frac{a}{\alpha}\right)^{\alpha'} + a\left(\frac{a}{\alpha}\right)^{\frac{1}{\alpha-1}} \\ &= -\left(\frac{a}{\alpha}\right)^{\alpha'} + \alpha\left(\frac{a}{\alpha}\right)^{\alpha'} = (\alpha-1)\left(\frac{a}{\alpha}\right)^{\alpha'}. \end{aligned}$$

$\square$

Corollary 29. *For $\alpha > 1$ consider $h : \mathbb{R}^d \to \mathbb{R}$,*

$$h(x) = -|x|^{\alpha} + cx \cdot y,$$

for a constant $c > 0$ and $y \in \mathbb{R}^d$. Then,

$$h(x) \leq (\alpha - 1)\left[\frac{c}{\alpha}\right]^{\alpha'} |y|^{\alpha'}, \textit{ for } x \in \mathbb{R}^d.$$

Proof. Note that by Cauchy-Schwarz inequality we have:

$$h(x) \leq -|x|^{\alpha} + c\,|x|\,|y|\,,$$

applying Proposition 9 with $a = c\,|y|$ we have:

$$h(x) \leq (\alpha - 1)\left[\frac{c}{\alpha}\right]^{\alpha'} |y|^{\alpha'}, \text{ for } x \in \mathbb{R}^d.$$

□

Conclusions and Comments

In this article, we proved the existence and smoothness of a solution of the Navier-Stokes Equation for viscosity large enough, it was possible after study remarkable spaces of functions $\mathcal{V}_{\alpha}(\mathcal{E})$ dominated by Fourier Caloric functions with initial condition in a space of functions $\mathcal{E}$ decreasing fast, furthermore we obtain as a byproduct the existence of a smooth curve of entire functions of order 2 for positive time that extend the solution (u, p) of the Navier-Stokes Equation to the complex domain.

Acknowledgments

BDVC acknowledges the support from the FSU2022-010 grant from Khalifa University, UAE.

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