Radial fields on the manifolds of symmetric positive definite matrices

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Abstract

On Hadamard manifolds, the radial fields, which are the negative gradients of the Busemann functions, can be used to designate a canonical sense of direction. This has many potential interesting applications to Hadamard manifold-valued data, for example in defining notions of quantiles or treatment effects. Some of the most commonly encountered Hadamard manifolds in statistics are the spaces of symmetric positive definite matrices, which are used in, for example, covariance matrix analysis and diffusion tensor imaging. In this paper, we derive an expression for the radial fields on these manifolds and demonstrate their smoothness.

Keywords: Positive definite matrices; Hadamard manifolds; geometric statistics; radial fields.

1 Introduction

In a metric space (M, d), two unit-speed geodesic rays $\gamma_1, \gamma_2 : [0, \infty) \to M$ are called asymptotic if $d(\gamma_1(t), \gamma_2(t)), t \in [0, \infty)$, is bounded; in the rest of this paper, we will refer to the metric space (M, d) as simply M, with its metric d being implicit. One can form an equivalence relation of geodesic rays in M on the basis of their being asymptotic; the set of all resulting equivalence classes is called the boundary at infinity ∂M, not to be confused with the topological boundary. There is a class of metric spaces called Hadamard spaces, equivalently complete CAT(0) spaces or global non-positive curvature spaces, on which for any $\xi \in \partial M$, there is exactly one member γ of this equivalence class for every $x \in M$ satisfying $\gamma(0) = x$; see Chapter II.8 of Bridson and Haefliger (1999) for more information.

Hadamard spaces that are also Riemannian manifolds are called Hadamard manifolds, which can equivalently be characterized as complete, simply connected Riemannian manifolds whose sectional curvatures are non-positive. By the Cartan–Hadamard theorem, an n-dimensional Hadamard manifold M is diffeomorphic to $\mathbb{R}^n$ via the exponential map $\exp_p : T_pM \cong \mathbb{R}^n \to M$ at any $p \in M$. For any $x \in M$ and $\xi \in \partial M$, denoting the unique member of ξ originating at x by γ_x, we can associate a unique unit vector $\xi_x := \gamma_x'(0)$ in T_xM with ξ. The vector fields on M defined by $x \mapsto \xi_x$ for $\xi \in \partial M$, which have been called radial fields (Heintze and Im Hof (1977), Shcherbakov (1983)), are also the negative gradients of the so-called Busemann functions $x \mapsto \lim_{t\to\infty} d(x, \gamma(t)) - t$, where γ is any member of ξ; thus the radial fields are normal to the level sets of these function, which are called horospheres. Proposition 3.3(a) of Shin and Oh (2023) shows that

$$\xi_x = \lim_{t\to\infty} \frac{\log_x(\gamma(t))}{d(x, \gamma(t))}, \tag{1}$$

where $\gamma : [0, \infty) \to M$ is any member of the equivalence class ξ, and Proposition 5.1 in that paper presents an expression for the radial fields in hyperbolic spaces. A note about the notation in this paper: exp and log with subscripts denote the Riemannian exponential maps and their inverses, respectively, while exp and log without subscripts denote the usual exponential and logarithm for real and positive numbers and Exp and Log denote the matrix exponential and logarithm.

Radial fields can be used to define a canonical sense of direction on Hadamard manifolds. That is, we can talk about ξ_x being the unit vector at x "in the direction of ξ". This is canonical in the sense that it does not require arbitrary decisions. On the other hand, one might try to define direction using parallel transport, but this is problematic because parallel transport between two points depends on the path taken between those points. One way to deal with this might be to choose a base point whose tangent space we transport vectors from other points, but in general, this choice of base point would be arbitrary.

Besides being interesting mathematical objects in their own right and as tools for studying the boundary at infinity, Busemann functions and horospheres, radial fields have many potential applications due to providing this sense of direction. As an example, Chaudhuri (1996) defined a quantile loss function for multivariate data by $\|x - p\| + \langle u, x - p\rangle$, where x is a data point and u is a fixed vector of norm less than 1; by conceptualizing $u = \|u\|(u/\|u\|)$ (if $u \neq 0$) and $x - p$ as tangent vectors at p, Shin and Oh (2023) generalized this loss function to Hadamard manifold-valued data as $d(p, x) + \langle \beta\xi_p, \log_p(x)\rangle$, where $\beta \in [0, 1)$ and $\xi \in \partial M$. Then other asymmetric loss functions, such as the expectile (Newey and Powell (1987), Hermann et al. (2018)) or M-quantile (Breckling and Chambers (1988)) loss functions can analogously be generalized to Hadamard manifolds using radial fields.

Other statistical tools that use vectors could conceivably also be generalized to Hadamard manifold-valued data by using non-negative numbers and radial fields to define magnitudes and directions, respectively, but because the use of radial fields for statistical inference on Hadamard manifolds is a new area of research, much of this vast potential is yet unexplored. For example, another possible application is in the area of causal inference. The most important parameter in causal inference is the average treatment effect (ATE) $E(r_T) - E(r_C)$, where r_T and r_C are the treatment and control random variables or vectors, respectively. Then on Hadamard manifolds, one could define the ATE to be the $(\beta, \xi) \in [0, \infty) \times \partial M$ for which $\exp_{r_C}(\beta\xi_{r_C})$ and r_T have the same Fréchet means, and the quantile and median treatment effects could be defined analogously.

An expression for the radial fields on the spaces of symmetric positive definite matrices specifically is needed because these are some of the most commonly encountered examples of Hadamard manifolds. Denote the space of real symmetric $m \times m$ matrices by $\mathcal{S}_m$ and the space of real symmetric positive-definite (SPD) $m \times m$ matrices by $\mathcal{P}_m$. The former is an $m(m + 1)/2$-dimensional vector space, and the latter can be considered a $m(m+1)/2$-dimensional smooth manifold on which the tangent space at each point is isomorphic to $\mathcal{S}_m$. This manifold is typically equipped with one of a handful of different Riemannian metrics, such as the Log-Cholesky metric of Lin (2019), but the most commonly used is the so-called trace, or affine invariant, metric, defined at $x \in \mathcal{P}_m$ by

$$\langle v_1, v_2 \rangle = \mathrm{tr}(x^{-1} v_1 x^{-1} v_2),$$

where $v_1, v_2 \in T_x\mathcal{P}_m \cong \mathcal{S}_m$. This Riemannian manifold is complete and simply connected with sectional curvatures in $[-1/2, 0]$ (see Proposition I.1 of Criscitiello and Boumal (2020)); therefore, this is a Hadamard manifold.

These spaces have many uses, and often, data take values in them. For example, diffusion tensor imaging (DTI), first proposed by Basser et al. (1994), is a methodology for modeling diffusion of water molecules in voxels of brain scans as 3×3 SPD matrices lying in $\mathcal{P}_3$. Crucially, these spaces need to be studied because covariance matrices (and their inverses, precision matrices), among the central objects of study in statistics and probability, are SPD matrices. Covariance matrices can be random $\mathcal{P}_m$-valued objects in their own right, for example, as sample covariance matrices or as parameters in a Bayesian framework, in which case the assigned prior is most commonly the inverse-Wishart distribution (see, for instance, Lee and Lee (2018)).

Our main contribution here is an expression for the radial fields on $\mathcal{P}_m$, which is much less forthcoming than in the case of hyperbolic space. We also demonstrate that the radial fields are smooth on $\mathcal{P}_m$, which is not true in general on Hadamard manifolds.

2 Radial fields on $\mathcal{P}_m$

Any $A \in \mathcal{S}_m$ has a real eigendecomposition

$$A = V \begin{pmatrix} d_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & d_m \end{pmatrix} V^T$$

Then, the matrix exponential of A is SPD and can be written as

$$\mathrm{Exp}(A) = V \begin{pmatrix} \exp(d_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \exp(d_m) \end{pmatrix} V^T. \tag{2}$$

Furthermore, if A is SPD, then A has a unique real SPD matrix logarithm

$$\mathrm{Log}(A) = V \begin{pmatrix} \log(d_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \log(d_m) \end{pmatrix} V^T \tag{3}$$

and, for any integer t, unique real SPD tth root

$$A^{1/t} = V \begin{pmatrix} d_1^{1/t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & d_m^{1/t} \end{pmatrix} V^T. \tag{4}$$

In this section, all matrices whose logarithms and tth roots are SPD, so $\mathrm{Log}(A)$ and $A^{1/t}$ will specifically refer to these unique real SPD matrices mentioned above.

On a practical note, there are computational issues with implementing matrix logarithms and square roots in code for general square matrices that make the process extremely slow, especially for batches of multiple matrices. In the specific case of real SPD matrices, however, the eigendecomposition method in the previous paragraph can be encoded to handle batches of matrices in parallel, for example, PyTorch, significantly accelerating the entire process.

The exponential maps, their inverses and parallel transport on $\mathcal{P}_m$ are given by

$$\exp_x(v) = x^{1/2}\mathrm{Exp}(x^{-1/2}vx^{-1/2})x^{1/2},$$
$$\log_x(p) = x^{1/2}\mathrm{Log}(x^{-1/2}px^{-1/2})x^{1/2}$$

and

$$\Gamma_{x\to p}(v) = x^{1/2}\mathrm{Exp}\left(\frac{1}{2}x^{-1/2}\log_x(p)x^{-1/2}\right)x^{-1/2}vx^{-1/2}\mathrm{Exp}\left(\frac{1}{2}x^{-1/2}\log_x(p)x^{-1/2}\right)x^{1/2},$$

where $x, p \in \mathcal{P}_m$ and $v \in T_x\mathcal{P}_m$ (see, for example, Section 3 of Sra and Hosseini (2015), 3.4 of Pennec et al. (2006), 5 of Ferreira et al. (2006) or IV.A of Jaquier and Calinon (2017)), and therefore, the distance between x and p is

$$d(x,p) = \|\mathrm{Log}(x^{-1/2}px^{-1/2})\|_F,$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

In the following theorem, we derive an expression for the radial field ξ_x.

Theorem 2.1. *For any $p \in \mathcal{P}_m$ and unit vector in $T_p\mathcal{P}_m$, let ξ be the unique point in $\partial\mathcal{P}_m$ such that the aforementioned unit vector equals ξ_p. Take any eigendecomposition VDV^T of $p^{-1/2}\xi_p p^{-1/2}$ satisfying $d_1 \geq \cdots \geq d_m$, where*

$$D = \begin{pmatrix} d_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & d_m \end{pmatrix}.$$

Then, denoting the columns of the matrix $W := x^{-1/2}p^{1/2}V$ by $w_1, \dots, w_m$ so that $W = [w_1, \dots, w_m]$, let $\{u_1, \dots, u_m\}$ be an orthonormal basis of $\mathbb{R}^m$ that results from applying the Gram-Schmidt orthonormalization process to $\{w_1, \dots, w_m\}$. Then $\xi_x = x^{1/2}UDU^Tx^{1/2}$, where $U := [u_1, \dots, u_m]$.

Proof. Throughout this proof, which makes extensive use of (2), (3), and (4), t is restricted to the positive integers. We will denote the Euclidean norm and inner product by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively, and the Frobenius norm by $\|\cdot\|_F$. The limit $\lim_{t\to\infty} d(x, \exp_p(t\xi))/t = 1$ since

$$(d(\exp_p(t\xi), p) - d(x,p))/t \leq d(x, \exp_p(t\xi))/t \leq (d(\exp_p(t\xi), p) + d(x,p))/t$$

by the triangle equality, so

$$
\begin{aligned}
\xi_x &= \lim_{t\to\infty} \frac{\log_x(\exp_p(t\xi))}{d(x, \exp_p(t\xi))} \\
&= \lim_{t\to\infty} \frac{x^{1/2} \operatorname{Log}(x^{-1/2} p^{1/2} \operatorname{Exp}(t p^{-1/2} \xi p^{-1/2}) p^{1/2} x^{-1/2}) x^{1/2}}{t} \\
&= x^{1/2} (\lim_{t\to\infty} \operatorname{Log}\big([x^{-1/2} p^{1/2} \operatorname{Exp}(t p^{-1/2} \xi p^{-1/2}) p^{1/2} x^{-1/2}]^{1/t}\big)) x^{1/2} \\
&= x^{1/2} \operatorname{Log}\left(\lim_{t\to\infty} [x^{-1/2} p^{1/2} \operatorname{Exp}(t p^{-1/2} \xi p^{-1/2}) p^{1/2} x^{-1/2}]^{1/t} \right) x^{1/2} \\
&= x^{1/2} \operatorname{Log}\left(\lim_{t\to\infty} [x^{-1/2} p^{1/2} V \operatorname{Exp}(tD) V^T p^{1/2} x^{-1/2}]^{1/t} \right) x^{1/2} \\
&= x^{1/2} \operatorname{Log}\left(\lim_{t\to\infty} \left[\sum_{i=1}^m e^{t d_i} w_i w_i^T \right]^{1/t} \right) x^{1/2};
\end{aligned} \tag{5}
$$

the first equality follows from (1), and the limit in the fourth exists because it must equal $\operatorname{Exp}(x^{-1/2}\xi_x x^{-1/2})$ by the continuity of Exp. Define

$$H(t) := \left[\sum_{i=1}^m e^{t d_i} w_i w_i^T \right]^{1/t}.$$

Let $S-1$ be the size of the set $\{j : d_j \neq d_{j+1}\} \subset \{1, \dots, m\}$, $n_1 < \cdots < n_{S-1}$ be the elements of this set, $n_0 = 0$ and $n_S = m$. For any $j \in \{1, \dots, m\}$, denote the jth largest eigenvalue of a matrix $A \in \mathcal{S}_m$ by $\alpha_j(A)$. Recall Weyl's inequality which states that for $A, B \in \mathcal{S}_m$, $\alpha_{i+j-1}(A+B) \leq \alpha_j(A) + \alpha_i(B) \leq \alpha_{i+j-m}(A+B)$; by letting $i = 1$ and N,

$$\alpha_j(A) + \alpha_m(B) \leq \alpha_j(A+B) \leq \alpha_j(A) + \alpha_1(B). \tag{6}$$

Also recall the minimax principle (Section I.10 of Kato (1995)) which states that for $A \in \mathcal{S}_m$,

$$\alpha_j(A) = \max_{\dim(\mathcal{T})=j} \min_{v\in\mathcal{T}, \|v\|_2=1} v^T A v, \tag{7}$$

where $\mathcal{T}$ is a j-dimensional subspace of $\mathbb{R}^m$. Denote by $\mathcal{P}'_m$ the space of $m \times m$ real symmetric positive semidefinite matrices. If $C \in \mathcal{S}_m$ and $A - C \in \mathcal{P}'_m$, $v^T C v = v^T A v - v^T (A - C) v \leq v^T A v$, so by (7),

$$\alpha_j(C) \leq \alpha_j(A). \tag{8}$$

Since W is invertible, $\{W_1, \ldots, W_m\}$ is indeed a basis for $\mathbb{R}^m$. Any j-dimensional subspace $\mathcal{T}$ of $\mathbb{R}^n$ contains some unit vector $u_{\mathcal{T}}$ which is orthogonal to each of $W_1, \ldots, W_{j-1}$ because the orthogonal complement of the span of $W_1, \ldots, W_{j-1}$ is of dimension $m-j+1$ and $\mathcal{T}$ is of dimension j; thus, if no such $u_{\mathcal{T}}$ exists, the union of two bases, one for each of these subspaces, is a linearly independent set of $m+1$ vectors in $\mathbb{R}^m$: a contradiction. Set s as the unique value in $1, \ldots, S-1$ for which $n_{s-1} < j \leq n_s$. For

$$A_s(t) := \sum_{i=1}^{n_s} \exp(t(d_i - d_{n_s}))w_i w_i^T,$$

taking this $u_{\mathcal{T}}$ gives

$$\begin{aligned} \alpha_j(A_s(t)) &\leq \max_{\dim(\mathcal{T})=j} u_{\mathcal{T}}^T A_s(t) u_{\mathcal{T}} \\ &= \max_{\dim(\mathcal{T})=j} \sum_{i=j}^{n_s} (u_{\mathcal{T}}^T w_i)^2 \\ &\leq \max_{\dim(\mathcal{T})=j} \sum_{i=j}^{n_s} (w_i^T w_i)(u_{\mathcal{T}}^T u_{\mathcal{T}}) \\ &= \sum_{i=j}^{n_s} w_i^T w_i \end{aligned} \tag{9}$$

by (7) and the Cauchy-Schwarz inequality. For $C_s(t) := \sum_{i=1}^{n_s} w_i w_i^T$, $A_s(t) - C_s(t) \in \mathcal{P}'_m$ and (8) holds; since $C_s(t)$ has rank $n_s \geq j$, $\alpha_j(C_s(t)) > 0$. Then, for

$$B_s(t) := \sum_{i=n_s+1}^{m} \exp(t(d_i - d_{n_s}))w_i w_i^T,$$

(6), (9), and (8) imply

$$\alpha_j(A_s(t) + B_s(t)) \in \left[\alpha_j(C_s(t)) + \alpha_m(B_s(t)), \sum_{i=j}^{n_s} w_i^T w_i + \alpha_1(B_s(t))\right], \tag{10}$$

and because

$$\lim_{t\to\infty} B_s(t) = 0 \tag{11}$$

and $\alpha_j(C_s(t))$ and $\sum_{i=j}^{n_s} w_i^T w_i$ are finite positive constants independent of t, the tth root of both bounds in this interval converges to 1 as $t \to \infty$. Thus,

$$\alpha_j(\lim_{t\to\infty} H(t)) = \exp(d_j) \lim_{t\to\infty} (\alpha_j(A_s(t) + B_s(t)))^{1/t} = \exp(d_j), \tag{12}$$

for each $j = 1, \ldots, m$, and the eigenvalues of $\lim_{t\to\infty} H(t)$ are $\exp(d_1), \ldots, \exp(d_m)$.

For $A, B \in \mathcal{S}_m$ and $a, b, \delta \in \mathbb{R}$, let E be a matrix whose columns constitute an orthonormal basis for the eigenspace of A associated with the eigenvalues contained in (a, b), and let L be a matrix whose columns constitute an orthonormal basis for the eigenspace of $A + B$ associated with the eigenvalues contained in $\mathbb{R}\backslash(a - \delta, b + \delta)$. Recall the Davis–Kahan $\sin(\Theta)$ theorem (see Section VII.3 of Bhatia (1996)) which states that

$$\|L^T E\|_F \leq \frac{\|B\|_F}{\delta}; \tag{13}$$

the norm can be any unitarily invariant norm, the Frobenius norm being one such example.

Let $A_s(t)$, $B_s(t)$ and $C_s(t)$ be as defined above. In this paragraph, we will give a high-level overview of the next part of the proof. The Davis-Kahan $\sin(\Theta)$ theorem can be used to show that the eigenspace of $H(t)$ corresponding to $\alpha_1(H(t)), \ldots, \alpha_{n_s}(H(t))$ converges in some sense to the eigenspace of $A_1(t)$ corresponding to non-zero eigenvalues, which is equivalently the span of $w_1, \ldots, w_{n_1}$ or of $u_1, \ldots, w_{n_1}$; this exploits the fact that $H(t)$ and $A_s(t) + B_s(t)$ have the exact same eigenvectors associated with corresponding eigenvalues thanks to (4). Then it can be shown that the eigenspace corresponding to $\alpha_{n_{s-1}+1}(H(t)), \ldots, \alpha_{n_s}(H(t))$ of $H(t)$ converges both to the span of $u_{n_{s-1}+1}, ..., u_{n_s}$ and to the eigenspace of $\lim_{t\to\infty} H(t)$ corresponding to $\exp(d_{n_s})$. This exploits the fact that

$$\lim_{t\to\infty} \alpha_k(H(t)) = \alpha_k(\lim_{t\to\infty} H(t)) \tag{14}$$

since the eigenvalues of a matrix are the roots of its characteristic polynomial, whose coefficients depend continuously on the entries of the matrix, and therefore, the ordered eigenvalues of a convergent sequence of matrices converge to the ordered eigenvalues of the limit of the matrices.

Letting $a = \alpha_{n_s}(C_s(t))/2 > 0$, $b = \sum_{i=1}^{n_s} w_i^T w_i + 1 < \infty$ and $\delta = \alpha_{n_s}(C_s(t))/4$, then $\alpha_i(A_s(t)) \in (a, b)$ precisely when $i = 1, \ldots, n_s$, thanks to (8), (9), and the fact that $m - n_s$ of the eigenvalues of $A_s(t)$ are 0 because $A_s(t)$ has rank n_s. The eigenspace associated with these eigenvalues is precisely the span of $\{w_1, \ldots, w_{n_s}\}$, and so we can let E in (13) be

$$E_s := [u_1, \ldots, u_{n_s}].$$

Denote an ordered orthonormal basis of $\mathbb{R}^m$ of eigenvectors of $A_s(t)+B_s(t)$ by $y_1(t), \ldots, y_m(t)$, and define $K_s(t) := [y_1(t), \ldots, y_{n_s}(t)]$ and $L_s(t) := [y_{n_s+1}(t), \ldots, y_m(t)]$; also define $K_0(t)$ to be the zero matrix. $H(t)$ and $A_s(t) + B_s(t)$ have the exact same eigenvectors so each $y_i(t)$ does not depend on s. For i in $n_s + 1, \ldots, m$, letting s' be the unique integer for which $n_{s'-1} < i \leq n_{s'}$, $\alpha_i(A_s(t)+B_s(t)) = \exp(t(d_i - d_{n_s}))\alpha_i(A_{s'}(t)+B_{s'}(t)) \to 0$ as $t \to 0$ by (10) since $d_i < d_j$. Therefore, $\alpha_{n_s+1}(A_s(t) + B_s(t)), \ldots, \alpha_m(A_s(t) + B_s(t)) \in \mathbb{R}\backslash(a - \delta, b + \delta)$ for sufficiently large t by (11) and we can choose L in (13) such that $y_{n_s+1}(t), \ldots, y_m(t)$ are among its columns.

For any $i = 1, \ldots, n_s$, $v_{s,i}(t) := K_s(t)K_s(t)^T u_i$, the projection of u_i onto the span of $y_1(t), \ldots, y_{n_s}(t)$, satisfies

$$\begin{aligned}
\|u_i - v_{s,i}(t)\|_2 &= \|(I - K_s(t)K_s(t)^T)u_i\|_2 \\
&= \|L_s(t)L_s(t)^T u_i\|_2 \\
&= (u_i^T L_s(t)L_s(t)^T L_s(t)L_s(t)^T u_i)^{1/2} \\
&= \|L_s(t)^T u_i\|_2 \\
&\to 0
\end{aligned} \tag{15}$$

as $t \to \infty$ by (11) and (13) since u_i is a column of E_s. If $s > 1$, $\{v_{s-1,l}\}$, $(l = 1, \ldots, n_{s-1})$, is a basis of the span of $y_1(t), \ldots, y_{n_{s-1}}(t)$ when t is sufficiently large because its elements are orthogonal and (15) ensures that they are eventually non-zero. Therefore, since $K_{s-1}(t)K_{s-1}(t)^T u_l$ and $K_{s-1}(t)K_{s-1}(t)^T u_i$ are orthogonal if $l \in \{1, \ldots, n_{s-1}\}$ and $i \in \{n_{s-1} + 1, \ldots, n_s\}$, $v_{s-1,i}(t) = 0$ and $v_{s,i}(t) = (K_s(t) - K_{s-1}(t))(K_s(t) - K_{s-1}(t))^T u_i$, the projection of u_i onto the span of $y_{n_{s-1}+1}(t), \ldots, y_{n_s}(t)$; this implies

$$\langle v_{s,i}(t), y_l(t)\rangle_2 = 0 \tag{16}$$

when $l \in \{1, \ldots, n_{s-1}\}$ in addition to when $l \in \{n_s + 1, \ldots, m\}$. So, keeping (12), (14), (15), and

(16) in mind,

$$
\begin{aligned}
\big(\lim_{t\to\infty} H(t)\big)u_i = \lim_{t\to\infty} H(t)\Big(\sum_{l=1}^{m}\langle u_i, y_l(t)\rangle_2 y_l(t)\Big) \\
&= \lim_{t\to\infty}\Big(\sum_{l=1}^{m}\alpha_l(H(t))\langle u_i, y_l(t)\rangle_2 y_l(t)\Big) \\
&= \lim_{t\to\infty}\Big(\sum_{l=n_{s-1}+1}^{n_s}\exp(d_{n_s})\langle v_{s,i}(t), y_l(t)\rangle_2 y_l(t)\Big) \\
&\quad + \lim_{t\to\infty}\Big(\sum_{l=n_{s-1}+1}^{n_s}[\alpha_l(H(t)) - \exp(d_{n_s})]\langle v_{s,i}(t), y_l(t)\rangle_2 y_l(t)\Big) \\
&\quad + \lim_{t\to\infty}\Big(\sum_{l=1}^{n_{s-1}}\alpha_l(H(t))\langle v_{s,i}(t), y_l(t)\rangle_2 y_l(t)\Big) \\
&\quad + \lim_{t\to\infty}\Big(\sum_{l=n_s+1}^{m}\alpha_l(H(t))\langle v_{s,i}(t), y_l(t)\rangle_2 y_l(t)\Big) \\
&\quad + \lim_{t\to\infty}\Big(\sum_{l=1}^{m}\alpha_k(H(t))\langle u_i - v_{s,i}(t), y_l(t)\rangle_2 y_l(t)\Big) \\
&= \exp(d_{n_s})\lim_{t\to\infty} v_{s,i}(t) \\
&= \exp(d_{n_s})u_i
\end{aligned}
$$

for $s = 1, \ldots, S$ and $i = n_{s-1}+1, \ldots, n_s$. The expression for ξ_x follows from (3) and (5).

□

One may wonder whether the radial fields are smooth on Hadamard manifolds. In fact, though they are known to be C^1 (see Proposition 3.1 of Heintze and Im Hof (1977)) they are not even guaranteed to be C^2. Green (1974) and Shcherbakov (1983) provide some conditions on the curvature of the manifold under which twice continuous differentiability can be guaranteed, but since they require the supremum of the sectional curvatures to be less than 0, these results do not apply to $\mathcal{P}_m$. However, we can show that the radial fields are, in fact, smooth in $\mathcal{P}_m$, just as Shin and Oh (2023) did in hyperbolic spaces.

Corollary 2.1. *The radial fields on $\mathcal{P}_m$ are smooth.*

Proof. Because $z \mapsto z/\|z\|_2$ on $\mathbb{R}^m\backslash\{0\} \to \mathbb{R}^m\backslash\{0\}$ is smooth, $W \mapsto U$ defined on $\mathrm{GL}_m(\mathbb{R}) \to \mathrm{GL}_m(\mathbb{R})$, which is diffeomorphic to an open subset of $\mathbb{R}^{m(m+1)/2}$, is also smooth. The map $z \mapsto z^{1/2}$

on $\mathcal{P}_m \to \mathcal{P}_m$, also diffeomorphic to an open subset of $\mathbb{R}^{m(m+1)/2}$, is also smooth, and therefore, so is $x \mapsto W$ on $\mathcal{P}_m \to \mathrm{GL}_m(\mathbb{R})$. Then, the smoothness of the map $x \mapsto \xi_x$ on $\mathcal{P}_m \to \mathcal{S}_m \cong \mathbb{R}^{m(m+1)/2}$ follows from Theorem 2.1. $\square$

This smoothness is important because, for example, it means that the joint asymptotic normality of quantiles of Theorem 4.2 and Corollaries 4.1 and 4.2 of Shin and Oh (2023) can be applied to quantiles on $\mathcal{P}_m$, and that the gradient of the quantile loss functions in that space can also be calculated using Jacobi fields as in hyperbolic spaces.

3 Concluding remarks

As detailed in the introduction, radial fields have the potential to generalize many statistical techniques to Hadamard manifolds by defining a canonical sense of direction. The results of this paper, namely an expression for the radial fields on $\mathcal{P}_m$, among the most commonly encountered Hadamard manifolds, and the smoothness of these fields, should be of great use to researchers looking to apply these techniques to $\mathcal{P}_m$.

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