SCHWARZ-PICK TYPE INEQUALITIES FROM AN OPERATOR THEORETICAL POINT OF VIEW

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Abstract. — We use (versions of) the von Neumann inequality for Hilbert space contractions to prove several Schwarz-Pick type inequalities. Specifically, we derive an alternate proof for a multi-point Schwarz-Pick inequality by Beardon and Minda, along with a generalized version for operators. Connections with model spaces and Peschl's invariant derivatives are established. Finally, Schwarz-Pick inequalities for analytic functions on polydisks and for higher order derivatives are discussed. An enhanced version of the Schwarz-Pick lemma, using the notion of distinguished variety, is obtained for the bidisk.

Contents

Notation	2
1. Introduction	2
2. A three points Schwarz-Pick lemma	5
2.A. Contractive three by three matrices	5
2.B. An operator-theoretical proof of Beardon-Minda's inequality	8
2.C. Connecting with model spaces theory	12
2.D. A Beardon-Minda type lemma for derivatives	13
3. Operator versions of Beardon-Minda's inequality	14
3.A. An operator version of the Schwarz-Pick inequality	15
3.B. An operator version of the Beardon-Minda inequality	15
4. Schwarz-Pick inequalities for the polydisk	17
4.A. Using von Neumann inequality for tuples of two by two matrices.	17
4.B. Peschl's invariant derivatives in several variables	18
4.C. Distinguished varieties and Schwarz-Pick inequalities	20
5. Higher order Schwarz-Pick inequalities	21
6. Appendix	23
References	26

The authors acknowledge support from the Labex CEMPI (ANR-11-LABX-0007-01).

²⁰⁰⁰ Mathematics Subject Classification. — 47A25, 47A30, 47A08, 30A10, 30C80, 15A60. Key words and phrases. — von Neumann inequality, Schwarz-Pick estimates, Beardon-Minda inequality, Peschl's invariant derivatives.

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Notation

\mathbb{D}	denotes the open unit disk
T	denotes the unit circle, $\mathbb{T} = \overline{\mathbb{D}} \backslash \mathbb{D}$
$\left\ \cdot\right\ _{\infty}$	for $\underline{\omega} = (\omega_1, \dots, \omega_n) \in \mathbb{C}^n$, we denote $\ \underline{\omega}\ _{\infty} = \sup\{ \omega_i : 1 \le i \le n\}$
$\mathcal{H}(\mathbb{D})$	is the set of functions that are holomorphic on \mathbb{D} , $\mathcal{H}(\mathbb{D}) = \mathcal{H}(\mathbb{D}, \mathbb{C})$
$\mathcal{H}(\mathbb{D},\mathbb{D})$	denotes the set of functions in $\mathcal{H}(\mathbb{D})$ mapping \mathbb{D} to \mathbb{D}
$\mathcal{A}(\overline{\mathbb{D}})$	is the disk algebra, $i.e.$ the set of functions that are holomorphic on $\mathbb D$ and continuous on $\overline{\mathbb D}$
(z,w)	denotes the complex pseudo-hyperbolic distance $(z, w) := \frac{z - w}{1 - \overline{w} z}$
$\rho(z,w)$	denotes the pseudo-hyperbolic distance $\rho(z,w):= (z,w) $
d(z, w)	denotes the hyperbolic distance $d(z, w) = \tanh^{-1} \frac{1 + \rho(z, w)}{1 - \rho(z, w)}$
$f^*(z,w)$	denotes the hyperbolic divided difference $f^*(z, w) := \frac{(f(z), f(w))}{(z, w)}$
$H^2(\mathbb{D})$	is the Hilbert-Hardy space of $\mathbb D$
$H^{\infty}(\mathbb{D})$	is the set of bounded holomorphic functions on $\mathbb D$
$\mathcal{B}(H,K)$	is the set of bounded linear operators from H to K , where H and K are two complex Hilbert spaces. $\mathcal{B}(H)$ is a short for $\mathcal{B}(H, H)$
$\ \cdot\ $	denotes the norm of an element of the Banach space under consideration. When $T \in \mathcal{B}(H, K)$, $ T $ denotes the operator norm
T^*	denotes the adjoint of T , where T is a Hilbert space operator
D_T	denotes the defect operator of a contraction $T \in \mathcal{B}(H)$, i.e. $ T \leq 1$. Thus $D_T = (\mathrm{Id} - T^*T)^{1/2}$, where Id is the identity operator
$\sigma(T)$	denotes the spectrum of $T \in \mathcal{B}(H)$
r(T)	denotes the spectral radius $r(T) = \sup\{ \lambda : \lambda \in \sigma(T)\}$ of T
$\operatorname{Ker}(T)$	denotes the kernel of T
$\operatorname{Im}(T)$	denotes the range (image) of T
$[\![1,n]\!]$	denotes the set of all integers j with $1 \le j \le n$.

1. Introduction

The Schwarz-Pick inequality, an invariant form of the Schwarz lemma, stands as a cornerstone in complex analysis. In geometric terms, it posits that a holomorphic map from the open unit disk into itself has the property of decreasing the distance between points in the hyperbolic metric. Equivalently, if $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$ and ω_1 and ω_2 are two points in \mathbb{D} , then

(1.1)
$$\rho(f(\omega_1), f(\omega_2)) = \left| \frac{f(\omega_1) - f(\omega_2)}{1 - \overline{f(\omega_1)} f(\omega_2)} \right| \le \left| \frac{\omega_1 - \omega_2}{1 - \overline{\omega_1} \omega_2} \right| = \rho(\omega_1, \omega_2)$$

Inequality (1.1) is strict for $\omega_1 \neq \omega_2$ unless f is a conformal automorphism of the unit disk. Moreover,

(1.2)
$$\frac{|f'(\omega)|}{1 - |f(\omega)|^2} \le \frac{1}{1 - |\omega|^2}.$$

The Schwarz-Pick inequalities (1.1) and (1.2) have been extended in various ways by many authors. A thorough overview of some of these advancements is provided in the comprehensive survey [17]. For the present study the following 'three points' version of (1.1) by Beardon and Minda [9] is pivotal. It involves the notion of hyperbolic divided difference $f^*(z, w)$ and states that if $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$, then

(1.3)
$$\rho\left(f^*(\omega_1,\omega_2),f^*(\omega_3,\omega_2)\right) \le \rho(\omega_1,\omega_3)$$

for three pairwise distinct points ω_1 , ω_2 and ω_3 in the unit disk. The Beardon-Minda inequality unifies in an elegant way many improvements of (1.1). An analogous theorem with more than three points has been proved in [8], where Baribeau, Rivard and Wegert also used iterated (hyperbolic) divided differences to give simpler conditions for the *n* points Nevanlinna–Pick interpolation problem. We refer also to [1, 35, 36] for related contributions.

Various operator-theoretical interpretations of the Schwarz-Pick inequality are possible. The most well-known interpretation, due to Sarason [41], exploits the equivalence of the Schwarz-Pick inequality with the Nevanlinna-Pick interpolation problem for two points. Consequently, the Schwarz-Pick inequality possesses an operator-theoretical significance concerning norm-preserving lifting of some operators that act on specific subspaces of the Hardy space $H^2(\mathbb{D})$. Additional generalizations can be derived using the commuting lifting theorem of Sz.-Nagy and Foias (see for instance [19]). It is noteworthy that the commutant lifting theorem is equivalent with Ando's dilation theorem ([29]). Further operator-theoretical interpretations of the Schwarz-Pick inequality have been explored in [5, 6, 22, 25, 27].

The starting point of this note was the natural question of looking for an operatortheoretical interpretation of the Beardon-Minda inequality. Notice that the Schwarz-Pick inequality can be obtained as a particular case of the von Neumann inequality for Hilbert space contractions. The von Neumann inequality states that if $T \in \mathcal{B}(H)$ is a bounded linear operator acting on a complex Hilbert space H with $||T|| \leq 1$ and f is a polynomial, then

(1.4)
$$||f(T)|| \le \sup\{|f(z)| : |z| \le 1\}.$$

This inequality extends to functions f in the disk algebra $\mathcal{A}(\overline{\mathbb{D}})$. The inequality (1.1) is obtained ([38, p.17], [31, exercices 2.17-2.18]) when applying von Neumann's inequality (1.4) to a polynomial (or an element in the disk algebra) f and a specific 2 × 2 matrix

acting on the 2-dimensional Hilbert space \mathbb{C}^2 . Then an approximation argument gives (1.1) for $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$. A similar derivation by Agler of the Schwarz-Pick inequality (1.1) is presented in [2] and [4, Chapter 8], where this concept is ingeniously applied in to establish an operator-theoretical proof of Lempert's theorem, demonstrating the equality of the Carathéodory and Kobayashi metrics on convex domains. In this case, von Neumann's inequality is substituted with the notion of spectral set.

The primary objective of this manuscript is to leverage (versions of) the von Neumann inequality to derive Schwarz-Pick type inequalities. Notably, when applying the von Neumann inequality to a remarkable 3×3 matrix (the matrix of the model operator in the Takenaka-Malmquist basis), the Beardon-Minda three-point Schwarz-Pick inequality is obtained. A similar proof is given for a Beardon-Minda type inequalities are obtained. We also consider some Schwarz-Pick related inequalities for higher derivatives. Additionally, we delve into the multivariable case, employing von Neumann's inequality on *n*-tuples of mutually commuting 2×2 or 3×3 matrices to prove Schwarz-Pick type inequalities for the polydisk. In the case of the bidisk an improvement can be given using the notion of distinguished variety and the refined version of Ando's inequality by Agler and McCarthy [3]. The reader is welcomed to notice that the multipoint Schwarz-Pick inequality of [8] can also be derived as a consequence of the von Neumann inequality. However, explicit computations with matrices become more intricate.

Outline. The manuscript is organized as follows. In the next section we use a theorem going back to Parrott to obtain criteria for scalar and operator 3×3 matrices to have (Hilbertian) operator norm no greater than one. This is applied to a specific matrix to obtain an alternate proof of the Beardon-Minda inequality. The significance of this specific matrix with model spaces is highlighted, and a discussion concerning the equality case in (1.1) is given. Also, a Beardon-Minda type inequality for derivatives, originally proved by Yamashita [44], is obtained as a consequence of the von Neumann inequality. This inequality can be rephrased in terms of Peschl's invariant derivatives.

In Section 3 we prove several operator versions of the Schwarz-Pick inequality and of the Beardon-Minda inequality. The Sylvester (operator) equation AX - XB = Y plays an important role in the proofs.

In the next section we consider the case of the polydisk. We give operator theoretical proofs of the analogues of (1.1) and (1.2) for the polydisk and discuss the Peschl's invariant derivatives in several variables. The proofs uses a result of Knese [25] that the (polydisk) von Neumann's inequality holds for *n*-tuples of 3×3 commuting contractive matrices. In the case of the bidisk we use a result of Agler and McCarthy [3] to obtained an enhanced version of the Schwarz-Pick inequality.

In Section 5 we give some Schwarz-Pick related inequalities for higher derivatives. An improvement of a classical result of F. Wiener is proved.

Parrott's theorem, which was essential in our proofs, is revisited in the Appendix. We hope that this will prove beneficial for readers interested in Schwarz-Pick inequalities who may not be extensively acquainted with operator theory.

2. A three points Schwarz-Pick lemma

In view of the preceding discussion, it is a natural question to apply the von Neumann inequality to 3×3 matrices.

2.A. Contractive three by three matrices. — First, we need an explicit criterion to determine whenever a 3×3 -upper triangular matrix is a contraction. Note that in view of Schur's decomposition theorem – which states that every square matrix is unitary equivalent to an upper-triangular matrix – it makes sense to restrict ourselves to that case. In essence, following the approach used for 2×2 matrices, one can calculate the operator norm of a 3×3 matrix acting on the Euclidean space \mathbb{C}^3 using the formula:

$$||T||^{2} = ||T^{*}T|| = r(T^{*}T) = \sup\{|\lambda| : \det(T^{*}T - \lambda \mathrm{Id}) = 0\}.$$

This computation of the operator norm ||T||, the largest singular value of T, leads to an equation of degree 3. However, the criterion derived from this observation holds limited practical interest. An alternative approach to obtain such a criterion uses the Schur parameters (cf. [13]). Adapting the argument in [21, Lemma 2.7], we follow here a different approach, based on a result about completion of matrices going back to Parrott (see [30, 19], [46, Theorem 12.22] and [7, 14]).

Theorem 2.1 (Parrott). — Let H_1, H_2, K_1, K_2 be Hilbert spaces, and suppose that the operators $\begin{bmatrix} A \\ C \end{bmatrix} \in \mathcal{B}(H_1, K_1 \oplus K_2)$ and $\begin{bmatrix} C & D \end{bmatrix} \in \mathcal{B}(H_1 \oplus H_2, K_2)$ are contractions. Then, $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : H_1 \oplus H_2 \to K_1 \oplus K_2$ is a contraction if and only if there exists a contraction $W \in \mathcal{B}(H_2, K_1)$ such that $B = D_{Z^*}WD_Y - ZC^*Y$, where $Z \in \mathcal{B}(H_1, K_1)$ and $Y \in \mathcal{B}(H_2, K_2)$ are contractions such that $D = D_{C^*}Y$ and $A = ZD_C$.

Moreover,

- 1. Y and Z can be chosen to be (respectively) Y_0 and Z_0 , the solutions of minimal operator norm among all solutions of the operator equations $D = D_{C^*}Y$ and $A = ZD_C$;
- 2. If T is a contraction, there exists a unique contraction W_0 such that

$$B = D_{Z_0^*} W_0 D_{Y_0} - Z_0 C^* Y_0 \text{ and } Im \left(D_{Z_0^*} \right)^{\perp} \subset Ker(W_0^*).$$

This operator satisfies

$$||W_0|| = \inf\{||W|| : B = D_{Z_0^*}WD_{Y_0} - Z_0C^*Y_0\}.$$

C. BADEA & A. RENARD

We shall call Y_0 and Z_0 the minimal solutions and we shall refer to W_0 as the minimal solution of the equation

$$B = D_{Z_0^*} W D_{Y_0} - Z_0 C^* Y_0.$$

A further discussion is given in the Appendix (Section 6). In particular, $T = \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix}$ is a contraction if and only if $B = (I - A_1 A_1^*)^{1/2} W (I - A_2^* A_2)^{1/2}$ for a certain contraction W and the scalar matrix

$$T = \begin{bmatrix} \omega_1 & a \\ 0 & \omega_2 \end{bmatrix}$$

has (Euclidean) norm no greater than one if and only if $|\omega_1| \leq 1$, $|\omega_2| \leq 1$ and $|a| \leq \sqrt{1-|\omega_1|^2}\sqrt{1-|\omega_2|^2}$.

The following result provides a criterion for determining whether a 3×3 operator matrix is a contraction, when the central entry of the matrix, W_2 , is a strict contraction.

Theorem 2.2. — Let H_1, H_2, H_3 be three Hilbert spaces. Let $W_i \in \mathcal{B}(H_i), 1 \leq i \leq 3$, be three contractions and denote

$$T = \begin{bmatrix} W_1 & A_1 & B \\ 0 & W_2 & A_2 \\ 0 & 0 & W_3 \end{bmatrix} \in \mathcal{B}(H_1 \oplus H_2 \oplus H_3).$$

Assume that $||W_2|| < 1$. Then, T is a contraction if and only if there exist three contractions $V_1 \in \mathcal{B}(H_2, H_1), V_2 \in \mathcal{B}(H_3, H_2), V_3 \in \mathcal{B}(H_3, H_1)$ such that :

(2.1)
(2.2)
$$\begin{cases} A_1 = D_{W_1^*} V_1 D_{W_2}, \\ A_2 = D_{W_2^*} V_2 D_{W_3}, \end{cases}$$

(2.3)
$$\begin{cases} B = \left[D_{W_1^*} (\mathrm{Id} - V_1 V_1^*) D_{W_1^*} \right]^{1/2} V_3 \left[D_{W_3} (\mathrm{Id} - V_2^* V_2) D_{W_3} \right]^{1/2} \\ -D_{W_1^*} V_1 W_2^* V_2 D_{W_3}. \end{cases}$$

Proof. — First, if T is a contraction, then $\begin{bmatrix} W_1 & A_1 \\ 0 & W_2 \end{bmatrix}$ and $\begin{bmatrix} W_2 & A_2 \\ 0 & W_3 \end{bmatrix}$ are also contractions, as they are compressions of T. Then Parrott's theorem implies that (2.1) and (2.2) are satisfied. In the following we assume that (2.1) and (2.2) are true.

Now, denote

$$A = \begin{bmatrix} W_1 & A_1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & W_2 \\ 0 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} A_2 \\ W_3 \end{bmatrix}.$$

By Parrott's theorem, T is a contraction if and only if :

(2.4)
$$B = (\mathrm{Id} - ZZ^*)^{1/2} V_3 (\mathrm{Id} - Y^*Y)^{1/2} - ZC^*Y,$$

for an arbitrary contraction $V_3 \in \mathcal{B}(H_3, H_1)$. Here Y and Z are contractions such that $D = (\mathrm{Id} - CC^*)^{1/2}Y$ and $A = Z(\mathrm{Id} - C^*C)^{1/2}$, the existence of which is ensured by Parrott's theorem for column (respectively row) matrix-operators. Indeed, $\begin{bmatrix} A \\ C \end{bmatrix}$ and $\begin{bmatrix} C & D \end{bmatrix}$ are

contractions. We have $\operatorname{Id} - CC^* = \begin{bmatrix} \operatorname{Id} - W_2 W_2^* & 0 \\ 0 & \operatorname{Id} \end{bmatrix}$ and $\operatorname{Id} - C^* C = \begin{bmatrix} \operatorname{Id} & 0 \\ 0 & \operatorname{Id} - W_2^* W_2 \end{bmatrix}$. Since $||W_2|| < 1$, these operators are invertible. Thus, we get

$$Z = AD_C^{-1} = \begin{bmatrix} W_1 & A_1 D_{W_2}^{-1} \end{bmatrix} = \begin{bmatrix} W_1 & D_{W_1^*} V_1 \end{bmatrix}$$
$$Y = D_{C^*} D = \begin{bmatrix} D_{W_2^*}^{-1} A_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} V_2 D_{W_3} \\ W_3 \end{bmatrix}.$$

It follows that $D_{Z^*} = \left[D_{W_1^*} (\mathrm{Id} - V_1 V_1^*) D_{W_1^*} \right]^{1/2}, D_Y = \left[D_{W_3} (\mathrm{Id} - V_2^* V_2) D_{W_3} \right]^{1/2}$ and $ZC^*Y = D_{W_1^*} V_1 W_2^* V_2 D_{W_3}$. Therefore, (2.4) is equivalent with (2.3).

We obtain the following general criterion in the scalar case.

Theorem 2.3. — Let $\omega_1, \omega_2, \omega_3 \in \overline{\mathbb{D}}$. Then, $T = \begin{pmatrix} \omega_1 & \alpha_1 & \beta \\ 0 & \omega_2 & \alpha_2 \\ 0 & 0 & \omega_3 \end{pmatrix}$ is a contraction when

acting on the Hilbert space \mathbb{C}^3 if and only if

(2.5) $|\omega_2| < 1,$

(2.6)
$$\begin{cases} |\alpha_i|^2 \le (1 - |\omega_i|^2)(1 - |\omega_{i+1}|^2), \ i = 1, 2, \\ |\beta(1 - |\omega_2|^2) + \alpha_1 \alpha_2 \overline{\omega_2}|^2 \le 1 \end{cases}$$

(2.7)
$$\left(\left[(1 - |\omega_1|^2)(1 - |\omega_2|^2) - |\alpha_1|^2 \right] \cdot \left[(1 - |\omega_2|^2)(1 - |\omega_3|^2) - |\alpha_2|^2 \right] \right) \right)$$

(2.8)
$$\boldsymbol{\zeta} |\omega_2| =$$

(2.8)
(2.9)
$$\begin{cases} |\omega_2| = 1, \\ \alpha_i = 0, \ i = 1, 2, \end{cases}$$

(2.10)
$$(|\beta|^2 \le (1 - |\omega_1|^2)(1 - |\omega_3|^2).$$

Proof. — As in the proof of Theorem 2.2, if T is a contraction, then the two dimensional compressions $\begin{bmatrix} \omega_1 & \alpha_1 \\ 0 & \omega_2 \end{bmatrix}$ and $\begin{bmatrix} \omega_2 & \alpha_2 \\ 0 & \omega_3 \end{bmatrix}$ are also contractions. Thus (2.6) is satisfied, and it will be assumed from now on. Note that if $|\omega_2| = 1$, this implies that $\alpha_1 = \alpha_2 = 0$.

We use similar notation as in the proof of Theorem 2.2, with

$$A = \begin{bmatrix} \omega_1 & \alpha_1 \end{bmatrix}, \quad B = \begin{bmatrix} \beta \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \omega_2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \alpha_2 \\ \omega_3 \end{bmatrix}.$$

By Theorem 2.1, T is a contraction if and only if :

(2.11)
$$B = (\mathrm{Id} - ZZ^*)^{1/2} V (\mathrm{Id} - Y^*Y)^{1/2} - ZC^*Y, \text{ for some contraction } V,$$

where Y and Z are contractions such that $D = (\mathrm{Id} - CC^*)^{1/2}Y$ and $A = Z(\mathrm{Id} - C^*C)^{1/2}$. We have

$$\mathrm{Id} - CC^* = \begin{bmatrix} 1 - |\omega_2|^2 & 0\\ 0 & 1 \end{bmatrix}$$

and

$$\mathrm{Id} - C^* C = \begin{bmatrix} 1 & 0\\ 0 & 1 - |\omega_2|^2 \end{bmatrix}.$$

First case. Assume first that $|\omega_2| < 1$. Then we can apply Theorem 2.2. An easy computation shows that

$$Y = (\mathrm{Id} - CC^*)^{-1/2} D = \begin{bmatrix} \frac{\alpha_2}{\sqrt{1 - |\omega_2|^2}} \\ \omega_3 \end{bmatrix}$$

and

$$Z = A \left(\operatorname{Id} - C^* C \right)^{-1/2} = \begin{bmatrix} \omega_1 & \frac{\alpha_1}{\sqrt{1 - |\omega_2|^2}} \end{bmatrix}$$

Thus, T is a contraction if and only if (2.11) is satisfied, that is

$$\beta + \frac{\alpha_1 \alpha_2 \overline{\omega_2}}{1 - |\omega_2|^2} = \left(1 - |\omega_1|^2 - \frac{|\alpha_1|^2}{1 - |\omega_2|^2}\right)^{1/2} V \left(1 - |\omega_3|^2 - \frac{|\alpha_2|^2}{1 - |\omega_2|^2}\right)^{1/2}$$

for some contraction V. This holds if and only if

$$\left\| \left(1 - |\omega_1|^2 - \frac{|\alpha_1|^2}{1 - |\omega_2|^2} \right)^{-1/2} \left(\beta + \frac{\alpha_1 \alpha_2 \overline{\omega_2}}{1 - |\omega_2|^2} \right) \left(1 - |\omega_3|^2 - \frac{|\alpha_2|^2}{1 - |\omega_2|^2} \right)^{-1/2} \right\| \le 1.$$

In can be easily shown that this is equivalent with the condition (2.7).

Second case. Assume now that $|\omega_2| = 1$. Let $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and $Z = \begin{bmatrix} z_1 & z_2 \end{bmatrix}$. As $D = (\mathrm{Id} - CC^*)^{1/2}Y$, we get $D^* = Y^*(\mathrm{Id} - CC^*)^{1/2}$. This holds if and only if $y_2 = \omega_3$. As Y^* can be chosen to be 0 on $\mathrm{Im}(D_C^*)^{\perp}$ (see the Appendix), we have $y_1 = 0$.

Similarly, $A = Z(\text{Id} - C^*C)^{1/2}$ holds if and only if $z_1 = \omega_1$ and, as before, we can choose $z_2 = 0$. We have $ZC^*Y = 0$. Therefore, T is a contraction if and only if $|\beta|^2 \leq (1 - |\omega_3|^2)(1 - |\omega_1|^2)$. This is equivalent with the condition (2.10).

2.B. An operator-theoretical proof of Beardon-Minda's inequality. — We refer to [42, Chapter 22] for the definition and basic properties of *divided differences* of n + 1(not necessarily distinct) points. We just recall here that for pairwise distinct points $z_0, z_1, \dots, z_n \in \mathbb{C}$, the divided differences of f at points z_0, z_1, \dots, z_n satisfy $[f(z_k)] = f(z_k)$ and the recurrence relation

$$[f(z_k), \cdots, f(z_{k+j})] = \frac{[f(z_{k+1}), \cdots, f(z_{k+j})] - [f(z_k), \cdots, f(z_{k+j-1})]}{z_{k+j} - z_k}$$

for $0 \le k \le j \le n$.

We also recall to the reader the following notation.

Definition 2.4. — Let $z, w \in \mathbb{D}$ and $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$. We define:

- 1. The complex pseudo-hyperbolic distance $(z, w) := \frac{z-w}{1-\overline{w}z}$;
- 2. The pseudo-hyperblic distance $\rho(z, w) := |(z, w)|;$

8

SCHWARZ-PICK TYPE INEQUALITIES

- 3. The hyperbolic distance $d(z, w) = \tanh^{-1} \frac{1+\rho(z,w)}{1-\rho(z,w)}$;
- 4. The hyperbolic divided difference $f^*(z, w) := \frac{(f(z), f(w))}{(z, w)}$.

We provide an operator-theoretic proof of the following result established by Beardon and Minda [9].

Theorem 2.5 (Beardon-Minda [9]). — Let $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$ and let ω_1, ω_2 and ω_3 be pairwise distinct points in \mathbb{D} . Then,

(2.12)
$$d(f^*(\omega_1, \omega_2), f^*(\omega_3, \omega_2)) \le d(\omega_1, \omega_3).$$

The proof in [9] requires an assumption that f is not a conformal automorphism of the unit disk. Such an assumption is unnecessary in the subsequent proof.

Proof of Theorem 2.5. — First, let us notice that Beardon-Minda's inequality (2.12) is equivalent with

(2.13)
$$\rho(f^*(\omega_1, \omega_2), f^*(\omega_3, \omega_2)) = \left| \frac{f^*(\omega_1, \omega_2) - f^*(\omega_3, \omega_2)}{1 - \overline{f^*(\omega_3, \omega_2)}} f^*(\omega_1, \omega_2) \right| \le \left| \frac{\omega_1 - \omega_3}{1 - \overline{\omega_3}\omega_1} \right|.$$

For $z, \omega \in \mathbb{D}$, we have

(2.14)
$$f^*(z,\omega) = \frac{f(z) - f(\omega)}{z - \omega} \cdot \frac{1 - \overline{\omega}z}{1 - \overline{f(\omega)}f(z)}$$

We also record the following important identity, valid for $u, v \in \mathbb{C}$. We have

(2.15)
$$S_{u,v} := (1 - |u|^2)(1 - |v|^2) = |1 - \overline{u}v|^2 - |u - v|^2.$$

Now, let $\omega_1, \omega_2, \omega_3 \in \mathbb{D}$, with $\omega_i \neq \omega_j$ $(i \neq j)$, and consider

$$T = \begin{pmatrix} \omega_1 & \alpha_1 & \beta \\ 0 & \omega_2 & \alpha_2 \\ 0 & 0 & \omega_3 \end{pmatrix},$$

with

$$\alpha_i = \sqrt{1 - |\omega_i|^2} \sqrt{(1 - |\omega_{i+1}|^2)}, \quad i = 1, 2,$$

and

$$\beta = \frac{-\overline{\omega_2}\alpha_1\alpha_2}{1-|\omega_2|^2} = -\overline{\omega_2}\sqrt{1-|\omega_1|^2}\sqrt{1-|\omega_3|^2}.$$

By Theorem 2.3, T is a contraction. Assume first that $f \in \mathcal{A}(\overline{\mathbb{D}})$ and $||f||_{\infty} \leq 1$. Then the matrix representation of f(T) can be expressed in terms of first order and second order divided differences as follows:

$$f(T) = \begin{pmatrix} f(\omega_1) & \alpha_1[f(\omega_1), f(\omega_2)] & \beta[f(\omega_1), f(\omega_3)] + \alpha_1 \alpha_2[f(\omega_1), f(\omega_2), f(\omega_3)] \\ 0 & f(\omega_2) & \alpha_2[f(\omega_2), f(\omega_3)] \\ 0 & 0 & f(\omega_3) \end{pmatrix}.$$

This can be verified directly by some direct computations for monomials and polynomials. The same formula extends to functions in the disk algebra $\mathcal{A}(\overline{\mathbb{D}})$. Assume that $f(\omega_i) \neq 0$ $f(\omega_j)$ whenever $i \neq j$ (otherwise, there is nothing to prove). As T is a contraction and $||f||_{\infty} \leq 1$, by von Neumann's inequality the operator f(T) is also a contraction.

Introducing the notation

$$\begin{split} \widetilde{\alpha_i} &= \alpha_i [f(\omega_i), f(\omega_{i+1})], \qquad i = 1, 2, \\ \widetilde{\beta} &= \beta [f(\omega_1), f(\omega_3)] + \alpha_1 \alpha_2 [f(\omega_1), f(\omega_2), f(\omega_3)], \end{split}$$

by Theorem 2.3, we have :

$$\left|\widetilde{\beta}\left(1-|f(\omega_2)|^2\right)+\widetilde{\alpha_1}\widetilde{\alpha_2}\overline{f(\omega_2)}\right|^2 \leq \left[S_{f(\omega_1),f(\omega_2)}-|\widetilde{\alpha_1}|^2\right] \times \left[S_{f(\omega_2),f(\omega_3)}-|\widetilde{\alpha_2}|^2\right].$$

If we multiply each side of this inequality by $|\omega_1 - \omega_3|^2$, we get

(2.16)
$$S_{\omega_1,\omega_3} \left| \left(1 - |f(\omega_2)|^2 \right) A + B \right|^2 \le |\omega_1 - \omega_3|^2 C_1 C_3,$$

where

$$\begin{split} A &:= -\overline{\omega_2} \left(f(\omega_1) - f(\omega_3) \right) + \left(1 - |\omega_2|^2 \right) \left(\frac{f(\omega_1) - f(\omega_2)}{\omega_1 - \omega_2} - \frac{f(\omega_2) - f(\omega_3)}{\omega_2 - \omega_3} \right), \\ B &:= \overline{f(\omega_2)} (1 - |\omega_2|^2) (\omega_1 - \omega_3) \cdot \frac{(f(\omega_1) - f(\omega_2))(f(\omega_2) - f(\omega_3))}{(\omega_1 - \omega_2)(\omega_2 - \omega_3)}, \\ C_i &:= S_{f(\omega_i), f(\omega_2)} - S_{\omega_i, \omega_2} \left| \frac{f(\omega_i) - f(\omega_2)}{\omega_i - \omega_2} \right|^2, \qquad i = 1, 3. \end{split}$$

We want to prove that (2.16) is equivalent with (2.13). The calculations are somewhat laborious; the key idea is to use (2.14) to make hyperbolic divided differences appear each time we see an expression of the form $f(z) - f(\omega)$. We provide additional details to assist the reader.

First of all, we have :

$$\begin{split} C_{i} &= S_{f(\omega_{i}),f(\omega_{2})} - S_{\omega_{i},\omega_{2}} \left| f^{*}(\omega_{i},\omega_{2}) \right|^{2} \times \left| \frac{1 - \overline{f(\omega_{2})}f(\omega_{i})}{1 - \overline{\omega_{2}}\omega_{i}} \right|^{2} \\ &= \left| 1 - \overline{f(\omega_{2})}f(\omega_{i}) \right|^{2} - \left| f^{*}(\omega_{i},\omega_{2}) \right|^{2} \times \left| \omega_{i} - \omega_{2} \right|^{2} \times \left| \frac{1 - \overline{f(\omega_{2})}f(\omega_{i})}{1 - \overline{\omega_{2}}\omega_{i}} \right|^{2} \\ &- \left| f^{*}(\omega_{i},\omega_{2}) \right|^{2} \times \left| 1 - \overline{f(\omega_{2})}f(\omega_{i}) \right|^{2} + \left| f^{*}(\omega_{i},\omega_{2}) \right|^{2} \times \left| \omega_{i} - \omega_{2} \right|^{2} \times \left| \frac{1 - \overline{f(\omega_{2})}f(\omega_{i})}{1 - \overline{\omega_{2}}\omega_{i}} \right|^{2} \\ &= \left| 1 - \overline{f(\omega_{2})}f(\omega_{i}) \right|^{2} \left(1 - \left| f^{*}(\omega_{i},\omega_{2}) \right|^{2} \right). \end{split}$$

Thus, we have

$$C_1C_3 = \left|1 - \overline{f(\omega_2)}f(\omega_1)\right|^2 \times \left|1 - \overline{f(\omega_2)}f(\omega_3)\right|^2 \times S_{f^*(\omega_1,\omega_2),f^*(\omega_3,\omega_2)}$$

10

Now, let us deal with the first member of the inequality. We have

$$A = f^*(\omega_1, \omega_2) \left(1 - \overline{f(\omega_2)} f(\omega_1) \right) - f^*(\omega_3, \omega_2) \left(1 - \overline{f(\omega_2)} f(\omega_3) \right)$$

= $\left(f^*(\omega_1, \omega_2) - f^*(\omega_3, \omega_2) \right) \left(1 - \overline{f(\omega_2)} f(\omega_1) \right) \left(1 - \overline{f(\omega_2)} f(\omega_3) \right) + \overline{f(\omega_2)} D,$

where $D := f^*(\omega_1, \omega_2) f(\omega_3) \left(1 - \overline{f(\omega_2)} f(\omega_1)\right) - f^*(\omega_3, \omega_2) f(\omega_1) \left(1 - \overline{f(\omega_2)} f(\omega_3)\right).$

This term can be written as follows, where we make appear the differences $f(\omega_3) - f(\omega_2)$ and $f(\omega_1) - f(\omega_2)$:

$$D = f^*(\omega_1, \omega_2) \left(f(\omega_3) - f(\omega_2) \right) \left(1 - \overline{f(\omega_2)} f(\omega_1) \right) - f^*(\omega_3, \omega_2) \left(f(\omega_1) - f(\omega_2) \right) \left(1 - \overline{f(\omega_2)} f(\omega_3) \right) + f(\omega_2) f^*(\omega_1, \omega_2) \left(1 - \overline{f(\omega_2)} f(\omega_1) \right) - f(\omega_2) f^*(\omega_3, \omega_2) \left(1 - \overline{f(\omega_2)} f(\omega_3) \right).$$

We obtain

$$D = f^{*}(\omega_{1}, \omega_{2})f^{*}(\omega_{3}, \omega_{2}) \left(\frac{1 - \overline{f(\omega_{2})}f(\omega_{3})}{1 - \overline{\omega_{2}}\omega_{3}}\right) (\omega_{3} - \omega_{2}) \left(1 - \overline{f(\omega_{2})}f(\omega_{1})\right) - f^{*}(\omega_{3}, \omega_{2})f^{*}(\omega_{1}, \omega_{2}) \left(\frac{1 - \overline{f(\omega_{2})}f(\omega_{1})}{1 - \overline{\omega_{2}}\omega_{1}}\right) (\omega_{1} - \omega_{2}) \left(1 - \overline{f(\omega_{2})}f(\omega_{3})\right) + f(\omega_{2})A = -(1 - |\omega_{2}|^{2})(\omega_{1} - \omega_{3}) \cdot \frac{(f(\omega_{1}) - f(\omega_{2}))(f(\omega_{2}) - f(\omega_{3}))}{(\omega_{1} - \omega_{2})(\omega_{2} - \omega_{3})} + f(\omega_{2})A.$$

Hence, we get

$$A = \left(f^*(\omega_1, \omega_2) - f^*(\omega_3, \omega_2)\right) \left(1 - \overline{f(\omega_2)}f(\omega_1)\right) \left(1 - \overline{f(\omega_2)}f(\omega_3)\right) - B + |f(\omega_2)|^2 A d\omega_2$$

Therefore

$$(1 - |f(\omega_2)|^2)A + B = (f^*(\omega_1, \omega_2) - f^*(\omega_3, \omega_2))\left(1 - \overline{f(\omega_2)}f(\omega_1)\right)\left(1 - \overline{f(\omega_2)}f(\omega_3)\right).$$

Combining all of these elements, the inequality represented by (2.16) transforms into

$$S_{\omega_1,\omega_3} |f^*(\omega_1,\omega_2) - f^*(\omega_3,\omega_2)|^2 \le |\omega_1 - \omega_3|^2 \times S_{f^*(\omega_1,\omega_2),f^*(\omega_3,\omega_2)},$$

which is equivalent with (2.13).

Beardon-Minda's inequality is thus proved for $f \in \mathcal{A}(\overline{\mathbb{D}})$. Now, for $f \in \mathcal{H}(\mathbb{D})$, we have $f_r : z \mapsto f(rz) \in \mathcal{A}(\overline{\mathbb{D}})$, for every $r \in]0, 1[$. Based on the preceding information, it can be concluded that Beardon-Minda's inequality is satisfied by the functions f_r , for all $r \in]0, 1[$, so it is also by f, by letting $r \to 1^-$.

The calculations in this proof can be somewhat simplified by assuming that $f(\omega_2) = 0$ and composing with a Möbius transformation at the end. However, this approach leads to a loss of symmetry in the formulas.

C. BADEA & A. RENARD

2.C. Connecting with model spaces theory. — In [9], it is further proved that if f does not represent an automorphism of the unit disk, equality holds in Theorem 2.5 if and only if f is a Blaschke product of degree no greater than 2. This inference can also be derived through operator theory considerations.

To achieve this, we must introduce certain concepts from model space theory. For a comprehensive introduction to these notions and more details, we direct the reader to [20] and [28].

Let $H^{\infty}(\mathbb{D})$ be the set of all holomorphic functions that are bounded on \mathbb{D} , and let $H^{2}(\mathbb{D})$ be the Hardy-Hilbert space of \mathbb{D} , which is the space of all holomorphic functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\sup_{0 < r < 1} \int_{\mathbb{T}} |f(\zeta)|^2 \mathrm{d}m(\zeta) < \infty$$

or, equivalently, such that

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty \quad \text{if } f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Let $S : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be the unilateral shift, defined by S(f)(z) = zf(z). For $f \in H^{\infty}(\mathbb{D})$, Fatou's theorem (see *e.g.* [20, theorem 1.10]) states that f has radial boundary values $f(\zeta)$, for almost every $\zeta \in \mathbb{T}$. A function $u \in H^{\infty}(\mathbb{D})$ is said to be inner if $|u(\zeta)| = 1$ almost everywhere on \mathbb{T} .

If u is an inner function, the corresponding model space \mathcal{K}_u is defined to be

$$\mathcal{K}_u := \left(u H^2(\mathbb{D}) \right)^{\perp} = \left\{ f \in H^2(\mathbb{D}) : \langle f, uh \rangle = 0, \, \forall \, h \in H^2(\mathbb{D}) \right\}.$$

We define the associated *compressed shift* by $S_u := P_u S|_{\mathcal{K}_u}$, where P_u is the orthogonal projection from $H^2(\mathbb{D})$ onto \mathcal{K}_u .

Now, let Θ be a finite Blaschke product with pairwise distinct zeros $\omega_1, \ldots, \omega_n \in \mathbb{D}$ and let $b_{\omega_k}(z) = \frac{z - \omega_k}{1 - \overline{\omega_k} z}$ denote a single Blaschke factor. Let $(\phi_1(z), \ldots, \phi_n(z))$ denote the Takenaka–Malmquist–Walsh orthonormal basis ([20, 28]) of \mathcal{K}_{Θ} , *i.e.*

$$\phi_1(z) = \frac{\sqrt{1 - |\omega_1|^2}}{1 - \overline{\omega_1} z} \quad \text{and} \quad \phi_k(z) = \left(\prod_{j=1}^{k-1} b_{\omega_j}\right) \frac{\sqrt{1 - |\omega_k|^2}}{1 - \overline{\omega_k} z} \quad k = 2, \dots, n.$$

Writing S_{Θ} with respect to the Takenaka–Malmquist basis gives the matrix representation M_{Θ} with entries

$$[M_{\Theta}]_{i,j} = \begin{cases} \omega_j & \text{if } i=j\\ \prod_{k=i+1}^{j-1} (-\overline{\omega_k}) \sqrt{1-|\omega_i|^2} \sqrt{1-|\omega_j|^2} & \text{if } ij. \end{cases}$$

It seems that the first appearance of this remarkable matrix was in [45]; see also [33, 34, 28, 19, 43]. In particular, for n = 2 and n = 3, we obtain the following matrices

12

$$\begin{split} T_2 &:= \begin{pmatrix} \omega_1 & \sqrt{1 - |\omega_1|^2} \sqrt{1 - |\omega_2|^2} \\ 0 & \omega_2 \end{pmatrix}, \\ T_3 &:= \begin{pmatrix} \omega_1 & \sqrt{1 - |\omega_1|^2} \sqrt{(1 - |\omega_2|^2} & -\overline{\omega_2} \sqrt{(1 - |\omega_1|^2} \sqrt{(1 - |\omega_3|^2)} \\ 0 & \omega_2 & \sqrt{1 - |\omega_2|^2} \sqrt{(1 - |\omega_3|^2)} \\ 0 & 0 & \omega_3 \end{pmatrix}, \end{split}$$

which have been used to obtain the Schwarz-Pick and Beardon-Minda inequalities.

We now show how to obtain the equality case in the Beardon-Minda inequality using the matrix T_3 . It follows from our proof of Theorem 2.5 that f satisfies (2.12) with equality if and only if $||f(T_3)|| = 1 = ||f||_{\infty}$. The fact that f is a finite Blaschke product of degree ≤ 2 has been proved by several authors, sometimes in relation to Crouzeix's conjecture. This can be generalized to n points. We refer to the discussion in [11, Theorem 3.1]. An explicit description of the matrix which diagonalize M_{Θ} is also given in [11].

We plan to return to the general Beardon-Minda type inequality $||f(M_{\Theta})|| \leq 1$ in a future paper.

2.D. A Beardon-Minda type lemma for derivatives. — We now investigate the case where $\omega_1 = \omega_2 = \omega_3 =: \omega$. For a holomorphic function f we use the notation

$$\Gamma(z,f) = \frac{(1-|z|^2)|f'(z)|}{1-|f(z)|^2}$$

The Schwarz-Pick inequality for derivatives (1.2) can then be expressed as $|\Gamma(z, f)| \leq 1$.

We give now an operator theoretical proof of the following result, proved by Yamashita in [44, Theorem 2].

Theorem 2.6. — Let
$$f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$$
 and let $\Gamma(z, f) = \frac{(1-|z|^2)|f'(z)|}{1-|f(z)|^2}$. Then, for every $\omega \in \mathbb{D}$,
(2.17) $\left|\frac{\partial\Gamma(\omega, f)}{\partial\omega}\right| \leq \frac{1-|\Gamma(\omega, f)|^2}{1-|\omega|^2}$.

Moreover, equality holds if and only if f is a Blaschke product of degree ≤ 2 .

Proof. — Let $\omega \in \mathbb{D}$, and let $T = \begin{pmatrix} \omega & \alpha & \beta \\ 0 & \omega & \alpha \\ 0 & 0 & \omega \end{pmatrix} \in \mathcal{M}_3(\mathbb{C})$, with $\alpha = 1 - |\omega|^2$ and $\beta = 0$

 $-\overline{\omega}(1-|\omega|^2)$. By Theorem 2.3, T is a contraction. Moreover, we can easily check that for f in the disk algebra we have

$$f(T) = \begin{pmatrix} f(\omega) & \alpha f'(\omega) & \frac{1}{2}\alpha^2 f''(\omega) + \beta f'(\omega) \\ 0 & f(\omega) & \alpha f'(\omega) \\ 0 & 0 & f(\omega) \end{pmatrix}$$

In this representation, the divided differences have been replaced in this limit case by first and second-order derivatives. By von Neumann's inequality, f(T) is a contraction. Using Theorem 2.3 we obtain:

$$\left| \left(\frac{1}{2} \alpha^2 f''(\omega) + \beta f'(\omega) \right) \left(1 - |f(\omega)|^2 \right) + \alpha^2 f'(\omega)^2 \overline{f(\omega)} \right| \le \left(1 - |f(\omega)|^2 \right)^2 - \left| \alpha f'(\omega) \right|^2,$$

which is equivalent with (2.17). A proof of the equality case can be obtained using model spaces, as discussed in the preceding subsection.

The inequality (2.17) can be rephrased in terms of *Peschl's invariant derivatives*. Let $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$, let $\omega \in \mathbb{D}$, and consider the mapping

(2.18)
$$g: z \in \mathbb{D} \mapsto \frac{f\left(\frac{z+\omega}{1+\overline{\omega}z}\right) - f(\omega)}{1 - \overline{f(\omega)}f\left(\frac{z+\omega}{1+\overline{\omega}z}\right)} \in \mathbb{C}.$$

Then g is analytic on \mathbb{D} and g(0) = 0. We have $g(z) = \sum_{n=1}^{\infty} \frac{D_n f(\omega)}{n!} z^n$, with $D_n f(z_0) := g^{(n)}(0)$. The quantities $D_n f(\omega)$ are called *Peschl's invariant derivatives* (see e.g. [24]).

The first two values of Peschl's invariant derivatives are explicitly computed as:

$$D_1 f(\omega) = \frac{(1 - |\omega|^2) f'(\omega)}{1 - |f(\omega)|^2},$$

$$D_2 f(\omega) = \frac{(1 - |\omega|^2)^2}{1 - |f(\omega)|^2} \left[f''(\omega) - \frac{2\overline{\omega} f'(\omega)}{1 - |\omega|^2} + \frac{2\overline{f(\omega)} f'(\omega)^2}{1 - |f(\omega)|^2} \right]$$

With these notations, the Schwarz-Pick inequality for derivatives (1.2) can be restated as $|D_1 f(\omega)| \leq 1$, while (2.17) can be written as $|D_2 f(\omega)| \leq 2(1 - |D_1 f(\omega)|^2)$.

We refer to [12, Proposition 3.4] for a different proof of (2.17) and to Section 4.B for a generalization to the polydisk.

3. Operator versions of Beardon-Minda's inequality

We move now to operator versions of the Schwarz-Pick and Beardon-Minda inequalities. The first operator generalization for the Schwarz-Pick inequality has been proved by Ky Fan in [18]; the following discussion has been inspired by the recent paper [22].

We recall the following theorem concerning the Sylvester equation AX - XB = Y, which has been studied *e.g.* in [10, 37].

Theorem 3.1 (Rosenblum,[10]). — Let H, K be two Hilbert spaces. Let $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}(K)$ be two operators with $\sigma(A) \cap \sigma(B) = \emptyset$. Then, for every $Y \in \mathcal{B}(H, K)$, the Sylvester equation AX - XB = Y has a unique solution X. Moreover, if Γ is a union of closed contours in the plane with total winding numbers 1 around $\sigma(A)$ and 0 around $\sigma(B)$, the solution can be expressed as

$$X = \frac{1}{2i\pi} \int_{\Gamma} (A - \xi)^{-1} Y (B - \xi)^{-1} d\xi.$$

 $\mathbf{14}$

3.A. An operator version of the Schwarz-Pick inequality. — The following result is a counterpart of [22, Theorem 3.5]. When specialized to scalars, it reduces to the Schwarz-Pick inequality for two distinct points.

Theorem 3.2. — Let H_1, H_2 be two Hilbert spaces. Consider three contractions $W_1 \in \mathcal{B}(H_1), W_2 \in \mathcal{B}(H_2)$ and $V \in \mathcal{B}(H_2, H_1)$. Assume that $\sigma(W_1) \cap \sigma(W_2) = \emptyset$, and that $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$ is holomorphic on an open neighborhood of $\sigma(W_1) \cup \sigma(W_2)$. We denote by $X = X_{W_1, W_2, V}$ the unique solution of Sylvester's equation

(3.1)
$$W_1 X - X W_2 = D_{W_1^*} V D_{W_2}.$$

Then, there exists a contraction $Y \in \mathcal{B}(H_2, H_1)$ such that

$$f(W_1)X - Xf(W_2) = D_{f(W_1)^*}YD_{f(W_2)}.$$

Proof. — Let $T = \begin{bmatrix} W_1 & D_{W_1^*} V D_{W_2} \\ 0 & W_2 \end{bmatrix}$. Denote $C = D_{W_1^*} V D_{W_2}$. By Parrott's theorem, T is a contraction. Moreover, using (3.1), we have

$$T = \begin{bmatrix} W_1 & C \\ 0 & W_2 \end{bmatrix} = \begin{bmatrix} \mathrm{Id} & -X \\ 0 & \mathrm{Id} \end{bmatrix} \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \begin{bmatrix} \mathrm{Id} & X \\ 0 & \mathrm{Id} \end{bmatrix}.$$

Notice that $\sigma(T) \subset \sigma(W_1) \cup \sigma(W_2)$. Indeed, for $\lambda \in \mathbb{C}$, we have

$$T - \lambda \mathrm{Id} = \begin{bmatrix} \mathrm{Id} & 0\\ 0 & W_2 - \lambda \mathrm{Id} \end{bmatrix} \begin{bmatrix} \mathrm{Id} & D_{W_1^*} V D_{W_2}\\ 0 & \mathrm{Id} \end{bmatrix} \begin{bmatrix} W_1 - \lambda \mathrm{Id} & 0\\ 0 & \mathrm{Id} \end{bmatrix}.$$

Therefore, if $\lambda \notin \sigma(W_1) \cup \sigma(W_2)$, then all factors in the previous decomposition are invertible and thus $\lambda \notin \sigma(T)$. So it makes sense to speak about f(T) and to write

$$f(T) = \begin{bmatrix} \mathrm{Id} & -X \\ 0 & \mathrm{Id} \end{bmatrix} \begin{bmatrix} f(W_1) & 0 \\ 0 & f(W_2) \end{bmatrix} \begin{bmatrix} \mathrm{Id} & X \\ 0 & \mathrm{Id} \end{bmatrix} = \begin{bmatrix} f(W_1) & f(W_1)X - Xf(W_2) \\ 0 & f(W_2) \end{bmatrix}$$

As $||f||_{\infty} \leq 1$, we have $||f(T)|| \leq 1$ by von Neumann's inequality. Thus, by Parrott's theorem, there exists a contraction $Y \in \mathcal{B}(H_2, H_1)$ such that $f(W_1)X - Xf(W_2) = D_{f(W_1)^*}YD_{f(W_2)}$.

3.B. An operator version of the Beardon-Minda inequality. — Utilizing the analogue proof framework as employed in Theorem 3.2, we can deduce the following outcome for 3×3 operator matrices.

Theorem 3.3. — Let H_1, H_2, H_3 be three Hilbert spaces. Consider three contractions $W_1 \in \mathcal{B}(H_1), W_2 \in \mathcal{B}(H_2)$ and $W_3 \in \mathcal{B}(H_3)$. Let $V_1 \in \mathcal{B}(H_2, H_1), V_2 \in \mathcal{B}(H_3, H_2)$, and $V_3 \in \mathcal{B}(H_3, H_1)$ be contractions. Assume that $||W_2|| < 1$ and that $\sigma(W_i) \cap \sigma(W_j) = \emptyset$, for all $1 \leq i < j \leq 3$. Suppose that $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$ is holomorphic on an open neighborhood of

 $\sigma(W_1) \cup \sigma(W_2) \cup \sigma(W_3)$. Let X_1, X_2, X_3 be respectively the unique solution of Sylvester's equations

(3.2) $W_1 X_1 - X_1 W_2 = D_{W_1^*} V_1 D_{W_2},$

$$(3.3) W_2 X_2 - X_2 W_3 = D_{W_2^*} V_2 D_{W_3} and$$

(3.4) $W_1X_3 - X_3W_3 = B - W_3X_1X_2 + X_1W_2X_2,$

where

$$B = \left[D_{W_1^*} (\mathrm{Id} - V_1 V_1^*) D_{W_1^*} \right]^{1/2} V_3 \left[D_{W_3} (\mathrm{Id} - V_2^* V_2) D_{W_3} \right]^{1/2} - D_{W_1^*} V_1 W_2^* V_2 D_{W_3}.$$

Then, there exist three contractions $Y_1 \in \mathcal{B}(H_2, H_1), Y_2 \in \mathcal{B}(H_3, H_2), Y_3 \in \mathcal{B}(H_3, H_1)$ such that :

$$\begin{array}{l} (3.5) \\ (3.6) \\ (3.6) \\ (3.7) \end{array} \begin{cases} f(W_1)X_1 - X_1f(W_2) = D_{f(W_1)^*}Y_1D_{f(W_2)}, \\ f(W_2)X_2 - X_2f(W_3) = D_{f(W_2)^*}Y_2D_{f(W_3)}, \\ f(W_1)X_3 - X_3f(W_3) = X_1f(W_2)X_2 - X_1X_2f(W_3) \\ + \left[D_{f(W_1)^*}(\operatorname{Id} - Y_1Y_1^*)D_{f(W_1)^*}\right]^{1/2}Y_3 \left[D_{f(W_3)}(\operatorname{Id} - Y_2^*Y_2)D_{f(W_3)}\right]^{1/2} \\ (3.7) \end{array}$$

Proof. — Denote $A_1 = D_{W_1^*}V_1D_{W_2}$ and $A_2 = D_{W_2^*}V_2D_{W_3}$. Then, according to Theorem 2.2, the operator

$$T = \begin{bmatrix} W_1 & A_1 & B \\ 0 & W_2 & A_2 \\ 0 & 0 & W_3 \end{bmatrix}$$

is a contraction. Notice also that $X_1 \in \mathcal{B}(H_2, H_1)$, $X_2 \in \mathcal{B}(H_3, H_2)$ and $X_3 \in \mathcal{B}(H_3, H_1)$ are respectively the unique solutions of Sylvester's equations $W_1X_1 - X_1W_2 = A_1$, $W_2X_2 - X_2W_3 = A_2$ and $W_1X_3 - X_3W_3 = B - W_3X_1X_2 + X_1W_2X_2$.

In analogy with some computations in the Heisenberg group, we can write

$$T = \begin{bmatrix} W_1 & A_1 & B \\ 0 & W_2 & A_2 \\ 0 & 0 & W_3 \end{bmatrix} = \begin{bmatrix} \mathrm{Id} & -X_1 & X_1 X_2 - X_3 \\ 0 & \mathrm{Id} & -X_2 \\ 0 & 0 & \mathrm{Id} \end{bmatrix} \begin{bmatrix} W_1 & 0 & 0 \\ 0 & W_2 & 0 \\ 0 & 0 & W_3 \end{bmatrix} \begin{bmatrix} \mathrm{Id} & X_1 & X_3 \\ 0 & \mathrm{Id} & X_2 \\ 0 & 0 & \mathrm{Id} \end{bmatrix}.$$

This diagonalization allows one to write the 3×3 operator matrix of f(T), which is a contraction by von Neumann's inequality:

$$f(T) = \begin{bmatrix} \mathrm{Id} & -X_1 & X_1 X_2 - X_3 \\ 0 & \mathrm{Id} & -X_2 \\ 0 & 0 & \mathrm{Id} \end{bmatrix} \begin{bmatrix} f(W_1) & 0 & 0 \\ 0 & f(W_2) & 0 \\ 0 & 0 & f(W_3) \end{bmatrix} \begin{bmatrix} \mathrm{Id} & X_1 & X_3 \\ 0 & \mathrm{Id} & X_2 \\ 0 & 0 & \mathrm{Id} \end{bmatrix}$$

Thus the matrix of f(T) is given by

$$\begin{bmatrix} f(W_1) & f(W_1)X_1 - X_1f(W_2) & f(W_1)X_3 - X_1f(W_2)X_2 + (X_1X_2 - X_3)f(W_3) \\ 0 & f(W_2) & f(W_2)X_2 - X_2f(W_3) \\ 0 & 0 & f(W_3) \end{bmatrix}.$$

We apply again Theorem 2.2.

16

In the scalar case, the condition (3.7) is equivalent with the Beardon-Minda inequality.

4. Schwarz-Pick inequalities for the polydisk

Let $n \in \mathbb{N}^*$. For $\underline{\omega} = (\omega_1, \cdots, \omega_n) \in \mathbb{C}^n$, we denote $\|\underline{\omega}\| = \sup_{1 \le i \le n} |\omega_i|$ the sup norm.

4.A. Using von Neumann inequality for tuples of two by two matrices. — It is a fascinating observation in operator theory that an analogue of the von Neumann inequality holds for the bidisk (Ando's theorem), but does not extend to the polydisk \mathbb{D}^n for $n \geq 3$. However, as proved by Drury [16] and Knese [26], there is an analogue for tuples of 2×2 and 3×3 commuting matrices.

Lemma 4.1 (Drury and Knese; see [16], [26]). — Let $T_1, ..., T_n$ be mutually commuting 2×2 or 3×3 contractions, and let $p \in \mathbb{C}[X_1, \dots, X_n]$. Then, we have

$$||p(T_1,...,T_n)|| \le ||p||_{\infty} := \sup\{|p(z_1,\cdots,z_n)| : \underline{z} \in \mathbb{D}^n\}.$$

This leads to operator theoretical proofs of the following known ([39, lemma 7.5.6]) Schwarz-Pick inequalities for the polydisk.

Theorem 4.2. (a) Let $f \in \mathcal{H}(\mathbb{D}^n, \mathbb{D})$ and let $\underline{a} = (a_1, \ldots, a_n), \underline{b} = (b_1, \ldots, b_n) \in \mathbb{D}^n$. Then

(4.1)
$$\left|\frac{f(a_1,\cdots,a_n)-f(b_1,\cdots,b_n)}{1-\overline{f(a_1,\cdots,a_n)}f(b_1,\cdots,b_n)}\right| \le \max_{1\le i\le n} \left|\frac{a_i-b_i}{1-\overline{a_i}b_i}\right|$$

(b) Let $f \in \mathcal{H}(\mathbb{D}^n, \mathbb{D})$ and let $\underline{a} = (a_2, \ldots, a_n) \in \mathbb{D}^n$. Then,

(4.2)
$$\sum_{i=1}^{n} (1 - |a_i|^2) \left| \frac{\partial f(\underline{a})}{\partial z_i} \right| \le 1 - |f(\underline{a})|^2.$$

Proof. — (a) We first observe that the result is obvious whenever $\underline{a} = \underline{b}$ or $f(\underline{a}) = f(\underline{b})$. Therefore, in the following, we assume $\underline{a} \neq \underline{b}$ and $f(\underline{a}) \neq f(\underline{b})$.

For $1 \leq i \leq n$, let

$$T_i = \begin{pmatrix} a_i & d(a_i - b_i) \\ 0 & b_i \end{pmatrix},$$

with

$$d = \min_{1 \le i \le n} \sqrt{\frac{(1 - |a_i|^2)(1 - |b_i|^2)}{|a_i - b_i|^2}}.$$

Here, whenever $a_i = b_i$, we make the convention that $\sqrt{\frac{(1-|a_i|^2)(1-|b_i|^2)}{|a_i-b_i|^2}} = +\infty$. As we assume that $\underline{a} \neq \underline{b}$, this cannot happen for all the indices *i*.

It can be easily verified that the matrices T_i are mutually commuting and that $||T_i|| \leq 1$. By induction it can be shown that for all $i \in [1, n]$, for all $k_i \in \mathbb{N}$,

$$T_i^{k_i} = \begin{pmatrix} a_i^{k_i} & d\left(a_i^{k_i} - b_i^{k_i}\right) \\ 0 & b_i^{k_i} \end{pmatrix}.$$

Let $p \in \mathbb{C}[X_1, \ldots, X_n]$ be a polynomial with $||p||_{\infty} < 1$. We have

$$p(T_1, \cdots, T_n) = \begin{pmatrix} p(\underline{a}) & d(p(\underline{a}) - p(\underline{b})) \\ 0 & p(\underline{b}) \end{pmatrix}.$$

Drury's result imply that $||p(T_1, \ldots, T_n)|| \leq 1$. As in the one variable case, a computation gives the Schwarz-Pick inequality (4.1) for p. By an approximation argument, (4.1) holds also for functions in the polydisk algebra. Now, if $f \in \mathcal{H}(\mathbb{D}^n, \mathbb{D})$, consider the family of functions $(f_r)_{0 < r < 1}$ defined by $f_r(z_1, \cdots, z_n) = f(rz_1, \ldots, rz_n)$. For all $r \in]0, 1[, f_r]$ is in the polydisk algebra and, thus, f_r satisfies (4.1). Then, let $r \to 1^-$ to conclude the proof.

(b) The proof follows the same method as that of Theorem 4.2. Let $\underline{a} = (a_2, \ldots, a_n) \in \mathbb{D}^n$ and let $p \in \mathbb{C}[X_1, \ldots, X_n]$ be a polynomial with $\|p\|_{\infty} < 1$. For $1 \leq k \leq n$, let $T_k = \begin{pmatrix} a_k & \gamma_k \\ 0 & a_k \end{pmatrix}$, where $\gamma_k = e^{i\theta_k}(1 - |a_k|^2)$, for some $\theta_k \in [0, 2\pi[$ to be chosen later on. For all $k \in [\![1, n]\!]$, $\|T_k\| \leq 1$, and, for all $k, l \in [\![1, n]\!]$, $T_k T_l = T_l T_k$. We have

$$p(T_1, \cdots, T_n) = \begin{pmatrix} p(\underline{a}) & \sum_{k=1}^n \gamma_k \frac{\partial p(\underline{a})}{\partial z_k} \\ 0 & p(\underline{a}) \end{pmatrix}.$$

Again, by Lemma 4.1, we get $||p(T_1, \ldots, T_n)|| \le 1$. Therefore

(4.3)
$$\left|\sum_{k=1}^{n} \gamma_k \frac{\partial p(\underline{a})}{\partial z_k}\right| \le 1 - |p(\underline{a})|^2.$$

Now, let $t_k = \frac{\partial p(a_1, a_2)}{\partial z_k}$, $0 \le k \le n$. We write $t_k = |t_k| e^{i\operatorname{Arg}(t_k)}$ and we set $\theta_k = -\operatorname{Arg}(t_k)$. With this choice we obtain $\gamma_k t_k = (1 - |a_k|^2) |t_k|$. Replacing in (4.3) we get

$$\sum_{i=1}^{n} (1-|a_i|^2) \left| \frac{\partial p(\underline{a})}{\partial z_i} \right| \le 1-|p(\underline{a})|^2.$$

We conclude by using an approximation argument.

Remark 4.3. — The study of the case of equality in the Schwarz-Pick inequalities for the polydisk is an interesting problem. Knese [25] studied the equality case in (4.2) using operator-theoretical methods (transfer functions) and described which functions play the role of automorphisms of the disk in this context-they turn out to be rational inner functions in the Schur-Agler class of the polydisk with an *added* symmetry constraint.

4.B. Peschl's invariant derivatives in several variables. — The inequalities from Section 2.D can be extended to analytic functions of several variables.

Let $n \in \mathbb{N}^*$, let $f \in \mathcal{H}(\mathbb{D}^n, \mathbb{D})$, and fix a vector $\underline{\omega} = (\omega_1, \cdots, \omega_n)$ in \mathbb{D}^n . Similarly as in the one variable case, we define

$$g: \underline{z} = (z_1, \cdots, z_n) \in \mathbb{D}^n \mapsto \frac{f\left(\frac{z_1 + \omega_1}{1 + \overline{\omega_1} z_1}, \cdots, \frac{z_n + \omega_n}{1 + \overline{\omega_n} z_n}\right) - f(\omega_1, \cdots, \omega_n)}{1 - \overline{f(\omega_1, \cdots, \omega_n)} f\left(\frac{z_1 + \omega_1}{1 + \overline{\omega_1} z_1}, \cdots, \frac{z_n + \omega_n}{1 + \overline{\omega_n} z_n}\right)} \in \mathbb{C}$$

 $\mathbf{18}$

and then write

$$g(z_1, \cdots, z_n) = \sum_{j_1, \cdots, j_n = 0}^{\infty} \frac{\partial^{j_1 + \cdots + j_n} g(0, \cdots, 0)}{\partial^{j_1} z_1 \cdots \partial^{j_n} z_n} z_1^{j_1} \cdots z_n^{j_n} = \sum_{j_1, \cdots, j_n = 0}^{\infty} a_{j_1, \cdots, j_n} z_1^{j_1} \cdots z_n^{j_n}.$$

For $k \in [\![1,n]\!]$, let $D_k f(\underline{w}) = \partial^k g(0,\ldots,0) = \sum_{j_1+\cdots+j_n=k} a_{j_1,\cdots,j_n}$. A straightforward computation gives :

$$D_1 f(\underline{\omega}) = \sum_{j=1}^n \frac{1 - |\omega_j|^2}{1 - |f(\underline{\omega})|^2} \cdot \frac{\partial f}{\partial z_j}(\underline{\omega}),$$

$$D_2 f(\underline{\omega}) = \sum_{j=1}^n \frac{\partial^2 g(0, \dots, 0)}{\partial^2 z_j} + 2 \sum_{1 \le j < k \le n} \frac{\partial^2 f(0, \dots, 0)}{\partial z_j \partial z_k}$$

$$= \sum_{j=1}^n \frac{(1 - |\omega_j|^2)^2}{1 - |f(\underline{\omega})|^2} \left(\frac{\partial^2 f(\underline{w})}{\partial^2 z_j} + \frac{2\overline{f(\underline{w})}}{1 - |f(\underline{w})|^2} - \frac{2\overline{\omega_j}}{1 - |\omega_j|^2} \cdot \frac{\partial f(\underline{\omega})}{\partial z_j} \right)$$

$$+ 2 \sum_{1 \le j < k \le n} \frac{(1 - |z_j|^2)(1 - |z_k|^2)}{1 - |f(\underline{\omega})|^2} \left(\frac{\partial f(\underline{\omega})}{\partial z_j \partial z_k} + \frac{2\overline{f(\underline{\omega})}}{1 - |f(\underline{\omega})|^2} \cdot \frac{\partial f(\underline{\omega})}{\partial z_j} \cdot \frac{\partial f(\underline{\omega})}{\partial z_j} \right).$$

With the same method of proof as before, we can arrive at the following result.

Theorem 4.4. — For $n \in \mathbb{N}^*$ let $\underline{w} = (\omega_1, \ldots, \omega_n) \in \mathbb{D}^n$ and consider $f \in \mathcal{H}(\mathbb{D}^n, \mathbb{D})$. Then, we have:

(4.4)
$$|D_2 f(\underline{\omega})| \le 2(1 - |D_1 f(\underline{\omega})|^2)$$

Proof. — For $1 \le k \le n$, let

$$T_k = \begin{pmatrix} \omega_k & \alpha_k & \beta_k \\ 0 & \omega_k & \alpha_k \\ 0 & 0 & \omega_k \end{pmatrix} \in \mathcal{M}_3(\mathbb{C}),$$

with $\alpha_k = 1 - |\omega_k|^2$ and $\beta_k = -\overline{\omega_k}(1 - |\omega_k|^2)$. By Theorem 2.3, T_k is a contraction, for all $k \in [\![1, n]\!]$. Moreover, for all $1 \le k, j \le n, T_j T_k = T_k T_j$. Therefore, by Knese's result, $p(T_1, \ldots, T_n)$ is a contraction, for every $p \in \mathbb{C}[X_1, \ldots, X_n]$ with $\|p\|_{\infty} < 1$. Moreover, it is easy to check that

$$p(T_1,\ldots,T_n) = \begin{pmatrix} p(\underline{\omega}) & \gamma_1 & \gamma_2 \\ 0 & p(\underline{\omega}) & \gamma_1 \\ 0 & 0 & p(\underline{\omega}) \end{pmatrix},$$

with

$$\gamma_1 = \sum_{j=1}^n \alpha_j \frac{\partial p(\underline{\omega})}{\partial z_j},$$

$$\gamma_2 = \frac{1}{2} \sum_{j=1}^n \alpha_j^2 \frac{\partial^2 p(\underline{w})}{\partial^2 z_j} + \sum_{1 \le j < k \le n} \alpha_j \alpha_k \frac{\partial^2 p(\underline{w})}{\partial z_j \partial z_k} + \sum_{j=1}^n \beta_j \frac{\partial p(\underline{w})}{\partial z_j}.$$

By Theorem 2.3, we obtain :

(4.5)
$$\left|\gamma_2\left(1-|p(\underline{\omega})|^2\right)+\gamma_1^2\overline{p(\underline{\omega})}\right| \le (1-|p(\underline{\omega})|^2)^2-|\gamma_1|^2,$$

C. BADEA & A. RENARD

which is equivalent with (4.4) for polynomials. The inequality extends to all functions $f \in \mathcal{H}(\mathbb{D}^n, \mathbb{D}).$

4.C. Distinguished varieties and Schwarz-Pick inequalities. — In the bidisk case, the refined version of Ando's inequality by Agler and McCarthy [3] results in corresponding enhancements of Schwarz-Pick type inequalities.

We start by recalling the notion of distinguished variety introduced in [3]. A distinguished variety is a set of the form $V \cap \overline{\mathbb{D}}^2$, where V is an algebraic set in \mathbb{C}^2 (so there is a polynomial $q \in \mathbb{C}[z, w]$ such that $V = \{(z, w) \in \mathbb{D}^2 : q(z, w) = 0\}$) with the property that $\overline{V} \cap \partial(\mathbb{D}^2) = \overline{V} \cap \mathbb{T}^2$.

Therefore a distinguished variety is the trace on \mathbb{D}^2 of a one-dimensional complex algebraic variety V in \mathbb{C}^2 such that V intersects \mathbb{D}^2 and exits the bidisk through its distinguished boundary, \mathbb{T}^2 , without intersecting any other part of its topological boundary. A distinguished variety has ([3]) the following determinantal representation

(4.6)
$$V \cap \mathbb{D}^2 = \left\{ (z, w) \in \mathbb{D}^2 : \det (\Psi(z) - w \operatorname{Id}) = 0 \right\}$$

for some matrix-valued rational function Ψ on the unit disc that is unitary on the unit circle.

Agler and McCarthy proved in [3] that for any pair of commuting contractive matrices (T_1, T_2) without unimodular eigenvalues, there is a distinguished variety $V \cap \mathbb{D}^2$ such that the von-Neumann inequality holds on $V \cap \mathbb{D}^2$ for any polynomial p in $\mathbb{C}[z_1, z_2]$, i.e.

(4.7)
$$||p(T_1, T_2)|| \le \sup_{(z_1, z_2) \in V \cap \mathbb{D}^2} |p(z_1, z_2)|.$$

Theorem 4.5. — (a) Let (a_1, a_2) and (b_1, b_2) be two points in the bidisk \mathbb{D}^2 . Then there is a distinguished variety $V \cap \mathbb{D}^2$ such that the Schwarz-Pick inequality

(4.8)
$$\left| \frac{f(a_1, a_2) - f(b_1, b_2)}{1 - \overline{f(a_1, a_2)} f(b_1, b_2)} \right| \le \max\left\{ \left| \frac{a_1 - b_1}{1 - \overline{a_1} b_1} \right|, \left| \frac{a_2 - b_2}{1 - \overline{a_2} b_2} \right| \right\}$$

holds for any function f which is holomorphic on the bidisk \mathbb{D}^2 and continuous on $\overline{\mathbb{D}}^2$ with

$$\sup_{z_1, z_2) \in V \cap \mathbb{D}^2} |f(z_1, z_2)| \le 1.$$

(b) Let (a_1, a_2) and (b_1, b_2) be two points in the bidisk \mathbb{D}^2 . Then there is a distinguished variety $V \cap \mathbb{D}^2$ such that the Schwarz-Pick inequality (4.8) holds for any function f which is holomorphic in the bidisk \mathbb{D}^2 and for which there is a sequence of positive real number (r_n) convergent to 1 with $r_n < 1$ such that

$$\sup_{n \ge 1, (z_1, z_2) \in V \cap \mathbb{D}^2} |f(r_n z_1, r_n z_2)| \le 1.$$

Proof. — Consider the matrices

$$T_1 = \begin{pmatrix} a_1 & d(a_1 - b_1) \\ 0 & b_1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} a_2 & d(a_2 - b_2) \\ 0 & b_2 \end{pmatrix},$$

 $\mathbf{20}$

with

$$d = \min\left\{\sqrt{\frac{(1-|a_1|^2)(1-|b_1|^2)}{|a_1-b_1|^2}}, \sqrt{\frac{(1-|a_2|^2)(1-|b_2|^2)}{|a_2-b_2|^2}}\right\},$$

with the same conventions as in the proof of Theorem 4.2, (a). Following [3], we can also assume that T_1 and T_2 are jointly diagonalizable (this is the first case in the proof of [3, Theorem 3.1]). It follows from the result proved in [3] (see also [4, p.211] for details and unexplained terminology) that there is a distinguished variety V such that $T = (T_1, T_2)$ can be extended to a pair of commuting unitaries $U = (U_1, U_2)$ with spectrum $\sigma(U) = \overline{V} \cap \partial(\mathbb{D}^2) = \overline{V} \cap \mathbb{T}^2$. As f is in the bidisk algebra, f(T) and f(U) are well-defined and f(T) is a restriction of f(U) to $\mathbb{C}^2 \times \mathbb{C}^2$. We obtain, as in [3], that

(4.9)
$$||f(T_1, T_2)|| \le \sup_{(z_1, z_2) \in V \cap \mathbb{D}^2} |f(z_1, f_2)|.$$

Therefore $f(T_1, T_2)$ is a contraction and the proof of Theorem 4.2, (a), implies that inequality (4.8) holds true. The second part, (b), follows from (a) applied to the functions $f(r_n z_1, r_n z_2)$ and then making $n \to \infty$.

The following result follows in a similar manner from the Agler and McCarthy result and the proof of Theorem 4.4.

Theorem 4.6. — Let $\underline{w} = (\omega_1, \omega_2) \in \mathbb{D}^2$. Then there exists a distinguished variety $V \cap \mathbb{D}^2$ such that

 $(4.10) |D_2 f(\underline{\omega})| \le 2(1 - |D_1 f(\underline{\omega})|^2)$

for every $f \in \mathcal{A}(\overline{\mathbb{D}}^2)$ with

$$\sup_{(z_1, z_2) \in V \cap \mathbb{D}^2} |f(z_1, z_2)| \le 1.$$

Some Nevanlinna–Pick interpolation problems on distinguished varieties in the bidisk have been studied in [23].

5. Higher order Schwarz-Pick inequalities

Let $f \in \mathcal{H}(\mathbb{D}, \mathbb{D})$ be an analytic function of \mathbb{D} into itself with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. It has been proved by F.W. Wiener that for each $k \geq 1$ we have

(5.1)
$$|a_k| \le 1 - |a_0|^2.$$

We refer for instance to [32] for an operator theoretical proof of this inequality and for applications to Bohr's phenomenon. For k = 1, the inequality (5.1) gives $|f'(0)| \leq 1 - |f(0)|^2$. Applying this inequality to $F(z) = f((\omega + z)/1 + \overline{\omega}z)$, for a fixed $\omega \in \mathbb{D}$, we obtain the Schwarz-Pick inequality (1.2). For an arbitrary k, a similar reasoning has been used by Ruscheveyeh [40] to obtain the following sharp higher-order inequality for an analytic function $f \in \mathcal{H}(\mathbb{D}, \mathbb{D}), z \in \mathbb{D}$ and $k \geq 1$:

(5.2)
$$|f^{(k)}(z)| \le \frac{k!(1-|f(z)|^2)}{(1-|z|)^k(1+|z|)}.$$

We prove in this section some results related to the estimate (5.1).

Theorem 5.1. — Let $f : \mathbb{D} \mapsto \overline{\mathbb{D}}$ be an analytic function. Assume that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (z \in \mathbb{D})$$

Then, for each $n \ge 1$ and each $k \ge 1$ we have

(5.3)
$$\left|a_{n+k}(1-|a_0^2|)+a_na_k\overline{a}_0\right|^2 \leq \left[(1-|a_0|^2)^2-|a_n|^2\right] \cdot \left[(1-|a_0|^2)^2-|a_k|^2\right].$$

Proof. — As $||f||_{\infty} \leq 1$, the multiplication operator M_f given by $M_f(g) = fg$ acts contractively on the Hardy space $H^2(\mathbb{D})$. Recall that $\{z^n : n \geq 0\}$ is an orthonormal basis of $H^2(\mathbb{D})$. The compression $T = P_K M_f | K$ of M_f to the 3-dimensional Euclidean space $K = \text{span}(1, z^n, z^{n+k})$ is also a contraction. The matrix of T is given by

$$T = \begin{pmatrix} a_0 & a_n & a_{n+k} \\ 0 & a_0 & a_k \\ 0 & 0 & a_0 \end{pmatrix}$$

Then (5.3) is a consequence of Theorem 2.3.

When $a_0 = 0$ we obtain the following consequence.

Corollary 5.2. — Let f be a analytic function of \mathbb{D} into $\overline{\mathbb{D}}$ with f(0) = 0 and $f(z) = \sum_{n=1}^{\infty} a_n z^n$ for $z \in \mathbb{D}$. Then

(5.4)
$$|a_{n+k}| \le \sqrt{1 - |a_n|^2} \cdot \sqrt{1 - |a_k|^2}.$$

For n = k = 1, and $a_0 = 0$, we obtain the inequality $|a_2| \le 1 - |a_1|^2$. Applying this inequality to (2.18) we obtain Yamashita's inequality $|D_2 f(\omega)| \le 2(1 - |D_1 f(\omega)|^2)$.

The following consequence is an improvement of Wiener's inequality (5.1).

Corollary 5.3. — Let f be a analytic function of \mathbb{D} into $\overline{\mathbb{D}}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in \mathbb{D}$. Then

$$1 - |a_0|^2 - |a_n| \ge \frac{|a_{2n}(1 - |a_0^2|) + a_n^2 \overline{a}_0|}{2(1 - |a_0|^2)}.$$

Proof. — Applying (5.3) for k = n we obtain

$$\left|a_{2n}(1-|a_0^2|)+a_n^2\overline{a}_0\right| \le \left[(1-|a_0|^2)^2-|a_n|^2\right].$$

Therefore

$$1 - |a_0|^2 - |a_n| \ge \frac{|a_{2n}(1 - |a_0^2|) + a_n^2 \overline{a}_0|}{1 - |a_0|^2 + |a_n|} \ge \frac{|a_{2n}(1 - |a_0^2|) + a_n^2 \overline{a}_0|}{2(1 - |a_0|^2)}.$$

The proof is complete.

SCHWARZ-PICK TYPE INEQUALITIES

6. Appendix

The objective of this Appendix is to revisit Parrott's theorem as stated in Theorem 2.1. We adopt the approach presented by Davis, Kahan, and Weinberger in [14], making some modifications, particularly regarding the selection of solutions with minimal norms.

This appendix is primarily intended for readers interested in Schwarz-Pick inequalities who may not have an extensive background in operator theory.

We start by recalling the following lemma.

Lemma 6.1 (Douglas [15]). — Let L, M_1, M_2 be Hilbert spaces. Suppose that $A \in \mathcal{B}(L, M_1)$, $B \in \mathcal{B}(L, M_2)$ and $c \geq 0$. Then, $B^*B \leq c^2 A^*A$ if and only if there exists $C \in \mathcal{B}(M_1, M_2)$ such that

(6.1)
$$\begin{cases} B = CA \\ \|C\| \le c \end{cases}$$

Moreover, if it is the case, there exists a unique operator C_0 satisfying (6.1) such that $Im(A)^{\perp} \subset Ker(C_0)$. The operator C_0 satisfies

$$||C_0||^2 = \inf\{||C||^2 : C \text{ satisfies } (6.1)\} = \inf\{\mu \ge 0 : B^*B \le \mu A^*A\}$$

and will thus be referred as the minimal solution of the equation B = CA.

From this lemma, we deduce the following result about column matrices. Recall that the defect operator of B is given by $D_B = (\mathrm{Id} - B^*B)^{1/2}$.

Proposition 6.2. — Let H, K_1, K_2 be Hilbert spaces. Suppose that $A \in \mathcal{B}(H, K_1)$ and $B \in \mathcal{B}(H, K_2)$ are contractions. Then, $\begin{bmatrix} A \\ B \end{bmatrix} : H_1 \to K_1 \oplus K_2$ is a contraction if and only if there exists a contraction $V \in \mathcal{B}(H, K_1)$ such that $A = VD_B$.

Moreover, if it is the case, there exists a unique contraction V_0 such that $A = V_0 D_B$ and $Im(D_B)^{\perp} \subset Ker(V_0)$. Then V_0 satisfies

$$||V_0|| = \inf\{ ||V|| : A = VD_B \}$$

and will thus be referred as the minimal solution of the equation $A = VD_B$.

Proof. — The column matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ is a contraction if and only if $A^*A \leq \mathrm{Id} - B^*B = D_B^*D_B$. Using Lemma 6.1, we obtain $A = VD_B$ with $||V|| \leq 1$.

Corollary 6.3. — Let H, K_1, K_2, A, B be as in Proposition 6.2, and let $U \in \mathcal{B}(H)$ be an arbitrary (but fixed) isometry. Then, $\begin{bmatrix} A \\ B \end{bmatrix} : H_1 \to K_1 \oplus K_2$ is a contraction if and only if there exists a contraction $V \in \mathcal{B}(H, K_1)$ such that $A = VUD_B$.

Moreover, if it is the case, there exists a unique contraction V_0 such that $A = V_0 U D_B$ and $Im(UD_B)^{\perp} \subset Ker(V_0)$. Then the operator V_0 satisfies

$$||V_0|| = \inf\{ ||V|| : A = VUD_B \}$$

and will thus be referred as the minimal solution of the equation $A = VUD_B$.

Proof. — It is enough to prove the sufficiency part. By Proposition 6.2, if $\begin{bmatrix} A \\ B \end{bmatrix}$ is a contraction, there exists a contraction $W \in \mathcal{B}(H, K_1)$ such that $A = WD_B$. Moreover, W can be chosen such that W = 0 on $\operatorname{Im}(D_B)^{\perp}$ (and in this case, the minimal solution W_0 is unique).

Now, let $V = WU^*$. As U is an isometry, it is easy to see that V is a contraction and that $VUD_B = WD_B = A$. Moreover, V = 0 on $\operatorname{Im}(UD_B)^{\perp}$. Indeed, let $x \in \operatorname{Im}(UD_B)^{\perp}$. For all $x' \in H$, $\langle x, UD_Bx' \rangle = 0$, which can be rewritten $\langle U^*x, D_Bx' \rangle = 0$. Thus, for $x \in \operatorname{Im}(UD_B)^{\perp}$, $U^*x \in \operatorname{Im}(D_B)^{\perp}$ and, then, $Vx = WU^*x = 0$ (by minimality of W). It is moreover easy to see that there exists a unique V such that $A = VUD_B$ and V = 0 on $\operatorname{Im}(D_B)^{\perp}$.

Corollary 6.4. — Let H_1, H_2, K be Hilbert spaces. Suppose that $A \in \mathcal{B}(H_1, K)$ and $B \in \mathcal{B}(H_2, K)$ are contractions. Then, $\begin{bmatrix} A & B \end{bmatrix} : H_1 \oplus H_2 \to K$ is a contraction if and only if there exists a contraction $V \in \mathcal{B}(H_1, K)$ such that $A = D_{B^*}V$.

Moreover, if it is the case, there exists a unique contraction V_0 such that $A = D_{B^*}V_0$ and $Im(D_{B^*})^{\perp} \subset Ker(V_0^*)$. We have

$$||V_0|| = \inf\{ ||V|| : A = D_{B^*}V \}.$$

The operator V_0 will be referred as the minimal solution of the equation $A = D_{B^*}V$.

Proof. — Observe that $\begin{bmatrix} A & B \end{bmatrix}$ is a contraction if and only if $\begin{bmatrix} A & B \end{bmatrix}^* = \begin{bmatrix} A^* \\ B^* \end{bmatrix}$ is a contraction, and then apply Proposition 6.2.

Proof of Theorem 2.1. — First of all, the existence of two contractions $Z \in \mathcal{B}(H_1, K_1)$ and $Y \in \mathcal{B}(H_2, K_2)$ such that $D = D_{C^*}Y$ and $A = ZD_C$ comes from Proposition 6.2 and Corollary 6.4, as $\begin{bmatrix} A \\ C \end{bmatrix}$ and $\begin{bmatrix} C & D \end{bmatrix}$ are contractions. We denote the minimal solutions by Y_0 , and respectively Z_0 .

 $\mathbf{24}$

Set $\mathbf{A} = \begin{bmatrix} A & B \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} C & D \end{bmatrix}$, so that we have $T = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}$, with $||\mathbf{B}|| \le 1$. Now, using that $TD_T = D_{T^*}T$, we have

$$\begin{aligned} \mathbf{Id}_{H_1 \oplus H_2} - \mathbf{B}^* \mathbf{B} &= \begin{bmatrix} \mathrm{Id}_{H_1} - C^* C & -C^* D \\ -D^* C & \mathrm{Id}_{H_2} - D^* D \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Id}_{H_1} - C^* C & -C^* D_{C^*} Y_0 \\ -Y_0^* D_{C^*} C & \mathrm{Id}_{H_2} - Y_0^* D_{C^*} D_{C^*} Y_0 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Id}_{H_1} - C^* C & -D_C C^* Y_0 \\ -Y_0^* C D_C & \mathrm{Id}_{H_2} - Y_0^* Y_0 + Y_0^* C C^* Y_0 \end{bmatrix} \\ &= \mathbf{S}^* \mathbf{S}, \end{aligned}$$

where $\mathbf{S} = \begin{bmatrix} D_C & -C^*Y_0 \\ 0 & D_{Y_0} \end{bmatrix}$.

For every $w \in H_1 \oplus H_2$, we have

$$\langle (\mathbf{Id}_{H_1 \oplus H_2} - \mathbf{B}^* \mathbf{B}) w, w \rangle = \langle \mathbf{S}^* \mathbf{S} w, w \rangle,$$

which is equivalent with $\|\mathbf{S}w\| = \|D_{\mathbf{B}}w\|$. Thus, there is an isometry $\mathbf{U} \in \mathcal{B}(H_1 \oplus H_2)$ such that $\mathbf{S} = \mathbf{U}D_{\mathbf{B}}$. Indeed, let $U : \operatorname{Im}(D_{\mathbf{B}}) \to H_1 \oplus H_2$, $D_{\mathbf{B}}x \mapsto \mathbf{S}x$. We extend \mathbf{U} by continuity to $\overline{\operatorname{Im}(D_{\mathbf{B}})}$, and we set $\mathbf{U} = \operatorname{Id}$ on $\operatorname{Im}(D_{\mathbf{B}})^{\perp}$.

Suppose that T is a contraction. Then, by Corollary 6.3, there exists a contraction

$$\mathbf{V} = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \in \mathcal{B}(H_1 \oplus H_2, K_1)$$

such that

(6.2)
$$\int \mathbf{A} = \mathbf{V} \mathbf{U} D_{\mathbf{B}},$$

(6.3)
$$\mathbf{V} = 0 \text{ on } \operatorname{Im}(\mathbf{S})^{\perp} .$$

By Corollary 6.4, there exists a contraction $W \in \mathcal{B}(H_2, K_1)$ such that $\mathbf{V} = \begin{bmatrix} V_1 & D_{V_1^*}W \end{bmatrix}$. The operator W can be chosen such that $\operatorname{Im}(D_{V_1^*}) \subset \operatorname{Ker}(W^*)$ (in that case, the minimal solution W_0 is unique).

Then, (6.2) is equivalent with

$$\begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} V_1 & D_{V_1^*}W \end{bmatrix} \begin{bmatrix} D_C & -C^*Y_0 \\ 0 & D_{Y_0} \end{bmatrix}$$

$$= \begin{bmatrix} V_1D_C & -V_1C^*Y_0 + D_{V_1^*}WD_{Y_0} \end{bmatrix}$$

In particular, we have $A = V_1 D_C$. We now show that $V_1 = Z_0$.

Fact 1. — $\operatorname{Im}(D_C)^{\perp} \oplus \{0\} \subset \operatorname{Im}(\mathbf{S})^{\perp}$.

Proof. — Let $v \in \operatorname{Im}(D_C)^{\perp} = \operatorname{Ker}(D_C)$. In order to prove that $\begin{bmatrix} v \\ 0 \end{bmatrix} \in \operatorname{Ker}(\mathbf{S}^*) = \operatorname{Im}(\mathbf{S})^{\perp}$, notice that we have

$$\mathbf{S}^* \begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -Y_0^* C v \end{bmatrix}.$$

As we know that $Y_0^* = 0$ on $\operatorname{Im}(D_{C^*})^{\perp} = \operatorname{Ker}(D_{C^*})$, it is enough to show $Cv \in \operatorname{Ker}(D_{C^*})$. Using again the identity $CD_C = D_{C^*}C$, we have

$$\|D_{C^*}Cv\|^2 = \langle D_{C^*}Cv, D_{C^*}Cv \rangle = \langle Cv, D_{C^*}^2Cv \rangle = \langle Cv, CD_C^2v \rangle = 0,$$

which completes the proof of the Fact 1.

Continuing the proof of Theorem 2.1, we can deduce from (6.3) that $V_1 = 0$ on $\text{Im}(D_C)^{\perp}$ and, thus, $V_1 = Z_0$. Finally, (6.4) is equivalent with

$$B = -Z_0 C^* Y_0 + D_{Z_0^*} W D_{Y_0}$$

Conversely, if there exists a contraction $W \in \mathcal{B}(H_2, K_1)$ such that $B = D_{Z_0^*} V D_{Y_0} - Z_0 C^* Y_0$, then it is easy to check that $\mathbf{A} = \mathbf{V}' \mathbf{S} = \mathbf{V}' \mathbf{U} \mathbf{D}_{\mathbf{B}}$, with $\mathbf{V}' = \begin{bmatrix} Z & D_{Z^*} W \end{bmatrix}$. As \mathbf{V}' is a contraction (Corollary 6.4), this implies that T is a contraction (Corollary 6.3). \Box

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 $\mathbf{28}$

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