# SCHWARZ-PICK TYPE INEQUALITIES FROM AN OPERATOR THEORETICAL POINT OF VIEW 

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#### Abstract

We use (versions of) the von Neumann inequality for Hilbert space contractions to prove several Schwarz-Pick type inequalities. Specifically, we derive an alternate proof for a multi-point Schwarz-Pick inequality by Beardon and Minda, along with a generalized version for operators. Connections with model spaces and Peschl's invariant derivatives are established. Finally, Schwarz-Pick inequalities for analytic functions on polydisks and for higher order derivatives are discussed. An enhanced version of the Schwarz-Pick lemma, using the notion of distinguished variety, is obtained for the bidisk.


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## Notation

$\mathbb{D} \quad$ denotes the open unit disk
$\mathbb{T} \quad$ denotes the unit circle, $\mathbb{T}=\overline{\mathbb{D}} \backslash \mathbb{D}$
$\|\cdot\|_{\infty} \quad$ for $\underline{\omega}=\left(\omega_{1}, \ldots, \omega_{n}\right) \in \mathbb{C}^{n}$, we denote $\|\underline{\omega}\|_{\infty}=\sup \left\{\left|\omega_{i}\right|: 1 \leq i \leq n\right\}$
$\mathcal{H}(\mathbb{D}) \quad$ is the set of functions that are holomorphic on $\mathbb{D}, \mathcal{H}(\mathbb{D})=\mathcal{H}(\mathbb{D}, \mathbb{C})$
$\mathcal{H}(\mathbb{D}, \mathbb{D}) \quad$ denotes the set of functions in $\mathcal{H}(\mathbb{D})$ mapping $\mathbb{D}$ to $\mathbb{D}$
$\mathcal{A}(\overline{\mathbb{D}}) \quad$ is the disk algebra, i.e. the set of functions that are holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$
$(z, w) \quad$ denotes the complex pseudo-hyperbolic distance $(z, w):=\frac{z-w}{1-\bar{w} z}$
$\rho(z, w) \quad$ denotes the pseudo-hyperbolic distance $\rho(z, w):=|(z, w)|$
$\mathrm{d}(z, w) \quad$ denotes the hyperbolic distance $\mathrm{d}(z, w)=\tanh ^{-1} \frac{1+\rho(z, w)}{1-\rho(z, w)}$
$f^{*}(z, w) \quad$ denotes the hyperbolic divided difference $f^{*}(z, w):=\frac{(f(z), f(w))}{(z, w)}$
$H^{2}(\mathbb{D}) \quad$ is the Hilbert-Hardy space of $\mathbb{D}$
$H^{\infty}(\mathbb{D}) \quad$ is the set of bounded holomorphic functions on $\mathbb{D}$
$\mathcal{B}(H, K) \quad$ is the set of bounded linear operators from $H$ to $K$, where $H$ and $K$ are two complex Hilbert spaces. $\mathcal{B}(H)$ is a short for $\mathcal{B}(H, H)$
$\|\cdot\| \quad$ denotes the norm of an element of the Banach space under consideration. When $T \in \mathcal{B}(H, K),\|T\|$ denotes the operator norm
$T^{*} \quad$ denotes the adjoint of $T$, where $T$ is a Hilbert space operator
$D_{T} \quad$ denotes the defect operator of a contraction $T \in \mathcal{B}(H)$, i.e. $\|T\| \leq 1$. Thus $D_{T}=\left(\operatorname{Id}-T^{*} T\right)^{1 / 2}$, where Id is the identity operator
$\sigma(T) \quad$ denotes the spectrum of $T \in \mathcal{B}(H)$
$r(T) \quad$ denotes the spectral radius $r(T)=\sup \{|\lambda|: \lambda \in \sigma(T)\}$ of $T$
$\operatorname{Ker}(T) \quad$ denotes the kernel of $T$
$\operatorname{Im}(T) \quad$ denotes the range (image) of $T$
$\llbracket 1, n \rrbracket \quad$ denotes the set of all integers $j$ with $1 \leq j \leq n$.

## 1. Introduction

The Schwarz-Pick inequality, an invariant form of the Schwarz lemma, stands as a cornerstone in complex analysis. In geometric terms, it posits that a holomorphic map from the
open unit disk into itself has the property of decreasing the distance between points in the hyperbolic metric. Equivalently, if $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$ and $\omega_{1}$ and $\omega_{2}$ are two points in $\mathbb{D}$, then

$$
\begin{equation*}
\rho\left(f\left(\omega_{1}\right), f\left(\omega_{2}\right)\right)=\left|\frac{f\left(\omega_{1}\right)-f\left(\omega_{2}\right)}{1-\overline{f\left(\omega_{1}\right)} f\left(\omega_{2}\right)}\right| \leq\left|\frac{\omega_{1}-\omega_{2}}{1-\overline{\omega_{1}} \omega_{2}}\right|=\rho\left(\omega_{1}, \omega_{2}\right) \tag{1.1}
\end{equation*}
$$

Inequality (1.1) is strict for $\omega_{1} \neq \omega_{2}$ unless $f$ is a conformal automorphism of the unit disk. Moreover,

$$
\begin{equation*}
\frac{\left|f^{\prime}(\omega)\right|}{1-|f(\omega)|^{2}} \leq \frac{1}{1-|\omega|^{2}} \tag{1.2}
\end{equation*}
$$

The Schwarz-Pick inequalities (1.1) and (1.2) have been extended in various ways by many authors. A thorough overview of some of these advancements is provided in the comprehensive survey [17]. For the present study the following 'three points' version of (1.1) by Beardon and Minda [9] is pivotal. It involves the notion of hyperbolic divided difference $f^{*}(z, w)$ and states that if $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$, then

$$
\begin{equation*}
\rho\left(f^{*}\left(\omega_{1}, \omega_{2}\right), f^{*}\left(\omega_{3}, \omega_{2}\right)\right) \leq \rho\left(\omega_{1}, \omega_{3}\right) \tag{1.3}
\end{equation*}
$$

for three pairwise distinct points $\omega_{1}, \omega_{2}$ and $\omega_{3}$ in the unit disk. The Beardon-Minda inequality unifies in an elegant way many improvements of (1.1). An analogous theorem with more than three points has been proved in [8], where Baribeau, Rivard and Wegert also used iterated (hyperbolic) divided differences to give simpler conditions for the $n$ points Nevanlinna-Pick interpolation problem. We refer also to [1, 35, 36] for related contributions.

Various operator-theoretical interpretations of the Schwarz-Pick inequality are possible. The most well-known interpretation, due to Sarason [41], exploits the equivalence of the Schwarz-Pick inequality with the Nevanlinna-Pick interpolation problem for two points. Consequently, the Schwarz-Pick inequality possesses an operator-theoretical significance concerning norm-preserving lifting of some operators that act on specific subspaces of the Hardy space $H^{2}(\mathbb{D})$. Additional generalizations can be derived using the commuting lifting theorem of Sz.-Nagy and Foias (see for instance [19]). It is noteworthy that the commutant lifting theorem is equivalent with Ando's dilation theorem ([29]). Further operator-theoretical interpretations of the Schwarz-Pick inequality have been explored in $[5,6,22,25,27]$.

The starting point of this note was the natural question of looking for an operatortheoretical interpretation of the Beardon-Minda inequality. Notice that the Schwarz-Pick inequality can be obtained as a particular case of the von Neumann inequality for Hilbert space contractions. The von Neumann inequality states that if $T \in \mathcal{B}(H)$ is a bounded linear operator acting on a complex Hilbert space $H$ with $\|T\| \leq 1$ and $f$ is a polynomial, then

$$
\begin{equation*}
\|f(T)\| \leq \sup \{|f(z)|:|z| \leq 1\} \tag{1.4}
\end{equation*}
$$

This inequality extends to functions $f$ in the disk algebra $\mathcal{A}(\overline{\mathbb{D}})$. The inequality (1.1) is obtained ([38, p.17], [31, exercices 2.17-2.18]) when applying von Neumann's inequality (1.4) to a polynomial (or an element in the disk algebra) $f$ and a specific $2 \times 2$ matrix
acting on the 2-dimensional Hilbert space $\mathbb{C}^{2}$. Then an approximation argument gives (1.1) for $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$. A similar derivation by Agler of the Schwarz-Pick inequality (1.1) is presented in [2] and [4, Chapter 8], where this concept is ingeniously applied in to establish an operator-theoretical proof of Lempert's theorem, demonstrating the equality of the Carathéodory and Kobayashi metrics on convex domains. In this case, von Neumann's inequality is substituted with the notion of spectral set.

The primary objective of this manuscript is to leverage (versions of) the von Neumann inequality to derive Schwarz-Pick type inequalities. Notably, when applying the von Neumann inequality to a remarkable $3 \times 3$ matrix (the matrix of the model operator in the Takenaka-Malmquist basis), the Beardon-Minda three-point Schwarz-Pick inequality is obtained. A similar proof is given for a Beardon-Minda type inequality for derivatives and operator versions of the Schwarz-Pick and Beardon-Minda type inequalities are obtained. We also consider some Schwarz-Pick related inequalities for higher derivatives. Additionally, we delve into the multivariable case, employing von Neumann's inequality on $n$-tuples of mutually commuting $2 \times 2$ or $3 \times 3$ matrices to prove Schwarz-Pick type inequalities for the polydisk. In the case of the bidisk an improvement can be given using the notion of distinguished variety and the refined version of Ando's inequality by Agler and McCarthy [3]. The reader is welcomed to notice that the multipoint Schwarz-Pick inequality of [8] can also be derived as a consequence of the von Neumann inequality. However, explicit computations with matrices become more intricate.

Outline. The manuscript is organized as follows. In the next section we use a theorem going back to Parrott to obtain criteria for scalar and operator $3 \times 3$ matrices to have (Hilbertian) operator norm no greater than one. This is applied to a specific matrix to obtain an alternate proof of the Beardon-Minda inequality. The significance of this specific matrix with model spaces is highlighted, and a discussion concerning the equality case in (1.1) is given. Also, a Beardon-Minda type inequality for derivatives, originally proved by Yamashita [44], is obtained as a consequence of the von Neumann inequality. This inequality can be rephrased in terms of Peschl's invariant derivatives.

In Section 3 we prove several operator versions of the Schwarz-Pick inequality and of the Beardon-Minda inequality. The Sylvester (operator) equation $A X-X B=Y$ plays an important role in the proofs.

In the next section we consider the case of the polydisk. We give operator theoretical proofs of the analogues of (1.1) and (1.2) for the polydisk and discuss the Peschl's invariant derivatives in several variables. The proofs uses a result of Knese [25] that the (polydisk) von Neumann's inequality holds for $n$-tuples of $3 \times 3$ commuting contractive matrices. In the case of the bidisk we use a result of Agler and McCarthy [3] to obtained an enhanced version of the Schwarz-Pick inequality.

In Section 5 we give some Schwarz-Pick related inequalities for higher derivatives. An improvement of a classical result of F. Wiener is proved.

Parrott's theorem, which was essential in our proofs, is revisited in the Appendix. We hope that this will prove beneficial for readers interested in Schwarz-Pick inequalities who may not be extensively acquainted with operator theory.

## 2. A three points Schwarz-Pick lemma

In view of the preceding discussion, it is a natural question to apply the von Neumann inequality to $3 \times 3$ matrices.
2.A. Contractive three by three matrices. - First, we need an explicit criterion to determine whenever a $3 \times 3$-upper triangular matrix is a contraction. Note that in view of Schur's decomposition theorem - which states that every square matrix is unitary equivalent to an upper-triangular matrix - it makes sense to restrict ourselves to that case. In essence, following the approach used for $2 \times 2$ matrices, one can calculate the operator norm of a $3 \times 3$ matrix acting on the Euclidean space $\mathbb{C}^{3}$ using the formula:

$$
\|T\|^{2}=\left\|T^{*} T\right\|=r\left(T^{*} T\right)=\sup \left\{|\lambda|: \operatorname{det}\left(T^{*} T-\lambda \mathrm{Id}\right)=0\right\}
$$

This computation of the operator norm $\|T\|$, the largest singular value of $T$, leads to an equation of degree 3. However, the criterion derived from this observation holds limited practical interest. An alternative approach to obtain such a criterion uses the Schur parameters (cf. [13]). Adapting the argument in [21, Lemma 2.7], we follow here a different approach, based on a result about completion of matrices going back to Parrott (see [30, 19], [46, Theorem 12.22] and [7, 14]).

Theorem 2.1 (Parrott). - Let $H_{1}, H_{2}, K_{1}, K_{2}$ be Hilbert spaces, and suppose that the operators $\left[\begin{array}{l}A \\ C\end{array}\right] \in \mathcal{B}\left(H_{1}, K_{1} \oplus K_{2}\right)$ and $\left[\begin{array}{ll}C & D\end{array}\right] \in \mathcal{B}\left(H_{1} \oplus H_{2}, K_{2}\right)$ are contractions. Then, $T=\left[\begin{array}{ll}A & B \\ C & D\end{array}\right]: H_{1} \oplus H_{2} \rightarrow K_{1} \oplus K_{2}$ is a contraction if and only if there exists a contraction $W \in \mathcal{B}\left(H_{2}, K_{1}\right)$ such that $B=D_{Z^{*}} W D_{Y}-Z C^{*} Y$, where $Z \in \mathcal{B}\left(H_{1}, K_{1}\right)$ and $Y \in$ $\mathcal{B}\left(H_{2}, K_{2}\right)$ are contractions such that $D=D_{C^{*}} Y$ and $A=Z D_{C}$.

Moreover,

1. $Y$ and $Z$ can be chosen to be (respectively) $Y_{0}$ and $Z_{0}$, the solutions of minimal operator norm among all solutions of the operator equations $D=D_{C^{*}} Y$ and $A=$ $Z D_{C}$;
2. If $T$ is a contraction, there exists a unique contraction $W_{0}$ such that

$$
B=D_{Z_{0}^{*}} W_{0} D_{Y_{0}}-Z_{0} C^{*} Y_{0} \quad \text { and } \quad \operatorname{Im}\left(D_{Z_{0}^{*}}\right)^{\perp} \subset \operatorname{Ker}\left(W_{0}^{*}\right)
$$

This operator satisfies

$$
\left\|W_{0}\right\|=\inf \left\{\|W\|: B=D_{Z_{0}^{*}} W D_{Y_{0}}-Z_{0} C^{*} Y_{0}\right\}
$$

We shall call $Y_{0}$ and $Z_{0}$ the minimal solutions and we shall refer to $W_{0}$ as the minimal solution of the equation

$$
B=D_{Z_{0}^{*}} W D_{Y_{0}}-Z_{0} C^{*} Y_{0}
$$

A further discussion is given in the Appendix (Section 6). In particular, $T=\left[\begin{array}{cc}A_{1} & B \\ 0 & A_{2}\end{array}\right]$ is a contraction if and only if $B=\left(I-A_{1} A_{1}^{*}\right)^{1 / 2} W\left(I-A_{2}^{*} A_{2}\right)^{1 / 2}$ for a certain contraction $W$ and the scalar matrix

$$
T=\left[\begin{array}{cc}
\omega_{1} & a \\
0 & \omega_{2}
\end{array}\right]
$$

has (Euclidean) norm no greater than one if and only if $\left|\omega_{1}\right| \leq 1,\left|\omega_{2}\right| \leq 1$ and $|a| \leq$ $\sqrt{1-\left|\omega_{1}\right|^{2}} \sqrt{1-\left|\omega_{2}\right|^{2}}$.

The following result provides a criterion for determining whether a $3 \times 3$ operator matrix is a contraction, when the central entry of the matrix, $W_{2}$, is a strict contraction.

Theorem 2.2. - Let $H_{1}, H_{2}, H_{3}$ be three Hilbert spaces. Let $W_{i} \in \mathcal{B}\left(H_{i}\right), 1 \leq i \leq 3$, be three contractions and denote

$$
T=\left[\begin{array}{ccc}
W_{1} & A_{1} & B \\
0 & W_{2} & A_{2} \\
0 & 0 & W_{3}
\end{array}\right] \in \mathcal{B}\left(H_{1} \oplus H_{2} \oplus H_{3}\right)
$$

Assume that $\left\|W_{2}\right\|<1$. Then, $T$ is a contraction if and only if there exist three contractions $V_{1} \in \mathcal{B}\left(H_{2}, H_{1}\right), V_{2} \in \mathcal{B}\left(H_{3}, H_{2}\right), V_{3} \in \mathcal{B}\left(H_{3}, H_{1}\right)$ such that :

$$
\left\{\begin{array}{l}
A_{1}=D_{W_{1}^{*}} V_{1} D_{W_{2}}  \tag{2.1}\\
A_{2}=D_{W_{2}^{*}} V_{2} D_{W_{3}} \\
B=\left[D_{W_{1}^{*}}\left(\operatorname{Id}-V_{1} V_{1}^{*}\right) D_{W_{1}^{*}}\right]^{1 / 2} V_{3}\left[D_{W_{3}}\left(\operatorname{Id}-V_{2}^{*} V_{2}\right) D_{W_{3}}\right]^{1 / 2} \\
-D_{W_{1}^{*}} V_{1} W_{2}^{*} V_{2} D_{W_{3}}
\end{array}\right.
$$

Proof. - First, if $T$ is a contraction, then $\left[\begin{array}{cc}W_{1} & A_{1} \\ 0 & W_{2}\end{array}\right]$ and $\left[\begin{array}{cc}W_{2} & A_{2} \\ 0 & W_{3}\end{array}\right]$ are also contractions, as they are compressions of $T$. Then Parrott's theorem implies that (2.1) and (2.2) are satisfied. In the following we assume that (2.1) and (2.2) are true.

Now, denote

$$
A=\left[\begin{array}{ll}
W_{1} & A_{1}
\end{array}\right], \quad C=\left[\begin{array}{cc}
0 & W_{2} \\
0 & 0
\end{array}\right] \quad \text { and } \quad D=\left[\begin{array}{c}
A_{2} \\
W_{3}
\end{array}\right]
$$

By Parrott's theorem, $T$ is a contraction if and only if :

$$
\begin{equation*}
B=\left(\operatorname{Id}-Z Z^{*}\right)^{1 / 2} V_{3}\left(\operatorname{Id}-Y^{*} Y\right)^{1 / 2}-Z C^{*} Y \tag{2.4}
\end{equation*}
$$

for an arbitrary contraction $V_{3} \in \mathcal{B}\left(H_{3}, H_{1}\right)$. Here $Y$ and $Z$ are contractions such that $D=\left(\operatorname{Id}-C C^{*}\right)^{1 / 2} Y$ and $A=Z\left(\operatorname{Id}-C^{*} C\right)^{1 / 2}$, the existence of which is ensured by Parrott's theorem for column (respectively row) matrix-operators. Indeed, $\left[\begin{array}{l}A \\ C\end{array}\right]$ and $\left[\begin{array}{ll}C & D\end{array}\right]$ are
contractions. We have $\operatorname{Id}-C C^{*}=\left[\begin{array}{cc}\operatorname{Id}-W_{2} W_{2}^{*} & 0 \\ 0 & \mathrm{Id}\end{array}\right]$ and $\operatorname{Id}-C^{*} C=\left[\begin{array}{cc}\operatorname{Id} & 0 \\ 0 & \mathrm{Id}-W_{2}^{*} W_{2}\end{array}\right]$.
Since $\left\|W_{2}\right\|<1$, these operators are invertible. Thus, we get

$$
\begin{aligned}
& Z=A D_{C}^{-1}=\left[\begin{array}{ll}
W_{1} & A_{1} D_{W_{2}}^{-1}
\end{array}\right]=\left[\begin{array}{ll}
W_{1} & D_{W_{1}^{*}} V_{1}
\end{array}\right], \\
& Y=D_{C^{*}} D=\left[\begin{array}{c}
D_{W_{2}^{*}}^{-1} A_{2} \\
W_{3}
\end{array}\right]=\left[\begin{array}{c}
V_{2} D_{W_{3}} \\
W_{3}
\end{array}\right] .
\end{aligned}
$$

It follows that $D_{Z^{*}}=\left[D_{W_{1}^{*}}\left(\operatorname{Id}-V_{1} V_{1}^{*}\right) D_{W_{1}^{*}}\right]^{1 / 2}, D_{Y}=\left[D_{W_{3}}\left(\operatorname{Id}-V_{2}^{*} V_{2}\right) D_{W_{3}}\right]^{1 / 2}$ and $Z C^{*} Y=D_{W_{1}^{*}} V_{1} W_{2}^{*} V_{2} D_{W_{3}}$. Therefore, (2.4) is equivalent with (2.3).

We obtain the following general criterion in the scalar case.
Theorem 2.3. - Let $\omega_{1}, \omega_{2}, \omega_{3} \in \overline{\mathbb{D}}$. Then, $T=\left(\begin{array}{ccc}\omega_{1} & \alpha_{1} & \beta \\ 0 & \omega_{2} & \alpha_{2} \\ 0 & 0 & \omega_{3}\end{array}\right)$ is a contraction when acting on the Hilbert space $\mathbb{C}^{3}$ if and only if

$$
\left\{\begin{array}{l}
\left|\omega_{2}\right|<1  \tag{2.5}\\
\left|\alpha_{i}\right|^{2} \leq\left(1-\left|\omega_{i}\right|^{2}\right)\left(1-\left|\omega_{i+1}\right|^{2}\right), i=1,2 \\
\left|\beta\left(1-\left|\omega_{2}\right|^{2}\right)+\alpha_{1} \alpha_{2} \overline{\omega_{2}}\right|^{2} \leq \\
{\left[\left(1-\left|\omega_{1}\right|^{2}\right)\left(1-\left|\omega_{2}\right|^{2}\right)-\left|\alpha_{1}\right|^{2}\right] \cdot\left[\left(1-\left|\omega_{2}\right|^{2}\right)\left(1-\left|\omega_{3}\right|^{2}\right)-\left|\alpha_{2}\right|^{2}\right]}
\end{array}\right.
$$

or

$$
\left\{\begin{array}{l}
\left|\omega_{2}\right|=1  \tag{2.8}\\
\alpha_{i}=0, i=1,2 \\
|\beta|^{2} \leq\left(1-\left|\omega_{1}\right|^{2}\right)\left(1-\left|\omega_{3}\right|^{2}\right)
\end{array}\right.
$$

Proof. - As in the proof of Theorem 2.2, if $T$ is a contraction, then the two dimensional compressions $\left[\begin{array}{cc}\omega_{1} & \alpha_{1} \\ 0 & \omega_{2}\end{array}\right]$ and $\left[\begin{array}{cc}\omega_{2} & \alpha_{2} \\ 0 & \omega_{3}\end{array}\right]$ are also contractions. Thus (2.6) is satisfied, and it will be assumed from now on. Note that if $\left|\omega_{2}\right|=1$, this implies that $\alpha_{1}=\alpha_{2}=0$.

We use similar notation as in the proof of Theorem 2.2, with

$$
A=\left[\begin{array}{ll}
\omega_{1} & \alpha_{1}
\end{array}\right], \quad B=[\beta], \quad C=\left[\begin{array}{cc}
0 & \omega_{2} \\
0 & 0
\end{array}\right] \quad \text { and } \quad D=\left[\begin{array}{l}
\alpha_{2} \\
\omega_{3}
\end{array}\right]
$$

By Theorem 2.1, $T$ is a contraction if and only if :

$$
\begin{equation*}
B=\left(\operatorname{Id}-Z Z^{*}\right)^{1 / 2} V\left(\operatorname{Id}-Y^{*} Y\right)^{1 / 2}-Z C^{*} Y, \text { for some contraction } V \tag{2.11}
\end{equation*}
$$

where $Y$ and $Z$ are contractions such that $D=\left(\operatorname{Id}-C C^{*}\right)^{1 / 2} Y$ and $A=Z\left(\operatorname{Id}-C^{*} C\right)^{1 / 2}$. We have

$$
\operatorname{Id}-C C^{*}=\left[\begin{array}{cc}
1-\left|\omega_{2}\right|^{2} & 0 \\
0 & 1
\end{array}\right]
$$

and

$$
\operatorname{Id}-C^{*} C=\left[\begin{array}{cc}
1 & 0 \\
0 & 1-\left|\omega_{2}\right|^{2}
\end{array}\right]
$$

First case. Assume first that $\left|\omega_{2}\right|<1$. Then we can apply Theorem 2.2. An easy computation shows that

$$
Y=\left(\operatorname{Id}-C C^{*}\right)^{-1 / 2} D=\left[\begin{array}{c}
\frac{\alpha_{2}}{\sqrt{1-\left|\omega_{2}\right|^{2}}} \\
\omega_{3}
\end{array}\right]
$$

and

$$
Z=A\left(\operatorname{Id}-C^{*} C\right)^{-1 / 2}=\left[\begin{array}{ll}
\omega_{1} & \frac{\alpha_{1}}{\sqrt{1-\left|\omega_{2}\right|^{2}}}
\end{array}\right] .
$$

Thus, $T$ is a contraction if and only if (2.11) is satisfied, that is

$$
\beta+\frac{\alpha_{1} \alpha_{2} \overline{\omega_{2}}}{1-\left|\omega_{2}\right|^{2}}=\left(1-\left|\omega_{1}\right|^{2}-\frac{\left|\alpha_{1}\right|^{2}}{1-\left|\omega_{2}\right|^{2}}\right)^{1 / 2} V\left(1-\left|\omega_{3}\right|^{2}-\frac{\left|\alpha_{2}\right|^{2}}{1-\left|\omega_{2}\right|^{2}}\right)^{1 / 2}
$$

for some contraction $V$. This holds if and only if

$$
\left\|\left(1-\left|\omega_{1}\right|^{2}-\frac{\left|\alpha_{1}\right|^{2}}{1-\left|\omega_{2}\right|^{2}}\right)^{-1 / 2}\left(\beta+\frac{\alpha_{1} \alpha_{2} \overline{\omega_{2}}}{1-\left|\omega_{2}\right|^{2}}\right)\left(1-\left|\omega_{3}\right|^{2}-\frac{\left|\alpha_{2}\right|^{2}}{1-\left|\omega_{2}\right|^{2}}\right)^{-1 / 2}\right\| \leq 1 .
$$

In can be easily shown that this is equivalent with the condition (2.7).
Second case. Assume now that $\left|\omega_{2}\right|=1$. Let $Y=\left[\begin{array}{l}y_{1} \\ y_{2}\end{array}\right]$ and $Z=\left[\begin{array}{ll}z_{1} & z_{2}\end{array}\right]$. As $D=$ $\left(\operatorname{Id}-C C^{*}\right)^{1 / 2} Y$, we get $D^{*}=Y^{*}\left(\operatorname{Id}-C C^{*}\right)^{1 / 2}$. This holds if and only if $y_{2}=\omega_{3}$. As $Y^{*}$ can be chosen to be 0 on $\operatorname{Im}\left(D_{C}^{*}\right)^{\perp}$ (see the Appendix), we have $y_{1}=0$.
Similarly, $A=Z\left(\operatorname{Id}-C^{*} C\right)^{1 / 2}$ holds if and only if $z_{1}=\omega_{1}$ and, as before, we can choose $z_{2}=0$. We have $Z C^{*} Y=0$. Therefore, $T$ is a contraction if and only if $|\beta|^{2} \leq$ $\left(1-\left|\omega_{3}\right|^{2}\right)\left(1-\left|\omega_{1}\right|^{2}\right)$. This is equivalent with the condition (2.10).
2.B. An operator-theoretical proof of Beardon-Minda's inequality. - We refer to [42, Chapter 22] for the definition and basic properties of divided differences of $n+1$ (not necessarily distinct) points. We just recall here that for pairwise distinct points $z_{0}, z_{1}, \cdots, z_{n} \in \mathbb{C}$, the divided differences of $f$ at points $z_{0}, z_{1}, \cdots, z_{n}$ satisfy $\left[f\left(z_{k}\right)\right]=$ $f\left(z_{k}\right)$ and the recurrence relation

$$
\left[f\left(z_{k}\right), \cdots, f\left(z_{k+j}\right)\right]=\frac{\left[f\left(z_{k+1}\right), \cdots, f\left(z_{k+j}\right)\right]-\left[f\left(z_{k}\right), \cdots, f\left(z_{k+j-1}\right)\right]}{z_{k+j}-z_{k}}
$$

for $0 \leq k \leq j \leq n$.
We also recall to the reader the following notation.
Definition 2.4. - Let $z, w \in \mathbb{D}$ and $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$. We define:

1. The complex pseudo-hyperbolic distance $(z, w):=\frac{z-w}{1-\bar{w} z}$;
2. The pseudo-hyperblic distance $\rho(z, w):=|(z, w)|$;
3. The hyperbolic distance $\mathrm{d}(z, w)=\tanh ^{-1} \frac{1+\rho(z, w)}{1-\rho(z, w)}$;
4. The hyperbolic divided difference $f^{*}(z, w):=\frac{(f(z), f(w))}{(z, w)}$.

We provide an operator-theoretic proof of the following result established by Beardon and Minda [9].

Theorem 2.5 (Beardon-Minda [9]). - Let $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$ and let $\omega_{1}$, $\omega_{2}$ and $\omega_{3}$ be pairwise distinct points in $\mathbb{D}$. Then,

$$
\begin{equation*}
\mathrm{d}\left(f^{*}\left(\omega_{1}, \omega_{2}\right), f^{*}\left(\omega_{3}, \omega_{2}\right)\right) \leq \mathrm{d}\left(\omega_{1}, \omega_{3}\right) \tag{2.12}
\end{equation*}
$$

The proof in [9] requires an assumption that $f$ is not a conformal automorphism of the unit disk. Such an assumption is unnecessary in the subsequent proof.

Proof of Theorem 2.5. - First, let us notice that Beardon-Minda's inequality (2.12) is equivalent with

$$
\begin{equation*}
\rho\left(f^{*}\left(\omega_{1}, \omega_{2}\right), f^{*}\left(\omega_{3}, \omega_{2}\right)\right)=\left|\frac{f^{*}\left(\omega_{1}, \omega_{2}\right)-f^{*}\left(\omega_{3}, \omega_{2}\right)}{1-\overline{f^{*}\left(\omega_{3}, \omega_{2}\right)} f^{*}\left(\omega_{1}, \omega_{2}\right)}\right| \leq\left|\frac{\omega_{1}-\omega_{3}}{1-\overline{\omega_{3}} \omega_{1}}\right| \tag{2.13}
\end{equation*}
$$

For $z, \omega \in \mathbb{D}$, we have

$$
\begin{equation*}
f^{*}(z, \omega)=\frac{f(z)-f(\omega)}{z-\omega} \cdot \frac{1-\bar{\omega} z}{1-\overline{f(\omega)} f(z)} \tag{2.14}
\end{equation*}
$$

We also record the following important identity, valid for $u, v \in \mathbb{C}$. We have

$$
\begin{equation*}
S_{u, v}:=\left(1-|u|^{2}\right)\left(1-|v|^{2}\right)=|1-\bar{u} v|^{2}-|u-v|^{2} \tag{2.15}
\end{equation*}
$$

Now, let $\omega_{1}, \omega_{2}, \omega_{3} \in \mathbb{D}$, with $\omega_{i} \neq \omega_{j}(i \neq j)$, and consider

$$
T=\left(\begin{array}{ccc}
\omega_{1} & \alpha_{1} & \beta \\
0 & \omega_{2} & \alpha_{2} \\
0 & 0 & \omega_{3}
\end{array}\right)
$$

with

$$
\alpha_{i}=\sqrt{1-\left|\omega_{i}\right|^{2}} \sqrt{\left(1-\left|\omega_{i+1}\right|^{2}\right.}, \quad i=1,2
$$

and

$$
\beta=\frac{-\overline{\omega_{2}} \alpha_{1} \alpha_{2}}{1-\left|\omega_{2}\right|^{2}}=-\overline{\omega_{2}} \sqrt{1-\left|\omega_{1}\right|^{2}} \sqrt{1-\left|\omega_{3}\right|^{2}}
$$

By Theorem 2.3, $T$ is a contraction. Assume first that $f \in \mathcal{A}(\overline{\mathbb{D}})$ and $\|f\|_{\infty} \leq 1$. Then the matrix representation of $f(T)$ can be expressed in terms of first order and second order divided differences as follows:

$$
f(T)=\left(\begin{array}{ccc}
f\left(\omega_{1}\right) & \alpha_{1}\left[f\left(\omega_{1}\right), f\left(\omega_{2}\right)\right] & \beta\left[f\left(\omega_{1}\right), f\left(\omega_{3}\right)\right]+\alpha_{1} \alpha_{2}\left[f\left(\omega_{1}\right), f\left(\omega_{2}\right), f\left(\omega_{3}\right)\right] \\
0 & f\left(\omega_{2}\right) & \alpha_{2}\left[f\left(\omega_{2}\right), f\left(\omega_{3}\right)\right] \\
0 & 0 & f\left(\omega_{3}\right)
\end{array}\right)
$$

This can be verified directly by some direct computations for monomials and polynomials. The same formula extends to functions in the disk algebra $\mathcal{A}(\overline{\mathbb{D}})$. Assume that $f\left(\omega_{i}\right) \neq$
$f\left(\omega_{j}\right)$ whenever $i \neq j$ (otherwise, there is nothing to prove). As $T$ is a contraction and $\|f\|_{\infty} \leq 1$, by von Neumann's inequality the operator $f(T)$ is also a contraction.

Introducing the notation

$$
\begin{aligned}
& \widetilde{\alpha_{i}}=\alpha_{i}\left[f\left(\omega_{i}\right), f\left(\omega_{i+1}\right)\right], \quad i=1,2, \\
& \widetilde{\beta}=\beta\left[f\left(\omega_{1}\right), f\left(\omega_{3}\right)\right]+\alpha_{1} \alpha_{2}\left[f\left(\omega_{1}\right), f\left(\omega_{2}\right), f\left(\omega_{3}\right)\right],
\end{aligned}
$$

by Theorem 2.3, we have :

$$
\left|\widetilde{\beta}\left(1-\left|f\left(\omega_{2}\right)\right|^{2}\right)+\widetilde{\alpha_{1}} \widetilde{\alpha_{2}} \overline{f\left(\omega_{2}\right)}\right|^{2} \leq\left[S_{f\left(\omega_{1}\right), f\left(\omega_{2}\right)}-\left|\widetilde{\alpha_{1}}\right|^{2}\right] \times\left[S_{f\left(\omega_{2}\right), f\left(\omega_{3}\right)}-\left|\widetilde{\alpha_{2}}\right|^{2}\right]
$$

If we multiply each side of this inequality by $\left|\omega_{1}-\omega_{3}\right|^{2}$, we get

$$
\begin{equation*}
S_{\omega_{1}, \omega_{3}}\left|\left(1-\left|f\left(\omega_{2}\right)\right|^{2}\right) A+B\right|^{2} \leq\left|\omega_{1}-\omega_{3}\right|^{2} C_{1} C_{3} \tag{2.16}
\end{equation*}
$$

where

$$
\begin{aligned}
A & :=-\overline{\omega_{2}}\left(f\left(\omega_{1}\right)-f\left(\omega_{3}\right)\right)+\left(1-\left|\omega_{2}\right|^{2}\right)\left(\frac{f\left(\omega_{1}\right)-f\left(\omega_{2}\right)}{\omega_{1}-\omega_{2}}-\frac{f\left(\omega_{2}\right)-f\left(\omega_{3}\right)}{\omega_{2}-\omega_{3}}\right) \\
B & :=\overline{f\left(\omega_{2}\right)}\left(1-\left|\omega_{2}\right|^{2}\right)\left(\omega_{1}-\omega_{3}\right) \cdot \frac{\left(f\left(\omega_{1}\right)-f\left(\omega_{2}\right)\right)\left(f\left(\omega_{2}\right)-f\left(\omega_{3}\right)\right)}{\left(\omega_{1}-\omega_{2}\right)\left(\omega_{2}-\omega_{3}\right)} \\
C_{i} & :=S_{f\left(\omega_{i}\right), f\left(\omega_{2}\right)}-S_{\omega_{i}, \omega_{2}}\left|\frac{f\left(\omega_{i}\right)-f\left(\omega_{2}\right)}{\omega_{i}-\omega_{2}}\right|^{2}, \quad i=1,3
\end{aligned}
$$

We want to prove that (2.16) is equivalent with (2.13). The calculations are somewhat laborious; the key idea is to use (2.14) to make hyperbolic divided differences appear each time we see an expression of the form $f(z)-f(\omega)$. We provide additional details to assist the reader.

First of all, we have :

$$
\begin{aligned}
C_{i}= & S_{f\left(\omega_{i}\right), f\left(\omega_{2}\right)}-S_{\omega_{i}, \omega_{2}}\left|f^{*}\left(\omega_{i}, \omega_{2}\right)\right|^{2} \times\left|\frac{1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{i}\right)}{1-\overline{\omega_{2}} \omega_{i}}\right|^{2} \\
= & \left|1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{i}\right)\right|^{2}-\left|f^{*}\left(\omega_{i}, \omega_{2}\right)\right|^{2} \times\left|\omega_{i}-\omega_{2}\right|^{2} \times\left|\frac{1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{i}\right)}{1-\overline{\omega_{2}} \omega_{i}}\right|^{2} \\
& -\left|f^{*}\left(\omega_{i}, \omega_{2}\right)\right|^{2} \times\left|1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{i}\right)\right|^{2}+\left|f^{*}\left(\omega_{i}, \omega_{2}\right)\right|^{2} \times\left|\omega_{i}-\omega_{2}\right|^{2} \times\left|\frac{1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{i}\right)}{1-\overline{\omega_{2}} \omega_{i}}\right|^{2} \\
= & \left|1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{i}\right)\right|^{2}\left(1-\left|f^{*}\left(\omega_{i}, \omega_{2}\right)\right|^{2}\right)
\end{aligned}
$$

Thus, we have

$$
C_{1} C_{3}=\left|1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{1}\right)\right|^{2} \times\left|1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{3}\right)\right|^{2} \times S_{f^{*}\left(\omega_{1}, \omega_{2}\right), f^{*}\left(\omega_{3}, \omega_{2}\right)}
$$

Now, let us deal with the first member of the inequality. We have

$$
\begin{aligned}
A & =f^{*}\left(\omega_{1}, \omega_{2}\right)\left(1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{1}\right)\right)-f^{*}\left(\omega_{3}, \omega_{2}\right)\left(1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{3}\right)\right) \\
& =\left(f^{*}\left(\omega_{1}, \omega_{2}\right)-f^{*}\left(\omega_{3}, \omega_{2}\right)\right)\left(1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{1}\right)\right)\left(1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{3}\right)\right)+\overline{f\left(\omega_{2}\right)} D
\end{aligned}
$$

where $D:=f^{*}\left(\omega_{1}, \omega_{2}\right) f\left(\omega_{3}\right)\left(1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{1}\right)\right)-f^{*}\left(\omega_{3}, \omega_{2}\right) f\left(\omega_{1}\right)\left(1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{3}\right)\right)$.
This term can be written as follows, where we make appear the differences $f\left(\omega_{3}\right)-f\left(\omega_{2}\right)$ and $f\left(\omega_{1}\right)-f\left(\omega_{2}\right)$ :

$$
\begin{aligned}
& D=f^{*}\left(\omega_{1}, \omega_{2}\right)\left(f\left(\omega_{3}\right)-f\left(\omega_{2}\right)\right)\left(1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{1}\right)\right) \\
& -f^{*}\left(\omega_{3}, \omega_{2}\right)\left(f\left(\omega_{1}\right)-f\left(\omega_{2}\right)\right)\left(1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{3}\right)\right)+f\left(\omega_{2}\right) f^{*}\left(\omega_{1}, \omega_{2}\right)\left(1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{1}\right)\right) \\
& -f\left(\omega_{2}\right) f^{*}\left(\omega_{3}, \omega_{2}\right)\left(1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{3}\right)\right)
\end{aligned}
$$

We obtain

$$
\begin{aligned}
D= & f^{*}\left(\omega_{1}, \omega_{2}\right) f^{*}\left(\omega_{3}, \omega_{2}\right)\left(\frac{1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{3}\right)}{1-\overline{\omega_{2}} \omega_{3}}\right)\left(\omega_{3}-\omega_{2}\right)\left(1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{1}\right)\right) \\
& -f^{*}\left(\omega_{3}, \omega_{2}\right) f^{*}\left(\omega_{1}, \omega_{2}\right)\left(\frac{1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{1}\right)}{1-\overline{\omega_{2}} \omega_{1}}\right)\left(\omega_{1}-\omega_{2}\right)\left(1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{3}\right)\right)+f\left(\omega_{2}\right) A \\
= & -\left(1-\left|\omega_{2}\right|^{2}\right)\left(\omega_{1}-\omega_{3}\right) \cdot \frac{\left(f\left(\omega_{1}\right)-f\left(\omega_{2}\right)\right)\left(f\left(\omega_{2}\right)-f\left(\omega_{3}\right)\right)}{\left(\omega_{1}-\omega_{2}\right)\left(\omega_{2}-\omega_{3}\right)}+f\left(\omega_{2}\right) A .
\end{aligned}
$$

Hence, we get

$$
A=\left(f^{*}\left(\omega_{1}, \omega_{2}\right)-f^{*}\left(\omega_{3}, \omega_{2}\right)\right)\left(1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{1}\right)\right)\left(1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{3}\right)\right)-B+\left|f\left(\omega_{2}\right)\right|^{2} A
$$

Therefore

$$
\left(1-\left|f\left(\omega_{2}\right)\right|^{2}\right) A+B=\left(f^{*}\left(\omega_{1}, \omega_{2}\right)-f^{*}\left(\omega_{3}, \omega_{2}\right)\right)\left(1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{1}\right)\right)\left(1-\overline{f\left(\omega_{2}\right)} f\left(\omega_{3}\right)\right) .
$$

Combining all of these elements, the inequality represented by (2.16) transforms into

$$
S_{\omega_{1}, \omega_{3}}\left|f^{*}\left(\omega_{1}, \omega_{2}\right)-f^{*}\left(\omega_{3}, \omega_{2}\right)\right|^{2} \leq\left|\omega_{1}-\omega_{3}\right|^{2} \times S_{f^{*}\left(\omega_{1}, \omega_{2}\right), f^{*}\left(\omega_{3}, \omega_{2}\right)}
$$

which is equivalent with (2.13).
Beardon-Minda's inequality is thus proved for $f \in \mathcal{A}(\overline{\mathbb{D}})$. Now, for $f \in \mathcal{H}(\mathbb{D})$, we have $f_{r}: z \mapsto f(r z) \in \mathcal{A}(\overline{\mathbb{D}})$, for every $\left.r \in\right] 0,1[$. Based on the preceding information, it can be concluded that Beardon-Minda's inequality is satisfied by the functions $f_{r}$, for all $\left.r \in\right] 0,1[$, so it is also by $f$, by letting $r \rightarrow 1^{-}$.

The calculations in this proof can be somewhat simplified by assuming that $f\left(\omega_{2}\right)=0$ and composing with a Möbius transformation at the end. However, this approach leads to a loss of symmetry in the formulas.
2.C. Connecting with model spaces theory. - In [9], it is further proved that if $f$ does not represent an automorphism of the unit disk, equality holds in Theorem 2.5 if and only if $f$ is a Blaschke product of degree no greater than 2. This inference can also be derived through operator theory considerations.

To achieve this, we must introduce certain concepts from model space theory. For a comprehensive introduction to these notions and more details, we direct the reader to [20] and [28].

Let $H^{\infty}(\mathbb{D})$ be the set of all holomorphic functions that are bounded on $\mathbb{D}$, and let $H^{2}(\mathbb{D})$ be the Hardy-Hilbert space of $\mathbb{D}$, which is the space of all holomorphic functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$
\sup _{0<r<1} \int_{\mathbb{T}}|f(\zeta)|^{2} \mathrm{~d} m(\zeta)<\infty
$$

or, equivalently, such that

$$
\sum_{n=0}^{\infty}\left|a_{n}\right|^{2}<\infty \quad \text { if } f(z)=\sum_{n=0}^{\infty} a_{n} z^{n}
$$

Let $S: H^{2}(\mathbb{D}) \rightarrow H^{2}(\mathbb{D})$ be the unilateral shift, defined by $S(f)(z)=z f(z)$. For $f \in$ $H^{\infty}(\mathbb{D})$, Fatou's theorem (see e.g. [20, theorem 1.10]) states that $f$ has radial boundary values $f(\zeta)$, for almost every $\zeta \in \mathbb{T}$. A function $u \in H^{\infty}(\mathbb{D})$ is said to be inner if $|u(\zeta)|=1$ almost everywhere on $\mathbb{T}$.

If $u$ is an inner function, the corresponding model space $\mathcal{K}_{u}$ is defined to be

$$
\mathcal{K}_{u}:=\left(u H^{2}(\mathbb{D})\right)^{\perp}=\left\{f \in H^{2}(\mathbb{D}):\langle f, u h\rangle=0, \forall h \in H^{2}(\mathbb{D})\right\}
$$

We define the associated compressed shift by $S_{u}:=\left.P_{u} S\right|_{\mathcal{K}_{u}}$, where $P_{u}$ is the orthogonal projection from $H^{2}(\mathbb{D})$ onto $\mathcal{K}_{u}$.

Now, let $\Theta$ be a finite Blaschke product with pairwise distinct zeros $\omega_{1}, \ldots, \omega_{n} \in \mathbb{D}$ and let $b_{\omega_{k}}(z)=\frac{z-\omega_{k}}{1-\overline{\omega_{k}} z}$ denote a single Blaschke factor. Let $\left(\phi_{1}(z), \ldots, \phi_{n}(z)\right)$ denote the Takenaka-Malmquist-Walsh orthonormal basis $([20,28])$ of $\mathcal{K}_{\Theta}$, i.e.

$$
\phi_{1}(z)=\frac{\sqrt{1-\left|\omega_{1}\right|^{2}}}{1-\overline{\omega_{1}} z} \quad \text { and } \quad \phi_{k}(z)=\left(\prod_{j=1}^{k-1} b_{\omega_{j}}\right) \frac{\sqrt{1-\left|\omega_{k}\right|^{2}}}{1-\overline{\omega_{k}} z} \quad k=2, \ldots, n
$$

Writing $S_{\Theta}$ with respect to the Takenaka-Malmquist basis gives the matrix representation $M_{\Theta}$ with entries

$$
\left[M_{\Theta}\right]_{i, j}=\left\{\begin{array}{cc}
\omega_{j} & \text { if } i=j \\
\prod_{k=i+1}^{j-1}\left(-\overline{\omega_{k}}\right) \sqrt{1-\left|\omega_{i}\right|^{2}} \sqrt{1-\left|\omega_{j}\right|^{2}} & \text { if } i<j \\
0 & \text { if } i>j
\end{array}\right.
$$

It seems that the first appearance of this remarkable matrix was in [45]; see also [33, 34, $28,19,43]$. In particular, for $n=2$ and $n=3$, we obtain the following matrices

$$
\begin{aligned}
T_{2} & :=\left(\begin{array}{cc}
\omega_{1} & \sqrt{1-\left|\omega_{1}\right|^{2}} \sqrt{1-\left|\omega_{2}\right|^{2}} \\
0 & \omega_{2}
\end{array}\right) \\
T_{3} & :=\left(\begin{array}{ccc}
\omega_{1} & \sqrt{1-\left|\omega_{1}\right|^{2}} \sqrt{\left(1-\left|\omega_{2}\right|^{2}\right.} & -\overline{\omega_{2}} \sqrt{\left(1-\left|\omega_{1}\right|^{2}\right.} \sqrt{\left(1-\left|\omega_{3}\right|^{2}\right)} \\
0 & \omega_{2} & \sqrt{1-\left|\omega_{2}\right|^{2}} \sqrt{\left(1-\left|\omega_{3}\right|^{2}\right.} \\
0 & 0 & \omega_{3}
\end{array}\right)
\end{aligned}
$$

which have been used to obtain the Schwarz-Pick and Beardon-Minda inequalities.
We now show how to obtain the equality case in the Beardon-Minda inequality using the matrix $T_{3}$. It follows from our proof of Theorem 2.5 that $f$ satisfies (2.12) with equality if and only if $\left\|f\left(T_{3}\right)\right\|=1=\|f\|_{\infty}$. The fact that $f$ is a finite Blaschke product of degree $\leq 2$ has been proved by several authors, sometimes in relation to Crouzeix's conjecture. This can be generalized to $n$ points. We refer to the discussion in [11, Theorem 3.1]. An explicit description of the matrix which diagonalize $M_{\Theta}$ is also given in [11].

We plan to return to the general Beardon-Minda type inequality $\left\|f\left(M_{\Theta}\right)\right\| \leq 1$ in a future paper.
2.D. A Beardon-Minda type lemma for derivatives. - We now investigate the case where $\omega_{1}=\omega_{2}=\omega_{3}=: \omega$. For a holomorphic function $f$ we use the notation

$$
\Gamma(z, f)=\frac{\left(1-|z|^{2}\right)\left|f^{\prime}(z)\right|}{1-|f(z)|^{2}} .
$$

The Schwarz-Pick inequality for derivatives (1.2) can then be expressed as $|\Gamma(z, f)| \leq 1$.
We give now an operator theoretical proof of the following result, proved by Yamashita in [44, Theorem 2].

Theorem 2.6. - Let $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$ and let $\Gamma(z, f)=\frac{\left(1-|z|^{2}\right)\left|f^{\prime}(z)\right|}{1-|f(z)|^{2}}$. Then, for every $\omega \in \mathbb{D}$,

$$
\begin{equation*}
\left|\frac{\partial \Gamma(\omega, f)}{\partial \omega}\right| \leq \frac{1-|\Gamma(\omega, f)|^{2}}{1-|\omega|^{2}} \tag{2.17}
\end{equation*}
$$

Moreover, equality holds if and only if $f$ is a Blaschke product of degree $\leq 2$.

Proof. - Let $\omega \in \mathbb{D}$, and let $T=\left(\begin{array}{ccc}\omega & \alpha & \beta \\ 0 & \omega & \alpha \\ 0 & 0 & \omega\end{array}\right) \in \mathcal{M}_{3}(\mathbb{C})$, with $\alpha=1-|\omega|^{2}$ and $\beta=$ $-\bar{\omega}\left(1-|\omega|^{2}\right)$. By Theorem 2.3, $T$ is a contraction. Moreover, we can easily check that for $f$ in the disk algebra we have

$$
f(T)=\left(\begin{array}{ccc}
f(\omega) & \alpha f^{\prime}(\omega) & \frac{1}{2} \alpha^{2} f^{\prime \prime}(\omega)+\beta f^{\prime}(\omega) \\
0 & f(\omega) & \alpha f^{\prime}(\omega) \\
0 & 0 & f(\omega)
\end{array}\right)
$$

In this representation, the divided differences have been replaced in this limit case by first and second-order derivatives. By von Neumann's inequality, $f(T)$ is a contraction. Using

Theorem 2.3 we obtain:

$$
\left|\left(\frac{1}{2} \alpha^{2} f^{\prime \prime}(\omega)+\beta f^{\prime}(\omega)\right)\left(1-|f(\omega)|^{2}\right)+\alpha^{2} f^{\prime}(\omega)^{2} \overline{f(\omega)}\right| \leq\left(1-|f(\omega)|^{2}\right)^{2}-\left|\alpha f^{\prime}(\omega)\right|^{2}
$$

which is equivalent with (2.17). A proof of the equality case can be obtained using model spaces, as discussed in the preceding subsection.

The inequality (2.17) can be rephrased in terms of Peschl's invariant derivatives. Let $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$, let $\omega \in \mathbb{D}$, and consider the mapping

$$
\begin{equation*}
g: z \in \mathbb{D} \mapsto \frac{f\left(\frac{z+\omega}{1+\bar{\omega} z}\right)-f(\omega)}{1-\overline{f(\omega)} f\left(\frac{z+\omega}{1+\bar{\omega} z}\right)} \in \mathbb{C} . \tag{2.18}
\end{equation*}
$$

Then $g$ is analytic on $\mathbb{D}$ and $g(0)=0$. We have $g(z)=\sum_{n=1}^{\infty} \frac{D_{n} f(\omega)}{n!} z^{n}$, with $D_{n} f\left(z_{0}\right):=$ $g^{(n)}(0)$. The quantities $D_{n} f(\omega)$ are called Peschl's invariant derivatives (see e.g. [24]).

The first two values of Peschl's invariant derivatives are explicitely computed as:

$$
\begin{aligned}
D_{1} f(\omega) & =\frac{\left(1-|\omega|^{2}\right) f^{\prime}(\omega)}{1-|f(\omega)|^{2}} \\
D_{2} f(\omega) & =\frac{\left(1-|\omega|^{2}\right)^{2}}{1-|f(\omega)|^{2}}\left[f^{\prime \prime}(\omega)-\frac{2 \bar{\omega} f^{\prime}(\omega)}{1-|\omega|^{2}}+\frac{2 \overline{f(\omega)} f^{\prime}(\omega)^{2}}{1-|f(\omega)|^{2}}\right]
\end{aligned}
$$

With these notations, the Schwarz-Pick inequality for derivatives (1.2) can be restated as $\left|D_{1} f(\omega)\right| \leq 1$, while $(2.17)$ can be written as $\left|D_{2} f(\omega)\right| \leq 2\left(1-\left|D_{1} f(\omega)\right|^{2}\right)$.

We refer to [12, Proposition 3.4] for a different proof of (2.17) and to Section 4.B for a generalization to the polydisk.

## 3. Operator versions of Beardon-Minda's inequality

We move now to operator versions of the Schwarz-Pick and Beardon-Minda inequalities. The first operator generalization for the Schwarz-Pick inequality has been proved by Ky Fan in [18]; the following discussion has been inspired by the recent paper [22].

We recall the following theorem concerning the Sylvester equation $A X-X B=Y$, which has been studied e.g. in [10, 37].

Theorem 3.1 (Rosenblum,[10]). - Let $H, K$ be two Hilbert spaces. Let $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}(K)$ be two operators with $\sigma(A) \cap \sigma(B)=\emptyset$. Then, for every $Y \in \mathcal{B}(H, K)$, the Sylvester equation $A X-X B=Y$ has a unique solution $X$. Moreover, if $\Gamma$ is a union of closed contours in the plane with total winding numbers 1 around $\sigma(A)$ and 0 around $\sigma(B)$, the solution can be expressed as

$$
X=\frac{1}{2 i \pi} \int_{\Gamma}(A-\xi)^{-1} Y(B-\xi)^{-1} d \xi
$$

3.A. An operator version of the Schwarz-Pick inequality. - The following result is a counterpart of [22, Theorem 3.5]. When specialized to scalars, it reduces to the Schwarz-Pick inequality for two distinct points.

Theorem 3.2. - Let $H_{1}, H_{2}$ be two Hilbert spaces. Consider three contractions $W_{1} \in$ $\mathcal{B}\left(H_{1}\right), W_{2} \in \mathcal{B}\left(H_{2}\right)$ and $V \in \mathcal{B}\left(H_{2}, H_{1}\right)$. Assume that $\sigma\left(W_{1}\right) \cap \sigma\left(W_{2}\right)=\emptyset$, and that $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$ is holomorphic on an open neighborhood of $\sigma\left(W_{1}\right) \cup \sigma\left(W_{2}\right)$. We denote by $X=X_{W_{1}, W_{2}, V}$ the unique solution of Sylvester's equation

$$
\begin{equation*}
W_{1} X-X W_{2}=D_{W_{1}^{*}} V D_{W_{2}} \tag{3.1}
\end{equation*}
$$

Then, there exists a contraction $Y \in \mathcal{B}\left(H_{2}, H_{1}\right)$ such that

$$
f\left(W_{1}\right) X-X f\left(W_{2}\right)=D_{f\left(W_{1}\right)^{*}} Y D_{f\left(W_{2}\right)}
$$

Proof. - Let $T=\left[\begin{array}{cc}W_{1} & D_{W_{1}^{*}} V D_{W_{2}} \\ 0 & W_{2}\end{array}\right]$. Denote $C=D_{W_{1}^{*}} V D_{W_{2}}$. By Parrott's theorem, $T$ is a contraction. Moreover, using (3.1), we have

$$
T=\left[\begin{array}{cc}
W_{1} & C \\
0 & W_{2}
\end{array}\right]=\left[\begin{array}{cc}
\mathrm{Id} & -X \\
0 & \mathrm{Id}
\end{array}\right]\left[\begin{array}{cc}
W_{1} & 0 \\
0 & W_{2}
\end{array}\right]\left[\begin{array}{cc}
\mathrm{Id} & X \\
0 & \mathrm{Id}
\end{array}\right] .
$$

Notice that $\sigma(T) \subset \sigma\left(W_{1}\right) \cup \sigma\left(W_{2}\right)$. Indeed, for $\lambda \in \mathbb{C}$, we have

$$
T-\lambda \operatorname{Id}=\left[\begin{array}{cc}
\operatorname{Id} & 0 \\
0 & W_{2}-\lambda \mathrm{Id}
\end{array}\right]\left[\begin{array}{cc}
\operatorname{Id} & D_{W_{1}^{*}} V D_{W_{2}} \\
0 & \mathrm{Id}
\end{array}\right]\left[\begin{array}{cc}
W_{1}-\lambda \mathrm{Id} & 0 \\
0 & \mathrm{Id}
\end{array}\right]
$$

Therefore, if $\lambda \notin \sigma\left(W_{1}\right) \cup \sigma\left(W_{2}\right)$, then all factors in the previous decomposition are invertible and thus $\lambda \notin \sigma(T)$. So it makes sense to speak about $f(T)$ and to write

$$
f(T)=\left[\begin{array}{cc}
\mathrm{Id} & -X \\
0 & \mathrm{Id}
\end{array}\right]\left[\begin{array}{cc}
f\left(W_{1}\right) & 0 \\
0 & f\left(W_{2}\right)
\end{array}\right]\left[\begin{array}{cc}
\mathrm{Id} & X \\
0 & \mathrm{Id}
\end{array}\right]=\left[\begin{array}{cc}
f\left(W_{1}\right) & f\left(W_{1}\right) X-X f\left(W_{2}\right) \\
0 & f\left(W_{2}\right)
\end{array}\right]
$$

As $\|f\|_{\infty} \leq 1$, we have $\|f(T)\| \leq 1$ by von Neumann's inequality. Thus, by Parrott's theorem, there exists a contraction $Y \in \mathcal{B}\left(H_{2}, H_{1}\right)$ such that $f\left(W_{1}\right) X-X f\left(W_{2}\right)=$ $D_{f\left(W_{1}\right)^{*}} Y D_{f\left(W_{2}\right)}$.
3.B. An operator version of the Beardon-Minda inequality. - Utilizing the analogue proof framework as employed in Theorem 3.2, we can deduce the following outcome for $3 \times 3$ operator matrices.

Theorem 3.3. - Let $H_{1}, H_{2}, H_{3}$ be three Hilbert spaces. Consider three contractions $W_{1} \in \mathcal{B}\left(H_{1}\right), W_{2} \in \mathcal{B}\left(H_{2}\right)$ and $W_{3} \in \mathcal{B}\left(H_{3}\right)$. Let $V_{1} \in \mathcal{B}\left(H_{2}, H_{1}\right), V_{2} \in \mathcal{B}\left(H_{3}, H_{2}\right)$, and $V_{3} \in \mathcal{B}\left(H_{3}, H_{1}\right)$ be contractions. Assume that $\left\|W_{2}\right\|<1$ and that $\sigma\left(W_{i}\right) \cap \sigma\left(W_{j}\right)=\emptyset$, for all $1 \leq i<j \leq 3$. Suppose that $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$ is holomorphic on an open neighborhood of
$\sigma\left(W_{1}\right) \cup \sigma\left(W_{2}\right) \cup \sigma\left(W_{3}\right)$. Let $X_{1}, X_{2}, X_{3}$ be respectively the unique solution of Sylvester's equations

$$
\begin{align*}
& W_{1} X_{1}-X_{1} W_{2}=D_{W_{1}^{*}} V_{1} D_{W_{2}}  \tag{3.2}\\
& W_{2} X_{2}-X_{2} W_{3}=D_{W_{2}^{*}} V_{2} D_{W_{3}} \text { and }  \tag{3.3}\\
& W_{1} X_{3}-X_{3} W_{3}=B-W_{3} X_{1} X_{2}+X_{1} W_{2} X_{2} \tag{3.4}
\end{align*}
$$

where

$$
B=\left[D_{\left.W_{1}^{*}\left(\operatorname{Id}-V_{1} V_{1}^{*}\right) D_{W_{1}^{*}}\right]^{1 / 2} V_{3}\left[D_{W_{3}}\left(\operatorname{Id}-V_{2}^{*} V_{2}\right) D_{W_{3}}\right]^{1 / 2}-D_{W_{1}^{*}} V_{1} W_{2}^{*} V_{2} D_{W_{3}} . . . . ~}\right.
$$

Then, there exist three contractions $Y_{1} \in \mathcal{B}\left(H_{2}, H_{1}\right), Y_{2} \in \mathcal{B}\left(H_{3}, H_{2}\right), Y_{3} \in \mathcal{B}\left(H_{3}, H_{1}\right)$ such that:

$$
\left\{\begin{array}{l}
f\left(W_{1}\right) X_{1}-X_{1} f\left(W_{2}\right)=D_{f\left(W_{1}\right)^{*}} Y_{1} D_{f\left(W_{2}\right)},  \tag{3.5}\\
f\left(W_{2}\right) X_{2}-X_{2} f\left(W_{3}\right)=D_{f\left(W_{2}\right)^{*} Y_{2} D_{f\left(W_{3}\right)},}^{f\left(W_{1}\right) X_{3}-X_{3} f\left(W_{3}\right)=X_{1} f\left(W_{2}\right) X_{2}-X_{1} X_{2} f\left(W_{3}\right)} \\
+\left[D_{f\left(W_{1}\right)^{*}}\left(\operatorname{Id}-Y_{1} Y_{1}^{*}\right) D_{f\left(W_{1}\right)^{*}}\right]^{1 / 2} Y_{3}\left[D_{f\left(W_{3}\right)}\left(\operatorname{Id}-Y_{2}^{*} Y_{2}\right) D_{f\left(W_{3}\right)}\right]^{1 / 2} \\
-D_{f\left(W_{1}\right)^{*}} Y_{1} f\left(W_{2}\right)^{*} Y_{2} D_{f\left(W_{3}\right)} .
\end{array}\right.
$$

Proof. - Denote $A_{1}=D_{W_{1}^{*}} V_{1} D_{W_{2}}$ and $A_{2}=D_{W_{2}^{*}} V_{2} D_{W_{3}}$. Then, according to Theorem 2.2, the operator

$$
T=\left[\begin{array}{ccc}
W_{1} & A_{1} & B \\
0 & W_{2} & A_{2} \\
0 & 0 & W_{3}
\end{array}\right]
$$

is a contraction. Notice also that $X_{1} \in \mathcal{B}\left(H_{2}, H_{1}\right), X_{2} \in \mathcal{B}\left(H_{3}, H_{2}\right)$ and $X_{3} \in \mathcal{B}\left(H_{3}, H_{1}\right)$ are respectively the unique solutions of Sylvester's equations $W_{1} X_{1}-X_{1} W_{2}=A_{1}, W_{2} X_{2}-$ $X_{2} W_{3}=A_{2}$ and $W_{1} X_{3}-X_{3} W_{3}=B-W_{3} X_{1} X_{2}+X_{1} W_{2} X_{2}$.

In analogy with some computations in the Heisenberg group, we can write

$$
T=\left[\begin{array}{ccc}
W_{1} & A_{1} & B \\
0 & W_{2} & A_{2} \\
0 & 0 & W_{3}
\end{array}\right]=\left[\begin{array}{ccc}
\mathrm{Id} & -X_{1} & X_{1} X_{2}-X_{3} \\
0 & \mathrm{Id} & -X_{2} \\
0 & 0 & \mathrm{Id}
\end{array}\right]\left[\begin{array}{ccc}
W_{1} & 0 & 0 \\
0 & W_{2} & 0 \\
0 & 0 & W_{3}
\end{array}\right]\left[\begin{array}{ccc}
\mathrm{Id} & X_{1} & X_{3} \\
0 & \mathrm{Id} & X_{2} \\
0 & 0 & \mathrm{Id}
\end{array}\right] .
$$

This diagonalization allows one to write the $3 \times 3$ operator matrix of $f(T)$, which is a contraction by von Neumann's inequality:

$$
f(T)=\left[\begin{array}{ccc}
\operatorname{Id} & -X_{1} & X_{1} X_{2}-X_{3} \\
0 & \mathrm{Id} & -X_{2} \\
0 & 0 & \mathrm{Id}
\end{array}\right]\left[\begin{array}{ccc}
f\left(W_{1}\right) & 0 & 0 \\
0 & f\left(W_{2}\right) & 0 \\
0 & 0 & f\left(W_{3}\right)
\end{array}\right]\left[\begin{array}{ccc}
\mathrm{Id} & X_{1} & X_{3} \\
0 & \mathrm{Id} & X_{2} \\
0 & 0 & \mathrm{Id}
\end{array}\right] .
$$

Thus the matrix of $f(T)$ is given by

$$
\left[\begin{array}{ccc}
f\left(W_{1}\right) & f\left(W_{1}\right) X_{1}-X_{1} f\left(W_{2}\right) & f\left(W_{1}\right) X_{3}-X_{1} f\left(W_{2}\right) X_{2}+\left(X_{1} X_{2}-X_{3}\right) f\left(W_{3}\right) \\
0 & f\left(W_{2}\right) & f\left(W_{2}\right) X_{2}-X_{2} f\left(W_{3}\right) \\
0 & 0 & f\left(W_{3}\right)
\end{array}\right] .
$$

We apply again Theorem 2.2.

In the scalar case, the condition (3.7) is equivalent with the Beardon-Minda inequality.

## 4. Schwarz-Pick inequalities for the polydisk

Let $n \in \mathbb{N}^{*}$. For $\underline{\omega}=\left(\omega_{1}, \cdots, \omega_{n}\right) \in \mathbb{C}^{n}$, we denote $\|\underline{\omega}\|=\sup _{1 \leq i \leq n}\left|\omega_{i}\right|$ the sup norm.
4.A. Using von Neumann inequality for tuples of two by two matrices. - It is a fascinating observation in operator theory that an analogue of the von Neumann inequality holds for the bidisk (Ando's theorem), but does not extend to the polydisk $\mathbb{D}^{n}$ for $n \geq 3$. However, as proved by Drury [16] and Knese [26], there is an analogue for tuples of $2 \times 2$ and $3 \times 3$ commuting matrices.

Lemma 4.1 (Drury and Knese; see [16], [26]). - Let $T_{1}, \ldots, T_{n}$ be mutually commuting $2 \times 2$ or $3 \times 3$ contractions, and let $p \in \mathbb{C}\left[X_{1}, \cdots, X_{n}\right]$. Then, we have

$$
\left\|p\left(T_{1}, \ldots, T_{n}\right)\right\| \leq\|p\|_{\infty}:=\sup \left\{\left|p\left(z_{1}, \cdots z_{n}\right)\right|: \underline{z} \in \mathbb{D}^{n}\right\}
$$

This leads to operator theoretical proofs of the following known ([39, lemma 7.5.6]) Schwarz-Pick inequalities for the polydisk.

Theorem 4.2.- (a) Let $f \in \mathcal{H}\left(\mathbb{D}^{n}, \mathbb{D}\right)$ and let $\underline{a}=\left(a_{1}, \ldots, a_{n}\right), \underline{b}=\left(b_{1}, \ldots, b_{n}\right) \in \mathbb{D}^{n}$. Then

$$
\begin{equation*}
\left|\frac{f\left(a_{1}, \cdots, a_{n}\right)-f\left(b_{1}, \cdots, b_{n}\right)}{1-\overline{f\left(a_{1}, \cdots, a_{n}\right)} f\left(b_{1}, \cdots, b_{n}\right)}\right| \leq \max _{1 \leq i \leq n}\left|\frac{a_{i}-b_{i}}{1-\overline{a_{i}} b_{i}}\right| \tag{4.1}
\end{equation*}
$$

(b) Let $f \in \mathcal{H}\left(\mathbb{D}^{n}, \mathbb{D}\right)$ and let $\underline{a}=\left(a_{2}, \ldots, a_{n}\right) \in \mathbb{D}^{n}$. Then,

$$
\begin{equation*}
\sum_{i=1}^{n}\left(1-\left|a_{i}\right|^{2}\right)\left|\frac{\partial f(\underline{a})}{\partial z_{i}}\right| \leq 1-|f(\underline{a})|^{2} \tag{4.2}
\end{equation*}
$$

Proof. - (a) We first observe that the result is obvious whenever $\underline{a}=\underline{b}$ or $f(\underline{a})=f(\underline{b})$. Therefore, in the following, we assume $\underline{a} \neq \underline{b}$ and $f(\underline{a}) \neq f(\underline{b})$.

For $1 \leq i \leq n$, let

$$
T_{i}=\left(\begin{array}{cc}
a_{i} & d\left(a_{i}-b_{i}\right) \\
0 & b_{i}
\end{array}\right)
$$

with

$$
d=\min _{1 \leq i \leq n} \sqrt{\frac{\left(1-\left|a_{i}\right|^{2}\right)\left(1-\left|b_{i}\right|^{2}\right)}{\left|a_{i}-b_{i}\right|^{2}}}
$$

Here, whenever $a_{i}=b_{i}$, we make the convention that $\sqrt{\frac{\left(1-\left|a_{i}\right|^{2}\right)\left(1-\left|b_{i}\right|^{2}\right)}{\left|a_{i}-b_{i}\right|^{2}}}=+\infty$. As we assume that $\underline{a} \neq \underline{b}$, this cannot happen for all the indices $i$.

It can be easily verified that the matrices $T_{i}$ are mutually commuting and that $\left\|T_{i}\right\| \leq 1$. By induction it can be shown that for all $i \in \llbracket 1, n \rrbracket$, for all $k_{i} \in \mathbb{N}$,

$$
T_{i}^{k_{i}}=\left(\begin{array}{cc}
a_{i}^{k_{i}} & d\left(a_{i}^{k_{i}}-b_{i}^{k_{i}}\right) \\
0 & b_{i}^{k_{i}}
\end{array}\right)
$$

Let $p \in \mathbb{C}\left[X_{1}, \ldots, X_{n}\right]$ be a polynomial with $\|p\|_{\infty}<1$. We have

$$
p\left(T_{1}, \cdots, T_{n}\right)=\left(\begin{array}{cc}
p(\underline{a}) & d(p(\underline{a})-p(\underline{b})) \\
0 & p(\underline{b})
\end{array}\right)
$$

Drury's result imply that $\left\|p\left(T_{1}, \ldots, T_{n}\right)\right\| \leq 1$. As in the one variable case, a computation gives the Schwarz-Pick inequality (4.1) for $p$. By an approximation argument, (4.1) holds also for functions in the polydisk algebra. Now, if $f \in \mathcal{H}\left(\mathbb{D}^{n}, \mathbb{D}\right)$, consider the family of functions $\left(f_{r}\right)_{0<r<1}$ defined by $f_{r}\left(z_{1}, \cdots, z_{n}\right)=f\left(r z_{1}, \ldots, r z_{n}\right)$. For all $\left.r \in\right] 0,1\left[, f_{r}\right.$ is in the polydisk algebra and, thus, $f_{r}$ satisfies (4.1). Then, let $r \rightarrow 1^{-}$to conclude the proof.
(b) The proof follows the same method as that of Theorem 4.2. Let $\underline{a}=\left(a_{2}, \ldots, a_{n}\right) \in$ $\mathbb{D}^{n}$ and let $p \in \mathbb{C}\left[X_{1}, \ldots, X_{n}\right]$ be a polynomial with $\|p\|_{\infty}<1$. For $1 \leq k \leq n$, let $T_{k}=\left(\begin{array}{cc}a_{k} & \gamma_{k} \\ 0 & a_{k}\end{array}\right)$, where $\gamma_{k}=e^{i \theta_{k}}\left(1-\left|a_{k}\right|^{2}\right)$, for some $\theta_{k} \in[0,2 \pi[$ to be chosen later on. For all $k \in \llbracket 1, n \rrbracket,\left\|T_{k}\right\| \leq 1$, and, for all $k, l \in \llbracket 1, n \rrbracket, T_{k} T_{l}=T_{l} T_{k}$. We have

$$
p\left(T_{1}, \cdots, T_{n}\right)=\left(\begin{array}{cc}
p(\underline{a}) & \sum_{k=1}^{n} \gamma_{k} \frac{\partial p(\underline{a})}{\partial z_{k}} \\
0 & p(\underline{a})
\end{array}\right) .
$$

Again, by Lemma 4.1, we get $\left\|p\left(T_{1}, \ldots, T_{n}\right)\right\| \leq 1$. Therefore

$$
\begin{equation*}
\left|\sum_{k=1}^{n} \gamma_{k} \frac{\partial p(\underline{a})}{\partial z_{k}}\right| \leq 1-|p(\underline{a})|^{2} \tag{4.3}
\end{equation*}
$$

Now, let $t_{k}=\frac{\partial p\left(a_{1}, a_{2}\right)}{\partial z_{k}}, 0 \leq k \leq n$. We write $t_{k}=\left|t_{k}\right| e^{i \operatorname{Arg}\left(t_{k}\right)}$ and we set $\theta_{k}=-\operatorname{Arg}\left(t_{k}\right)$. With this choice we obtain $\gamma_{k} t_{k}=\left(1-\left|a_{k}\right|^{2}\right)\left|t_{k}\right|$. Replacing in (4.3) we get

$$
\sum_{i=1}^{n}\left(1-\left|a_{i}\right|^{2}\right)\left|\frac{\partial p(\underline{a})}{\partial z_{i}}\right| \leq 1-|p(\underline{a})|^{2}
$$

We conclude by using an approximation argument.

Remark 4.3. - The study of the case of equality in the Schwarz-Pick inequalities for the polydisk is an interesting problem. Knese [25] studied the equality case in (4.2) using operator-theoretical methods (transfer functions) and described which functions play the role of automorphisms of the disk in this context-they turn out to be rational inner functions in the Schur-Agler class of the polydisk with an added symmetry constraint.
4.B. Peschl's invariant derivatives in several variables. - The inequalities from Section 2.D can be extended to analytic functions of several variables.

Let $n \in \mathbb{N}^{*}$, let $f \in \mathcal{H}\left(\mathbb{D}^{n}, \mathbb{D}\right)$, and fix a vector $\underline{\omega}=\left(\omega_{1}, \cdots, \omega_{n}\right)$ in $\mathbb{D}^{n}$. Similarly as in the one variable case, we define

$$
g: \underline{z}=\left(z_{1}, \cdots, z_{n}\right) \in \mathbb{D}^{n} \mapsto \frac{f\left(\frac{z_{1}+\omega_{1}}{1+\overline{\omega_{1}} z_{1}}, \cdots, \frac{z_{n}+\omega_{n}}{1+\overline{\omega_{n}} z_{n}}\right)-f\left(\omega_{1}, \cdots, \omega_{n}\right)}{1-\overline{f\left(\omega_{1}, \cdots, \omega_{n}\right)} f\left(\frac{z_{1}+\omega_{1}}{1+\overline{\omega_{1}} z_{1}}, \cdots, \frac{z_{n}+\omega_{n}}{1+\bar{\omega}_{n} z_{n}}\right)} \in \mathbb{C}
$$

and then write

$$
g\left(z_{1}, \cdots, z_{n}\right)=\sum_{j_{1}, \cdots, j_{n}=0}^{\infty} \frac{\partial^{j_{1}+\cdots+j_{n}} g(0, \cdots, 0)}{\partial^{j_{1}} z_{1} \cdots \partial^{j_{n}} z_{n}} z_{1}^{j_{1}} \cdots z_{n}^{j_{n}}=\sum_{j_{1}, \cdots, j_{n}=0}^{\infty} a_{j_{1}, \cdots, j_{n}} z_{1}^{j_{1}} \cdots z_{n}^{j_{n}} .
$$

For $k \in \llbracket 1, n \rrbracket$, let $D_{k} f(\underline{w})=\partial^{k} g(0, \ldots, 0)=\sum_{j_{1}+\cdots+j_{n}=k} a_{j_{1}, \cdots, j_{n}}$. A straightforward computation gives :

$$
\begin{aligned}
D_{1} f(\underline{\omega})= & \sum_{j=1}^{n} \frac{1-\left|\omega_{j}\right|^{2}}{1-|f(\underline{\omega})|^{2}} \cdot \frac{\partial f}{\partial z_{j}}(\underline{\omega}), \\
D_{2} f(\underline{\omega})= & \sum_{j=1}^{n} \frac{\partial^{2} g(0, \cdots, 0)}{\partial^{2} z_{j}}+2 \sum_{1 \leq j<k \leq n} \frac{\partial^{2} f(0, \cdots, 0)}{\partial z_{j} \partial z_{k}} \\
= & \sum_{j=1}^{n} \frac{\left(1-\left|\omega_{j}\right|^{2}\right)^{2}}{1-|f(\underline{\omega})|^{2}}\left(\frac{\partial^{2} f(\underline{\omega})}{\partial^{2} z_{j}}+\frac{2 \overline{f(\underline{w})}}{1-|f(\underline{w})|^{2}}-\frac{2 \overline{\omega_{j}}}{1-\left|\omega_{j}\right|^{2}} \cdot \frac{\partial f(\underline{\omega})}{\partial z_{j}}\right) \\
& +2 \sum_{1 \leq j<k \leq n} \frac{\left(1-\left|z_{j}\right|^{2}\right)\left(1-\left|z_{k}\right|^{2}\right)}{1-|f(\underline{\omega})|^{2}}\left(\frac{\partial f(\underline{\omega})}{\partial z_{j} \partial z_{k}}+\frac{2 \overline{f(\underline{\omega})}}{1-|f(\underline{\omega})|^{2}} \cdot \frac{\partial f(\underline{\omega})}{\partial z_{j}} \cdot \frac{\partial f(\underline{\omega})}{\partial z_{k}}\right) .
\end{aligned}
$$

With the same method of proof as before, we can arrive at the following result.

Theorem 4.4. - For $n \in \mathbb{N}^{*}$ let $\underline{w}=\left(\omega_{1}, \ldots, \omega_{n}\right) \in \mathbb{D}^{n}$ and consider $f \in \mathcal{H}\left(\mathbb{D}^{n}, \mathbb{D}\right)$. Then, we have:

$$
\begin{equation*}
\left|D_{2} f(\underline{\omega})\right| \leq 2\left(1-\left|D_{1} f(\underline{\omega})\right|^{2}\right) . \tag{4.4}
\end{equation*}
$$

Proof. - For $1 \leq k \leq n$, let

$$
T_{k}=\left(\begin{array}{ccc}
\omega_{k} & \alpha_{k} & \beta_{k} \\
0 & \omega_{k} & \alpha_{k} \\
0 & 0 & \omega_{k}
\end{array}\right) \in \mathcal{M}_{3}(\mathbb{C})
$$

with $\alpha_{k}=1-\left|\omega_{k}\right|^{2}$ and $\beta_{k}=-\overline{\omega_{k}}\left(1-\left|\omega_{k}\right|^{2}\right)$. By Theorem 2.3, $T_{k}$ is a contraction, for all $k \in \llbracket 1, n \rrbracket$. Moreover, for all $1 \leq k, j \leq n, T_{j} T_{k}=T_{k} T_{j}$. Therefore, by Knese's result, $p\left(T_{1}, \ldots, T_{n}\right)$ is a contraction, for every $p \in \mathbb{C}\left[X_{1}, \ldots, X_{n}\right]$ with $\|p\|_{\infty}<1$. Moreover, it is easy to check that

$$
p\left(T_{1}, \ldots, T_{n}\right)=\left(\begin{array}{ccc}
p(\underline{\omega}) & \gamma_{1} & \gamma_{2} \\
0 & p(\underline{\omega}) & \gamma_{1} \\
0 & 0 & p(\underline{\omega})
\end{array}\right)
$$

with

$$
\begin{aligned}
& \gamma_{1}=\sum_{j=1}^{n} \alpha_{j} \frac{\partial p(\underline{\underline{\omega}})}{\partial z_{j}}, \\
& \gamma_{2}=\frac{1}{2} \sum_{j=1}^{n} \alpha_{j}^{2} \frac{\partial^{2} p(\underline{w})}{\partial^{2} z_{j}}+\sum_{1 \leq j<k \leq n} \alpha_{j} \alpha_{k} \frac{\partial^{2} p(\underline{w})}{\partial z_{j} \partial z_{k}}+\sum_{j=1}^{n} \beta_{j} \frac{\partial p(\underline{w})}{\partial z_{j}} .
\end{aligned}
$$

By Theorem 2.3, we obtain :

$$
\begin{equation*}
\left|\gamma_{2}\left(1-|p(\underline{\omega})|^{2}\right)+\gamma_{1}^{2} \overline{p(\underline{\omega})}\right| \leq\left(1-|p(\underline{\omega})|^{2}\right)^{2}-\left|\gamma_{1}\right|^{2} \tag{4.5}
\end{equation*}
$$

which is equivalent with (4.4) for polynomials. The inequality extends to all functions $f \in \mathcal{H}\left(\mathbb{D}^{n}, \mathbb{D}\right)$.
4.C. Distinguished varieties and Schwarz-Pick inequalities. - In the bidisk case, the refined version of Ando's inequality by Agler and McCarthy [3] results in corresponding enhancements of Schwarz-Pick type inequalities.

We start by recalling the notion of distinguished variety introduced in [3]. A distinguished variety is a set of the form $V \cap \overline{\mathbb{D}}^{2}$, where $V$ is an algebraic set in $\mathbb{C}^{2}$ (so there is a polynomial $q \in \mathbb{C}[z, w]$ such that $\left.V=\left\{(z, w) \in \mathbb{D}^{2}: q(z, w)=0\right\}\right)$ with the property that

$$
\bar{V} \cap \partial\left(\mathbb{D}^{2}\right)=\bar{V} \cap \mathbb{T}^{2}
$$

Therefore a distinguished variety is the trace on $\mathbb{D}^{2}$ of a one-dimensional complex algebraic variety $V$ in $\mathbb{C}^{2}$ such that $V$ intersects $\mathbb{D}^{2}$ and exits the bidisk through its distinguished boundary, $\mathbb{T}^{2}$, without intersecting any other part of its topological boundary. A distinguished variety has ([3]) the following determinantal representation

$$
\begin{equation*}
V \cap \mathbb{D}^{2}=\left\{(z, w) \in \mathbb{D}^{2}: \quad \operatorname{det}(\Psi(z)-w \mathrm{Id})=0\right\} \tag{4.6}
\end{equation*}
$$

for some matrix-valued rational function $\Psi$ on the unit disc that is unitary on the unit circle.

Agler and McCarthy proved in [3] that for any pair of commuting contractive matrices $\left(T_{1}, T_{2}\right)$ without unimodular eigenvalues, there is a distinguished variety $V \cap \mathbb{D}^{2}$ such that the von-Neumann inequality holds on $V \cap \mathbb{D}^{2}$ for any polynomial $p$ in $\mathbb{C}\left[z_{1}, z_{2}\right]$, i.e.

$$
\begin{equation*}
\left\|p\left(T_{1}, T_{2}\right)\right\| \leq \sup _{\left(z_{1}, z_{2}\right) \in V \cap \mathbb{D}^{2}}\left|p\left(z_{1}, z_{2}\right)\right| \tag{4.7}
\end{equation*}
$$

Theorem 4.5.-(a) Let $\left(a_{1}, a_{2}\right)$ and $\left(b_{1}, b_{2}\right)$ be two points in the bidisk $\mathbb{D}^{2}$. Then there is a distinguished variety $V \cap \mathbb{D}^{2}$ such that the Schwarz-Pick inequality

$$
\begin{equation*}
\left|\frac{f\left(a_{1}, a_{2}\right)-f\left(b_{1}, b_{2}\right)}{1-\overline{f\left(a_{1}, a_{2}\right)} f\left(b_{1}, b_{2}\right)}\right| \leq \max \left\{\left|\frac{a_{1}-b_{1}}{1-\overline{a_{1}} b_{1}}\right|,\left|\frac{a_{2}-b_{2}}{1-\overline{a_{2}} b_{2}}\right|\right\} \tag{4.8}
\end{equation*}
$$

holds for any function $f$ which is holomorphic on the bidisk $\mathbb{D}^{2}$ and continuous on $\overline{\mathbb{D}}^{2}$ with

$$
\sup _{\left(z_{1}, z_{2}\right) \in V \cap \mathbb{D}^{2}}\left|f\left(z_{1}, z_{2}\right)\right| \leq 1
$$

(b) Let $\left(a_{1}, a_{2}\right)$ and $\left(b_{1}, b_{2}\right)$ be two points in the bidisk $\mathbb{D}^{2}$. Then there is a distinguished variety $V \cap \mathbb{D}^{2}$ such that the Schwarz-Pick inequality (4.8) holds for any function $f$ which is holomorphic in the bidisk $\mathbb{D}^{2}$ and for which there is a sequence of positive real number $\left(r_{n}\right)$ convergent to 1 with $r_{n}<1$ such that

$$
\sup _{n \geq 1,\left(z_{1}, z_{2}\right) \in V \cap \mathbb{D}^{2}}\left|f\left(r_{n} z_{1}, r_{n} z_{2}\right)\right| \leq 1
$$

Proof. - Consider the matrices

$$
T_{1}=\left(\begin{array}{cc}
a_{1} & d\left(a_{1}-b_{1}\right) \\
0 & b_{1}
\end{array}\right), \quad T_{2}=\left(\begin{array}{cc}
a_{2} & d\left(a_{2}-b_{2}\right) \\
0 & b_{2}
\end{array}\right)
$$

with

$$
d=\min \left\{\sqrt{\frac{\left(1-\left|a_{1}\right|^{2}\right)\left(1-\left|b_{1}\right|^{2}\right)}{\left|a_{1}-b_{1}\right|^{2}}}, \sqrt{\frac{\left(1-\left|a_{2}\right|^{2}\right)\left(1-\left|b_{2}\right|^{2}\right)}{\left|a_{2}-b_{2}\right|^{2}}}\right\}
$$

with the same conventions as in the proof of Theorem 4.2, (a). Following [3], we can also assume that $T_{1}$ and $T_{2}$ are jointly diagonalizable (this is the first case in the proof of [3, Theorem 3.1]). It follows from the result proved in [3] (see also [4, p.211] for details and unexplained terminology) that there is a distinguished variety $V$ such that $T=\left(T_{1}, T_{2}\right)$ can be extended to a pair of commuting unitaries $U=\left(U_{1}, U_{2}\right)$ with spectrum $\sigma(U)=\bar{V} \cap \partial\left(\mathbb{D}^{2}\right)=\bar{V} \cap \mathbb{T}^{2}$. As $f$ is in the bidisk algebra, $f(T)$ and $f(U)$ are well-defined and $f(T)$ is a restriction of $f(U)$ to $\mathbb{C}^{2} \times \mathbb{C}^{2}$. We obtain, as in [3], that

$$
\begin{equation*}
\left\|f\left(T_{1}, T_{2}\right)\right\| \leq \sup _{\left(z_{1}, z_{2}\right) \in V \cap \mathbb{D}^{2}}\left|f\left(z_{1}, f_{2}\right)\right| \tag{4.9}
\end{equation*}
$$

Therefore $f\left(T_{1}, T_{2}\right)$ is a contraction and the proof of Theorem 4.2, (a), implies that inequality (4.8) holds true. The second part, (b), follows from (a) applied to the functions $f\left(r_{n} z_{1}, r_{n} z_{2}\right)$ and then making $n \rightarrow \infty$.

The following result follows in a similar manner from the Agler and McCarthy result and the proof of Theorem 4.4.

Theorem 4.6. - Let $\underline{w}=\left(\omega_{1}, \omega_{2}\right) \in \mathbb{D}^{2}$. Then there exists a distinguished variety $V \cap \mathbb{D}^{2}$ such that

$$
\begin{equation*}
\left|D_{2} f(\underline{\omega})\right| \leq 2\left(1-\left|D_{1} f(\underline{\omega})\right|^{2}\right) \tag{4.10}
\end{equation*}
$$

for every $f \in \mathcal{A}\left(\overline{\mathbb{D}}^{2}\right)$ with

$$
\sup _{\left(z_{1}, z_{2}\right) \in V \cap \mathbb{D}^{2}}\left|f\left(z_{1}, z_{2}\right)\right| \leq 1
$$

Some Nevanlinna-Pick interpolation problems on distinguished varieties in the bidisk have been studied in [23].

## 5. Higher order Schwarz-Pick inequalities

Let $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$ be an analytic function of $\mathbb{D}$ into itself with $f(z)=\sum_{n=0}^{\infty} a_{n} z^{n}$. It has been proved by F.W. Wiener that for each $k \geq 1$ we have

$$
\begin{equation*}
\left|a_{k}\right| \leq 1-\left|a_{0}\right|^{2} \tag{5.1}
\end{equation*}
$$

We refer for instance to [32] for an operator theoretical proof of this inequality and for applications to Bohr's phenomenon. For $k=1$, the inequality (5.1) gives $\left|f^{\prime}(0)\right| \leq 1-$ $|f(0)|^{2}$. Applying this inequality to $F(z)=f((\omega+z) / 1+\bar{\omega} z)$, for a fixed $\omega \in \mathbb{D}$, we obtain the Schwarz-Pick inequality (1.2). For an arbitrary $k$, a similar reasoning has been used by Ruscheveyeh [40] to obtain the following sharp higher-order inequality for an analytic function $f \in \mathcal{H}(\mathbb{D}, \mathbb{D}), z \in \mathbb{D}$ and $k \geq 1$ :

$$
\begin{equation*}
\left|f^{(k)}(z)\right| \leq \frac{k!\left(1-|f(z)|^{2}\right)}{(1-|z|)^{k}(1+|z|)} \tag{5.2}
\end{equation*}
$$

We prove in this section some results related to the estimate (5.1).

Theorem 5.1.- Let $f: \mathbb{D} \mapsto \overline{\mathbb{D}}$ be an analytic function. Assume that

$$
f(z)=\sum_{n=0}^{\infty} a_{n} z^{n} \quad(z \in \mathbb{D})
$$

Then, for each $n \geq 1$ and each $k \geq 1$ we have

$$
\begin{equation*}
\left|a_{n+k}\left(1-\left|a_{0}^{2}\right|\right)+a_{n} a_{k} \bar{a}_{0}\right|^{2} \leq\left[\left(1-\left|a_{0}\right|^{2}\right)^{2}-\left|a_{n}\right|^{2}\right] \cdot\left[\left(1-\left|a_{0}\right|^{2}\right)^{2}-\left|a_{k}\right|^{2}\right] . \tag{5.3}
\end{equation*}
$$

Proof. - As $\|f\|_{\infty} \leq 1$, the multiplication operator $M_{f}$ given by $M_{f}(g)=f g$ acts contractively on the Hardy space $H^{2}(\mathbb{D})$. Recall that $\left\{z^{n}: n \geq 0\right\}$ is an orthonormal basis of $H^{2}(\mathbb{D})$. The compression $T=P_{K} M_{f} \mid K$ of $M_{f}$ to the 3-dimensional Euclidean space $K=\operatorname{span}\left(1, z^{n}, z^{n+k}\right)$ is also a contraction. The matrix of $T$ is given by

$$
T=\left(\begin{array}{ccc}
a_{0} & a_{n} & a_{n+k} \\
0 & a_{0} & a_{k} \\
0 & 0 & a_{0}
\end{array}\right)
$$

Then (5.3) is a consequence of Theorem 2.3.

When $a_{0}=0$ we obtain the following consequence.

Corollary 5.2. - Let $f$ be a analytic function of $\mathbb{D}$ into $\overline{\mathbb{D}}$ with $f(0)=0$ and $f(z)=$ $\sum_{n=1}^{\infty} a_{n} z^{n}$ for $z \in \mathbb{D}$. Then

$$
\begin{equation*}
\left|a_{n+k}\right| \leq \sqrt{1-\left|a_{n}\right|^{2}} \cdot \sqrt{1-\left|a_{k}\right|^{2}} \tag{5.4}
\end{equation*}
$$

For $n=k=1$, and $a_{0}=0$, we obtain the inequality $\left|a_{2}\right| \leq 1-\left|a_{1}\right|^{2}$. Applying this inequality to (2.18) we obtain Yamashita's inequality $\left|D_{2} f(\omega)\right| \leq 2\left(1-\left|D_{1} f(\omega)\right|^{2}\right)$.

The following consequence is an improvement of Wiener's inequality (5.1).

Corollary 5.3. - Let $f$ be a analytic function of $\mathbb{D}$ into $\overline{\mathbb{D}}$ with $f(z)=\sum_{n=0}^{\infty} a_{n} z^{n}$ for $z \in \mathbb{D}$. Then

$$
1-\left|a_{0}\right|^{2}-\left|a_{n}\right| \geq \frac{\left|a_{2 n}\left(1-\left|a_{0}^{2}\right|\right)+a_{n}^{2} \bar{a}_{0}\right|}{2\left(1-\left|a_{0}\right|^{2}\right)}
$$

Proof. - Applying (5.3) for $k=n$ we obtain

$$
\left|a_{2 n}\left(1-\left|a_{0}^{2}\right|\right)+a_{n}^{2} \bar{a}_{0}\right| \leq\left[\left(1-\left|a_{0}\right|^{2}\right)^{2}-\left|a_{n}\right|^{2}\right] .
$$

Therefore

$$
1-\left|a_{0}\right|^{2}-\left|a_{n}\right| \geq \frac{\left|a_{2 n}\left(1-\left|a_{0}^{2}\right|\right)+a_{n}^{2} \bar{a}_{0}\right|}{1-\left|a_{0}\right|^{2}+\left|a_{n}\right|} \geq \frac{\left|a_{2 n}\left(1-\left|a_{0}^{2}\right|\right)+a_{n}^{2} \bar{a}_{0}\right|}{2\left(1-\left|a_{0}\right|^{2}\right)}
$$

The proof is complete.

## 6. Appendix

The objective of this Appendix is to revisit Parrott's theorem as stated in Theorem 2.1. We adopt the approach presented by Davis, Kahan, and Weinberger in [14], making some modifications, particularly regarding the selection of solutions with minimal norms.

This appendix is primarily intended for readers interested in Schwarz-Pick inequalities who may not have an extensive background in operator theory.

We start by recalling the following lemma.

Lemma 6.1 (Douglas [15]). - Let $L, M_{1}, M_{2}$ be Hilbert spaces. Suppose that $A \in$ $\mathcal{B}\left(L, M_{1}\right), B \in \mathcal{B}\left(L, M_{2}\right)$ and $c \geq 0$. Then, $B^{*} B \leq c^{2} A^{*} A$ if and only if there exists $C \in \mathcal{B}\left(M_{1}, M_{2}\right)$ such that

$$
\left\{\begin{array}{l}
B=C A  \tag{6.1}\\
\|C\| \leq c
\end{array}\right.
$$

Moreover, if it is the case, there exists a unique operator $C_{0}$ satisfying (6.1) such that $\operatorname{Im}(A)^{\perp} \subset \operatorname{Ker}\left(C_{0}\right)$. The operator $C_{0}$ satisfies

$$
\left\|C_{0}\right\|^{2}=\inf \left\{\|C\|^{2}: C \text { satisfies }(6.1)\right\}=\inf \left\{\mu \geq 0: B^{*} B \leq \mu A^{*} A\right\}
$$

and will thus be referred as the minimal solution of the equation $B=C A$.

From this lemma, we deduce the following result about column matrices. Recall that the defect operator of $B$ is given by $D_{B}=\left(\operatorname{Id}-B^{*} B\right)^{1 / 2}$.

Proposition 6.2. - Let $H, K_{1}, K_{2}$ be Hilbert spaces. Suppose that $A \in \mathcal{B}\left(H, K_{1}\right)$ and $B \in \mathcal{B}\left(H, K_{2}\right)$ are contractions. Then, $\left[\begin{array}{l}A \\ B\end{array}\right]: H_{1} \rightarrow K_{1} \oplus K_{2}$ is a contraction if and only if there exists a contraction $V \in \mathcal{B}\left(H, K_{1}\right)$ such that $A=V D_{B}$.

Moreover, if it is the case, there exists a unique contraction $V_{0}$ such that $A=V_{0} D_{B}$ and $\operatorname{Im}\left(D_{B}\right)^{\perp} \subset \operatorname{Ker}\left(V_{0}\right)$. Then $V_{0}$ satisfies

$$
\left\|V_{0}\right\|=\inf \left\{\|V\|: A=V D_{B}\right\}
$$

and will thus be referred as the minimal solution of the equation $A=V D_{B}$.
Proof. - The column matrix $\left[\begin{array}{l}A \\ B\end{array}\right]$ is a contraction if and only if $A^{*} A \leq \operatorname{Id}-B^{*} B=D_{B}^{*} D_{B}$. Using Lemma 6.1, we obtain $A=V D_{B}$ with $\|V\| \leq 1$.

Corollary 6.3. - Let $H, K_{1}, K_{2}, A, B$ be as in Proposition 6.2, and let $U \in \mathcal{B}(H)$ be an arbitrary (but fixed) isometry. Then, $\left[\begin{array}{l}A \\ B\end{array}\right]: H_{1} \rightarrow K_{1} \oplus K_{2}$ is a contraction if and only if there exists a contraction $V \in \mathcal{B}\left(H, K_{1}\right)$ such that $A=V U D_{B}$.

Moreover, if it is the case, there exists a unique contraction $V_{0}$ such that $A=V_{0} U D_{B}$ and $\operatorname{Im}\left(U D_{B}\right)^{\perp} \subset \operatorname{Ker}\left(V_{0}\right)$. Then the operator $V_{0}$ satisfies

$$
\left\|V_{0}\right\|=\inf \left\{\|V\|: A=V U D_{B}\right\}
$$

and will thus be referred as the minimal solution of the equation $A=V U D_{B}$.

Proof. - It is enough to prove the sufficiency part. By Proposition 6.2, if $\left[\begin{array}{l}A \\ B\end{array}\right]$ is a contraction, there exists a contraction $W \in \mathcal{B}\left(H, K_{1}\right)$ such that $A=W D_{B}$. Moreover, $W$ can be chosen such that $W=0$ on $\operatorname{Im}\left(D_{B}\right)^{\perp}$ (and in this case, the minimal solution $W_{0}$ is unique).

Now, let $V=W U^{*}$. As $U$ is an isometry, it is easy to see that $V$ is a contraction and that $V U D_{B}=W D_{B}=A$. Moreover, $V=0$ on $\operatorname{Im}\left(U D_{B}\right)^{\perp}$. Indeed, let $x \in \operatorname{Im}\left(U D_{B}\right)^{\perp}$. For all $x^{\prime} \in H,\left\langle x, U D_{B} x^{\prime}\right\rangle=0$, which can be rewritten $\left\langle U^{*} x, D_{B} x^{\prime}\right\rangle=0$. Thus, for $x \in \operatorname{Im}\left(U D_{B}\right)^{\perp}, U^{*} x \in \operatorname{Im}\left(D_{B}\right)^{\perp}$ and, then, $V x=W U^{*} x=0$ (by minimality of $W$ ). It is moreover easy to see that there exists a unique $V$ such that $A=V U D_{B}$ and $V=0$ on $\operatorname{Im}\left(D_{B}\right)^{\perp}$.

Corollary 6.4. - Let $H_{1}, H_{2}, K$ be Hilbert spaces. Suppose that $A \in \mathcal{B}\left(H_{1}, K\right)$ and $B \in \mathcal{B}\left(H_{2}, K\right)$ are contractions. Then, $\left[\begin{array}{ll}A & B\end{array}\right]: H_{1} \oplus H_{2} \rightarrow K$ is a contraction if and only if there exists a contraction $V \in \mathcal{B}\left(H_{1}, K\right)$ such that $A=D_{B^{*}} V$.

Moreover, if it is the case, there exists a unique contraction $V_{0}$ such that $A=D_{B^{*}} V_{0}$ and $\operatorname{Im}\left(D_{B^{*}}\right)^{\perp} \subset \operatorname{Ker}\left(V_{0}^{*}\right)$. We have

$$
\left\|V_{0}\right\|=\inf \left\{\|V\|: A=D_{B^{*}} V\right\}
$$

The operator $V_{0}$ will be referred as the minimal solution of the equation $A=D_{B^{*}} V$.

Proof. - Observe that $\left[\begin{array}{ll}A & B\end{array}\right]$ is a contraction if and only if $\left[\begin{array}{ll}A & B\end{array}\right]^{*}=\left[\begin{array}{l}A^{*} \\ B^{*}\end{array}\right]$ is a contraction, and then apply Proposition 6.2.

Proof of Theorem 2.1. - First of all, the existence of two contractions $Z \in \mathcal{B}\left(H_{1}, K_{1}\right)$ and $Y \in \mathcal{B}\left(H_{2}, K_{2}\right)$ such that $D=D_{C^{*}} Y$ and $A=Z D_{C}$ comes from Proposition 6.2 and Corollary 6.4, as $\left[\begin{array}{l}A \\ C\end{array}\right]$ and $\left[\begin{array}{ll}C & D\end{array}\right]$ are contractions. We denote the minimal solutions by $Y_{0}$, and respectively $Z_{0}$.

Set $\mathbf{A}=\left[\begin{array}{ll}A & B\end{array}\right]$ and $\mathbf{B}=\left[\begin{array}{ll}C & D\end{array}\right]$, so that we have $T=\left[\begin{array}{l}\mathbf{A} \\ \mathbf{B}\end{array}\right]$, with $\|\mathbf{B}\| \leq 1$. Now, using that $T D_{T}=D_{T^{*}} T$, we have

$$
\begin{aligned}
\mathbf{I d}_{H_{1} \oplus H_{2}}-\mathbf{B}^{*} \mathbf{B} & =\left[\begin{array}{cc}
\operatorname{Id}_{H_{1}}-C^{*} C & -C^{*} D \\
-D^{*} C & \operatorname{Id}_{H_{2}}-D^{*} D
\end{array}\right] \\
& =\left[\begin{array}{cc}
\operatorname{Id}_{H_{1}}-C^{*} C & -C^{*} D_{C^{*}} Y_{0} \\
-Y_{0}^{*} D_{C^{*}} C & \operatorname{Id}_{H_{2}}-Y_{0}^{*} D_{C^{*}} D_{C^{*}} Y_{0}
\end{array}\right] \\
& =\left[\begin{array}{cc}
\operatorname{Id}_{H_{1}}-C^{*} C & -D_{C} C^{*} Y_{0} \\
-Y_{0}^{*} C D_{C} & \operatorname{Id}_{H_{2}}-Y_{0}^{*} Y_{0}+Y_{0}^{*} C C^{*} Y_{0}
\end{array}\right] \\
& =\mathbf{S}^{*} \mathbf{S},
\end{aligned}
$$

where $\mathbf{S}=\left[\begin{array}{cc}D_{C} & -C^{*} Y_{0} \\ 0 & D_{Y_{0}}\end{array}\right]$.
For every $w \in H_{1} \oplus H_{2}$, we have

$$
\left\langle\left(\mathbf{I d}_{H_{1} \oplus H_{2}}-\mathbf{B}^{*} \mathbf{B}\right) w, w\right\rangle=\left\langle\mathbf{S}^{*} \mathbf{S} w, w\right\rangle
$$

which is equivalent with $\|\mathbf{S} w\|=\left\|D_{\mathbf{B}} w\right\|$. Thus, there is an isometry $\mathbf{U} \in \mathcal{B}\left(H_{1} \oplus H_{2}\right)$ such that $\mathbf{S}=\mathbf{U} D_{\mathbf{B}}$. Indeed, let $U: \operatorname{Im}\left(D_{\mathbf{B}}\right) \rightarrow H_{1} \oplus H_{2}, D_{\mathbf{B}} x \mapsto \mathbf{S} x$. We extend $\mathbf{U}$ by continuity to $\overline{\operatorname{Im}\left(D_{\mathbf{B}}\right)}$, and we set $\mathbf{U}=\operatorname{Id}$ on $\operatorname{Im}\left(D_{\mathbf{B}}\right)^{\perp}$.

Suppose that $T$ is a contraction. Then, by Corollary 6.3 , there exists a contraction

$$
\mathbf{V}=\left[\begin{array}{ll}
V_{1} & V_{2}
\end{array}\right] \in \mathcal{B}\left(H_{1} \oplus H_{2}, K_{1}\right)
$$

such that

$$
\left\{\begin{array}{l}
\mathbf{A}=\mathbf{V U} D_{\mathbf{B}}  \tag{6.2}\\
\mathbf{V}=0 \text { on } \operatorname{Im}(\mathbf{S})^{\perp}
\end{array}\right.
$$

By Corollary 6.4, there exists a contraction $W \in \mathcal{B}\left(H_{2}, K_{1}\right)$ such that $\mathbf{V}=\left[\begin{array}{ll}V_{1} & D_{V_{1}^{*}} W\end{array}\right]$. The operator $W$ can be chosen such that $\operatorname{Im}\left(D_{V_{1}^{*}}\right) \subset \operatorname{Ker}\left(W^{*}\right)$ (in that case, the minimal solution $W_{0}$ is unique).

Then, (6.2) is equivalent with

$$
\begin{align*}
{\left[\begin{array}{cc}
A & B
\end{array}\right] } & =\left[\begin{array}{ll}
V_{1} & D_{V_{1}^{*}} W
\end{array}\right]\left[\begin{array}{cc}
D_{C} & -C^{*} Y_{0} \\
0 & D_{Y_{0}}
\end{array}\right] \\
& =\left[\begin{array}{ll}
V_{1} D_{C} & -V_{1} C^{*} Y_{0}+D_{V_{1}^{*}} W D_{Y_{0}}
\end{array}\right] \tag{6.4}
\end{align*}
$$

In particular, we have $A=V_{1} D_{C}$. We now show that $V_{1}=Z_{0}$.

Fact 1. $-\operatorname{Im}\left(D_{C}\right)^{\perp} \oplus\{0\} \subset \operatorname{Im}(\mathbf{S})^{\perp}$.

Proof. - Let $v \in \operatorname{Im}\left(D_{C}\right)^{\perp}=\operatorname{Ker}\left(D_{C}\right)$. In order to prove that $\left[\begin{array}{l}v \\ 0\end{array}\right] \in \operatorname{Ker}\left(\mathbf{S}^{*}\right)=\operatorname{Im}(\mathbf{S})^{\perp}$, notice that we have

$$
\mathbf{S}^{*}\left[\begin{array}{l}
v \\
0
\end{array}\right]=\left[\begin{array}{c}
0 \\
-Y_{0}^{*} C v
\end{array}\right]
$$

As we know that $Y_{0}^{*}=0$ on $\operatorname{Im}\left(D_{C^{*}}\right)^{\perp}=\operatorname{Ker}\left(D_{C^{*}}\right)$, it is enough to show $C v \in \operatorname{Ker}\left(D_{C^{*}}\right)$. Using again the identity $C D_{C}=D_{C^{*}} C$, we have

$$
\left\|D_{C^{*}} C v\right\|^{2}=\left\langle D_{C^{*}} C v, D_{C^{*}} C v\right\rangle=\left\langle C v, D_{C^{*}}^{2} C v\right\rangle=\left\langle C v, C D_{C}^{2} v\right\rangle=0
$$

which completes the proof of the Fact 1.

Continuing the proof of Theorem 2.1, we can deduce from (6.3) that $V_{1}=0$ on $\operatorname{Im}\left(D_{C}\right)^{\perp}$ and, thus, $V_{1}=Z_{0}$. Finally, (6.4) is equivalent with

$$
B=-Z_{0} C^{*} Y_{0}+D_{Z_{0}^{*}} W D_{Y_{0}}
$$

Conversely, if there exists a contraction $W \in \mathcal{B}\left(H_{2}, K_{1}\right)$ such that $B=D_{Z_{0}^{*}} V D_{Y_{0}}-$ $Z_{0} C^{*} Y_{0}$, then it is easy to check that $\mathbf{A}=\mathbf{V}^{\prime} \mathbf{S}=\mathbf{V}^{\prime} \mathbf{U} \mathbf{D}_{\mathbf{B}}$, with $\mathbf{V}^{\prime}=\left[\begin{array}{ll}Z & D_{Z^{*}} W\end{array}\right]$. As $\mathbf{V}^{\prime}$ is a contraction (Corollary 6.4), this implies that $T$ is a contraction (Corollary 6.3).

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