

# SATURATION RANK FOR NILRADICAL OF PARABOLIC SUBALGEBRAS IN TYPE A

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ABSTRACT. Let  $\mathfrak{p}(d)$  be a standard parabolic subalgebra of  $\mathfrak{sl}_{n+1}(K)$  and  $\mathfrak{u}$  be the corresponding nilradical defined over an algebraically closed field  $K$  of characteristic  $p > 0$ . We construct a finite connected quiver  $Q(d)$ , through which we provide a combinatorial characterization of the centralizer  $c_{\mathfrak{u}}(x(d))$  of the Richardson element  $x(d)$ . We specifically focus on the centralizer when the Levi factor of  $\mathfrak{p}(d)$  is determined by either one or two simple roots. This allows us to demonstrate that, under certain mild restrictions, the saturation rank of  $\mathfrak{u}$  equals the semisimple rank of the algebraic  $K$ -group  $SL_{n+1}(K)$ .

## 1. INTRODUCTION

Let  $(\mathfrak{g}, [p])$  be a finite-dimensional restricted Lie algebra defined over an algebraically closed field  $K$  of characteristic  $p > 0$ . The restricted nullcone of  $\mathfrak{g}$  is the fiber of zero of the  $[p]$ -map, which is  $V(\mathfrak{g}) = \{x \in \mathfrak{g} \mid x^{[p]} = 0\}$ . The subset

$$\mathbb{E}(r, \mathfrak{g}) := \{\mathfrak{e} \in \text{Gr}_r(\mathfrak{g}) \mid [\mathfrak{e}, \mathfrak{e}] = 0, \mathfrak{e} \subset V(\mathfrak{g})\}$$

of the Grassmannian  $\text{Gr}_r(\mathfrak{g})$  of  $r$ -planes, introduced in [4] is closed and hence a projective variety. We write the union of elements of  $\mathbb{E}(r, \mathfrak{g})$  as  $V_{\mathbb{E}(r, \mathfrak{g})} := \bigcup_{\mathfrak{e} \in \mathbb{E}(r, \mathfrak{g})} \mathfrak{e}$ , which is contained in the conical variety  $V(\mathfrak{g})$ . We consider an important invariant of restricted Lie algebras  $\mathfrak{g}$ , denoted as  $\text{srk}(\mathfrak{g})$ , and call it the saturation rank. This rank is defined by

$$\text{srk}(\mathfrak{g}) := \max\{r \in \mathbb{N} \mid V(\mathfrak{g}) = V_{\mathbb{E}(r, \mathfrak{g})}\}.$$

When restricting a  $\mathfrak{g}$ -module to elements of  $\mathbb{E}(r, \mathfrak{g})$  for a certain rank  $r$  within a restricted Lie algebra  $\mathfrak{g}$ , it is crucial to ensure that no information is lost in comparison to its restricted nullcone  $V(\mathfrak{g})$ . This is the pivotal moment where the saturation rank takes center stage. As demonstrated in [7], it has been established that the Carlson module  $L_{\zeta}$  remains indecomposable when the saturation rank of  $\mathfrak{g}$  satisfies  $\text{srk}(\mathfrak{g}) \geq 2$ . A prototypical case occurs when  $\mathfrak{g}$  is the algebraic Lie algebras of reductive algebraic groups  $G$ , which implies that  $\text{srk}(\mathfrak{g}) = \text{rk}_{ss}(G)$  is the semisimple rank of a reductive algebraic group under some mild restrictions (cf. [8]). In other cases, such as the algebraic Lie algebras of non-reductive groups, the saturation rank remains unknown.

Let  $G$  be a reductive algebraic group over an algebraically closed field  $K$  and let  $P$  be a parabolic subgroup of  $G$  with unipotent radical  $U$ . We write  $\mathfrak{g}$ ,  $\mathfrak{p}$  and  $\mathfrak{u}$  for the Lie algebras of  $G, P$  and  $U$  respectively. The established fact that  $G$  exhibits

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finitely many nilpotent orbits within the Lie algebra  $\mathfrak{g}$  is widely acknowledged: this was initially proved by Richardson under the condition of either  $\text{char}K$  is zero or good for  $G$ ; we direct interested readers to consult [6] for an overview of the result in bad characteristic. It follows that there is a unique nilpotent orbit  $G \cdot e$  which intersects  $\mathfrak{u}$  in an open dense subvariety. Richardson's dense orbit theorem tells us that the intersection  $G \cdot e \cap \mathfrak{u} = P \cdot e$  is a single  $P$ -orbit (we may assume that  $e \in \mathfrak{u}$ ). The  $P$ -orbit  $P \cdot e$  is called the Richardson orbit and its elements are called Richardson elements (cf. [9]).

The purpose of this paper is to determine the saturation rank of the nilpotent radical  $\mathfrak{u}$  in case  $G = \text{SL}_{n+1}(K)$ . For any parabolic subgroup  $P$  of  $G$ , there is a unique dimension vector  $d$  such that  $P$  is conjugate to  $P(d)$ , where  $d$  gives the sizes of the blocks in the Levi subgroup of  $P$  containing maximal torus  $T$ . Therefore, in what follows it suffices to just consider parabolic subgroups of the form  $P(d)$ . In view of Lemma 4.2, under some mild restrictions, the saturation rank  $\text{srk}(\mathfrak{u})$  is determined by the local saturation rank of Richardson elements. The construction work of such elements has been elucidated by Brüstle et al. in [3]. Furthermore, Baur et al. describe a normal form for Richardson elements in the classical case in [1, 2].

We write  $\mathfrak{p} = \mathfrak{p}(d)$  as the Lie algebra of  $P(d)$  where  $d = (d_1, \dots, d_r)$  is the corresponding dimension vector. The Richardson element which is obtained through a horizontal line diagram  $L_h(d)$  (see [1] for more details) is now expressed by

$$x(d) = x(L_h(d)) = \sum_{i \rightarrow j} e_{i,j}.$$

We conclude this introduction with a succinct overview of the contents covered in our paper. In Section 2, we present an alternative characterization of the nilradical  $\mathfrak{u}$  of  $\mathfrak{p}(d)$  and introduce a method for identifying elements that commute with the Richardson element  $x(d)$  for a given dimension vector  $d$ . Section 3 deals with the centralizer of  $x(d)$  when the Levi factor of  $\mathfrak{p}(d)$  is determined by either one or two simple roots. Finally we show that, in Section 4, the saturation rank of  $\mathfrak{u}$  coincides with the semisimple rank of the group  $G = \text{SL}_{n+1}(K)$  subject to certain constraints. Consequently, we conclude that the Carlson module  $L_\zeta$ , acting as a module over  $U_0(\mathfrak{u})$ , remains indecomposable for  $n \geq 2$  and when the characteristic of  $K$  is greater than or equal to  $n + 1$ .

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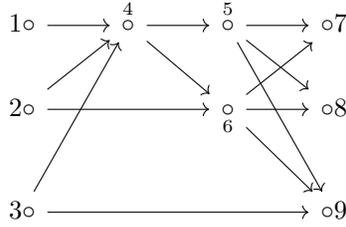
## 2. FINITE QUIVER ARISING FROM THE DIMENSION VECTOR

Let  $G = \text{SL}_{n+1}(K)$  be the special linear group over an algebraically closed field  $K$ . Let  $T$  be maximal torus of  $G$  and  $P(d)$  a standard parabolic subgroup of  $G$  containing  $T$ . Let  $\Phi$  be the root system of  $G$  with respect to  $T$ ,  $\Delta = \{\alpha_1, \dots, \alpha_n\}$  be the base of  $\Phi$  and  $\Phi^+$  be the set of positive roots. We consider the parabolic

subalgebra  $\mathfrak{p}(d) = \text{Lie}(P(d))$  with its decomposition  $\mathfrak{p}(d) = \mathfrak{m} \oplus \mathfrak{u}$ . Let  $\Phi(\mathfrak{m}) \subseteq \Phi$  be the closed subsystem of  $\Phi$  determined by the levi factor  $\mathfrak{m}$  and  $\Delta(\mathfrak{m}) \subseteq \Delta$  be the base of  $\Phi(\mathfrak{m})$ . Then the nilpotent radical  $\mathfrak{u}$  can be written as  $\mathfrak{u} = \bigoplus_{\alpha \in \Phi^+ \setminus \Phi(\mathfrak{m})^+} \mathfrak{g}_\alpha$  where  $\mathfrak{g}_\alpha$  is the root subspace of  $\mathfrak{g}$  corresponding to  $\alpha$ . In this section, our study centers on quivers  $\Gamma := (\Gamma_0, \Gamma_1)$  that exhibit a lack of loops or multiple arrows where  $\Gamma_0 = \{1, 2, \dots, n+1\}$ . For such quivers, the set of arrows is represented as a subset  $\Gamma_1 \subseteq \Gamma_0 \times \Gamma_0$ , stemming from the Cartesian product of the set of vertices  $\Gamma_0$ , and two maps  $s, t : \Gamma_1 \mapsto \Gamma_0$  which associated to each arrow  $\alpha \in \Gamma_1$  its source  $s(\alpha) \in \Gamma_0$  and its target  $t(\alpha) \in \Gamma_0$ , respectively.

**Definition 2.1.** Let  $d = (d_1, \dots, d_r)$  be a dimension vector associated to  $\mathfrak{p}(d)$ . Arrange  $r$  columns of  $d_i$  dots, top-adjusted. Given  $(a, b) \in \Gamma_0 \times \Gamma_0$ , there is an arrow  $\alpha : a \rightarrow b$  if  $a - b$  is a horizontal line in  $L_h(d)$  or  $a, b$  are from two adjacent columns and the column where  $a$  lies in is on the left. Let  $\Gamma_1$  be the set of arrows, then  $(\Gamma_0, \Gamma_1)$  is a locally connected finite quiver, denoted by  $Q(d)$ .

**Example 2.2.** Considering the parabolic subalgebra  $\mathfrak{p}(d)$  of  $\mathfrak{sl}_9(K)$  with dimension vector  $d = (3, 1, 2, 3)$ , then  $Q(d)$  is as follows:



**Theorem 2.1.** Let  $\mathfrak{u}$  be the nilpotent radical of a standard parabolic subalgebra  $\mathfrak{p}(d)$ . There exists an admissible ideal  $J$  of path algebra  $KQ(d)$  generated by all commutativity relations  $\omega_1 - \omega_2$  such that  $\mathfrak{u} \cong \text{rad}KQ(d)/J$  as Lie algebras.

*Proof.* We first construct an algebra homomorphism

$$\begin{aligned} \varphi : \text{rad}KQ(d) &\longrightarrow \mathfrak{u} \\ \rho &\longmapsto e_{s(\rho), t(\rho)} \end{aligned}$$

Recall that  $\mathfrak{u} = \bigoplus_{\alpha \in \Phi^+ \setminus \Phi(\mathfrak{m})^+} \mathfrak{g}_\alpha$ , the indices  $i, j$  of the root vector  $x_\alpha = e_{i,j}$  for  $\alpha = \epsilon_i - \epsilon_j$  in  $\Phi^+ \setminus \Phi(\mathfrak{m})^+$  existing as vertices in  $Q(d)$  is connected by a path  $\rho$  since  $i, j$  are from different columns. We claim that  $\varphi$  is well-defined and surjective. Given two paths  $\omega_1$  and  $\omega_2$ , if  $\varphi(\omega_1) = \varphi(\omega_2)$ , then  $s(\omega_1) = s(\omega_2)$  and  $t(\omega_1) = t(\omega_2)$ . Then the map  $\varphi$  admits an admissible ideal generated by all commutativity relations  $\omega_1 - \omega_2$  as a kernel. Hence we conclude that the map  $\varphi$  defined in the statement is an isomorphism.

The Lie structure on  $\text{rad}KQ(d)$ , which is defined as  $[x, y]_Q = xy - yx$  for  $x, y \in \text{rad}KQ(d)$ , where  $xy$  and  $yx$  are obtained by the product of two paths in path algebra  $KQ(d)$ . It is obvious that  $\varphi$  is compatible with Lie brackets, i.e.  $\varphi[x, y]_Q = [\varphi(x), \varphi(y)]$ , so the map  $\varphi$  is an isomorphism as Lie algebras.  $\square$

**Example 2.3.** Let  $d = (1, 2, 1)$  be the dimension vector and  $Q(d)$  be the corresponding quiver

$$\begin{array}{ccccc} 1\circ & \xrightarrow{\alpha} & \overset{2}{\circ} & \xrightarrow{\beta} & \circ 4 \\ & \searrow \gamma & & \nearrow \delta & \\ & & \underset{3}{\circ} & & \end{array}$$

The  $K$ -algebra homomorphism  $\varphi : \text{rad}KQ(d) \rightarrow \mathfrak{u}$  is defined by

$$\begin{aligned} \varphi(\alpha) &= e_{1,2}, & \varphi(\beta) &= e_{2,4}, & \varphi(\gamma) &= e_{1,3}, \\ \varphi(\delta) &= e_{3,4}, & \varphi(\alpha\beta) &= \varphi(\gamma\delta) &= e_{1,4}. \end{aligned}$$

Here, we see that  $\varphi$  is a surjection and  $\text{Ker } \varphi = \langle \alpha\beta - \gamma\delta \rangle = J$ . Hence,  $\text{rad}KQ(d)/J \cong \mathfrak{u}$ .

Given a vertex  $x \in \Gamma_0$  in  $Q(d)$ , we put

$$x^+ := \{y \in \Gamma_0 \mid x \rightarrow y \text{ is a horizontal arrow in } Q(d)\}$$

$$x^- := \{y \in \Gamma_0 \mid y \rightarrow x \text{ is a horizontal arrow in } Q(d)\},$$

so that  $x^+$  and  $x^-$  are the subsets of successor and predecessor of the vertex  $x$  in  $Q(d)$ , respectively. Obviously,  $x^+$  (resp.  $x^-$ ) is a singleton or an empty set, so we may identify the set  $x^+$  (resp.  $x^-$ ) with its element when it is non-empty without any ambiguity. If we have two vertices  $a, b \in \Gamma_0$  in  $Q(d)$  with  $a < b$ , we use  $l(a)$  (resp.  $l(b)$ ) to indicate the line number in which  $a$  (resp.  $b$ ) lies in the quiver  $Q(d)$ . If  $l(a) = l(b)$ , then  $e_{a,b}$  is a summand of  $x(d)$ , which allows us to rewrite  $x(d) = x_1 + x_2 + \cdots + x_s$  for some integer  $s$  with

$$x_i = \sum_{\substack{a-b \\ l(a)=l(b)=i}} e_{a,b}$$

**Lemma 2.2.** Let  $x(d)$  be a Richardson element of  $\mathfrak{p}(d)$  written as  $x(d) = x_1 + x_2 + \cdots + x_s$ . Given an element

$$x = \sum_{\epsilon_i - \epsilon_j \in \Phi^+ \setminus \Phi(\mathfrak{m})^+} k_{i,j} e_{i,j}$$

of  $\mathfrak{u}$  with  $k_{a,b} \neq 0$  and  $l(a) \neq l(b)$ . If  $[x(d), x] = 0$ , then we have the following three statements:

- (1) If  $a^- \neq \emptyset$ , then  $b^- \neq \emptyset$  and  $k_{a^-, b^-} = k_{a,b}$ .
- (2) If  $b^+ \neq \emptyset$ , then  $a^+ \neq \emptyset$  and  $k_{a^+, b^+} = k_{a,b}$ .
- (3) If  $a^- = b^+ = \emptyset$ , then  $[x(d), x - k_{a,b} e_{a,b}] = 0$ .

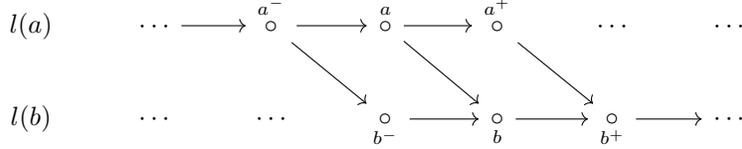
*Proof.* We first have the statement:  $[x_i, e_{a,b}] = 0$  for  $i \neq l(a), l(b)$ . Then we have

$$[x(d), x] = [x_{l(a)}, k_{a,b} e_{a,b}] + [x_{l(b)}, k_{a,b} e_{a,b}] + [x(d), x'] = 0$$

where  $x' = x - k_{a,b} e_{a,b}$ . If  $a^- \neq \emptyset$ , then  $[x_{l(a)}, k_{a,b} e_{a,b}] = k_{a,b} e_{a^-, b}$ . Since  $-k_{a,b} e_{a^-, b}$  cannot appear as a summand of  $[x_{l(b)}, k_{a,b} e_{a,b}]$ , which enforces the term  $k_{a,b} e_{a^-, b}$  existing as a summand of  $x' \cdot x(d)$ . Specifically, we should have  $k_{a^-, c} e_{a^-, c} \cdot e_{c,b} = k_{a,b} e_{a^-, b}$  for some  $c$  where  $k_{a^-, c} e_{a^-, c}$  is a term of  $x'$  and  $e_{c,b}$  is a term of  $x(d)$ . By

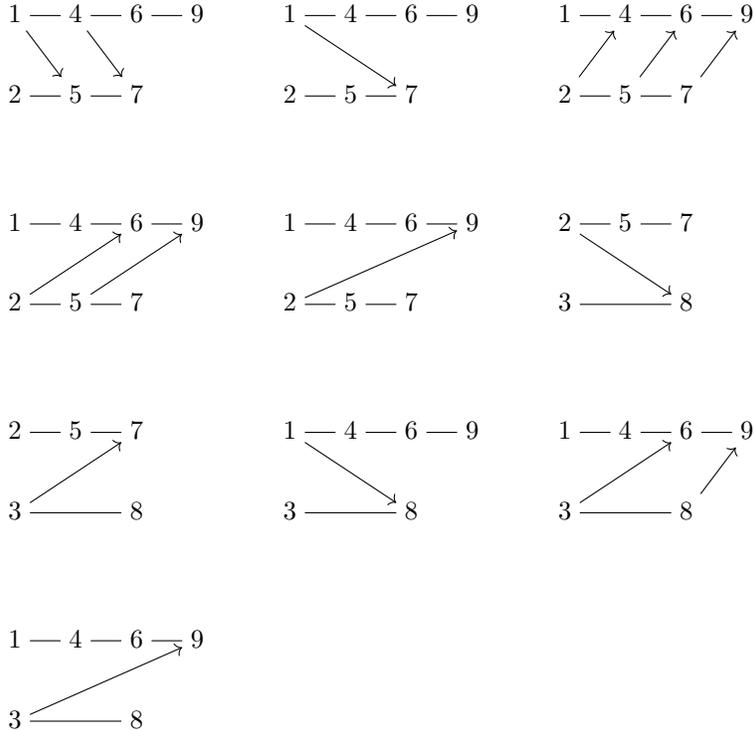
the construction of  $x(d)$ , we have  $c = b^-$ . Therefore,  $b^- \neq \emptyset$  and  $k_{a^-, b^-} = k_{a, b}$ , which proves statement (1).

If  $b^+ \neq \emptyset$ , then  $[x_{l(b)}, k_{a, b}e_{a, b}] = -k_{a, b}e_{a, b^+}$ . By the same token, we shall have  $a^+ \neq \emptyset$  and  $k_{a^+, b^+} = k_{a, b}$ . Instead of offering the proof of assertion (2), we opt to present a diagram in case  $l(a) < l(b)$ .



If  $a^- = b^+ = \emptyset$ , then  $[x_{l(a)}, k_{a, b}e_{a, b}] = [x_{l(b)}, k_{a, b}e_{a, b}] = 0$ , implying  $[x(d), x'] = 0$ , which proves statement (3).  $\square$

**Remark 2.4.** Theorem 2.1 and Lemma 2.2 provides us with a method to identify all elements that commute with the Richardson element  $x(d)$ . For example, let's consider the parabolic subalgebra  $\mathfrak{p}(d) \subseteq \mathfrak{sl}_9(K)$  defined by the dimension vector  $d = (3, 2, 3, 1)$ . In this case,  $\Delta(\mathfrak{m}) = \{\alpha_1, \alpha_2, \alpha_4, \alpha_6, \alpha_7\}$ . Apart from powers of the Richardson element  $x(d)$ , the following ten diagrams provide the generators in  $\mathfrak{u}$  that commute with  $x(d)$ :



The corresponding elements, from left to right and from top to bottom, are

$$\begin{array}{cccccc}
 e_{1,5} + e_{4,7} & e_{1,7} & e_{2,4} + e_{5,6} + e_{7,9} & e_{2,6} + e_{5,9} & e_{2,9} \\
 e_{2,8} & e_{3,7} & e_{1,8} & e_{3,6} + e_{8,9} & e_{3,9}
 \end{array}$$

### 3. CENTRALIZERS OF RICHARDSON ELEMENTS WITH $|\Delta(\mathbf{m})| \leq 2$

Given  $x(d)$  as a Richardson element of  $\mathfrak{p}(d)$ , through the natural embedding of  $\mathfrak{sl}_{n+1}(K) = \text{Lie}(\text{SL}_{n+1}(K))$  in  $\mathfrak{gl}(\mathbb{V})$  with  $\dim \mathbb{V} = n + 1$ , each Richardson element  $x(d)$  becomes a nilpotent element in  $\mathfrak{gl}(\mathbb{V})$ , and therefore a nilpotent endomorphism of  $\mathbb{V}$ . We can therefore associate to  $x(d)$  a partition  $\pi$ , written sometimes in the form  $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0)$ , and sometimes as  $[1^{r_1} 2^{r_2} 3^{r_3} \dots]$ , given by the sizes of the blocks in the Jordan normal form of  $x(d)$ .

**Lemma 3.1.** *The partition of the Richardson element  $x(d)$  is, up to a permutation,*

- (1)  $\pi = [1, n]$  provided  $|\Delta(\mathbf{m})| = 1$ .
- (2)  $\pi = [2, n - 1]$  or  $\pi = [1^2, n - 1]$  provided  $|\Delta(\mathbf{m})| = 2$ .

*Proof.* (1) Since  $|\Delta(\mathbf{m})| = 1$ , we may assume that  $\Delta(\mathbf{m}) = \{\alpha_i\}$  for some  $i$ . Then the associated dimension vector is  $d = (1, \dots, 2, \dots, 1)$ , with the  $i$ -th coordinate is 2, and the remaining coordinates are 1. The number of edges in the horizontal line diagram  $L_h(d)$  is  $n - 1$ , which means the corresponding Richardson element  $x(d)$  is the sum of  $n - 1$  elementary matrices. As a result, the rank of  $x(d)$  is  $n - 1$ . Since the eigenvalue of  $x(d)$  is 0, the number of Jordan blocks is the geometric multiplicity of 0, which is  $n + 1 - \text{rank}(x(d)) = 2$ .

The rank of  $x(d)^i$  is  $n - i$  for  $1 \leq i \leq n$ , as well as  $\text{rank}(x(d)^0) = n + 1$  and  $\text{rank}(x(d)^{n+1}) = 0$ . We have

$$\text{rank}(x(d)^{m-1}) - 2\text{rank}(x(d)^m) + \text{rank}(x(d)^{m+1}) = \begin{cases} 1, & \text{if } m = 1 \text{ or } n, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the partition of  $x(d)$  is  $\pi = [1, n]$  proving the first statement.

(2) If  $|\Delta(\mathbf{m})| = 2$ , then the rank of  $x(d)$  is either  $n - 1$  or  $n - 2$ . The remaining proof is similar to part (1).  $\square$

If  $|\Delta(\mathbf{m})| = q$ , then there exists an injective map from set of labeled horizontal line diagrams  $L_h(d)$  to set  $\mathbb{N}^{n+1-q}$ , defined by

$$\begin{aligned} \varphi : \{\text{linear diagram } L_h(d)\} &\longrightarrow \mathbb{N}^{n+1-q} \\ L_h(d) &\mapsto (D_1, D_2, \dots, D_{n+1-q}) \end{aligned}$$

where  $D_1 = 1$  and  $D_i = D_1 + \sum_{j=1}^{i-1} d_j$  for  $1 < i \leq n + 1 - q$ . Thereafter, we may write  $L_h(d) = (D_1, D_2, \dots, D_{n+1-q} \mid \widetilde{D}_1, \widetilde{D}_2, \dots, \widetilde{D}_q)$  alternatively where  $\widetilde{D}_j$  for  $1 \leq j \leq q$  represents the remaining labeled numbers arranged from left to right and proceeds in a top-down manner.

Recall that if  $x(d)$  has partition  $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0)$ , then there exist  $v_1, v_2, \dots, v_r \in \mathbb{V}$  such that all  $x(d)^j v_i$  with  $1 \leq i \leq r$  and  $0 \leq j < \lambda_i$  are a basis for  $\mathbb{V}$  and such that  $x(d)^{\lambda_i} v_i = 0$  for all  $i$ . For each integer  $m \geq 1$ , denote by  $J_m$  the  $(m \times m)$ -matrix where the  $(i, i + 1)$  entries with  $1 \leq i < m$  are equal to 1 and all remaining entries are equal to 0. In what follows, we shall give the  $v_i$  for certain  $x(d)$ .

**Lemma 3.2.** *Let  $x(d)$  be a Richardson element of  $\mathfrak{p}(d)$  determined by  $\Delta(\mathbf{m})$ .*

(1) If  $\Delta(\mathbf{m}) = \{\alpha_r\}$  and  $L_h(d) = (D_1, D_2, \dots, D_n \mid \widetilde{D}_1)$ , then  $\mathbb{V}$  has a basis

$$\{x(d)^{n-1}v_1, \dots, x(d)v_1, v_1, v_2\}$$

where  $v_1 = e_{D_n}, v_2 = e_{\widetilde{D}_1}$  and the action of  $x(d)$  on  $v_1$  can be represented as a diagram like

$$\begin{array}{ccccccc} e_{D_1} & \longleftarrow & e_{D_2} & \longleftarrow & \cdots & \longleftarrow & e_{D_{n-1}} & \longleftarrow & e_{D_n} \\ & \text{curved arrow} & & \text{curved arrow} & & \text{curved arrow} & & \text{curved arrow} & \\ & x(d) & & x(d) & & x(d) & & x(d) & \\ x(d)^{n-1}v_1 & & x(d)^{n-2}v_1 & & \cdots & & x(d)v_1 & & v_1 \end{array}$$

(2) If  $\Delta(\mathbf{m}) = \{\alpha_r, \alpha_{r+1}\}$  and  $L_h(d) = (D_1, D_2, \dots, D_{n-1} \mid \widetilde{D}_1, \widetilde{D}_2)$ , then  $\mathbb{V}$  has a basis

$$\{x(d)^{n-2}v_1, \dots, x(d)v_1, v_1, v_2, v_3\}$$

where  $v_1 = e_{D_{n-1}}, v_2 = e_{\widetilde{D}_1}, v_3 = e_{\widetilde{D}_2}$  and the action of  $x(d)$  on  $v_1$  can be represented as a diagram like

$$\begin{array}{ccccccc} e_{D_1} & \longleftarrow & e_{D_2} & \longleftarrow & \cdots & \longleftarrow & e_{D_{n-2}} & \longleftarrow & e_{D_{n-1}} \\ & \text{curved arrow} & & \text{curved arrow} & & \text{curved arrow} & & \text{curved arrow} & \\ & x(d) & & x(d) & & x(d) & & x(d) & \\ x(d)^{n-2}v_1 & & x(d)^{n-3}v_1 & & \cdots & & x(d)v_1 & & v_1 \end{array}$$

(3) If  $\Delta(\mathbf{m}) = \{\alpha_r, \alpha_s\}$  with  $r < s - 1$  and  $L_h(d) = (D_1, D_2, \dots, D_{n-1} \mid \widetilde{D}_1, \widetilde{D}_2)$ , then  $\mathbb{V}$  has a basis

$$\{x(d)^{n-2}v_1, \dots, x(d)v_1, v_1, x(d)v_2, v_2\}$$

where  $v_1 = e_{D_{n-1}}, v_2 = e_{\widetilde{D}_2}$  and the action of  $x(d)$  on  $v_1, v_2$  can be represented as a diagram like

$$\begin{array}{ccccccc} e_{D_1} & \longleftarrow & e_{D_2} & \longleftarrow & \cdots & \longleftarrow & e_{D_{n-2}} & \longleftarrow & e_{D_{n-1}} \\ & \text{curved arrow} & & \text{curved arrow} & & \text{curved arrow} & & \text{curved arrow} & \\ & x(d) & & x(d) & & x(d) & & x(d) & \\ x(d)^{n-2}v_1 & & x(d)^{n-3}v_1 & & \cdots & & x(d)v_1 & & v_1 \end{array}$$

$$\begin{array}{ccc} e_{\widetilde{D}_1} & \longleftarrow & e_{\widetilde{D}_2} \\ & \text{curved arrow} & \\ & x(d) & \\ x(d)v_2 & & v_2 \end{array}$$

*Proof.* We exclusively demonstrate the veracity of statement (1), noting the analogous nature of proofs for statements (2) and (3). Assume  $x(d)$  is the Richardson element given by  $\Delta(\mathbf{m}) = \{\alpha_r\}$  and  $L_h(d) = (D_1, D_2, \dots, D_n \mid \widetilde{D}_1)$ , then  $x(d) = \sum_{i=1}^{n-1} e_{D_i, D_{i+1}}$ . Given such a diagram  $L_h(d)$ , we let  $\sigma$  be the permutation of the set  $\Gamma_0$ , where

$$\sigma(i) = \begin{cases} D_i, & \text{if } 1 \leq i \leq n, \\ \widetilde{D}_1, & \text{if } i = n + 1. \end{cases}$$

We now define  $P = \prod_{i=1}^n E(i, \sigma(i))$  as the product of elementary matrices. Then we have  $P = (e_{\sigma(1)}, e_{\sigma(2)}, \dots, e_{\sigma(n+1)}) = (e_{D_1}, e_{D_2}, \dots, e_{D_n}, e_{\widetilde{D}_1})$  and further

$$x(d)(e_{\sigma(1)}, e_{\sigma(2)}, \dots, e_{\sigma(n+1)}) = (e_{\sigma(1)}, e_{\sigma(2)}, \dots, e_{\sigma(n+1)}) \begin{pmatrix} J_n & 0 \\ 0 & J_1 \end{pmatrix},$$

implying  $x(d)e_{\sigma(i+1)} = e_{\sigma(i)}$  for  $1 \leq i \leq n-1$  and  $x(d)e_{\sigma(n+1)} = 0$ . As a result, we have  $v_1 = e_{\sigma(n)} = e_{D_n}$ ,  $v_2 = e_{\sigma(n+1)} = e_{\widetilde{D}_1}$  and the action of  $x(d)$  on  $v_1$  described in the diagram, as desired.  $\square$

**Example 3.1.** Consider the restricted Lie algebra  $\mathfrak{g} = \mathfrak{sl}_6(K)$ . We give three examples to illustrate Lemma 3.2:

- (1) Suppose  $\Delta(\mathfrak{m}) = \{\alpha_3\}$ . Then the dimension vector is  $d = (1, 1, 2, 1, 1)$  and the line diagram  $L_h(d)$

$$1 \text{ --- } 2 \text{ --- } 3 \text{ --- } 5 \text{ --- } 6$$

4

can be written as  $(1, 2, 3, 5, 6 \mid 4)$ . Let  $\sigma = (456)$  be the permutation, then Lemma 3.2 provides that  $v_1 = e_{\sigma(5)} = e_6$ ,  $x(d)^i v_1 = e_{\sigma(5-i)}$  ( $1 \leq i \leq 4$ ) and  $v_2 = e_{\sigma(6)} = e_4$ .

- (2) Let  $\Delta(\mathfrak{m}) = \{\alpha_2, \alpha_3\}$ . Then the dimension vector is  $d = (1, 3, 1, 1)$  and the line diagram  $L_h(d)$

$$1 \text{ --- } 2 \text{ --- } 5 \text{ --- } 6$$

3

4

can be written as  $(1, 2, 5, 6 \mid 3, 4)$ . Let  $\sigma = (35)(46)$  be the permutation, then Lemma 3.2 provides that  $v_1 = e_{\sigma(4)} = e_6$ ,  $x(d)v_1 = e_5$ ,  $x(d)^2 v_1 = e_2$ ,  $x(d)^3 v_1 = e_1$ ,  $v_2 = e_{\sigma(5)} = e_3$  and  $v_3 = e_{\sigma(6)} = e_4$ .

- (3) Let  $\Delta(\mathfrak{m}) = \{\alpha_1, \alpha_3\}$ . Then the dimension vector is  $d = (2, 2, 1, 1)$  and the line diagram  $L_h(d)$

$$1 \text{ --- } 3 \text{ --- } 5 \text{ --- } 6$$

$$2 \text{ --- } 4$$

can be written as  $(1, 3, 5, 6 \mid 2, 4)$ . Let  $\sigma = (23564)$  be the permutation, by Lemma 3.2, we have  $v_1 = e_{\sigma(5)} = e_6$ ,  $x(d)v_1 = e_5$ ,  $x(d)^2 v_1 = e_3$ ,  $x(d)^3 v_1 = e_1$ ,  $v_2 = e_{\sigma(6)} = e_4$  and  $x(d)v_2 = e_2$ .

Let  $c_{\mathfrak{gl}(\mathbb{V})}(x(d))$  be the centralizer of  $x(d)$  in  $\mathfrak{gl}(\mathbb{V})$ , and  $c_{\mathfrak{u}}(x(d)) := c_{\mathfrak{gl}(\mathbb{V})}(x(d)) \cap \mathfrak{u}$  be the centralizer of  $x(d)$  in  $\mathfrak{u}$ . Each  $Z \in c_{\mathfrak{gl}(\mathbb{V})}(x(d))$  is determined by the  $Z(v_i)$  for  $1 \leq i \leq r$  because  $Z(x(d)^k v_i) = x(d)^k Z(v_i)$  for all  $i$  and  $k$ . Further we have to have  $x(d)^{\lambda_i} Z(v_i) = 0$  for all  $i$ . Using this one checks that  $Z(v_i)$  has the form

$$Z(v_i) = \sum_{j=1}^r \sum_{k=\max(0, \lambda_j - \lambda_i)}^{\lambda_j - 1} a_{k,j;i} x(d)^k v_j.$$

When  $|\Delta(\mathfrak{m})| = 0$ , then  $x(d)$  is the regular nilpotent element of  $\mathfrak{sl}_{n+1}(K)$ . We refer the interested reader to [8] for relevant results. In the following, we will determine the centralizer  $c_{\mathfrak{u}}(x(d))$  of  $x(d)$  and its center  $Z(c_{\mathfrak{u}}(x(d)))$  when  $1 \leq |\Delta(\mathfrak{m})| \leq 2$ .

**Theorem 3.3.** *If  $|\Delta(\mathfrak{m})| = 1$ , the centralizer  $c_{\mathfrak{u}}(x(d))$  is characterized as follows:*

(1) *If  $\Delta(\mathfrak{m}) = \{\alpha_1\}$ , then elements*

$$e_{2,n+1}, x(d)^k (1 \leq k \leq n-1)$$

*form a basis of  $c_{\mathfrak{u}}(x(d))$ .*

(2) *If  $\Delta(\mathfrak{m}) = \{\alpha_s\}$  with  $1 < s < n$ , then elements*

$$e_{1,s+1}, e_{s+1,n+1}, x(d)^k (1 \leq k \leq n-1)$$

*form a basis of  $c_{\mathfrak{u}}(x(d))$ .*

(3) *If  $\Delta(\mathfrak{m}) = \{\alpha_n\}$ , then elements*

$$e_{1,n+1}, x(d)^k (1 \leq k \leq n-1)$$

*form a basis of  $c_{\mathfrak{u}}(x(d))$ .*

*Proof.* When  $|\Delta(\mathfrak{m})| = 1$ , the partition of  $x(d)$  is  $[1, n]$  according to Lemma 3.1. Then there exist two elements  $v_1$  and  $v_2$  such that  $x(d)^i v_1$  for  $0 \leq i \leq n-1$  together with  $v_2$  form a basis of  $\mathbb{V}$ . Let  $Z \in c_{\mathfrak{u}}(x(d))$ . The first observation of  $c_{\mathfrak{u}}(x(d)) = c_{\mathfrak{gl}(\mathbb{V})}(x(d)) \cap \mathfrak{u}$ , implying  $Z \in c_{\mathfrak{gl}(\mathbb{V})}(x(d))$  and further

$$Z(v_1) = \sum_{k=0}^{n-1} a_{k,1;1} x(d)^k v_1 + a_{0,2;1} v_2$$

$$Z(v_2) = a_{n-1,1;2} x(d)^{n-1} v_1 + a_{0,2;2} v_2.$$

Additionally, the fact that  $Z \in \mathfrak{u}$  gives  $a_{0,1;1} = 0$  and  $a_{0,2;2} = 0$ . By assumption, we let  $\Delta(\mathfrak{m}) = \{\alpha_s\}$  for  $1 \leq s \leq n$ . According to Lemma 3.2, we have

$$v_1 = \begin{cases} e_{n+1}, & \text{if } 1 \leq s < n, \\ e_n, & \text{if } s = n \end{cases}$$

and  $v_2 = e_{s+1}$ .

Case 1.  $\Delta(\mathfrak{m}) = \{\alpha_1\}$ . Since  $x(d)^{n-1} v_1 = e_1$ , which gives  $a_{n-1,1;2} = 0$ . Let  $Z_1 \in c_{\mathfrak{u}}(x(d))$  with  $Z_1(v_1) = v_2$ . Then  $Z_1 = e_{2,n+1} + Z'_1$ . By Lemma 2.2,  $Z'_1 \in c_{\mathfrak{u}}(x(d))$ , implying that  $Z'_1(v_1) = Z'_1(v_2) = 0$ , so  $Z'_1 = 0$ . Therefore,

$$Z = \sum_{k=1}^{n-1} a_{k,1;1} x(d)^k + a_{0,2;1} e_{2,n+1}.$$

Case 2.  $\Delta(\mathfrak{m}) = \{\alpha_s\}$  for  $1 < s < n+1$ . Let  $Z_2 \in c_{\mathfrak{u}}(x(d))$  with  $Z_2(v_1) = v_2$ , then  $Z_2 = e_{s+1,n+1} + Z'_2$ . By Lemma 2.2,  $Z'_2 \in c_{\mathfrak{u}}(x(d))$ , giving  $Z'_2(v_1) = 0$ . If  $Z'_2(v_2) = e_1$ , then  $Z'_2 = e_{1,s+1}$ . Hence,

$$Z = \sum_{k=1}^{n-1} a_{k,1;1} x(d)^k + a_{0,2;1} e_{s+1,n+1} + a_{n-1,1;2} e_{1,s+1}.$$

Case 3.  $\Delta(\mathbf{m}) = \{\alpha_n\}$ . Given that  $v_1 = e_n$  and  $v_2 = e_{n+1}$ , it can be deduced that  $a_{0,2;1} = 0$ . Let  $Z_3 \in c_u(x(d))$  with  $Z_3(v_2) = x(d)^{n-1}v_1 = e_1$ . Then  $Z_3 = e_{1,n+1} + Z'_3$  and  $Z'_3 \in c_u(x(d))$  by Lemma 2.2. Hence,

$$Z = \sum_{k=1}^{n-1} a_{k,1;1} x(d)^k + a_{n-1,1;2} e_{1,n+1}.$$

□

**Theorem 3.4.** *If  $|\Delta(\mathbf{m})| = 2$  and  $x(d)$  is of partition  $[1^2, n-1]$ , then the centralizer  $c_u(x(d))$  is characterized as follows:*

(1) *If  $\Delta(\mathbf{m}) = \{\alpha_1, \alpha_2\}$ , then elements*

$$e_{2,n+1}, e_{3,n+1}, x(d)^k (1 \leq k \leq n-2)$$

*form a basis for  $c_u(x(d))$ .*

(2) *If  $\Delta(\mathbf{m}) = \{\alpha_r, \alpha_{r+1}\}$  with  $1 < r < n-1$ , then elements*

$$e_{1,r+1}, e_{1,r+2}, e_{r+1,n+1}, e_{r+2,n+1}, x(d)^k (1 \leq k \leq n-2)$$

*form a basis for  $c_u(x(d))$ .*

(3) *If  $\Delta(\mathbf{m}) = \{\alpha_{n-1}, \alpha_n\}$ , then elements*

$$e_{1,n}, e_{1,n+1}, x(d)^k (1 \leq k \leq n-2)$$

*form a basis for  $c_u(x(d))$ .*

*Proof.* If  $|\Delta(\mathbf{m})| = 2$  and the partition of  $x(d)$  is  $[1^2, n-1]$ , then there exist three elements  $v_1, v_2$  and  $v_3$  such that  $x(d)^i v_1$  for  $0 \leq i \leq n-2$  together with  $v_2, v_3$  are a basis of  $\mathbb{V}$ . Let  $Z \in c_u(x(d))$ . Then  $Z \in c_{\mathfrak{gl}(\mathbb{V})}(x(d))$  and

$$\begin{aligned} Z(v_1) &= \sum_{k=0}^{n-2} a_{k,1;1} x(d)^k v_1 + a_{0,2;1} v_2 + a_{0,3;1} v_3 \\ Z(v_2) &= a_{n-2,1;2} x(d)^{n-2} v_1 + a_{0,2;2} v_2 + a_{0,3;2} v_3 \\ Z(v_3) &= a_{n-2,1;3} x(d)^{n-2} v_1 + a_{0,2;3} v_2 + a_{0,3;3} v_3. \end{aligned}$$

Since  $Z \in \mathfrak{u}$ , we have  $a_{0,1;1} = a_{0,2;2} = a_{0,3;3} = 0$ . By assumption, we may let  $\Delta(\mathbf{m}) = \{\alpha_r, \alpha_{r+1}\}$  for  $1 \leq r < n$ . By Lemma 3.2, we have

$$v_1 = \begin{cases} e_{n-1}, & \text{if } r = n-1, \\ e_{n+1}, & \text{if } r < n-1, \end{cases}$$

$v_2 = e_{r+1}$  and  $v_3 = e_{r+2}$ . The remainder of the proof exhibits similarity to the demonstration of Theorem 3.3. □

**Theorem 3.5.** *If  $|\Delta(\mathbf{m})| = 2$  and  $x(d)$  is of partition  $[2, n-1]$ , then the centralizer  $c_u(x(d))$  is characterized as follows:*

(1) *If  $\Delta(\mathbf{m}) = \{\alpha_1, \alpha_n\}$ , then elements*

$$e_{1,n+1}, e_{2,n}, x(d)^k (1 \leq k \leq n-2)$$

*form a basis for  $c_u(x(d))$ .*

(2) If  $\Delta(\mathbf{m}) = \{\alpha_1, \alpha_s\}$  with  $2 < s < n$ , then elements

$$e_{1,s+1}, e_{2,n+1}, e_{2,n} + e_{s+1,n+1}, x(d)^k (1 \leq k \leq n-2)$$

form a basis for  $c_u(x(d))$ .

(3) If  $\Delta(\mathbf{m}) = \{\alpha_r, \alpha_n\}$  with  $1 < r < n-1$ , then elements

$$e_{1,n+1}, e_{r+1,n}, e_{1,r+1} + e_{2,n+1}, x(d)^k (1 \leq k \leq n-2)$$

form a basis for  $c_u(x(d))$ .

(4) If  $\Delta(\mathbf{m}) = \{\alpha_r, \alpha_s\}$  with  $1 < r < s-1 < n-1$ , then elements

$$e_{1,s+1}, e_{r+1,n+1}, e_{1,r+1} + e_{2,s+1}, e_{r+1,n} + e_{s+1,n+1}, x(d)^k (1 \leq k \leq n-2)$$

form a basis for  $c_u(x(d))$ .

*Proof.* When  $x(d)$  has partition  $[2, n-1]$ , there exist two elements  $v_1$  and  $v_2$  such that  $x(d)^i v_1$  for  $0 \leq i \leq n-2$  together with  $v_2, x(d)v_2$  form a basis of  $\mathbb{V}$ . If  $Z \in c_u(x(d))$ , then

$$Z(v_1) = \sum_{k=1}^{n-2} a_{k,1;1} x(d)^k v_1 + a_{0,2;1} v_2 + a_{1,2;1} x(d) v_2$$

$$Z(v_2) = a_{n-3,1;2} x(d)^{n-3} v_1 + a_{n-2,1;2} x(d)^{n-2} v_1 + a_{1,2;2} x(d) v_2$$

with  $a_{1,1;1} = a_{1,2;2}$ . Since  $|\Delta(\mathbf{m})| = 2$ , we may assume that  $\Delta(\mathbf{m}) = \{\alpha_r, \alpha_s\}$  where  $1 \leq r < s \leq n$  and  $r+1 < s$ . By Lemma 3.2, we have

$$v_1 = \begin{cases} e_n, & \text{if } s = n, \\ e_{n+1}, & \text{if } s < n, \end{cases}$$

and  $v_2 = e_{s+1}$ .

Case 1.  $\Delta(\mathbf{m}) = \{\alpha_1, \alpha_n\}$ . Then  $a_{0,2;1} = a_{n-3,1;2} = 0$ . Let  $Z' \in c_u(x(d))$  with  $Z'(v_1) = x(d)v_2$  and  $Z'(v_2) = 0$ , we have  $Z' = e_{2,n}$ . Let  $Z'' \in c_u(x(d))$  with  $Z''(v_1) = 0$  and  $Z''(v_2) = x(d)^{n-2}v_1$ , we have  $Z'' = e_{1,n+1}$ . In this case, we have

$$Z = \sum_{k=1}^{n-2} a_{k,1;1} x(d)^k + a_{1,2;1} e_{2,n} + a_{n-2,1;2} e_{1,n+1}.$$

Case 2.  $\Delta(\mathbf{m}) = \{\alpha_1, \alpha_s\}$  with  $2 < s < n$ . Let  $x \in c_u(x(d))$  with  $x = e_{a,b} + x'$  where  $l(a) \neq l(b)$ . If  $a^- = b^+ = \emptyset$ , then  $e_{a,b} = e_{1,s+1}$  or  $e_{a,b} = e_{2,n+1}$ , and  $x' \in c_u(x(d))$  by Lemma 2.2. Further, we have  $e_{1,s+1}(v_1) = 0$  and  $e_{1,s+1}(v_2) = x(d)^{n-2}v_1$ . By the same token,  $e_{2,n+1}(v_1) = x(d)v_2$  and  $e_{2,n+1}(v_2) = 0$ . If  $a^- \neq \emptyset$  or  $b^+ \neq \emptyset$ , then  $e_{a,b} = e_{2,n}$  and  $x - e_{2,n} - e_{s+1,n+1} \in c_u(x(d))$  by Lemma 2.2. Moreover, we have

$$(e_{2,n} + e_{s+1,n+1})(v_1) = v_2, \quad (e_{2,n} + e_{s+1,n+1})(v_2) = 0.$$

Since  $e_{a,b} \neq e_{3,s+1}$ , which implies that  $a_{n-3,1;2} = 0$ . In this case, we have

$$Z = \sum_{k=1}^{n-2} a_{k,1;1} x(d)^k + a_{n-2,1;2} e_{1,s+1} + a_{1,2;1} e_{2,n+1} + a_{0,2;1} (e_{2,n} + e_{s+1,n+1}).$$

Case 3.  $\Delta(\mathbf{m}) = \{\alpha_r, \alpha_n\}$  with  $1 < r < n-1$ . Then  $a_{0,2;1} = 0$ . Let  $x \in c_u(x(d))$  with  $x = e_{a,b} + x'$ . If  $a^- = b^+ = \emptyset$ , then  $e_{a,b} = e_{1,n+1}$  or  $e_{a,b} = e_{r+1,n}$ , and  $x' \in c_u(x(d))$  by Lemma 2.2. Further, we have  $e_{1,n+1}(v_1) = 0$  and  $e_{1,n+1}(v_2) = x(d)^{n-2}v_1$ . By the same token,  $e_{r+1,n}(v_1) = x(d)v_2$  and  $e_{r+1,n}(v_2) = 0$ . If  $a^- \neq \emptyset$  or  $b^+ \neq \emptyset$ , then  $e_{a,b} = e_{2,n+1}$  and  $x - e_{2,n+1} - e_{1,r+1} \in c_u(x(d))$  by Lemma 2.2. Moreover, we have

$$(e_{2,n+1} + e_{1,r+1})(v_1) = 0, \quad (e_{2,n+1} + e_{1,r+1})(v_2) = x(d)^{n-3}v_1.$$

In this case, we have

$$Z = \sum_{k=1}^{n-2} a_{k,1;1}x(d)^k + a_{n-2,1;2}e_{1,n+1} + a_{1,2;1}e_{r+1,n} + a_{n-3,1;2}(e_{2,n+1} + e_{1,r+1}).$$

Case 4.  $\Delta(\mathbf{m}) = \{\alpha_r, \alpha_s\}$  with  $1 < r < s-1 < n-1$ . Let  $x \in c_u(x(d))$  with  $x = e_{a,b} + x'$ . If  $a^- = b^+ = \emptyset$ , then  $e_{a,b} = e_{1,s+1}$  or  $e_{a,b} = e_{r+1,n+1}$ , and  $x' \in c_u(x(d))$  by Lemma 2.2. Further, we have  $e_{1,s+1}(v_1) = 0$  and  $e_{1,s+1}(v_2) = x(d)^{n-2}v_1$ . By the same token,  $e_{r+1,n+1}(v_1) = x(d)v_2$  and  $e_{r+1,n+1}(v_2) = 0$ . If  $a^- \neq \emptyset$  or  $b^+ \neq \emptyset$ , then  $e_{a,b} = e_{1,r+1}$  and  $x - e_{1,r+1} - e_{2,s+1} \in c_u(x(d))$  or  $e_{a,b} = e_{r+1,n}$  and  $x - e_{r+1,n} - e_{s+1,n+1} \in c_u(x(d))$  by Lemma 2.2. Moreover, we have

$$(e_{1,r+1} + e_{2,s+1})(v_1) = 0, \quad (e_{1,r+1} + e_{2,s+1})(v_2) = x(d)^{n-3}v_1$$

and

$$(e_{r+1,n} + e_{s+1,n+1})(v_1) = v_2, \quad (e_{r+1,n} + e_{s+1,n+1})(v_2) = 0$$

In this case, we have

$$\begin{aligned} Z = & \sum_{k=1}^{n-2} a_{k,1;1}x(d)^k + a_{n-2,1;2}e_{1,s+1} + a_{1,2;1}e_{r+1,n+1} \\ & + a_{n-3,1;2}(e_{1,r+1} + e_{2,s+1}) + a_{0,2;1}(e_{r+1,n} + e_{s+1,n+1}). \end{aligned}$$

□

**Corollary 3.6.** *Let  $x(d)$  be a Richardson element determined by  $|\Delta(\mathbf{m})| \leq 2$  and  $Z(c_u(x(d)))$  be the center of its centralizer  $c_u(x(d))$ . Then*

- (1)  $Z(c_u(x(d))) = c_u(x(d))$ , provided that  $\Delta(\mathbf{m})$  is one of the following cases
 
$$\{\alpha_1\}, \{\alpha_n\}, \{\alpha_1, \alpha_2\}, \{\alpha_1, \alpha_n\}, \{\alpha_{n-1}, \alpha_n\}.$$
- (2)  $Z(c_u(x(d)))$  is spanned by the power  $x(d)^k$  of  $x(d)$ , provided that  $\Delta(\mathbf{m})$  is one of the following cases
 
$$\{\alpha_s\}, \{\alpha_r, \alpha_{r+1}\} (1 < r < n-1), \{\alpha_r, \alpha_s\} (1 < r < s-1 < n-1).$$
- (3)  $Z(c_u(x(d)))$  is spanned by  $e_{2,n+1}$  together with  $x(d)^k$  ( $1 \leq k \leq n-2$ ), provided that  $\Delta(\mathbf{m}) = \{\alpha_1, \alpha_s\}$  and  $2 < s < n$ .
- (4)  $Z(c_u(x(d)))$  is spanned by  $e_{1,n+1}$  together with  $x(d)^k$  ( $1 \leq k \leq n-2$ ), provided that  $\Delta(\mathbf{m}) = \{\alpha_r, \alpha_n\}$  and  $1 < r < n-1$ .

Moreover, the centralizer  $c_u(x(d))$  of  $x(d)$  can be characterized by the following Table 1

$\Delta(\mathfrak{m})$	<i>restrictions</i>	$c_{\mathfrak{u}}(x(d))$	<i>dimension</i>
$\alpha_1$	<i>none</i>	<i>abelian</i>	$n$
$\alpha_s$	$1 < s < n$	<i>not abelian</i>	$n + 1$
$\alpha_n$	<i>none</i>	<i>abelian</i>	$n$
$\alpha_1, \alpha_2$	<i>none</i>	<i>abelian</i>	$n$
$\alpha_{n-1}, \alpha_n$	<i>none</i>	<i>abelian</i>	$n$
$\alpha_r, \alpha_{r+1}$	$1 < r < n - 1$	<i>not abelian</i>	$n + 2$
$\alpha_1, \alpha_n$	<i>none</i>	<i>abelian</i>	$n$
$\alpha_r, \alpha_n$	$1 < r < n - 1$	<i>not abelian</i>	$n + 1$
$\alpha_1, \alpha_s$	$2 < s < n$	<i>not abelian</i>	$n + 1$
$\alpha_r, \alpha_s$	$1 < r < s - 1 < n - 1$	<i>not abelian</i>	$n + 2$

TABLE 1. Summarization for  $c_{\mathfrak{u}}(x(d))$

#### 4. SATURATION RANK FOR NILPOTENT RADICAL

Keep the notations as above, that is  $G = \mathrm{SL}_{n+1}(K)$ ,  $P = P(d)$  the standard parabolic subgroup of  $G$  given by the set  $\Delta(\mathfrak{m})$ ,  $\mathfrak{p}(d) = \mathrm{Lie}(P(d))$  the Lie algebra of  $P(d)$  and  $\mathfrak{u}$  the nilpotent ideal of  $\mathfrak{p}(d)$ . For any arbitrary element  $x$  in  $V(\mathfrak{u})$ , we define the set

$$\mathbb{E}(r, \mathfrak{u})_x := \{\mathfrak{e} \in \mathbb{E}(r, \mathfrak{u}) \mid x \in \mathfrak{e}\}$$

as a subset of  $\mathbb{E}(r, \mathfrak{u})$  consisting of elements that contain  $x$ . Since  $\mathbb{E}(1, \mathfrak{u})_x \neq \emptyset$ , the number

$$\mathrm{rk}(\mathfrak{u})_x := \max\{r \in \mathbb{N} \mid \mathbb{E}(r, \mathfrak{u})_x \neq \emptyset\},$$

is called the *local saturation rank* of  $x$ . The first step towards the determination of the saturation rank of  $\mathfrak{u}$  is (see Sect. 3.1 in [8]):

**Lemma 4.1.** *Let  $\mathrm{rk}_{\min}(\mathfrak{u}) = \min\{\mathrm{rk}(\mathfrak{u})_x \mid x \in V(\mathfrak{u})\}$ . Then  $\mathrm{srk}(\mathfrak{u}) = \mathrm{rk}_{\min}(\mathfrak{u})$ .*

We consider the set

$$\mathcal{O}_{\min}(\mathfrak{u}) := \{x \in V(\mathfrak{u}) \mid \mathrm{rk}(\mathfrak{u})_x = \mathrm{rk}_{\min}(\mathfrak{u})\},$$

which is an open subset of  $V(\mathfrak{u})$  (See Sect. 3.1 in [8]). Lemma 4.1 does not give a complete determination of the saturation rank of  $\mathfrak{u}$  because it does not say what the possible elements of  $\mathcal{O}_{\min}(\mathfrak{u})$  are. Recall that the nilpotent ideal  $\mathfrak{u}$  is the union of its intersection with the nilpotent orbits in  $\mathfrak{g}$ . The finiteness of the number of nilpotent orbits ensures that there is a unique orbit  $\mathcal{O}$ , such that  $\mathcal{O} \cap \mathfrak{u}$  is a

$P(d)$ -orbit and open dense in  $\mathfrak{u}$ . We call  $\mathcal{O}$  the Richardson orbit corresponding to  $P(d)$ .

**Lemma 4.2.** *If  $V(\mathfrak{u}) = \mathfrak{u}$ , then the saturation rank of  $\mathfrak{u}$  is determined by the local saturation rank of elements in  $\mathcal{O} \cap \mathfrak{u}$ ; that is  $\text{srk}(\mathfrak{u}) = \text{rk}(\mathfrak{u})_e, \forall e \in \mathcal{O} \cap \mathfrak{u}$ .*

*Proof.* For any  $e \in \mathcal{O} \cap \mathfrak{u}$ , the orbit  $\mathcal{O} \cap \mathfrak{u} = P(d) \cdot e$  forms an open subset of  $\mathfrak{u}$ . The fact that  $\mathcal{O}_{\text{min}}$  is open in  $V(\mathfrak{u})$  implies that it is also open in  $\mathfrak{u}$  since  $V(\mathfrak{u}) = \mathfrak{u}$ . It is important to note that  $\mathfrak{u}$  is irreducible, and therefore, the intersection  $P(d) \cdot e \cap \mathcal{O}_{\text{min}}$  is indeed non-empty. It is observed that the adjoint action of  $P(d)$  on  $\mathfrak{u}$  remains within  $\mathfrak{u}$ , thereby implying that the local saturation rank  $\text{rk}(\mathfrak{u})_e$  of  $e$  is equal to that of  $p \cdot e$  for any  $p \in P(d)$ . As a result,  $P(d) \cdot e \subset \mathcal{O}_{\text{min}}$ , and  $\text{srk}(\mathfrak{u}) = \text{rk}(\mathfrak{u})_e$  for any  $e \in \mathcal{O} \cap \mathfrak{u}$ .  $\square$

**Lemma 4.3.** *Let  $x(d)$  be a Richardson element and  $c_{\mathfrak{u}}(x(d))$  be its centralizer in  $\mathfrak{u}$ . Assume that  $V(\mathfrak{u}) = \mathfrak{u}$ , then*

- (1) *If  $c_{\mathfrak{u}}(x(d))$  is abelian, then  $\text{srk}(\mathfrak{u}) = \dim c_{\mathfrak{u}}(x(d))$ .*
- (2) *If  $c_{\mathfrak{u}}(x(d))$  is not abelian, then  $\dim Z(c_{\mathfrak{u}}(x(d))) \leq \text{srk}(\mathfrak{u}) < \dim c_{\mathfrak{u}}(x(d))$ .*

*Proof.* It is observed that any  $\mathfrak{e} \in \mathbb{E}(\text{rk}(\mathfrak{u})_{x(d)}, \mathfrak{u})_{x(d)}$  is contained in  $c_{\mathfrak{u}}(x(d))$  and contains the center  $Z(c_{\mathfrak{u}}(x(d)))$ . Thus, we have

$$\dim Z(c_{\mathfrak{u}}(x(d))) \leq \text{rk}(\mathfrak{u})_{x(d)} \leq \dim c_{\mathfrak{u}}(x(d)).$$

We immediately deduce the results in viewing of  $\text{srk}(\mathfrak{u}) = \text{rk}(\mathfrak{u})_{x(d)}$  by Lemma 4.2 since  $x(d)$  belongs to  $\mathcal{O} \cap \mathfrak{u}$ .  $\square$

We say a standard parabolic subgroup  $P(d)$  of  $G$  is restricted provided that  $\mathfrak{u} \subseteq V(\mathfrak{g})$ , or, equivalently, that  $\mathcal{O} \subseteq V(\mathfrak{g})$  (cf. [5]).

**Theorem 4.4.** *Let  $\mathfrak{p}(d)$  be a parabolic subalgebra of  $\mathfrak{sl}_{n+1}(K)$  with  $|\Delta(\mathfrak{m})| \leq 2$ , and  $\mathfrak{u}$  be the nilpotent ideal of  $\mathfrak{p}(d)$ . If  $p \geq n + 1$  or  $P(d)$  is restricted, then the following statements hold:*

- (1) *The saturation rank of  $\mathfrak{u}$  is  $n$ .*
- (2) *Any maximal elementary subalgebra associated with a Richardson element  $x(d)$  is either unique or parametrized by points of  $\mathbb{P}^1$  or  $\mathbb{P}^1 \times \mathbb{P}^1$ .*
- (3) *The variety  $\mathbb{E}(n, \mathfrak{u})_{x(d)}$  has a dimension of at most 2.*

Moreover, we give the characterization of the variety  $\mathbb{E}(n, \mathfrak{u})_{x(d)}$  in Table 2.

*Proof.* In light of Corollary 3.6, there are five types of  $\Delta(\mathfrak{m})$  for which the centralizer  $c_{\mathfrak{u}}(x(d))$  is abelian and has dimension  $n$ . Consequently, the maximal elementary subalgebra containing  $x(d)$  is  $c_{\mathfrak{u}}(x(d))$  itself, and by Lemma 4.3,  $\text{srk}(\mathfrak{u}) = n$  when  $\Delta(\mathfrak{m})$  is as follows:

$$\{\alpha_1\}, \{\alpha_n\}, \{\alpha_1, \alpha_2\}, \{\alpha_1, \alpha_n\}, \{\alpha_{n-1}, \alpha_n\}.$$

For the remaining cases, we will examine each one individually.

$\Delta(\mathbf{m})$	<i>restrictions</i>	$\mathbb{E}(n, \mathbf{u})_{x(d)}$	<i>dimension</i>
$\alpha_1$	<i>none</i>	<i>singleton</i>	0
$\alpha_s$	$1 < s < n$	$\mathbb{P}^1$	1
$\alpha_n$	<i>none</i>	<i>singleton</i>	0
$\alpha_1, \alpha_2$	<i>none</i>	<i>singleton</i>	0
$\alpha_{n-1}, \alpha_n$	<i>none</i>	<i>singleton</i>	0
$\alpha_r, \alpha_{r+1}$	$1 < r < n - 1$	$\mathbb{P}^1 \times \mathbb{P}^1$	2
$\alpha_1, \alpha_n$	<i>none</i>	<i>singleton</i>	0
$\alpha_r, \alpha_n$	$1 < r < n - 1$	$\mathbb{P}^1$	1
$\alpha_1, \alpha_s$	$2 < s < n$	$\mathbb{P}^1$	1
$\alpha_r, \alpha_s$	$1 < r < s - 1 < n - 1$	$\mathbb{P}^1 \cup \mathbb{P}^1$	1

TABLE 2. Summarization for  $\mathbb{E}(n, \mathbf{u})_{x(d)}$

Case 1.  $\Delta(\mathbf{m}) = \{\alpha_s\}$  with  $1 < s < n$ . The maximal elementary subalgebras containing  $x(d)$  are

$$\bigoplus_{i=1}^{n-1} Kx(d)^i \oplus K(ae_{1,s+1} + be_{s+1,n+1})$$

which are parametrized by points  $(a : b) \in \mathbb{P}^1$ . Consequently, the saturation rank is  $n$ , and the variety  $\mathbb{E}(n, \mathbf{u})_{x(d)}$  is irreducible with dimension 1.

Case 2.  $\Delta(\mathbf{m}) = \{\alpha_r, \alpha_{r+1}\}$  with  $1 < r < n - 1$ . The maximal elementary subalgebras containing  $x(d)$  are

$$\bigoplus_{i=1}^{n-2} Kx(d)^i \oplus K(ae_{1,r+1} + be_{r+1,n+1}) \oplus K(a'e_{1,r+2} + b'e_{r+2,n+1})$$

where  $(a : b), (a' : b') \in \mathbb{P}^1$  and  $\mathbb{E}(n, \mathbf{u})_{x(d)}$  is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ . So, the saturation rank is  $n$  and the variety  $\mathbb{E}(n, \mathbf{u})_{x(d)}$  is irreducible with dimension 2.

Case 3.  $\Delta(\mathbf{m}) = \{\alpha_r, \alpha_n\}$  with  $1 < r < n - 1$ . The maximal elementary subalgebras containing  $x(d)$  are

$$\bigoplus_{i=1}^{n-2} Kx(d)^i \oplus K(a(e_{1,r+1} + e_{2,n+1}) + be_{r+1,n}) \oplus Ke_{1,n+1}$$

which are parametrized by points  $(a : b) \in \mathbb{P}^1$ . Hence, the saturation rank is  $n$  and the variety  $\mathbb{E}(n, \mathbf{u})_{x(d)}$  is irreducible with dimension 1.

Case 4.  $\Delta(\mathbf{m}) = \{\alpha_1, \alpha_s\}$  with  $2 < s < n$ . The maximal elementary subalgebras containing  $x(d)$  are

$$\bigoplus_{i=1}^{n-2} Kx(d)^i \oplus K(a(e_{2,n} + e_{s+1,n+1}) + be_{1,s+1}) \oplus Ke_{2,n+1}$$

which are parametrized by points  $(a : b) \in \mathbb{P}^1$ . Hence, the saturation rank is  $n$  and the variety  $\mathbb{E}(n, \mathbf{u})_{x(d)}$  is irreducible with dimension 1.

Case 5.  $\Delta(\mathbf{m}) = \{\alpha_r, \alpha_s\}$  with  $1 < r < s - 1 < n - 1$ . There exist two types of  $n$ -dimensional maximal elementary subalgebras

$$\begin{aligned} & \bigoplus_{i=1}^{n-2} Kx(d)^i \oplus K(a(e_{1,r+1} + e_{2,s+1}) + be_{r+1,n+1}) \oplus Ke_{1,s+1}, \\ & \bigoplus_{i=1}^{n-2} Kx(d)^i \oplus K(a'(e_{r+1,n} + e_{s+1,n+1}) + b'e_{1,s+1}) \oplus Ke_{r+1,n+1} \end{aligned}$$

containing  $x(d)$ , both parametrized by points in  $\mathbb{P}^1$ . Therefore, the saturation rank is  $n$ , and the variety  $\mathbb{E}(n, \mathbf{u})_{x(d)}$  can be viewed as the union of two copies of  $\mathbb{P}^1$ , denoted as  $\mathbb{P}^1 \cup \mathbb{P}^1$ .

□

**Remark 4.1.** We give two remarks concerning Theorem 4.4.

1. As illustrated in Remark 2.4, for  $\mathfrak{p}(d) \subseteq \mathfrak{sl}_9(K)$  and  $d = (3, 2, 3, 1)$ , the Richardson element is  $x(d) = e_{1,4} + e_{4,6} + e_{6,9} + e_{2,5} + e_{5,7} + e_{3,8}$  and there exists a maximal elementary subalgebra

$$\bigoplus_{i=1}^3 Kx(d)^i \oplus K(e_{1,5} + e_{4,7}) \oplus Ke_{1,7} \oplus Ke_{1,8} \oplus Ke_{2,8} \oplus Ke_{2,9} \oplus Ke_{3,7} \oplus Ke_{3,9}$$

with dimension 10. Thereby, Theorem 4.4(1) may not hold for  $|\Delta(\mathbf{m})| > 2$ .

2. Our fundamental application of saturation rank determined in Theorem 4.4 is given by the indecomposability of Carlson modules  $L_\zeta$ . Let  $(P_n, d_n)_{n \geq 0}$  be a minimal projective resolution of the trivial  $U_0(\mathfrak{u})$ -module  $K$ . Then

$$\text{Hom}(\Omega^n(K), K) \rightarrow H^n(\mathfrak{u}, K), \quad \hat{\zeta} \mapsto [\hat{\zeta} \circ d_n],$$

is an isomorphism. If  $\zeta := [\hat{\zeta} \circ d_n] \neq 0$ , then the Carlson module is defined as  $L_\zeta := \text{Ker } \hat{\zeta} \subseteq \Omega^n(K)$ . By virtue of Theorem 6.4.4 in [7], under the conditions given in Theorem 4.4, if  $n \geq 2$  and  $\zeta \neq 0$  has odd degree, then  $L_\zeta$  is indecomposable.

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