# Ergodicity for 2D Navier-Stokes equations with a degenerate pure jump noise 

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#### Abstract

In this paper, we establish the ergodicity for stochastic 2D NavierStokes equations driven by a highly degenerate pure jump Lévy noise. The noise could appear in as few as four directions. This gives an affirmative anwser to a longstanding problem. The case of Gaussian noise was treated in Hairer and Mattingly [Ann. of Math., 164(3):993-1032, 2006]. To obtain the uniqueness of invariant measure, we use Malliavin calculus and anticipating stochastic calculus to establish the equi-continuity of the semigroup, the so-called e-property, and prove some weak irreducibility of the solution process.


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## 1 Introduction and Main results

### 1.1 Introduction

In the theory of turbulence, the study of the equations of fluid mechanics driven by degenerate noise forcing, that is, the driving noise does not act directly on all the determining modes of the flow, is ubiquitous; see e.g., [Eyi96, Nov65, Sta88,

[^0]VKF79]. And in the physics literature when discussing the behavior of stochastic fluid dynamics in the turbulent regime, the main assumptions usually made are ergodicity and statistical translational invariance of the stationary state. The uniqueness of an invariant measure and the ergodicity of the randomly forced dissipative partial differential equations(PDEs) driven by degenerate noise have been the problem of central concern for many years.

Because of the complexity and the difficulty of the problem, it is much less understood and there are only a few works on this topic. In this paper, we confine ourselves to stochastic 2D Navier-Stokes equations. In [HM06, HM11] the authors studied the 2D Navier-Stokes equations on the torus and the sphere and established the exponential mixing, provided that the random perturbation is white in time and contains several Fourier modes. In [Shi15], the exponential mixing was established for the 2D Navier-Stokes system perturbed by a spacetime localised smooth stochastic forcing. In the paper [KNS20] the authors proved a similar result in the situation when random forces are localised in the Fourier space and coloured in time. The problem of mixing for the NavierStokes system with a random perturbation acting through the boundary has been studied in [Shi21]. The authors in [KS12] proved the polynomial mixing of the 2D Navier-Stokes equations driven by a compound Poisson process. We remark that the volume of the intensity measure of compound Poisson process is finite.

So far, there are no results for the case when the random perturbation is pure jump Lévy noise of infinite activity, that is, the volume of the intensity measure of the Lévy noise is infinite. This is the subject of the present article. We point out that there are no results even for the non-degenerate case of pure jump Lévy noise of infinite activity, i.e., all determining modes of the unforced PDE are directly affected by the noise.

Now, let us give a brief introduction to the main result. The Navier-Stokes (NS) equations on the torus $\mathbb{T}^{2}=[-\pi, \pi]^{2}$ are given by

$$
\begin{equation*}
\partial_{t} u+(u \cdot \nabla) u=\nu \Delta u-\nabla p+\xi, \operatorname{div} u=0 \tag{1.1}
\end{equation*}
$$

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where $u=u(x, t) \in \mathbb{R}^{2}$ denotes the value of the velocity field at time $t$ and position $x=\left(x_{1}, x_{2}\right), p(x, t)$ denotes the pressure, $\nu>0$ is the viscosity and $\xi=\xi(x, t)$ is an external force field acting on the fluid. Since the velocity and vorticity formulations are equivalent in this setting, we choose to use the vorticity equation as this simplifies the exposition. For a divergence-free velocity field $u$, we define the vorticity $w$ by $w=\nabla \wedge u=\partial_{2} u_{1}-\partial_{1} u_{2}$. Note that $u$ can be recovered from $w$ and the condition $\operatorname{div} u=0$. With these notations we rewrite the NS system (1.1) in the vorticity formulation:

$$
\begin{equation*}
\partial_{t} w=\nu \Delta w+B(\mathcal{K} w, w)+\partial_{t} \eta,\left.\quad w\right|_{t=0}=w_{0} \tag{1.2}
\end{equation*}
$$

$\square$
where $\eta=\eta(x, t)$ is an external random force, $B(u, w)=-(u \cdot \nabla) w$, and $\mathcal{K}$ is the Biot-Savart operator which is defined in Fourier space by $(\mathcal{K} w)_{k}=-i w_{k} k^{\perp} /|k|^{2}$, where $k^{\perp}=\left(k_{1}, k_{2}\right)^{\perp}=\left(-k_{2}, k_{1}\right)$, and $w_{k}$ is the scalar product of $w$ with $\frac{1}{2 \pi} e^{i k \cdot x}$.

The Biot-Savart operator has the property that the divergence of $\mathcal{K} w$ vanishes and that $w=\nabla \wedge(\mathcal{K} w)$.

In this paper, we prove that there exists a unique statistically invariant state of the system (1.2). Roughly speaking, we establish the following result. For rigorous statement and general version of the result, please see Theorem 1.6 below.

Consider the system (1.2) with noise
$\eta(x, t)=b_{1} \sin \left(x_{1}\right) L_{1}(t)+b_{2} \cos \left(x_{1}\right) L_{2}(t)+b_{3} \sin \left(x_{1}+x_{2}\right) L_{3}(t)+b_{4} \cos \left(x_{1}+x_{2}\right) L_{4}(t)$,
where $b_{1}, \cdots, b_{4}$ are non-zero constants, $L_{t}=\left(L_{1}(t), \cdots, L_{4}(t)\right)$ is a 4-dimensional pure jump process with Lévy measure $\nu_{L}$ :

$$
\begin{equation*}
\nu_{L}(\mathrm{~d} z)=\int_{0}^{\infty}(2 \pi u)^{-2} e^{-\frac{|z|^{2}}{2 u}} \nu_{S}(\mathrm{~d} u) \mathrm{d} z \tag{1.3}
\end{equation*}
$$

here $\nu_{S}$ is a measure on $(0, \infty)$ satisfying

$$
\nu_{S}((0, \infty))=\infty \text { and } \int_{0}^{\infty}\left(e^{\zeta u}-1\right) \nu_{S}(\mathrm{~d} u)<\infty \text { for some } \zeta>0
$$

Then the main results imply that the Markov semigroup generated by the system (1.2) possesses a unique invariant measure $\mu^{*}$ on the space $H=\left\{w \in L^{2}\left(\mathbb{T}^{2}, \mathbb{R}\right)\right.$ : $\left.\int_{\mathbb{T}^{2}} w(x) \mathrm{d} x=0\right\}$.

There are now empirical data which shows that Lévy processes are more suitable to realistically represent external forces in statistical physics(c.f. [Nov65]), climatology(c.f. [IP06]) and mathematics of finance(c.f. [KT13]). Therefore, the mathematical analysis of stochastic partial differential equations driven by Lévy processes becomes very important. This motivates the study of this paper.

To prove the main result, we will use Malliavin calculus and anticipating stochastic calculus to establish an equi-continuity of the semigroup, the so-called e-property, and prove some weak irreducibility for the solution process of the system (1.2).

To deal with the setting of highly degenerate noises, we need some quantitative control of the Malliavin matrix, and it is inevitable to use the "future information". Hence, some anticipating stochastic analysis are necessary. However, Malliavin calculus associated with Poisson random measures is much less effective than that of the Wiener case. In this paper, we assume that the driving noise is a subordinated Brownian motion. Introducing a sort of time change, we borrow the nonadapted stochastic analysis associated with Brownian motion when dealing with the "future information".

Because of the strong intensity of the jumps and also the unbounded jumps, we need to introduce a new set of ideas and techniques to establish the uniqueness of invariant measures in comparison to the case of Gaussian noise (see [HM06] [HM11][FGRT15]). Now we highlight some of them.

- In HM06], the authors gave preliminary estimates for the solutions in [HM06, Lemma 4.10], which plays an essential role in controlling various terms during the proof of the asymptotically strong Feller. The proof of [HM06, Lemma 4.10] strongly relies on the Girsanov transformation and exponential martingale estimate of the Gaussian noise. However, the Girsanov transformation and exponential martingale inequality associated with Poisson random measures is expressed in terms of complicated exponential type nonlinear transformations, which seems very hard (if not impossible) to control; see [App09, Theorem 5.2.9]. In the setting of pure jump Lévy noise, to overcome the above difficulty, we design a sequence of stopping times $\sigma_{k}$ (see (2.7) and (2.8)), and build new preliminary estimates, see Lemma 2.2 in this paper which is totally different from the ones for the Gaussian case, i.e., [HM06, Lemma 4.10]. We point out that Lemma 2.2 seems not possible to be proved by the exponential martingale inequalities and the Itô formula for Poisson random measures.
With the help of the stopping times $\sigma_{k}$, we could identify the "bad part" of the sample space $\Omega$, denoted by $\{\omega \in \Omega: \Theta>M\}$; see (4.2) and (4.3) for the definition of the random variable $\Theta$; and the "bad part" could be controlled by the strong law of large numbers, which means that $\lim _{M \rightarrow \infty} \mathbb{P}(\{\omega \in \Omega: \Theta>M\})=0$. On the "good part" of the sample space, we could obtain something like the asymptotic strong Feller property; see (4.33). Combining the two parts, we obtain the $e$-property.
- Let $\mathcal{M}_{0, t}$ be the Malliavin matrix of $w_{t}$ and $\mathcal{S}_{\alpha, N_{N}}$ be some subspace of $H$ (For the definition of $\mathcal{M}_{0, t}$, see Section 4.2 in [HM06] or (2.23)-(2.24) below). To obtain the ergodicity via Malliavin calculus, one key ingredient is to show

$$
\begin{equation*}
\mathbb{P}\left(\inf _{\phi \in \mathcal{S}_{\alpha, N}}\left\langle\mathcal{M}_{0,1} \phi, \phi\right\rangle<\varepsilon\right) \leq C\left(\left\|w_{0}\right\|\right) r(\varepsilon), \forall \varepsilon \in(0,1) \text { and } w_{0} \in H \tag{1.4}
\end{equation*}
$$

where $\left\|w_{0}\right\|$ denotes the $L^{2}$ norm of $w_{0}, C$ is some function from $[0, \infty)$ to $[0, \infty)$ and $r$ is a function on $(0,1)$ with $\lim _{\varepsilon \rightarrow 0} r(\varepsilon)=0$. In the existing literatures $[\mathrm{HM} 06][\mathrm{HM} 11][$ FGRT15] etc., the properties of Gaussian polynomials(see, e.g., [HM11, Theorem 7.1]) play very essential roles for the estimate of the left side of (1.4). Similar arguments do not work in the case of pure jump processes. In this paper, using the fact that the jump times of pure jump noise with infinite activity are dense in any time interval $[a, b]$ with $0 \leq a<b$, we find a new way to get something like (1.4). First, we prove

$$
\begin{equation*}
\mathbb{P}\left(\inf _{\phi \in \mathcal{S}_{\alpha, N}}\left\langle\mathcal{M}_{0, \sigma_{1}} \phi, \phi\right\rangle=0\right)=0, \forall w_{0} \in H \tag{1.5}
\end{equation*}
$$

where $\sigma_{1}$ is a positive stopping time. Then, with the help of (1.5) and the dissipative property of Navier-Stokes system, we derive a weaker version of (1.4) which is sufficient for our purpose. In a word, our method of proving something like (1.4) is totally different from that of the Gaussian case. See Proposition 3.4 and Proposition 3.5 in Section 3 for more details.

Finally, we point out that there are not many results on the ergodicity of stochastic partial differential equations driven by pure jump Lévy noise. And we list them here for readers who are interested. For the case that the driving noise is non-degenerate, we refer to [PZ11, PXZ11, PSXZ12, Xu13, WXX17, WX18, DXZ14, DWX20, BHR16, FHR16, WYZZ22]. For the case that the driving noise is degenerate, we refer to [SXX19, WYZZ24, MR10].

### 1.2 Main results

We consider the system (1.2) in the Hilbert space:

$$
\begin{equation*}
H=\left\{w \in L^{2}\left(\mathbb{T}^{2}, \mathbb{R}\right): \int_{\mathbb{T}^{2}} w(x) \mathrm{d} x=0\right\} \tag{1.6}
\end{equation*}
$$

endowed with the $L^{2}$-scalar product $\langle\cdot, \cdot\rangle$ and the corresponding $L^{2}$-norm $\|\cdot\|$.
In order to describe the noise $\eta$, we introduce the following notation. Denote

$$
\mathbb{Z}_{+}^{2}=\left\{\left(k_{1}, k_{2}\right) \in \mathbb{Z}^{2}: k_{2}>0\right\} \cup\left\{\left(k_{1}, 0\right) \in \mathbb{Z}^{2}: k_{1}>0\right\}
$$

and $\mathbb{Z}_{-}^{2}=\left\{\left(k_{1}, k_{2}\right):-\left(k_{1}, k_{2}\right) \in \mathbb{Z}_{+}^{2}\right\}$. For any $k=\left(k_{1}, k_{2}\right) \in \mathbb{Z}_{*}^{2}:=\mathbb{Z}^{2} \backslash(0,0)$, set

$$
e_{k}=e_{k}(x)= \begin{cases}\sin \langle k, x\rangle & \text { if }\left(k_{1}, k_{2}\right) \in \mathbb{Z}_{+}^{2} \\ \cos \langle k, x\rangle & \text { if }\left(k_{1}, k_{2}\right) \in \mathbb{Z}_{-}^{2}\end{cases}
$$

We assume that $\eta$ is a white-in-time noise of the form

$$
\begin{equation*}
\eta(x, t)=\sum_{k \in \mathcal{Z}_{0}} b_{k} W_{S_{t}}^{k} e_{k}(x), \tag{1.7}
\end{equation*}
$$

where $\mathcal{Z}_{0} \subset \mathbb{Z}_{*}^{2}$ is a finite set, $b_{k}, k \in \mathcal{Z}_{0}$ are non-zero constants, $W_{S_{t}}=\left(W_{S_{t}}^{k}\right)_{k \in \mathcal{Z}_{0}}$ is a $\left|\mathcal{Z}_{0}\right|$-dimensional subordinated Brownian motion which will be specified below. For convenience, we always denote $\left|\mathcal{Z}_{0}\right|$ by $d$. Assume the canonical basis of $\mathbb{R}^{d}$ is $\left\{\theta_{k}\right\}_{k \in \mathcal{Z}_{0}}$ and the linear operator $Q: \mathbb{R}^{d} \rightarrow H$ is defined in the following way:

$$
Q z=\sum_{k \in \mathcal{Z}_{0}} b_{k} z_{k} e_{k}, \forall z=\sum_{k \in \mathcal{Z}_{0}} z_{k} \theta_{k} \in \mathbb{R}^{d}
$$

then, $\eta(t)=Q W_{S_{t}}$.
Now let us give the details for the subordinated Brownian motion $W_{S_{t}}$. Let $\left(\mathbb{W}, \mathbb{H}, \mathbb{P}^{\mu_{W}}\right)$ be the classical Wiener space, i.e., $\mathbb{W}$ is the space of all continuous functions from $\mathbb{R}^{+}$to $\mathbb{R}^{d}$ with vanishing values at starting point $0, \mathbb{H} \subseteq \mathbb{W}$ is the Cameron-Martin space consisting of all absolutely continuous functions with square integrable derivatives, $\mathbb{P}^{\mu_{\mathbb{W}}}$ is the Wiener measure so that the coordinate process $W_{t}(\mathrm{w}):=\mathrm{w}_{t}$ is a $d$-dimensional standard Brownian motion. Let $\mathbb{S}$ be the space of all càdlàg increasing functions $\ell$ from $\mathbb{R}^{+}$to $\mathbb{R}^{+}$with $\ell_{0}=0$. Suppose that $\mathbb{S}$ is endowed with the Skorohod metric and a probability measure $\mathbb{P}^{\mu_{s}}$ so
that the coordinate process $S_{t}(\ell):=\ell_{t}$ is a pure jump subordinator with Lévy measure $\nu_{S}$ satisfying

$$
\int_{0}^{\infty}(1 \wedge u) \nu_{S}(d u)<\infty
$$

Consider now the following product probability space $(\Omega, \mathcal{F}, \mathbb{P}):=(\mathbb{W} \times \mathbb{S}, \mathcal{B}(\mathbb{W}) \times$ $\left.\mathcal{B}(\mathbb{S}), \mathbb{P}^{\mu_{W}} \times \mathbb{P}^{\mu_{\mathbb{S}}}\right)$, and define for $\omega=(\mathrm{w}, \ell) \in \mathbb{W} \times \mathbb{S}, L_{t}(\omega):=\mathrm{w}_{\ell_{t}}$. Then, $\left(L_{t}=W_{S_{t}}\right)_{t \geq 0}$ is a $d$-dimensional pure jump Lévy process with Lévy measure $\nu_{L}$ given by

$$
\begin{equation*}
\nu_{L}(E)=\int_{0}^{\infty} \int_{E}(2 \pi u)^{-\frac{d}{2}} e^{-\frac{|z|^{2}}{2 u}} \mathrm{~d} z \nu_{S}(\mathrm{~d} u), \quad E \in \mathcal{B}\left(\mathbb{R}^{d}\right) \tag{1.8}
\end{equation*}
$$

To formulate the main result, let us recall that a set $\mathcal{Z}_{0} \subset \mathbb{Z}_{*}^{2}$ is a generator if any element of $\mathbb{Z}_{*}^{2}$ is a finite linear combination of elements of $\mathcal{Z}_{0}$ with integer coefficients. In what follows, we assume that the following two conditions are in place.

16-5 Condition 1.1. The set $\mathcal{Z}_{0} \subset \mathbb{Z}_{*}^{2}$ appeared in (1.7) is a finite, symmetric (i.e., $-\mathcal{Z}_{0}=\mathcal{Z}_{0}$ ) generator that contains at least two non-parallel vectors $m$ and $n$ such that $|m| \neq|n|$.

This is the condition under which the ergodicity of the NS system is established in [HM06, HM11] in the case of a white-in-time noise and in [KNS20] in the case of a bounded noise. The set

$$
\mathcal{Z}_{0}=\{(1,0),(-1,0),(1,1),(-1,-1)\} \subset \mathbb{Z}_{*}^{2}:=\mathbb{Z}^{2} \backslash\{(0,0)\}
$$

is an example satisfying this condition.
$14-2$ Condition 1.2. Assume that $\nu_{S}$ satisfies

$$
\int_{0}^{\infty}\left(e^{\zeta u}-1\right) \nu_{S}(\mathrm{~d} u)<\infty \text { for some } \zeta>0
$$

and

$$
\begin{equation*}
\nu_{S}((0, \infty))=\infty \tag{1.9}
\end{equation*}
$$

Remark 1.3. If $\nu_{S}(\mathrm{~d} u)=u^{-1-\frac{\alpha}{2}} I_{\{0<u \leq \aleph\}} \mathrm{d} u$ for some $\alpha \in[0,2)$ and $\aleph>0$, then condition 1.2 is satisfied. In this case, $\nu_{L}(\mathrm{~d} z)=\zeta(z) \mathrm{d} z$ and $\zeta(z)$ satisfies

$$
\begin{align*}
& \frac{C_{1}}{|z|^{d+\alpha}} \leq \zeta(z) \leq \frac{C_{2}}{|z|^{d+\alpha}}, \quad \forall|z| \leq 1 \\
& \zeta(z) \leq \frac{C_{3}}{|z|^{d+\alpha}} \exp \left\{-\frac{|z|^{2}}{4 \aleph}\right\}, \quad \forall|z| \geq 1 \tag{1.10}
\end{align*}
$$

where $C_{1}, C_{2}, C_{3}$ are positive constants depending on $\alpha, d$ and $\aleph$. Thus, the appearance of the small jumps of $L_{t}$ will behave like $\alpha$-stable processes and the appearance of big jumps will be very rare.

We denote by $P_{t}\left(w_{0}, \cdot\right)$ the transition probabilities of the solution of the stochastic Navier-Stokes equation (1.2), i.e,

$$
P_{t}\left(w_{0}, A\right)=\mathbb{P}\left(w(t) \in A \mid w(0)=w_{0}\right)
$$

for every Borel set $A \subseteq H$ and

$$
P_{t} f\left(w_{0}\right)=\int_{H} f(w) P_{t}\left(w_{0}, \mathrm{~d} w\right), \quad P_{t}^{*} \mu(A)=\int_{H} P_{t}\left(w_{0}, A\right) \mu\left(\mathrm{d} w_{0}\right)
$$

for every $f: H \rightarrow \mathbb{R}$ and probability measure $\mu$ on $H$.
Before we state the main theorem in this paper, we present two propositions which are the essential ingredients in the proof of the main result.

Proposition 1.4. Under the Condition 1.1 and Condition 1.2, the Markov semigroup $\left\{P_{t}\right\}_{t \geq 0}$ has the e-property, i.e., for any bounded and Lipschitz continuous function $f, w_{0} \in H$ and $\varepsilon>0$, there exists a $\delta>0$ such that

$$
\left|P_{t} f\left(w_{0}^{\prime}\right)-P_{t} f\left(w_{0}\right)\right|<\varepsilon, \forall t \geq 0 \text { and } w_{0}^{\prime} \text { with }\left\|w_{0}^{\prime}-w_{0}\right\|<\delta .
$$

The proof of this proposition is given in Section 4 based on Malliavin calculus, which constitutes a major part of the paper.

Since there are many constants appearing in the proof, we adopt the following convention. Without otherwise specified, the letters $C, C_{1}, C_{2}, \cdots$ are always used to denote unessential constants that may change from line to line and implicitly depend on the data of the system (1.2), i.e., $\nu,\left\{b_{k}\right\}_{k \in \mathcal{Z}_{0}}, \nu_{S}$ and $d=\left|\mathcal{Z}_{0}\right|$. Also, we usually do not explicitly indicate the dependencies on the parameters $\nu,\left\{b_{k}\right\}_{k \in \mathcal{Z}_{0}}, \nu_{S}$ and $d=\left|\mathcal{Z}_{0}\right|$ on every occasion. The proof of the proposition below is almost the same as that in [EM01, Lemma 3.1]; for the convenience of the readers, we give its short proof in Section 5 .

16-6 Proposition 1.5. (Weak Irreducibility) For any $\mathcal{C}, \gamma>0$, there exists a $T=$ $T(\mathcal{C}, \gamma)>0$ such that

$$
\inf _{\left\|w_{0}\right\| \leq \mathcal{C}} P_{T}\left(w_{0}, \mathcal{B}_{\gamma}\right)>0
$$

where $\mathcal{B}_{\gamma}=\{w \in H:\|w\| \leq \gamma\}$.
After we state Proposition 1.4 and Proposition 1.5, we have the following main result of the paper.

16-8 Theorem 1.6. Consider the 2D Navier-Stokes equation (1.2) with a degenerate pure jump noise (1.7). Under the Condition 1.1 and Condition 1.2, there exists a unique invariant measure $\mu^{*}$ for the system (1.2), i.e., $\mu^{*}$ is a unique probability measure on $H$ such that $P_{t}^{*} \mu^{*}=\mu^{*}$ for every $t \geq 0$.

Proof. We first prove the existence. By Lemma 2.1 below, it holds that

$$
\frac{\nu}{2} \mathbb{E} \int_{0}^{t}\left\|w_{s}\right\|_{1}^{2} \mathrm{~d} s \leq\left\|w_{0}\right\|^{2}+C t
$$

here the definition of $\|\cdot\|_{1}$ is introduced in (2.1). Following the arguments in the proof of [DX09, Theorem 5.1] and using Krylov-Bogoliubov criteria, we obtain the existence of invariant measure.

Now we prove the uniqueness. Assume that there are two distinct invariant probability measures $\mu_{1}$ and $\mu_{2}$ for $\left\{P_{t}\right\}_{t \geq 0}$. By Proposition 1.4 and $\{\mathrm{KSS} 12$, Theorem 1] (or [GL15, Proposition 1.10]), one has

$$
\begin{equation*}
\text { Supp } \mu_{1} \cap \operatorname{Supp} \mu_{2}=\emptyset \tag{1.11}
\end{equation*}
$$

On the other hand, by (2.3) in Lemma 2.1 below, for every invariant measure $\mu$, the following priori bound

$$
\int_{H}\|w\|^{2} \mu(\mathrm{~d} w) \leq C
$$

holds(See [EMS01, Lemma B.2]). Following the arguments in the proof of [HM06, Corollary 4.2] and using Proposition 1.5, for every invariant measure $\mu$, we have $0 \in \operatorname{Supp} \mu$. This contradicts (1.11). We complete the proof of uniqueness.

The rest of the paper is organised as follows. In Section 2, we provide some estimates for the solution $w_{t}$ and introduce the essential ingredients of the Malliavin calculus for the solution. Moreover, we give all the necessary estimates associated with the Malliavin matrix. Section 3 is devoted to the proof of the invertibility of the Malliavin matrix of the solution $w_{t}$ which plays a key role in the proof of Proposition 1.4. In Section 4, we give the proof of Proposition 1.4. The proof of Proposition 1.5 is given In Section 5. Some of the technical proofs are put in the Appendix.

## 2 Preliminaries

### 2.1 Notations

In this paper, we use the following notations. Let $H_{N}=\operatorname{span}\left\{e_{j}: j \in\right.$ $\mathbb{Z}_{*}^{2}$ and $\left.|j| \leq N\right\} . P_{N}$ denotes the orthogonal projections from $H$ onto $H_{N}$. Define $Q_{N} u:=u-P_{N} u, \forall u \in H$.
For $\alpha \in \mathbb{R}$ and a smooth function $w \in H$, we define the norm $\|w\|_{\alpha}$ by

$$
\begin{equation*}
\|w\|_{\alpha}^{2}=\sum_{k \in \mathbb{Z}_{*}^{2}}|k|^{2 \alpha} w_{k}^{2}, \tag{2.1}
\end{equation*}
$$

where $w_{k}$ denotes the Fourier mode with wavenumber $k$. When $\alpha=0$, as stated in Section 1.2, we also denote this norm $\|\cdot\|_{\alpha}$ by $\|\cdot\|$. For any $\left(s_{1}, s_{2}, s_{3}\right) \in$ $\mathbb{R}_{+}^{3}$ with $\sum_{i=1}^{3} s_{i} \geq 1$ and $\left(s_{1}, s_{2}, s_{3}\right) \neq(1,0,0),(0,1,0),(0,0,1)$, the following
relations will be used frequently in this paper(c.f. [CF88]):

$$
\begin{align*}
\langle B(u, v), w\rangle & =-\langle B(u, w), v\rangle, \quad \text { if } \nabla \cdot u=0 \\
|\langle B(u, v), w\rangle| & \leq C\|u\|_{s_{1}}\|v\|_{1+s_{2}}\|w\|_{s_{3}} \\
\|\mathcal{K} u\|_{\alpha} & =\|u\|_{\alpha-1}  \tag{2.2}\\
\|w\|_{1 / 2}^{2} & \leq\|w\|\|w\|_{1} .
\end{align*}
$$

$L^{\infty}(H)$ is the space of bounded Borel-measurable functions $\psi: H \rightarrow \mathbb{R}$ with the norm $\|\psi\|_{\infty}=\sup _{u \in H}|\psi(u)| . C_{b}(H)$ is the space of continuous functions. $C_{b}^{1}(H)$ is the space of functions $\psi \in C_{b}(H)$ that are continuously Fréchet differentiable with bounded derivatives. $\mathcal{L}(X, Y)$ is the space of bounded linear operators from Banach spaces $X$ into Banach space $Y$ endowed with the natural norm $\|\cdot\|_{\mathcal{L}(X, Y)}$. If there are no confusions, we always write the operator norm $\|\cdot\|_{\mathcal{L}(X, Y)}$ as $\|\cdot\|$.

Let $N_{L}(\mathrm{~d} t, \mathrm{~d} z)$ be the Poisson random measure associated with the Lévy process $L_{t}=W_{S_{t}}$, i.e.,

$$
N_{L}((0, t] \times U)=\sum_{s \leq t} I_{U}\left(L_{t}-L_{t-}\right), U \in \mathcal{B}\left(\mathbb{R}^{d} \backslash\{0\}\right)
$$

Let $\tilde{N}_{L}(\mathrm{~d} t, \mathrm{~d} z)$ denote the compensated Poisson random measure associated with $N_{L}(\mathrm{~d} t, \mathrm{~d} z)$, i.e.,

$$
\tilde{N}_{L}(\mathrm{~d} t, \mathrm{~d} z)=N_{L}((0, t] \times U)-\mathrm{d} t \nu_{L}(\mathrm{~d} z)
$$

Similar notation also apply to $N_{S}(\mathrm{~d} t, \mathrm{~d} z)$ and $\tilde{N}_{S}(\mathrm{~d} t, \mathrm{~d} z)$. As the measure $\nu_{L}(\mathrm{~d} z)$ is symmetric, the Lévy process $L_{t}$ admits the following representation:

$$
L_{t}=\int_{0}^{t} \int_{\mathbb{R}^{d} \backslash\{0\}} z \tilde{N}_{L}(\mathrm{~d} s, \mathrm{~d} z)
$$

Let $F=F(\mathrm{w}, \ell)$ be a random variable on the space $(\Omega, \mathcal{F}, \mathbb{P})$. We use $\mathbb{E}^{\mu_{\mathbb{W}}} F$ to denote the expectation of $F$ when we take the element $\ell$ as fixed, i.e,

$$
\mathbb{E}^{\mu_{\mathbb{W}}} F=\int_{\mathbb{W}} F(\mathrm{w}, \ell) \mathbb{P}^{\mu_{\mathbb{W}}}(\mathrm{dw})
$$

The notation $\mathbb{E}^{\mu_{s}} F$ has the similar meaning. We use $\mathbb{E} F$ to denote the expectation of $F$ under the measure $\mathbb{P}=\mathbb{P}^{\mu_{\mathbb{W}}} \times \mathbb{P}^{\mu_{\mathrm{s}}}$.
The filtration used in this paper is

$$
\mathcal{F}_{t}:=\sigma\left(W_{S_{s}}, S_{s}: s \leq t\right)
$$

For any fixed $\ell \in \mathbb{S}$ and positive number $a=a(\ell)$ which is independent of the Brownian motion $\left(W_{t}\right)_{t \geq 0}$, the filtration $\mathcal{F}_{a}^{W}$ is defined by

$$
\mathcal{F}_{a}^{W}:=\sigma\left(W_{s}: s \leq a\right)
$$

If $\tau: \Omega \rightarrow[0, \infty]$ is a stopping time with respect to the filtration $\mathcal{F}_{t}, \mathcal{F}_{\tau}$ denotes the past $\sigma$-field defined by

$$
\mathcal{F}_{\tau}=\left\{A \in \mathcal{F}: \forall t \geq 0, A \cap\{\tau \leq t\} \in \mathcal{F}_{t}\right\}
$$

### 2.2 Priori estimates on the solutions

166-3 Lemma 2.1. Let $w_{t}$ be the solution to equation (1.2) with initial value $w_{0}$. Then, there exist positive constants $C_{1}, C_{2}$, which depend on the parameters $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$, such that

$$
\begin{align*}
\mathbb{E}\left\|w_{t}\right\|^{2} & \leq e^{-\nu t}\left\|w_{0}\right\|^{2}+C_{1}, \forall t \geq 0  \tag{2.3}\\
\frac{\nu}{2} \mathbb{E} \int_{0}^{t}\left\|w_{s}\right\|_{1}^{2} \mathrm{~d} s & \leq\left\|w_{0}\right\|^{2}+C_{2} t, \forall t \geq 0 \tag{2.4}
\end{align*}
$$

Proof. Let $\nu_{L}$ be the intensity measure defined in (1.8). We claim that

$$
\begin{equation*}
\int_{|z| \leq 1}|z|^{2} \nu_{L}(\mathrm{~d} z)+\int_{|z| \geq 1}|z|^{n} \nu_{L}(\mathrm{~d} z)<\infty, \quad \forall n \geq 2 \tag{2.5}
\end{equation*}
$$

We only prove that $\int_{|z| \geq 1}|z|^{n} \nu_{L}(\mathrm{~d} z)<\infty, \forall n \geq 2$, the first term in (2.5) can be treated similarly. By definition, we have

$$
\begin{aligned}
\int_{|z| \geq 1}|z|^{n} \nu_{L}(\mathrm{~d} z) & =\int_{|z| \geq 1}|z|^{n}\left[\int_{0}^{\infty}(2 \pi u)^{-d / 2} e^{-\frac{|z|^{2}}{2 u}} \nu_{S}(\mathrm{~d} u)\right] \mathrm{d} z \\
& =C_{d} \int_{1}^{\infty} r^{n+d-1}\left[\int_{0}^{\infty} u^{-d / 2} e^{-\frac{r^{2}}{2 u}} \nu_{S}(\mathrm{~d} u)\right] \mathrm{d} r \\
& =C_{d} \int_{0}^{\infty} \nu_{S}(\mathrm{~d} u) \int_{1}^{\infty} r^{n+d-1} e^{-\frac{r^{2}}{2 u}} u^{-d / 2} \mathrm{~d} r \\
& =C_{d} \int_{0}^{\infty} \nu_{S}(\mathrm{~d} u) \int_{1 /(2 u)}^{\infty}(2 u x)^{\frac{n+d-1}{2}} e^{-x} u^{-d / 2} \frac{2 u}{2 \sqrt{2 u x}} \mathrm{~d} x \\
& \leq C_{d, n} \int_{0}^{\infty} u^{n / 2} \nu_{S}(\mathrm{~d} u)<\infty
\end{aligned}
$$

Now, we prove (2.3) and (2.4). Applying Itô's formula to $\left\|w_{t}\right\|^{2}$, we obtain

$$
\begin{align*}
\mathrm{d}\left\|w_{t}\right\|^{2}= & -2 \nu\left\|w_{t}\right\|_{1}^{2} \mathrm{~d} t+2 \int_{z \in \mathbb{R}^{d} \backslash\{0\}}\left\langle w_{t}, Q z\right\rangle \tilde{N}_{L}(\mathrm{~d} t, \mathrm{~d} z)  \tag{2.6}\\
& +\int_{z \in \mathbb{R}^{d} \backslash\{0\}}\|Q z\|^{2} N_{L}(\mathrm{~d} t, \mathrm{~d} z) .
\end{align*}
$$

Set $C=\int_{z \in \mathbb{R}^{d} \backslash\{0\}}\|Q z\|^{2} \nu_{L}(\mathrm{~d} z)$, which is a constant only depending on $\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$. By (2.6) and standard arguments,

$$
\mathbb{E}\left\|w_{t}\right\|^{2}+2 \nu \int_{0}^{t} \mathbb{E}\left\|w_{s}\right\|_{1}^{2} \mathrm{~d} s=\left\|w_{0}\right\|^{2}+C t
$$

which yields the desired results (2.3) and (2.4).
Let $\sigma_{0}=0$ and $\mathfrak{B}_{0}=\sum_{j \in \mathcal{Z}_{0}} b_{j}^{2}$. For any $\kappa>0, k \in \mathbb{N}$ and $\ell \in \mathbb{S}$, we define

$$
\begin{equation*}
\sigma=\sigma(\ell)=\sigma_{1}(\ell)=\inf \left\{t \geq 0: \nu t-8 \mathfrak{B}_{0} \kappa \ell_{t}>1\right\} \tag{2.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\sigma_{k}=\sigma_{k}(\ell)=\inf \left\{t \geq \sigma_{k-1}, \nu\left(t-\sigma_{k-1}\right)-8 \mathfrak{B}_{0} \kappa\left(\ell_{t}-\ell_{\sigma_{k-1}}\right)>1\right\} \tag{2.8}
\end{equation*}
$$

For the solutions to equation (1.2) and theses stopping times $\sigma_{k}$ (with respect to $\mathcal{F}_{t}$ ), we have the following moment estimates.
qu-2 Lemma 2.2. There exists a constant $\kappa_{0} \in(0, \nu]$ only depending on $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}$ and $d=\left|\mathcal{Z}_{0}\right|$ such that the following statements hold:
(1) For any $\kappa \in\left(0, \kappa_{0}\right]$ and the stopping time $\sigma$ defined in (2.7), we have

$$
\begin{equation*}
\mathbb{E}^{\mu_{S}} \exp \{10 \nu \sigma\} \leq C_{\kappa}, \tag{2.9}
\end{equation*}
$$

where $C_{\kappa}$ is a constant depending on $\kappa$ and $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$. Hence

$$
\begin{equation*}
\mathbb{E} \exp \{10 \nu \sigma\} \leq C_{\kappa} \tag{2.10}
\end{equation*}
$$

(2) For any $\kappa \in\left(0, \kappa_{0}\right]$, almost all $\ell \in \mathbb{S}$ (under the measure $\mathbb{P}^{\mu_{\mathbb{s}}}$ ) and the stopping times $\sigma_{k}$ defined in (2.7) and (2.8), we have

$$
\begin{align*}
\mathbb{E}^{\mu_{W}}[\exp & \left\{\kappa\left\|w_{\sigma_{k}}\right\|^{2}-\kappa\left\|w_{\sigma_{k-1}}\right\|^{2} e^{-1}\right. \\
& +\nu \kappa \int_{\sigma_{k-1}}^{\sigma_{k}} e^{-\nu\left(\sigma_{k}-s\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma_{k}}-\ell_{s}\right)}\left\|w_{s}\right\|_{1}^{2} \mathrm{~d} s  \tag{2.11}\\
& \left.\left.-\kappa \mathfrak{B}_{0}\left(\ell_{\sigma_{k}}-\ell_{\sigma_{k-1}}\right)\right\} \mid \mathcal{F}_{\ell_{\sigma_{k-1}}}^{W}\right] \leq C,
\end{align*}
$$

where $C$ is a constant only depending on $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}$ and d. Moreover, the following statements hold:

$$
\begin{align*}
\mathbb{E}[\exp \{\kappa \| & \left\|w_{\sigma_{k}}\right\|^{2}-\kappa\left\|w_{\sigma_{k-1}}\right\|^{2} e^{-1} \\
& +\nu \kappa \int_{\sigma_{k-1}}^{\sigma_{k}} e^{-\nu\left(\sigma_{k}-s\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma_{k}}-\ell_{s}\right)}\left\|w_{s}\right\|_{1}^{2} \mathrm{~d} s  \tag{2.12}\\
& \left.\left.-\kappa \mathfrak{B}_{0}\left(\ell_{\sigma_{k}}-\ell_{\sigma_{k-1}}\right)\right\} \mid \mathcal{F}_{\sigma_{k-1}}\right] \leq C
\end{align*}
$$

where $C$ is the constant appearing in (2.11).
(3) For any $\kappa \in\left(0, \kappa_{0}\right]$ and $k \in \mathbb{N}$, one has

$$
\begin{equation*}
\mathbb{E}\left[\exp \left\{\kappa\left\|w_{\sigma_{k+1}}\right\|^{2}\right\} \mid \mathcal{F}_{\sigma_{k}}\right] \leq C_{\kappa} \exp \left\{\kappa e^{-1}\left\|w_{\sigma_{k}}\right\|^{2}\right\} \tag{2.13}
\end{equation*}
$$

where $C_{\kappa}$ is a constant depending on $\kappa$ and $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$.
(4) For any $\kappa \in\left(0, \kappa_{0}\right]$, there exists a $C_{\kappa}>0$ depending on $\kappa$ and $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$ such that for any $n \in \mathbb{N}$ and $w_{0} \in H$, one has

$$
\begin{equation*}
\mathbb{E}_{w_{0}} \exp \left\{\kappa \sum_{i=1}^{n}\left\|w_{\sigma_{i}}\right\|^{2}-C_{\kappa} n\right\} \leq e^{a \kappa\left\|w_{0}\right\|^{2}} \tag{2.14}
\end{equation*}
$$

where $a=\frac{1}{1-e^{-1}}$. In this paper, we use the notation $\mathbb{E}_{w_{0}}$ for expectations under the measure $\mathbb{P}$ with respect to solutions to (1.2) with initial condition $w_{0}$.
(5) For any $\kappa \in\left(0, \kappa_{0}\right], w_{0} \in H$ and $n \in \mathbb{N}$, one has

$$
\begin{equation*}
\mathbb{E}_{w_{0}} \sup _{s \in[0, \sigma]}\left\|w_{s}\right\|^{2 n} \leq C_{n, \kappa}\left(1+\left\|w_{0}\right\|^{2 n}\right) \tag{2.15}
\end{equation*}
$$

where $C_{n, \kappa}$ is a constant depending on $n, \kappa$ and $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$.
The proof of the above lemma is long, we leave it in Appendix A. Throughout this paper, $\kappa_{0}$ is the constant appearing in Lemma 2.2.

### 2.3 Elements of Malliavin calculus

Let $d=\left|\mathcal{Z}_{0}\right|$ and denote the canonical basis of $\mathbb{R}^{d}$ by $\left\{\theta_{j}\right\}_{j \in \mathcal{Z}_{0}}$. We have defined the linear operator $Q: \mathbb{R}^{d} \rightarrow H$ in the following way: for any $z=\sum_{j \in \mathcal{Z}_{0}} z_{j} \theta_{j} \in$ $\mathbb{R}^{d}$,

$$
Q z=\sum_{j \in \mathcal{Z}_{0}} b_{j} z_{j} e_{j}
$$

The adjoint of $Q$ is given by $Q^{*}: H \rightarrow \mathbb{R}^{d}$ :

$$
Q^{*} \xi=\left(b_{j}\left\langle\xi, e_{j}\right\rangle\right)_{j \in \mathcal{Z}_{0}} \in \mathbb{R}^{d}, \text { for } \xi \in H
$$

For any $0 \leq s \leq t$ and $\xi \in H$, let $J_{s, t} \xi$ be the solution of the linearised problem:

$$
\begin{align*}
\partial_{t} J_{s, t} \xi-\nu \Delta J_{s, t} \xi-\tilde{B}\left(w_{t}, J_{s, t} \xi\right) & =0  \tag{2.16}\\
J_{s, s} \xi & =\xi
\end{align*}
$$

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where $\tilde{B}(w, v)=B(\mathcal{K} w, v)+B(\mathcal{K} v, w)$.
For any $0 \leq t \leq T$ and $\xi \in H$, let $K_{t, T}$ be the adjoint of $J_{t, T}$, i.e., $\varrho_{t}:=K_{t, T} \xi$ satisfies the following backward equation

$$
\begin{align*}
\partial_{t} \varrho_{t}+\nu \Delta \varrho_{t}+D \tilde{B}^{*}\left(w_{t}\right) \varrho_{t} & =0  \tag{2.17}\\
\varrho_{T} & =\xi
\end{align*}
$$

$\square$
where $\left\langle D \tilde{B}^{*}(w) \rho, \psi\right\rangle=\langle\rho, D \tilde{B}(w) \psi\rangle$ and $D \tilde{B}(w) \psi=B(\mathcal{K} w, \psi)+B(\mathcal{K} \psi, w)$.
Denote by $J_{s, t}^{(2)}(\phi, \psi)$ the second derivative of $w_{t}$ with respect to initial value $w_{0}$ in the directions of $\phi$ and $\psi$. Then

$$
\left\{\begin{array}{l}
\partial_{t} J_{s, t}^{(2)}(\phi, \psi)=\nu \Delta J_{s, t}^{(2)}(\phi, \psi)+B\left(\mathcal{K} J_{s, t} \phi, J_{s, t} \psi\right)+B\left(\mathcal{K} J_{s, t} \psi, J_{s, t} \phi\right)  \tag{2.18}\\
\quad+B\left(\mathcal{K} w_{t}, J_{s, t}^{(2)}(\phi, \psi)\right)+B\left(\mathcal{K} J_{s, t}^{(2)}(\phi, \psi), w_{t}\right), \quad \text { for } t>s \\
\\
J_{s, s}^{(2)}(\phi, \psi)=0
\end{array}\right.
$$

For given $\ell \in \mathbb{S}, t>0$, let $\Phi(t, W)$ be a $\mathcal{F}_{\ell_{t}}^{W}$-measurable random variable. For $v \in L^{2}\left(\left[0, \ell_{t}\right] ; \mathbb{R}^{d}\right)$, the Malliavin derivative of $\Phi$ in the direction $v$ is defined by

$$
\mathcal{D}^{v} \Phi(t, W)=\lim _{\varepsilon \rightarrow 0} \frac{1}{\varepsilon}\left(\Phi\left(t, w_{0}, W+\varepsilon \int_{0} v \mathrm{~d} s\right)-\Phi\left(t, w_{0}, W\right)\right)
$$

where the limit holds almost surely (e.g., see the book Nua06] for finite-dimensional setting or the papers [MP06, HM06, HM11, FGRT15] for Hilbert space case). In the definition of Malliavin derivative, the element $\ell$ is taken as fixed. Then, $\mathcal{D}^{v} w_{t}$ satisfies the following equation:

$$
\mathrm{d} \mathcal{D}^{v} w_{t}=\nu \Delta \mathcal{D}^{v} w_{t} \mathrm{~d} t+\tilde{B}\left(\mathcal{D}^{v} w_{t}, w_{t}\right) \mathrm{d} t+Q \mathrm{~d}\left(\int_{0}^{\ell_{t}} v_{s} \mathrm{~d} s\right)
$$

By the Riesz representation theorem, there is a linear operator $\mathcal{D}: L^{2}(\Omega, H) \rightarrow$ $L^{2}\left(\Omega ; L^{2}\left(\left[0, \ell_{t}\right] ; \mathbb{R}^{d}\right) \otimes H\right)$ such that

$$
\begin{equation*}
\mathcal{D}^{v} w_{t}=\langle\mathcal{D} w, v\rangle_{L^{2}\left(\left[0, \ell_{t}\right] ; \mathbb{R}^{d}\right)}, \forall v \in L^{2}\left(\left[0, \ell_{t}\right] ; \mathbb{R}^{d}\right) \tag{2.19}
\end{equation*}
$$

Actually, we have the following lemma.
$17-1$ Lemma 2.3. For any $\ell \in \mathbb{S}$ and $v \in L^{2}\left(\left[0, \ell_{t}\right] ; \mathbb{R}^{d}\right)$, we have

$$
\mathcal{D}^{v} w_{t}=\int_{0}^{\ell_{t}} J_{\gamma_{u}, t} Q v_{u} \mathrm{~d} u
$$

here $\gamma_{u}$ is defined by $\gamma_{u}=\inf \left\{t \geq 0, S_{t}(\ell) \geq u\right\}$. Hence, we also have

$$
\mathcal{D}_{u}^{j} w_{t}=J_{\gamma_{u}, t} Q \theta_{j}, \forall u \in\left[0, \ell_{t}\right]
$$

where $\mathcal{D}_{u}^{j}$ denotes the Malliavin derivative with respect to the $j$ th component of the noise at time $u$.
Proof. We need to prove that for any $v \in L^{2}\left(\left[0, \ell_{t}\right], \mathbb{R}^{d}\right)$,

$$
\begin{equation*}
\mathcal{D}^{v} w_{t}=\int_{0}^{t} J_{r, t} Q \mathrm{~d}\left(\int_{0}^{\ell_{r}} v_{s} \mathrm{~d} s\right)=\int_{0}^{\ell_{t}} J_{\gamma_{u}, t} Q v_{u} \mathrm{~d} u \tag{2.20}
\end{equation*}
$$

The first equality in (2.20) follows from the formula of constant variations or Fubini's theorem; see [Zh16, Page 370 , lines 1-5] for example. So we give a proof of the second equality in (2.20). Obviously, we have

$$
\begin{equation*}
\int_{0}^{t} J_{r, t} Q \mathrm{~d}\left(\int_{0}^{\ell_{r}} v_{s} \mathrm{~d} s\right)=\sum_{r \leq t} J_{r, t} Q \int_{\ell_{r-}}^{\ell_{r}} v_{s} \mathrm{~d} s \tag{2.21}
\end{equation*}
$$

Since $\gamma_{u}=r, u \in\left(\ell_{r-}, \ell_{r}\right)$, it holds that

$$
\sum_{r \leq t} J_{r, t} Q \int_{\ell_{r-}}^{\ell_{r}} v_{u} \mathrm{~d} u=\sum_{r \leq t} \int_{\ell_{r_{-}}}^{\ell_{r}} J_{\gamma_{u}, t} Q v_{u} \mathrm{~d} u=\int_{0}^{\ell_{t}} J_{\gamma_{u}, t} Q v_{u} \mathrm{~d} u
$$

Combining this with (2.21), we complete the proof of the second equality in (2.20).

For any $s \leq t$ and $\ell \in \mathbb{S}$, define the linear operator $\mathcal{A}_{s, t}: L^{2}\left(\left[\ell_{s}, \ell_{t}\right] ; \mathbb{R}^{d}\right) \rightarrow H$ by

$$
\begin{equation*}
\mathcal{A}_{s, t} v=\int_{\ell_{s}}^{\ell_{t}} J_{\gamma_{u}, t} Q v_{u} \mathrm{~d} u, \quad v \in L^{2}\left(\left[\ell_{s}, \ell_{t}\right] ; \mathbb{R}^{d}\right) \tag{2.22}
\end{equation*}
$$

The adjoint of $\mathcal{A}_{s, t}, \mathcal{A}_{s, t}^{*}: H \rightarrow L^{2}\left(\left[\ell_{s}, \ell_{t}\right] ; \mathbb{R}^{d}\right)$, is given by

$$
\mathcal{A}_{s, t}^{*} \phi=\left(b_{j}\left\langle\phi, J_{\gamma_{u}, t} e_{j}\right\rangle\right)_{j \in \mathcal{Z}_{0}, u \in\left[\ell_{s}, \ell_{t}\right]} .
$$

The Malliavin matrix $\mathcal{M}_{s, t}: H \rightarrow H$ is defined by

$$
\begin{equation*}
\mathcal{M}_{s, t} \phi=\mathcal{A}_{s, t} \mathcal{A}_{s, t}^{*} \phi \tag{2.23}
\end{equation*}
$$

By a simple calculation, we have(c.f. [Zh14, Lemma 2.2]) ${ }^{1}$

$$
\begin{equation*}
\left\langle\mathcal{M}_{s, t} \phi, \phi\right\rangle=\sum_{j \in \mathcal{Z}_{0}} b_{j}^{2} \int_{s}^{t}\left\langle K_{r, t} \phi, e_{j}\right\rangle^{2} \mathrm{~d} \ell_{r} \tag{2.24}
\end{equation*}
$$

(For any function $f:[a, b] \rightarrow \mathbb{R}$, the integral $\int_{a}^{b} f(s) \mathrm{d} \ell_{s}$ is interpreted as $\int_{(a, b]} f(s) \mathrm{d} \ell_{s}$.)

In the rest of this subsection, we will provide some estimates for $J_{s, t}, J_{s, t}^{(2)}, \mathcal{A}_{s, t}$, $\mathcal{A}_{s, t}^{*}, \mathcal{M}_{s, t}$ and their Malliavin derivatives, which will be used in subsequent sections.

15-4 Lemma 2.4. There exists a constant $\mathcal{C}_{0}$ only depending on $\nu$ such that for any $\xi, \phi, \psi \in H$ and $0 \leq s \leq t \leq T, J_{s, t}$ and $J_{s, t}^{(2)}$ satisfy almost surely

$$
\begin{align*}
\sup _{t \in[s, T]}\left\|J_{s, t} \xi\right\|^{2} & \leq \mathcal{C}_{0}\|\xi\|^{2} e^{\mathcal{C}_{0} \int_{s}^{T}\left\|w_{r}\right\|_{1}^{4 / 3} \mathrm{~d} r}  \tag{2.25}\\
\int_{s}^{t}\left\|J_{s, r} \xi\right\|_{1}^{2} \mathrm{~d} r & \leq \mathcal{C}_{0}\|\xi\|^{2} e^{\mathcal{C}_{0} \int_{s}^{t}\left\|w_{r}\right\|_{1}^{4 / 3} \mathrm{~d} r}  \tag{2.26}\\
\sup _{t \in[s, T]}\left\|J_{s, t}^{(2)}(\phi, \psi)\right\|^{2} & \leq \mathcal{C}_{0}\|\phi\|^{2}\|\psi\|^{2} e^{\mathcal{C}_{0} \int_{s}^{T}\left\|w_{r}\right\|_{1}^{4 / 3} \mathrm{~d} r} . \tag{2.27}
\end{align*}
$$

Furthermore, for any $\kappa>0$ and $0 \leq s \leq T$, it also holds that

$$
\begin{align*}
\left\|J_{s, T} \xi\right\|^{2} \leq \mathcal{C}_{0} \exp & \left\{\frac{\nu \kappa}{120} \int_{s}^{T}\left\|w_{s}\right\|_{1}^{2} e^{-\nu(T-s)+8 \mathfrak{B}_{0} \kappa\left(\ell_{T}-\ell_{s}\right)} \mathrm{d} s\right.  \tag{2.28}\\
& \left.+C_{\kappa} \int_{s}^{T} e^{2 \nu(T-s)-16 \mathfrak{B}_{0} \kappa\left(\ell_{T}-\ell_{s}\right)} \mathrm{d} s\right\}\|\xi\|^{2}
\end{align*}
$$

where $\mathcal{C}_{0}$ is taken from (2.25)-(2.27), and $C_{\kappa}$ is a constant depending on $\kappa, \nu$.

$$
\begin{aligned}
{ }^{1} \text { Actually, we have, } \\
\begin{aligned}
\left\langle\mathcal{M}_{s, t} \phi, \phi\right\rangle & =\sum_{j \in \mathcal{Z}_{0}} b_{j}^{2} \int_{\ell_{s}}^{\ell_{t}}\left\langle K_{\gamma_{u}, t} \phi, e_{j}\right\rangle^{2} \mathrm{~d} u=\sum_{j \in \mathcal{Z}_{0}} b_{j}^{2} \sum_{r \in(s, t]} \int_{\ell_{r-}}^{\ell_{r}}\left\langle K_{\gamma_{u}, t} \phi, e_{j}\right\rangle^{2} \mathrm{~d} u \\
& =\sum_{j \in \mathcal{Z}_{0}} b_{j}^{2} \sum_{r \in(s, t]}\left\langle K_{r, t} \phi, e_{j}\right\rangle^{2}\left(\ell_{r}-\ell_{r-}\right)=\sum_{j \in \mathcal{Z}_{0}} b_{j}^{2} \int_{s}^{t}\left\langle K_{r, t} \phi, e_{j}\right\rangle^{2} \mathrm{~d} \ell_{r} .
\end{aligned}
\end{aligned}
$$

Remark 2.5. In the other places of this paper, we need to use the constants appeared in (2.25)-(2.27), so we use a special notation $\mathcal{C}_{0}$ to denote them.

Proof. By (2.2),

$$
\begin{align*}
& \left\langle B\left(\mathcal{K} J_{s, t} \xi, w_{t}\right), J_{s, t} \xi\right\rangle \leq C\left\|w_{t}\right\|_{1}\left\|J_{s, t} \xi\right\|_{1 / 2}\left\|J_{s, t} \xi\right\| \\
& \leq \frac{\nu}{4}\left\|J_{s, t} \xi\right\|_{1}^{2}+C\left\|w_{t}\right\|_{1}^{4 / 3}\left\|J_{s, t} \xi\right\|^{2}, \tag{2.29}
\end{align*}
$$

where $C=C(\nu)$. Therefore, applying the chain rule to $\left\|J_{s, t} \xi\right\|^{2}$, one arrives at

$$
\mathrm{d}\left\|J_{s, t} \xi\right\|^{2} \leq-\nu\left\|J_{s, t} \xi\right\|_{1}^{2} \mathrm{~d} t+C\left\|w_{t}\right\|_{1}^{4 / 3}\left\|J_{s, t} \xi\right\|^{2} \mathrm{~d} t
$$

which implies

$$
\left\|J_{s, t} \xi\right\|^{2} \leq C\|\xi\|^{2} e^{C \int_{s}^{t}\left\|w_{r}\right\|_{1}^{4 / 3} \mathrm{~d} r}, \quad \forall 0 \leq s \leq t
$$

and

$$
\begin{aligned}
& \nu \int_{s}^{t}\left\|J_{s, r} \xi\right\|_{1}^{2} \mathrm{~d} r \leq\|\xi\|^{2}+C \int_{s}^{t}\left\|w_{r}\right\|_{1}^{4 / 3}\left\|J_{s, r} \xi\right\|^{2} \mathrm{~d} r \\
& \leq\|\xi\|^{2}+C\|\xi\|^{2} \int_{s}^{t}\left\|w_{r}\right\|_{1}^{4 / 3} \mathrm{~d} r e^{C \int_{s}^{t}\left\|w_{r}\right\|_{1}^{4 / 3} \mathrm{~d} r} \leq C\|\xi\|^{2} e^{C \int_{s}^{t}\left\|w_{r}\right\|_{1}^{4 / 3} \mathrm{~d} r} .
\end{aligned}
$$

The proof of (2.25) and (2.26) is complete.
Now we prove (2.27). As in (2.29), we have
$\left\langle B\left(\mathcal{K} J_{s, t}^{(2)}(\phi, \psi), w_{t}\right), J_{s, t}^{(2)}(\phi, \psi)\right\rangle \leq \frac{\nu}{4}\left\|J_{s, t}^{(2)}(\phi, \psi)\right\|_{1}^{2}+C\left\|w_{t}\right\|_{1}^{4 / 3}\left\|J_{s, t}^{(2)}(\phi, \psi)\right\|^{2}$.
Moreover,

$$
\begin{aligned}
& \left\langle B\left(\mathcal{K} J_{s, t} \phi, J_{s, t} \psi\right), J_{s, t}^{(2)}(\phi, \psi)\right\rangle \leq C\left\|J_{s, t}^{(2)}(\phi, \psi)\right\|_{1}\left\|J_{s, t} \phi\right\|_{1 / 2}\left\|J_{s, t} \psi\right\| \\
& \leq \frac{\nu}{4}\left\|J_{s, t}^{(2)}(\phi, \psi)\right\|_{1}^{2}+C\left\|J_{s, t} \phi\right\|_{1 / 2}^{2}\left\|J_{s, t} \psi\right\|^{2} .
\end{aligned}
$$

Hence, applying the chain rule to $\left\|J_{s, t}^{(2)}(\phi, \psi)\right\|^{2}$, with the help of $(2.25)-(2.26)$, we get

$$
\begin{aligned}
& \left\|J_{s, t}^{(2)}(\phi, \psi)\right\|^{2} \leq C e^{C \int_{s}^{t}\left\|w_{r}\right\|_{1}^{4 / 3} \mathrm{~d} r} \int_{s}^{t}\left[\left\|J_{s, r} \phi\right\|_{1 / 2}^{2}\left\|J_{s, r} \psi\right\|^{2}+\left\|J_{s, r} \psi\right\|_{1 / 2}^{2}\left\|J_{s, r} \phi\right\|^{2}\right] \mathrm{d} r \\
& \leq C e^{C \int_{s}^{t}\left\|w_{r}\right\|_{1}^{4 / 3} \mathrm{~d} r}\|\phi\|^{2}\|\psi\|^{2}
\end{aligned}
$$

(2.28) is easily obtained by Young's inequality and (2.25).

Recall that $P_{N}$ is the orthogonal projection from $H$ into $H_{N}=\operatorname{span}\left\{e_{j} ; j \in\right.$ $\left.Z_{*}^{2},|j| \leq N\right\}$ and $Q_{N}=I-P_{N}$. For any $N \in \mathbb{N}, t \geq 0$ and $\xi \in H$, denote $\xi_{t}^{h}=Q_{N} J_{0, t} \xi, \xi_{t}^{\iota}=P_{N} J_{0, t} \xi$ and $\xi_{t}=J_{0, t} \xi$.

1340-3 Lemma 2.6. For any $t \geq 0$ and $\xi \in H$, one has

$$
\left\|\xi_{t}^{h}\right\|^{2} \leq \exp \left\{-\nu N^{2} t\right\}\|\xi\|^{2}+\frac{C\|\xi\|^{2}}{\sqrt{N}} \exp \left\{C \int_{0}^{t}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\} \sup _{s \in[0, t]}\left\|w_{s}\right\|
$$

where $C$ is a constant depending on $\nu$.
Proof. Note that $\left\|\xi_{t}^{h}\right\|_{1}^{2} \geq N^{2}\left\|\xi_{t}^{h}\right\|^{2}$ and that

$$
\begin{aligned}
& \left\langle B\left(\mathcal{K} \xi_{t}, w_{t}\right), \xi_{t}^{h}\right\rangle+\left\langle B\left(\mathcal{K} w_{t}, \xi_{t}\right), \xi_{t}^{h}\right\rangle \leq C\left\|\xi_{t}^{h}\right\|_{1}\left\|w_{t}\right\|_{1 / 2}\left\|\xi_{t}\right\| \\
& \leq \frac{\nu}{4}\left\|\xi_{t}^{h}\right\|_{1}^{2}+C\left\|w_{t}\right\|_{1 / 2}^{2}\left\|\xi_{t}\right\|^{2}
\end{aligned}
$$

applying the chain rule to $\left\|\xi_{t}^{h}\right\|^{2}$, we find

$$
\begin{aligned}
& \left\|\xi_{t}^{h}\right\|^{2} \leq \exp \left\{-\nu N^{2} t\right\}\|\xi\|^{2}+C \int_{0}^{t} \exp \left\{-\nu N^{2}(t-s)\right\}\left\|w_{s}\right\|_{1 / 2}^{2}\left\|\xi_{s}\right\|^{2} \mathrm{~d} s \\
& \leq \exp \left\{-\nu N^{2} t\right\}\|\xi\|^{2} \\
& +C \exp \left\{C \int_{0}^{t}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\}\|\xi\|^{2} \sup _{s \in[0, t]}\left\|w_{s}\right\| \int_{0}^{t} \exp \left\{-\nu N^{2}(t-s)\right\}\left\|w_{s}\right\|_{1} \mathrm{~d} s \\
& \leq \exp \left\{-\nu N^{2} t\right\}\|\xi\|^{2}+C \exp \left\{C \int_{0}^{t}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\}\|\xi\|^{2} \sup _{s \in[0, t]}\left\|w_{s}\right\| \\
& \quad \times\left(\int_{0}^{t} \exp \left\{-4 \nu N^{2}(t-s)\right\} \mathrm{d} s\right)^{1 / 4}\left(\int_{0}^{t}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right)^{3 / 4}
\end{aligned}
$$

where we have used (2.25) in the second inequality. The above inequality implies the desired result.

Lemma 2.7. Assume that $\xi_{0}^{\iota}=0$, then for any $t \geq 0$,

$$
\left\|\xi_{t}^{\iota}\right\|^{2} \leq C\|\xi\|^{2} \exp \left\{C \int_{0}^{t}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\} \sup _{s \in[0, t]}\left(\left\|w_{s}\right\|^{5 / 2}+1\right) \frac{1+t}{N^{1 / 4}}
$$

where $C$ is a constant depending on $\nu$. Furthermore, combining the above inequality with Lemma 2.6, for any $\xi \in H$ and $t \geq 0$, we have

$$
\begin{aligned}
& \left\|J_{0, t} Q_{N} \xi\right\|^{2} \\
& \leq C\left(e^{-\nu N^{2} t}+\frac{1+t}{N^{1 / 4}}\right) \exp \left\{C \int_{0}^{t}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\} \sup _{s \in[0, t]}\left(\left\|w_{s}\right\|^{5 / 2}+1\right)\|\xi\|^{2}
\end{aligned}
$$

Proof. In view of (2.2), one has

$$
\left\langle B\left(\mathcal{K} \xi_{t}^{\iota}, w_{t}\right), \xi_{t}^{\iota}\right\rangle \leq \frac{\nu}{4}\left\|\xi_{t}^{\iota}\right\|_{1}^{2}+C\left\|w_{t}\right\|_{1}^{4 / 3}\left\|\xi_{t}^{\iota}\right\|^{2}
$$

(See (2.29) for similar arguments) and

$$
\left\langle B\left(\mathcal{K} w_{t}, \xi_{t}^{h}\right), \xi_{t}^{\iota}\right\rangle+\left\langle B\left(\mathcal{K} \xi_{t}^{h}, w_{t}\right), \xi_{t}^{\iota}\right\rangle
$$

$$
\leq C\left\|\xi_{t}^{\iota}\right\|_{1}\left\|w_{t}\right\|\left\|\xi_{t}^{h}\right\|_{1 / 2} \leq \frac{\nu}{4}\left\|\xi_{t}^{\iota}\right\|_{1}^{2}+C\left\|w_{t}\right\|^{2}\left\|\xi_{t}^{h}\right\|_{1 / 2}^{2}
$$

Thus, applying the chain rule to $\left\|\xi_{t}^{\iota}\right\|^{2}$, we have

$$
\begin{aligned}
& \left\|\xi_{t}^{h}\right\|^{2} \leq C \exp \left\{C \int_{0}^{t}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\} \int_{0}^{t}\left\|w_{s}\right\|^{2}\left\|\xi_{s}^{h}\right\|_{1 / 2}^{2} \mathrm{~d} s \\
& \leq \\
& \leq C \exp \left\{C \int_{0}^{t}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\} \sup _{s \in[0, t]}\left\|w_{s}\right\|^{2} \int_{0}^{t}\left\|\xi_{s}^{h}\right\|\left\|\xi_{s}^{h}\right\|_{1} \mathrm{~d} s \\
& \leq \\
& \leq C \exp \left\{C \int_{0}^{t}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\} \sup _{s \in[0, t]}\left\|w_{s}\right\|^{2}\left(\int_{0}^{t}\left\|\xi_{s}^{h}\right\|^{2} \mathrm{~d} s\right)^{1 / 2}\left(\int_{0}^{t}\left\|\xi_{s}^{h}\right\|_{1}^{2} \mathrm{~d} s\right)^{1 / 2} \\
& \leq C\|\xi\| \exp \left\{C \int_{0}^{t}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\} \sup _{s \in[0, t]}\left\|w_{s}\right\|^{2}\left(\int_{0}^{t}\left\|\xi_{s}^{h}\right\|^{2} \mathrm{~d} s\right)^{1 / 2} \\
& \leq C\|\xi\| \exp \left\{C \int_{0}^{t}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\} \sup _{s \in[0, t]}\left\|w_{s}\right\|^{2} \\
& \\
& \times\left(\int_{0}^{t}\left[\exp \left\{-\nu N^{2} s\right\}\|\xi\|^{2}+\frac{C\|\xi\|^{2}}{\sqrt{N}} \exp \left\{C \int_{0}^{s}\left\|w_{r}\right\|_{1}^{4 / 3} \mathrm{~d} r\right\} \sup _{r \in[0, s]}\left\|w_{r}\right\|\right] \mathrm{d} s\right)^{1 / 2} \\
& \leq C\|\xi\|^{2} \exp \left\{C \int_{0}^{t}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\} \sup _{s \in[0, t]}\left\|w_{s}\right\|^{2}\left(\frac{1}{N}+\frac{\sup _{s \in[0, t]}\left\|w_{r}\right\|^{1 / 2} \sqrt{t}}{N^{1 / 4}}\right) \\
& \leq C\|\xi\|^{2} \exp \left\{C \int_{0}^{t}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\} \sup _{s \in[0, t]}\left(\left\|w_{s}\right\|^{5 / 2}+1\right) \frac{1+t}{N^{1 / 4}},
\end{aligned}
$$

where in the fourth inequality we have used (2.26) for the fourth inequality and Lemma 2.6 for the fifth inequality.

Using the similar arguments as that in [HM06, Section 4.8] or [FGRT-2015, Lemma A.6], we have the following lemma.

L:2.2 Lemma 2.8. There is a constant $C=C\left(\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, d\right)>0$ such that for any $0 \leq s<t$ and $\beta>0$, we have

$$
\begin{gather*}
\left\|\mathcal{A}_{s, t}\right\|_{\mathcal{L}\left(L^{2}\left(\left[\ell_{s}, \ell_{t}\right] ; \mathbb{R}^{d}\right), H\right)}^{2} \leq C \int_{\ell_{s}}^{\ell_{t}}\left\|J_{\gamma_{u}, t}\right\|_{\mathcal{L}(H, H)}^{2} \mathrm{~d} u  \tag{2.30}\\
\left\|\mathcal{A}_{s, t}^{*}\left(\mathcal{M}_{s, t}+\beta \mathbb{I}\right)^{-1 / 2}\right\|_{\mathcal{L}\left(H, L^{2}\left(\left[\ell_{s}, \ell_{t}\right] ; \mathbb{R}^{d}\right)\right)} \leq 1  \tag{2.31}\\
\left\|\left(\mathcal{M}_{s, t}+\beta \mathbb{I}\right)^{-1 / 2} \mathcal{A}_{s, t}\right\|_{\mathcal{L}\left(L^{2}\left(\left[\ell_{s}, \ell_{t}\right] ; \mathbb{R}^{d}\right), H\right)} \leq 1  \tag{2.32}\\
\left\|\left(\mathcal{M}_{s, t}+\beta \mathbb{I}\right)^{-1 / 2}\right\|_{\mathcal{L}(H, H)} \leq \beta^{-1 / 2} \tag{2.33}
\end{gather*}
$$

$\square$
2.8
2.9
2.10

15-5 Lemma 2.9. For any $0 \leq s \leq t, j \in\{1, \cdots, d\}$ and $u \in\left[0, \ell_{t}\right]$, we have

$$
\mathcal{D}_{u}^{j} J_{s, t} \xi=\left\{\begin{array}{l}
J_{\gamma_{u}, t}^{(2)}\left(Q \theta_{j}, J_{s, \gamma_{u}} \xi\right), u \in\left[\ell_{s}, \ell_{t}\right] \\
J_{s, t}^{(2)}\left(J_{\gamma_{u}, s} Q \theta_{j}, \xi\right) \quad \text { if } u<\ell_{s}
\end{array}\right.
$$

Proof. In view of Lemma 2.3, the proof is the same as that in [HM06, (4.29)]. We omit the details.

As in $[\mathrm{HM} 06]$, if $A: \mathcal{H}_{1} \rightarrow \mathcal{H}_{2}$ is a random linear map between two Hilbert spaces, we denote by $\mathcal{D}_{s}^{i} A: \mathcal{H}_{1} \rightarrow \mathcal{H}_{2}$ the random linear map defined by

$$
\left(\mathcal{D}_{s}^{i} A\right) h=\left\langle\mathcal{D}_{s}(A h), \theta_{i}\right\rangle
$$

L:2.3 Lemma 2.10. The operators $J_{s, t}, \mathcal{A}_{s, t}$, and $\mathcal{A}_{s, t}^{*}$ are Malliavin differentiable, and for any $r>0$, the following inequalities hold

$$
\begin{align*}
& {\left[\left\|\mathcal{D}_{r}^{i} J_{0, \sigma}\right\|_{\mathcal{L}(H, H)}\right.}\left.\leq C_{\kappa} \exp \left\{\mathcal{C}_{0} \int_{0}^{\sigma}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\}\right]  \tag{2.34}\\
&\left\|\mathcal{D}_{r}^{i} \mathcal{A}_{0, \sigma}\right\|_{\mathcal{L}\left(L^{2}\left(\left[0, \ell_{\sigma}\right] ; \mathbb{R}^{d}\right), H\right)} \leq C_{\kappa}(\sigma+1) \exp \left\{\mathcal{C}_{0} \int_{0}^{\sigma}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\}  \tag{2.35}\\
&\left\|\mathcal{D}_{r}^{i} \mathcal{A}_{0, \sigma}^{*}\right\|_{\mathcal{L}\left(H, L^{2}\left(\left[0, \ell_{\sigma}\right] ; \mathbb{R}^{d}\right)\right)} \leq C_{\kappa}(\sigma+1) \exp \left\{\mathcal{C}_{0} \int_{0}^{\sigma}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\} \tag{2.36}
\end{align*}
$$

2.11
2.12
2.13
where $\mathcal{C}_{0}$ is the same constant as that appeared in Lemma 2.4, $C_{\kappa}$ is a constant depending on $\kappa, \nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}$,d (Recall that $\sigma$ depends on $\kappa$.).

Proof. The inequality (2.34) is derived from Lemma 2.4 and Lemma 2.9. (2.35) is obtained by Lemmas 2.4, 2.9, Cauchy-Schwarz inequality and the fact that

$$
\mathcal{A}_{0, \sigma} v=\int_{0}^{\ell_{\sigma}} J_{\gamma_{u}, \sigma} Q v(u) \mathrm{d} u, \quad \ell_{\sigma} \leq \frac{\nu \sigma}{8 \mathfrak{B}_{0} \kappa}
$$

The inequality (2.36) is a consequence of (2.35).

## 3 The invertibility of the Malliavin matrix $\mathcal{M}_{0, t}$.

Before stating the main results in this section, we prepare two lemmas. Lemma 3.2 can be seen as the pure jump version of Theorem 7.1 in [HM11], which deal with the Wiener case. In this section, we use $\Delta f(s)$ to denote $f(s)-f(s-)$.

Lemma 2024 Number jump
Lemma 3.1. Consider a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, and a given Lévy process $\widetilde{L}(t), t \geq 0$ on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, which takes values in a topological vector space $(\mathcal{T}, B(\mathcal{T}))$. Suppose that the Lévy process $\widetilde{L}(t), t \geq 0$ has a $\sigma$-finite intensity measure $\widetilde{\nu}$. For any $G \in B(\mathcal{T})$, define $N_{\widetilde{L}}^{G}\left(\left(t_{1}, t_{2}\right]\right)=\sharp\left\{s \in\left(t_{1}, t_{2}\right]: \Delta \widetilde{L}(s) \in G\right\}$. If $\widetilde{\nu}(G)=\infty$, then there exists a $\widetilde{\Omega}_{0} \in \widetilde{\mathcal{F}}$ with $\widetilde{\mathbb{P}}\left(\widetilde{\Omega}_{0}\right)=1$ such that for any $\widetilde{\omega} \in \widetilde{\Omega}_{0}$ and any $0 \leq t_{1}<t_{2}$,

$$
N_{\widetilde{L}}^{G}\left(\left(t_{1}, t_{2}\right]\right)(\widetilde{\omega})=\infty
$$

Moreover, $\{s \in[0, \infty): \Delta \widetilde{L}(s)(\widetilde{\omega}) \in G\}$ is dense on $[0, \infty)$.

Proof. Let $\mathbb{Q}$ denote the set of all rational number on $\mathbb{R}$. It is sufficient to prove that for any $0 \leq t_{1}<t_{2}$ with $t_{1}, t_{2} \in \mathbb{Q}$, we have $\left.\widetilde{\mathbb{P}}\left(N_{\widetilde{L}}^{G}\left(\left(t_{1}, t_{2}\right]\right)=\infty\right)\right)=1$. Let $K_{n}, n \geq 1$ be an increasing sequence of measurable subsets of $\mathcal{T}$ such that $K_{n} \uparrow$ $\mathcal{T}$ and $\widetilde{\nu}\left(K_{n}\right)<\infty$. It is well known that $N_{\widetilde{L}}^{G \cap K_{n}}\left(\left(t_{1}, t_{2}\right]\right)$ is a Poisson random variable with parameter $\widetilde{\nu}\left(G \cap K_{n}\right)\left(t_{2}-t_{1}\right)$. To prove $\left.\widetilde{\mathbb{P}}\left(N_{\widetilde{L}}^{G}\left(\left(t_{1}, t_{2}\right]\right)=\infty\right)\right)=1$, it suffices to show that for any positive integer $\left.M, \widetilde{\mathbb{P}}\left(N_{\widetilde{L}}^{G}\left(\left(t_{1}, t_{2}\right]\right)>M\right)\right)=1$. Indeed,

$$
\begin{aligned}
& \widetilde{\mathbb{P}}\left(N_{\widetilde{L}}^{G}\left(\left(t_{1}, t_{2}\right]\right)>M\right)=\lim _{n \rightarrow \infty} \widetilde{\mathbb{P}}\left(N_{\widetilde{L}}^{G \cap K_{n}}\left(\left(t_{1}, t_{2}\right]\right)>M\right) \\
= & 1-\lim _{n \rightarrow \infty} \widetilde{\mathbb{P}}\left(N_{\widetilde{L}}^{G \cap K_{n}}\left(\left(t_{1}, t_{2}\right]\right) \leq M\right) \\
= & 1-\lim _{n \rightarrow \infty} \sum_{m=0}^{M} \exp \left(-\nu\left(G \cap K_{n}\right)\left(t_{2}-t_{1}\right)\right) \frac{\left(\widetilde{\nu}\left(G \cap K_{n}\right)\left(t_{2}-t_{1}\right)\right)^{m}}{m!}=1,
\end{aligned}
$$

where we used the fact that $\widetilde{\nu}\left(G \cap K_{n}\right)\left(t_{2}-t_{1}\right) \rightarrow \infty$, as $n \rightarrow \infty$.
Consider the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, and a $\mathbb{R}^{d}$-valued Lévy process $\tilde{L}(t)=\left(\tilde{L}^{1}(t), \tilde{L}^{2}(t), \ldots, \tilde{L}^{d}(t)\right), t \geq 0$ with a $\sigma$-finite intensity measure $\widetilde{\nu}$. For any $n \in \mathbb{N}$ and $1 \leq i \leq d$, let

$$
G_{n}^{i}=\left\{y \in \mathbb{R}^{d} \backslash\{0\}: \max _{j \in\{1, \cdots, d\} \text { with } j \neq i}\left|y_{j}\right|<\frac{\left|y_{i}\right|}{n}\right\}
$$

Assume that $\widetilde{\nu}$ satisfies the following condition:

$$
\widetilde{\nu}\left(G_{n}^{i}\right)=\infty, \quad \forall n \in \mathbb{N} \text { and } 1 \leq i \leq d
$$

By Lemma 3.1, there exists a $\Omega_{n}^{i} \in \tilde{\mathcal{F}}$ with $\tilde{\mathbb{P}}\left(\Omega_{n}^{i}\right)=1$ such that for any $\omega \in \Omega_{n}^{i}$, the set $\left\{s \in[0, \infty): \Delta \tilde{L}(\omega, s) \in G_{n}^{i}\right\}$ is dense in $[0, \infty)$. Let $\Omega_{0}:=\cap_{i=1}^{d} \cap_{n \in \mathbb{N}} \Omega_{n}^{i}$, then $\tilde{\mathbb{P}}\left(\Omega_{0}\right)=1$, and for any $\omega \in \Omega_{0}$ the set $\left\{s \in[0, \infty): \Delta \tilde{L}(\omega, s) \in G_{n}^{i}\right\}$ is dense in $[0, \infty)$ for any $1 \leq i \leq d$ and $n \in \mathbb{N}$. We stress that $\Omega_{0}$ only depends on the Lévy process $\tilde{L}(t), t \geq 0$.

31-1 Lemma 3.2. If for some $\omega_{0} \in \Omega_{0}$, the following three conditions are satisfied:
(1) $a\left(\omega_{0}\right), b\left(\omega_{0}\right) \in[0, \infty)$ and $a\left(\omega_{0}\right)<b\left(\omega_{0}\right)$;
(2) $g_{i}\left(\omega_{0}, \cdot\right):\left[a\left(\omega_{0}\right), b\left(\omega_{0}\right)\right] \rightarrow \mathbb{R}, 0 \leq i \leq d$, are continuous functions;
(3)

$$
\begin{equation*}
g_{0}\left(\omega_{0}, r\right)+\sum_{i=1}^{d} g_{i}\left(\omega_{0}, r\right) \tilde{L}^{i}\left(\omega_{0}, r\right)=0, \quad \forall r \in\left[a\left(\omega_{0}\right), b\left(\omega_{0}\right)\right] \tag{3.1}
\end{equation*}
$$

Then

$$
g_{i}\left(\omega_{0}, r\right)=0, \quad \forall r \in\left[a\left(\omega_{0}\right), b\left(\omega_{0}\right)\right], \quad 0 \leq i \leq d
$$

Proof. By (3.1), it is sufficient to show that

$$
\begin{equation*}
g_{i}\left(\omega_{0}, r\right)=0, \quad \forall r \in\left[a\left(\omega_{0}\right), b\left(\omega_{0}\right)\right] \tag{3.2}
\end{equation*}
$$

for $i=1, \cdots, d$. Let us prove

$$
g_{1}\left(\omega_{0}, r\right)=0, \quad r \in\left[a\left(\omega_{0}\right), b\left(\omega_{0}\right)\right] .
$$

The proofs of the other cases with $i=2, \cdots, d$ are similar.
The conditions (2) and (3) imply that

$$
\begin{equation*}
0=\sum_{i=1}^{d} \Delta \tilde{L}^{i}\left(\omega_{0}, r\right) g_{i}\left(\omega_{0}, r\right), \quad r \in\left[a\left(\omega_{0}\right), b\left(\omega_{0}\right)\right] \tag{3.3}
\end{equation*}
$$

Fix $n \in \mathbb{N}$. For any $s \in\left\{s \in[0, \infty): \Delta \tilde{L}\left(\omega_{0}, s\right) \in G_{n}^{1}\right\} \cap\left[a\left(\omega_{0}\right), b\left(\omega_{0}\right)\right]$, by (3.3) and the definition of $G_{n}^{1}$, one has

$$
\begin{aligned}
0 & =\left|\sum_{i=1}^{d} \Delta \tilde{L}^{i}\left(\omega_{0}, s\right) g_{i}\left(\omega_{0}, s\right)\right| \\
& \geq\left|\Delta \tilde{L}^{1}\left(\omega_{0}, s\right)\right| \cdot\left|g_{1}\left(\omega_{0}, s\right)\right|-\sum_{i=2}^{d}\left|\Delta \tilde{L}^{i}\left(\omega_{0}, s\right)\right| \cdot\left|g_{i}\left(\omega_{0}, s\right)\right| \\
& \geq\left|\Delta \tilde{L}^{1}\left(\omega_{0}, s\right)\right| \cdot\left|g_{1}\left(\omega_{0}, s\right)\right|-\frac{d}{n}\left|\Delta \tilde{L}^{1}\left(\omega_{0}, s\right)\right| \cdot \sum_{i=2}^{d}\left|g_{i}\left(\omega_{0}, s\right)\right|
\end{aligned}
$$

which implies

$$
\left|g_{1}\left(\omega_{0}, s\right)\right| \leq \frac{d}{n} \sum_{i=2}^{d}\left|g_{i}\left(\omega_{0}, s\right)\right|
$$

where we have used the fact that $\left|\Delta \tilde{L}^{1}\left(\omega_{0}, s\right)\right|>0$. Recall that the definition of $\Omega_{0}$ implies that the set $\left\{s \in[0, \infty): \Delta \tilde{L}\left(\omega_{0}, s\right) \in G_{n}^{1}\right\}$ is dense on $[0, \infty)$ for any $n \in \mathbb{N}$. By the continuity of $g_{i}\left(\omega_{0}, \cdot\right), i=1,2, \ldots, d$, we obtain

$$
\sup _{s \in\left[a\left(\omega_{0}\right), b\left(\omega_{0}\right)\right]}\left|g_{1}\left(\omega_{0}, s\right)\right| \leq \frac{d}{n} \sup _{s \in\left[a\left(\omega_{0}\right), b\left(\omega_{0}\right)\right]} \sum_{i=2}^{d}\left|g_{i}\left(\omega_{0}, s\right)\right|, \quad \forall n \in \mathbb{N} .
$$

Since $n$ is arbitrary, we obtain that

$$
g_{1}\left(\omega_{0}, s\right)=0, \quad \forall s \in\left[a\left(\omega_{0}\right), b\left(\omega_{0}\right)\right] .
$$

The proof is complete.
Recall the assumption (1.9): $\nu_{S}((0, \infty))=\infty$. By Lemma 3.1, for the process $S_{t}, t \geq 0$, we have the following result.

## 25-1 Lemma 3.3.

$$
\mathbb{P}^{\mu_{s}}\left(\ell:\left\{s: \Delta S_{s}(\ell)>0\right\} \text { is dense in }(0, \infty)\right)=1
$$

Recall that

$$
\left\langle\mathcal{M}_{0, \sigma} \phi, \phi\right\rangle=\sum_{j \in \mathcal{Z}_{0}} b_{j}^{2} \int_{0}^{\sigma}\left\langle K_{r, \sigma} \phi, e_{j}\right\rangle^{2} \mathrm{~d} \ell_{r}
$$

The first objective this section is to prove the following proposition.
1-66 Proposition 3.4. For any $\alpha \in(0,1], N \in \mathbb{N}$ and $w_{0} \in H$, one has

$$
\begin{equation*}
\mathbb{P}\left(\inf _{\phi \in \mathcal{S}_{\alpha, N}}\left\langle\mathcal{M}_{0, \sigma} \phi, \phi\right\rangle=0\right)=0 \tag{3.4}
\end{equation*}
$$

where $\mathcal{S}_{\alpha, N}=\left\{\phi:\left\|P_{N} \phi\right\| \geq \alpha,\|\phi\|=1\right\}$.
We will prove the following stronger result than (3.4) for later use:

$$
\begin{equation*}
\mathbb{P}\left(\omega=(\mathrm{w}, \ell): \inf _{\phi \in \mathcal{S}_{\alpha, N}} \sum_{j \in \mathcal{Z}_{0}} b_{j}^{2} \int_{\sigma / 2}^{\sigma}\left\langle K_{r, \sigma} \phi, e_{j}\right\rangle^{2} \mathrm{~d} \ell_{r}=0\right)=0 \tag{3.5}
\end{equation*}
$$

Proof. To prove (3.5), we first make some preparations.
For every $i \in \mathcal{Z}_{0}$ and $n \in \mathbb{N}$, set

$$
G_{n}^{i}=\left\{y=\left(y_{j}, j \in \mathcal{Z}_{0}\right) \in \mathbb{R}^{d} \backslash\{0\}: \max _{j \in \mathcal{Z}_{0} \text { with } j \neq i}\left|y_{j}\right|<\frac{\left|y_{i}\right|}{n}\right\}
$$

Then, for any $i \in \mathcal{Z}_{0}$ and $u>0$, we have

$$
\begin{aligned}
& \int_{G_{n}^{i}}(2 \pi u)^{-d / 2} e^{-\frac{|y|^{2}}{2 u}} \mathrm{~d} y=\int_{G_{n}^{1}}(2 \pi u)^{-d / 2} e^{-\frac{|y|^{2}}{2 u}} \mathrm{~d} y \\
= & C_{d} \int_{0}^{\infty} \mathrm{d} z_{1} \int_{0}^{z_{1} / n} \mathrm{~d} z_{2} \cdots \int_{0}^{z_{1} / n} \mathrm{~d} z_{d}(2 \pi u)^{-d / 2} e^{-\frac{\sum_{k=1}^{d} z_{k}^{2}}{2 u}} \\
\geq & C_{d} n^{-d+1}(2 \pi)^{-d / 2} \int_{0}^{\infty} u^{-d / 2} e^{-\frac{d z_{1}^{2}}{2 u}} z_{1}^{d-1} \mathrm{~d} z_{1} \\
= & C_{d} n^{-d+1}(2 \pi)^{-d / 2} \int_{0}^{\infty} e^{-\frac{d x^{2}}{2}} x^{d-1} \mathrm{~d} x \\
= & C_{d, n}>0 .
\end{aligned}
$$

Hence, for every $i \in \mathcal{Z}_{0}$ and $n \in \mathbb{N}$,

$$
\begin{aligned}
\nu_{L}\left(G_{n}^{i}\right) & =\int_{G_{n}^{i}}\left(\int_{0}^{\infty}(2 \pi u)^{-d / 2} e^{-\frac{|y|^{2}}{2 u}} \nu_{S}(\mathrm{~d} u)\right) \mathrm{d} y \\
& =\int_{0}^{\infty} \nu_{S}(\mathrm{~d} u) \int_{G_{n}^{i}}(2 \pi u)^{-d / 2} e^{-\frac{|y|^{2}}{2 u}} \mathrm{~d} y
\end{aligned}
$$

$$
\geq C_{d, n} \nu_{S}((0, \infty))=\infty
$$

where $\nu_{L}$ is the Lévy measure of $L(t)=W_{S_{t}}$ given in (1.8). Therefore, there exists a $\Omega_{0}^{1} \in \mathcal{F}$ with $\mathbb{P}\left(\Omega_{0}^{1}\right)=1$ and on the set $\Omega_{0}^{1}$, Lemma 3.2 applies to the Lévy process $\tilde{L}(t)=L(t)=W_{S_{t}}$.

By Lemma 3.3 , there exists a set $\mathbb{S}_{0} \subseteq \mathbb{S}$ with $\mathbb{P}^{\mu_{s}}\left(\mathbb{S}_{0}\right)=1$ and for any $\ell \in \mathbb{S}_{0}$, $\left\{s: \Delta S_{s}(\ell):=\ell_{s}-\ell_{s-}>0\right\}$ is dense in $(0, \infty)$, which implies that if $f$ is a nonnegative continuous function on some time interval $[a, b]$ and $\int_{a}^{b} f(s) \mathrm{d} \ell_{s}=0$, then $f(s)=0, s \in[a, b]$. Denote $\Omega_{0}^{2}=\mathbb{W} \times \mathbb{S}_{0} \subseteq \Omega$. Obviously, $\mathbb{P}\left(\Omega_{0}^{2}\right)=1$.

We are now in the position to prove (3.5). Set

$$
\begin{equation*}
\mathcal{L}:=\left\{\omega: \inf _{\phi \in \mathcal{S}_{\alpha, N}} \sum_{j \in \mathcal{Z}_{0}} b_{j}^{2} \int_{\sigma / 2}^{\sigma}\left\langle K_{r, \sigma} \phi, e_{j}\right\rangle^{2} \mathrm{~d} \ell_{r}=0\right\} \cap \Omega_{0}^{1} \cap \Omega_{0}^{2} \tag{3.6}
\end{equation*}
$$

In the following, we will prove $\mathcal{L}=\varnothing$, completing the proof of (3.5).
Assume that $\mathcal{L} \neq \varnothing$ and $\omega=(\mathrm{w}, \ell)$ belongs to the event $\mathcal{L}$. Then, for some $\phi$ with

$$
\begin{equation*}
\left\|P_{N} \phi\right\| \geq \alpha \tag{3.7}
\end{equation*}
$$

one has $\int_{\sigma / 2}^{\sigma}\left\langle K_{r, \sigma} \phi, e_{j}\right\rangle^{2} \mathrm{~d} \ell_{r}=0$ for $j \in \mathcal{Z}_{0}$. By the property of $\ell \in \mathbb{S}_{0}$ stated above and the continuity of $\left\langle K_{r, \sigma} \phi, e_{j}\right\rangle$ with respect to $r$, it holds that

$$
\begin{equation*}
\sup _{r \in[\sigma / 2, \sigma]}\left|\left\langle K_{r, \sigma} \phi, e_{j}\right\rangle\right|=0, \quad \forall j \in \mathcal{Z}_{0} \tag{3.8}
\end{equation*}
$$

With the help of (2.17), $\varrho_{t}:=\left\langle K_{t, \sigma} \phi, e_{j}\right\rangle$ satisfies the following equation:

$$
\begin{aligned}
& \partial_{t} \varrho_{t}+c_{j} \varrho_{t}+\left\langle K_{t, \sigma} \phi, B\left(\mathcal{K} w_{t}, e_{j}\right)+B\left(\mathcal{K} e_{j}, w_{t}\right)\right\rangle=0, \\
& \varrho_{\sigma}=\left\langle\phi, e_{j}\right\rangle
\end{aligned}
$$

where $c_{j}$ is a constant depending on $j$. Combining the above equation with (3.8), we deduce that for any $t \in[\sigma / 2, \sigma]$,

$$
\left\langle K_{t, \sigma} \phi, B\left(\mathcal{K} w_{t}, e_{j}\right)+B\left(\mathcal{K} e_{j}, w_{t}\right)\right\rangle=0
$$

Let $v_{t}=w_{t}-\sum_{i \in \mathcal{Z}_{0}} b_{i} W_{S_{t}}^{i} e_{i}$. Then, the above equation becomes

$$
\left\langle K_{t, \sigma} \phi, \tilde{B}\left(v_{t}+\sum_{i \in \mathcal{Z}_{0}} b_{i} W_{S_{t}}^{i} e_{i}, e_{j}\right)\right\rangle=0
$$

That is

$$
f(t)+\sum_{i \in \mathcal{Z}_{0}} W_{S_{t}}^{i} b_{i}\left\langle K_{t, \sigma} \phi, \tilde{B}\left(e_{i}, e_{j}\right)\right\rangle=0, \forall t \in[\sigma / 2, \sigma]
$$

where $f(t):=\left\langle K_{t, \sigma} \phi, \tilde{B}\left(v_{t}, e_{j}\right)\right\rangle$ is a continuous stochastic process. By the assumption (3.6), the above equality and the fact that Lemma 3.2 holds for $\omega \in \Omega_{0}^{1}$, one arrives at that

$$
\left\langle K_{t, \sigma} \phi, \tilde{B}\left(e_{i}, e_{j}\right)\right\rangle=0, \quad \forall t \in[\sigma / 2, \sigma]
$$

Checking through the above arguments, we actually proved that

$$
\begin{align*}
& \left\langle K_{t, \sigma} \phi, e_{j}\right\rangle=0, \forall t \in[\sigma / 2, \sigma] \\
& \Rightarrow\left\langle K_{t, \sigma} \phi, \tilde{B}\left(e_{i}, e_{j}\right)\right\rangle=0, \forall i \in \mathcal{Z}_{0} \text { and } t \in[\sigma / 2, \sigma] \tag{3.9}
\end{align*}
$$

Define the set $\mathcal{Z}_{n} \subseteq \mathbb{Z}_{*}^{2}$ recursively:

$$
\mathcal{Z}_{n}=\left\{i+j \mid j \in \mathcal{Z}_{0}, i \in \mathcal{Z}_{n-1} \text { with }\left\langle i^{\perp}, j\right\rangle \neq 0,|i| \neq|j|\right\}
$$

where $i^{\perp}=\left(i_{2},-i_{1}\right)$. Assume that we have proved

$$
\left\langle K_{t, \sigma} \phi, e_{j}\right\rangle=0, \forall j \in \mathcal{Z}_{n-1} \text { and } t \in[\sigma / 2, \sigma] .
$$

Then, by (3.9), it follows that

$$
\left\langle K_{t, \sigma} \phi, \tilde{B}\left(e_{j}, e_{i}\right)\right\rangle=0, \forall j \in \mathcal{Z}_{n-1}, i \in \mathcal{Z}_{0} \text { and } t \in[\sigma / 2, \sigma] .
$$

It is easy to verify that $\mathcal{Z}_{m}$ is symmetric for any $m \geq 0$, i.e. $\mathcal{Z}_{m}=-\mathcal{Z}_{m}$. Also by the definition of $\mathcal{Z}_{n}$, one can see that

$$
\left\{e_{j}, j \in \mathcal{Z}_{n}\right\} \subseteq \operatorname{span}\left\{\tilde{B}\left(e_{i}, e_{j}\right): j \in \mathcal{Z}_{0}, i \in \mathcal{Z}_{n-1}\right\}
$$

Hence,

$$
\left\langle K_{t, \sigma} \phi, e_{j}\right\rangle=0, \forall j \in \mathcal{Z}_{n} \text { and } t \in[\sigma / 2, \sigma] .
$$

By this recursion,

$$
\left\langle K_{t, \sigma} \phi, e_{j}\right\rangle=0, \quad \forall j \in \cup_{n=1}^{\infty} \mathcal{Z}_{n}=\mathbb{Z}_{*}^{2} \text { and } t \in[\sigma / 2, \sigma]
$$

(Here, we have used [HM06, Proposition 4.4].) Let $t \rightarrow \sigma$ to get $\phi=0$, which contradicts (3.7). Therefore, $\mathcal{L}=\varnothing$.

The proof of (3.5) is complete.

Proposition 3.4 is not sufficient for the proof of Proposition 1.4, we need a stronger statement. For $\alpha \in(0,1], w_{0} \in H, N \in \mathbb{N}, \mathfrak{R}>0$ and $\varepsilon>0$, let $^{2}$

$$
\begin{equation*}
X^{w_{0}, \alpha, N}=\inf _{\phi \in \mathcal{S}_{\alpha, N}}\left\langle\mathcal{M}_{0, \sigma} \phi, \phi\right\rangle \tag{3.10}
\end{equation*}
$$

and denote

$$
\begin{equation*}
r(\varepsilon, \alpha, \mathfrak{R}, N)=\sup _{\left\|w_{0}\right\| \leq \Re} \mathbb{P}\left(X^{w_{0}, \alpha, N}<\varepsilon\right) \tag{3.11}
\end{equation*}
$$

Based on (3.5) and the dissipative property of Navier-Stokes system, we obtain the following result whose proof is given in Appendix B.
Proposition 3.5. For $\alpha \in(0,1], \Re>0$ and $N \in \mathbb{N}$, we have

$$
\lim _{\varepsilon \rightarrow 0} r(\varepsilon, \alpha, \Re, N)=0 .
$$

[^1]
## 4 Proof of Proposition 1.4.

54-1
Let us take $f \in C_{b}^{1}(H)$ and $\xi \in H$ with $\|\xi\|=1$. Compute the derivative of $\mathbb{E}_{w_{0}} f\left(w_{t}\right)$ with respect to $w_{0}$ in the direction $\xi$ :

$$
\begin{equation*}
\nabla_{\xi} \mathbb{E}_{w_{0}} f\left(w_{t}\right)=\mathbb{E} \nabla f\left(w_{t}\right) J_{0, t} \xi \tag{4.1}
\end{equation*}
$$

In the papers [HM06, HM11], their ideas of proof of the asymptotic strong Feller property is to approximate the perturbation $J_{0, t} \xi$ caused by the variation of the initial condition with a variation, $\mathcal{A}_{0, t} v=\mathcal{D}^{v} w_{t}$, of the noise by an appropriate process $v$. Denote by $\rho_{t}$ the residual error between $J_{0, t} \xi$ and $\mathcal{A}_{0, t} v$ :

$$
\rho_{t}=J_{0, t} \xi-\mathcal{A}_{0, t} v
$$

The proof of Proposition 1.4 is much more involved than that in [HM06, HM11] since we even don't have $\mathbb{E}\left\|J_{0, t} \xi\right\|<\infty$.

Let us first explain the main ideas of the proof of Proposition 1.4. Let $\kappa_{0}=\kappa_{0}\left(\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d\right)$ be the constant appeared in the statement of Lemma 2.2. Recall that, for any $\kappa \in\left(0, \kappa_{0}\right.$ ], the stopping times $\sigma_{k}$ are defined in (2.7)(2.8). For any $\kappa \in\left(0, \kappa_{0}\right]$ and $n \in \mathbb{N}$, we define the following random variables on $\mathbb{S}$ :

$$
\begin{equation*}
X_{n}=\int_{\sigma_{n}}^{\sigma_{n+1}} e^{2 \nu\left(\sigma_{n+1}-s\right)-16 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma_{n+1}}-\ell_{s}\right)} \mathrm{d} s, \quad Y_{n}=\ell_{\sigma_{n+1}}-\ell_{\sigma_{n}} \tag{4.2}
\end{equation*}
$$

By the strong law of large numbers, Lemma 2.2 and the definitions of $\sigma_{k}$, we have ${ }^{3}$

$$
\lim _{n \rightarrow \infty} \frac{\sum_{i=0}^{n-1} X_{i}}{n}<\infty, \quad \text { almost surely }
$$

and

$$
\lim _{n \rightarrow \infty} \frac{\sum_{i=0}^{n-1} Y_{i}}{n} \leq \frac{\nu}{8 \mathfrak{B}_{0} \kappa} \lim _{n \rightarrow \infty} \frac{\sum_{i=0}^{n-1}\left(\sigma_{i+1}-\sigma_{i}\right)}{n}<\infty, \quad \text { almost surely }
$$

Therefore, with probability one, we have

$$
\begin{equation*}
\Theta:=\sup _{n \geq 1} \frac{\sum_{i=0}^{n-1} X_{i}}{n}+\sup _{n \geq 1} \frac{\sum_{i=0}^{n-1} Y_{i}}{n}<\infty \tag{4.3}
\end{equation*}
$$

For any $\Upsilon, M>0, f \in C_{b}^{1}(H)$ and $w_{0}, w_{0}^{\prime} \in B_{H}(\Upsilon):=\{w \in H,\|w\|<\Upsilon\}$, one has

$$
\begin{align*}
& \left|P_{t} f\left(w_{0}\right)-P_{t} f\left(w_{0}^{\prime}\right)\right|=\left|\mathbb{E} f\left(w_{t}^{w_{0}}\right)-\mathbb{E} f\left(w_{t}^{w_{0}^{\prime}}\right)\right| \\
& \leq\left|\mathbb{E} f\left(w_{t}^{w_{0}}\right) I_{\{\Theta \leq M\}}-\mathbb{E} f\left(w_{t}^{w_{0}^{\prime}}\right) I_{\{\Theta \leq M\}}\right|+2\|f\|_{\infty} \mathbb{P}(\Theta \geq M) \\
& :=I_{1}+I_{2} \tag{4.4}
\end{align*}
$$

[^2]where $w_{t}^{w_{0}}$ is the solution of equation (1.2) with initial value $w_{0}$.
One can choose the constant $M$ sufficiently large, independent of the initial condition $w_{0}$ and time $t$, to make $I_{2}$ arbitrarily small. The main difficulty lies in the estimate of $I_{1}$. Denote $P_{t}^{M} f\left(w_{0}\right)=\mathbb{E}\left[f\left(w_{t}^{w_{0}}\right) I_{\{\Theta \leq M\}}\right]$. For any process $v \in L^{2}\left(\Omega \times[0, \infty) ; \mathbb{R}^{d}\right)=L^{2}\left(\mathbb{W} \times \mathbb{S} \times[0, \infty) ; \mathbb{R}^{d}\right)$, we write
\[

$$
\begin{align*}
& \left|\nabla_{\xi} P_{t}^{M} f\left(w_{0}\right)\right|=\left|\mathbb{E} \nabla f\left(w_{t}\right) J_{0, t} \xi I_{\{\Theta \leq M\}}\right| \\
& =\left|\mathbb{E}\left[\nabla f\left(w_{t}\right) \mathcal{D}^{v} w_{t} I_{\{\Theta \leq M\}}\right]+\mathbb{E}\left[\nabla f\left(w_{t}\right) \rho_{t} I_{\{\Theta \leq M\}}\right]\right| \tag{4.5}
\end{align*}
$$
\]

For any fixed $\ell \in \mathbb{S}$, the process $v=v^{\ell}$ in the above will be chosen such that $v^{\ell} \in L^{2}\left(\mathbb{W} \times\left[0, \ell_{t}\right] ; \mathbb{R}^{d}\right)$ and that $v^{\ell}$ is Skorokhod integrable with respect to the Brownian motion $W$. Since $\{\ell: \Theta(\ell) \leq M\}$ is independent of the Brownian motion $W_{t}$, it holds that

$$
\begin{align*}
& \mathbb{E}\left[\nabla f\left(w_{t}\right) \mathcal{D}^{v} w_{t} I_{\{\Theta \leq M\}}\right]=\mathbb{E}^{\mu_{\mathbb{S}}}\left[I_{\{\Theta \leq M\}} \mathbb{E}^{\mu_{\mathbb{W}}}\left(\nabla f\left(w_{t}\right) \mathcal{D}^{v} w_{t}\right)\right] \\
& =\mathbb{E}^{\mu_{\mathbb{S}}}\left[I_{\{\Theta \leq M\}} \mathbb{E}^{\mu_{\mathbb{W}}}\left(f\left(w_{t}\right) \int_{0}^{\ell_{t}} v(s) \mathrm{d} W(s)\right)\right] \\
& =\mathbb{E}\left[f\left(w_{t}\right) \int_{0}^{\ell_{t}} v(s) \mathrm{d} W(s) I_{\{\Theta \leq M\}}\right] \tag{4.6}
\end{align*}
$$

In the above, for any fixed $\ell \in \mathbb{S}$, the integral $\int_{0}^{\ell_{t}} v(s) \mathrm{d} W(s)$ is interpreted as the Skohorod integral. In a word, the estimate of $I_{1}$ is obtained through some gradient estimates of $\nabla_{\xi} P_{t}^{M} f\left(w_{0}\right)$. In order to do this, by (4.5)-(4.6), we need to select suitable direction $v$ and do some moment estimates for $\rho_{t}$ and $\int_{0}^{\ell_{t}} v(s) \mathrm{d} W(s)$. This will be done in subsections 4.1-4.3. In subsection 4.4, we complete the proof of Proposition 1.4.

### 4.1 The choice of $v$.

In this section, we always assume that $\|\xi\|=1$. For any $\kappa>0$, recall that the stopping times $\sigma_{k}$ are defined in (2.7)-(2.8). For any $\ell \in \mathbb{S}$ and $\kappa>0$, we will define the perturbation $v$ to be 0 on all intervals of the type $\left[\ell_{\sigma_{n+1}}, \ell_{\sigma_{n+2}}\right], n \in 2 \mathbb{N}$, and by some $v_{\sigma_{n}, \sigma_{n+1}} \in L^{2}\left(\left[\ell_{\sigma_{n}}, \ell_{\sigma_{n+1}}\right], H\right), n \in 2 \mathbb{N}$ on the remaining intervals. For fixed $\ell \in \mathbb{S}$ and $n \in 2 \mathbb{N}$, define the infinitesimal variation:

$$
\begin{align*}
& v_{\sigma_{n}, \sigma_{n+1}}(r)=\mathcal{A}_{\sigma_{n}, \sigma_{n+1}}^{*}\left(\mathcal{M}_{\sigma_{n}, \sigma_{n+1}}+\beta \mathbb{I}\right)^{-1} J_{\sigma_{n}, \sigma_{n+1}} \rho_{\sigma_{n}}, r \in\left[\ell_{\sigma_{n}}, \ell_{\sigma_{n+1}}\right]  \tag{4.7}\\
& v_{\sigma_{n+1}, \sigma_{n+2}}(r)=0, r \in\left[\ell_{\sigma_{n+1}}, \ell_{\sigma_{n+2}}\right]
\end{align*}
$$

where $\rho_{\sigma_{n}}$ is the residual of the infinitesimal displacement at time $\sigma_{n}$. Set

$$
v(r)=\left\{\begin{array}{l}
v_{\sigma_{n}, \sigma_{n+1}}(r), \quad r \in\left[\ell_{\sigma_{n}}, \ell_{\sigma_{n+1}}\right] \text { and } n \in 2 \mathbb{N},  \tag{4.8}\\
v_{\sigma_{n+1}, \sigma_{n+2}}(r), \quad r \in\left[\ell_{\sigma_{n+1}}, \ell_{\sigma_{n+2}}\right] \text { and } n \in 2 \mathbb{N} .
\end{array}\right.
$$

Here and after, we use $v_{a, b}$ to denote the function $v$ restricted on the interval [ $\ell_{a}, \ell_{b}$ ] and the constant $\beta$ in (4.7) will be decided later. Obviously, $\rho_{0}=J_{0,0} \xi-$ $\mathcal{A}_{0,0} v=\xi$.

Similar to [HM06], we have the following recursions for $\rho_{\sigma_{n}}$.

Lemma 4.1. For any $\beta>0$, if we definethe direction $v$ according to (4.8), then

$$
\rho_{\sigma_{n+2}}=J_{\sigma_{n+1}, \sigma_{n+2}} \beta\left(\mathcal{M}_{\sigma_{n}, \sigma_{n+1}}+\beta \mathbb{I}\right)^{-1} J_{\sigma_{n}, \sigma_{n+1}} \rho_{\sigma_{n}}, \quad \forall n \in 2 \mathbb{N} .
$$

Proof. By a straightforward calculation,

$$
\begin{aligned}
& \rho_{\sigma_{n+2}}=J_{0, \sigma_{n+2}} \xi-\mathcal{A}_{0, \sigma_{n+2}} v=J_{0, \sigma_{n+2}} \xi-\int_{0}^{\ell_{\sigma_{n+2}}} J_{\gamma_{u}, \sigma_{n+2}} Q v_{u} \mathrm{~d} u \\
& =J_{\sigma_{n+1}, \sigma_{n+2}} J_{0, \sigma_{n+1}} \xi-J_{\sigma_{n+1}, \sigma_{n+2}} \int_{0}^{\ell \sigma_{n+1}} J_{\gamma_{u}, \sigma_{n+1}} Q v_{u} \mathrm{~d} u \\
& =J_{\sigma_{n+1}, \sigma_{n+2}} \rho_{\sigma_{n+1}}
\end{aligned}
$$

and

$$
\begin{aligned}
\rho_{\sigma_{n+1}} & =J_{0, \sigma_{n+1}} \xi-\int_{0}^{\ell_{\sigma_{n+1}}} J_{\gamma_{u}, \sigma_{n+1}} Q v_{u} \mathrm{~d} u \\
& =J_{0, \sigma_{n+1}} \xi-\int_{0}^{\ell_{\sigma_{n}}} J_{\gamma_{u}, \sigma_{n+1}} Q v_{u} \mathrm{~d} u-\int_{\ell_{\sigma_{n}}}^{\ell_{\sigma_{n+1}}} J_{\gamma_{u}, \sigma_{n+1}} Q v_{u} \mathrm{~d} u \\
& =J_{\sigma_{n}, \sigma_{n+1}} \rho_{\sigma_{n}}-\mathcal{A}_{\sigma_{n}, \sigma_{n+1}} \mathcal{A}_{\sigma_{n}, \sigma_{n+1}}^{*}\left(\mathcal{M}_{\sigma_{n}, \sigma_{n+1}}+\beta \mathbb{I}\right)^{-1} J_{\sigma_{n}, \sigma_{n+1}} \rho_{\sigma_{n}} \\
& =\beta\left(\mathcal{M}_{\sigma_{n}, \sigma_{n+1}}+\beta \mathbb{I}\right)^{-1} J_{\sigma_{n}, \sigma_{n+1}} \rho_{\sigma_{n}},
\end{aligned}
$$

which yields the desired result.

### 4.2 The control of $\rho_{\sigma_{n}}$.

Let $A_{\varepsilon}=A_{\varepsilon, w_{0}, \alpha, N}=\left\{X^{w_{0}, \alpha, N} \geq \varepsilon\right\}$, where the random variable $X^{w_{0}, \alpha, N}$ is defined in (3.10). To provide an estimate for $\left\|\rho_{\sigma_{n}}\right\|$, we start with some preparations.

3-3 Lemma 4.2. (c.f. [HM11, Lemma 5.4]) For any positive constants $\beta, \varepsilon, \alpha \in$ $(0,1], N \in \mathbb{N}$ and $\xi \in H$, the following inequality holds with probability 1 :

$$
\begin{equation*}
\beta\left\|P_{N}\left(\beta \mathbb{I}+\mathcal{M}_{0, \sigma}\right)^{-1} \xi\right\| \leq\|\xi\|\left(\alpha \vee \sqrt{\frac{\beta}{\varepsilon}}\right) I_{A_{\varepsilon}}+\|\xi\| I_{A_{\varepsilon}^{c}} \tag{4.9}
\end{equation*}
$$

Proof. On the event $A_{\varepsilon}^{c}$, the inequality (4.9) obviously holds. On the event $A_{\varepsilon}$, this inequality is proved in [HM11, Lemma 5.14], so we omit the details.

Let $\mathcal{R}_{\sigma_{n}, \sigma_{n+1}}^{\beta}=\beta\left(\mathcal{M}_{\sigma_{n}, \sigma_{n+1}}+\beta \mathbb{I}\right)^{-1}$. We have the following estimate for $\mathcal{R}_{\sigma_{n}, \sigma_{n+1}}^{\beta}$.

3-9 Lemma 4.3. For any $\kappa>0, \delta \in(0,1], p \geq 1$ and $N \in \mathbb{N}$, there exists a $\beta=$ $\beta(\kappa, \delta, p, N)>0$ such that

$$
\begin{equation*}
\mathbb{E}\left[\left\|P_{N} \mathcal{R}_{\sigma_{n}, \sigma_{n+1}}^{\beta}\right\|^{p} \mid \mathcal{F}_{\sigma_{n}}\right] \leq \delta e^{\kappa\left\|w_{\sigma_{n}}\right\|^{2}}, \quad \forall n \in \mathbb{N} \tag{4.10}
\end{equation*}
$$

Proof. Denote $\sigma_{1}$ by $\sigma$. We here give a proof for the case $n=0$ and $p \geq 2$. The other cases can be proved similarly. Let $\mathfrak{R}=\mathfrak{R}_{\delta, \kappa}$ be a positive constant such that $\exp \left\{\kappa \mathfrak{R}^{2}\right\} \geq \frac{1}{\delta}$. We divide into the following two cases to prove (4.10).

Case 1: $\left\|w_{0}\right\| \geq \mathfrak{R}$. In this case,

$$
\mathbb{E}\left[\left\|P_{N} \mathcal{R}_{0, \sigma}^{\beta}\right\|^{p} \mid \mathcal{F}_{0}\right] \leq 1 \leq \delta e^{\kappa\left\|w_{0}\right\|^{2}}
$$

Case 2: $\left\|w_{0}\right\| \leq \mathfrak{R}$. For any positive constants $\varepsilon, \beta$ and $\alpha \in(0,1]$, by Lemma 4.2, we have

$$
\mathbb{E}\left[\left\|P_{N} \mathcal{R}_{0, \sigma}^{\beta}\right\|^{p} \mid \mathcal{F}_{0}\right] \leq C_{p}\left(\alpha \vee \sqrt{\frac{\beta}{\varepsilon}}\right)^{p}+C_{p} r(\varepsilon, \alpha, \mathfrak{R}, N)
$$

where $C_{p}$ is a constant only depending on $p$, and $r(\varepsilon, \alpha, \Re, N)$ is defined in (3.11). Choose now $\alpha=\alpha(p)$ small enough such that

$$
C_{p} \alpha^{p} \leq \frac{\delta}{2}
$$

By Proposition 3.5, $\lim _{\varepsilon \rightarrow 0} r(\varepsilon, \alpha, \Re, N)=0$. Pick a small constant $\varepsilon$ such that

$$
C_{p} r(\varepsilon, \alpha, \mathfrak{R}, N) \leq \frac{\delta}{2}
$$

Finally, we choose $\beta$ small enough so that

$$
C_{p}(\sqrt{\beta / \varepsilon})^{p}<\frac{\delta}{2}
$$

Putting the above steps together, we see that $\mathbb{E}\left[\left\|P_{N} \mathcal{R}_{0, \sigma}^{\beta}\right\|^{p} \mid \mathcal{F}_{0}\right] \leq \delta e^{\kappa\left\|w_{0}\right\|^{2}}$.
By Lemma 4.1, for any $n \in 2 \mathbb{N}, \beta>0$ and $N \in \mathbb{N}$, one easily sees that

$$
\begin{align*}
& \rho_{\sigma_{n+2}}=J_{\sigma_{n+1}, \sigma_{n+2}} \mathcal{R}_{\sigma_{n}, \sigma_{n+1}}^{\beta} J_{\sigma_{n}, \sigma_{n+1}} \rho_{\sigma_{n}} \\
& =J_{\sigma_{n+1}, \sigma_{n+2}} Q_{N} \mathcal{R}_{\sigma_{n}, \sigma_{n+1}}^{\beta} J_{\sigma_{n}, \sigma_{n+1}} \rho_{\sigma_{n}}+J_{\sigma_{n+1}, \sigma_{n+2}} P_{N} \mathcal{R}_{\sigma_{n}, \sigma_{n+1}}^{\beta} J_{\sigma_{n}, \sigma_{n+1}} \rho_{\sigma_{n}} \\
& :=\rho_{\sigma_{n+2}}^{(1)}+\rho_{\sigma_{n+2}}^{(2)} . \tag{4.11}
\end{align*}
$$

The values of $\beta$ and $N$ will be decided later.
To estimate $\left\|\rho_{\sigma_{n+2}}\right\|$, we first consider the term $\rho_{\sigma_{n+2}}^{(1)}$. For any $\kappa>0, \xi \in H$ and $n \in \mathbb{N}$, by Lemma 2.7 and Young's inequality, we have

$$
\begin{align*}
& \left\|J_{\sigma_{n+1}, \sigma_{n+2}} Q_{N} \xi\right\|^{2} \\
& \leq C\left(e^{-\nu N^{2}\left(\sigma_{n+2}-\sigma_{n+1}\right)}+\frac{1+\sigma_{n+2}-\sigma_{n+1}}{N^{1 / 4}}\right) \sup _{r \in\left[\sigma_{n+1}, \sigma_{n+2}\right]}\left(\left\|w_{r}\right\|^{5 / 2}+1\right)\|\xi\|^{2} \\
& \quad \times \exp \left\{\frac{\nu \kappa}{120} \int_{\sigma_{n+1}}^{\sigma_{n+2}}\left\|w_{r}\right\|_{1}^{2} e^{-\nu\left(\sigma_{n+2}-r\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma_{n+2}}-\ell_{r}\right)} \mathrm{d} r\right. \\
& \left.\quad+C_{\kappa} \int_{\sigma_{n+1}}^{\sigma_{n+2}} e^{2 \nu\left(\sigma_{n+2}-r\right)-16 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma_{n+2}}-\ell_{r}\right)} \mathrm{d} r\right\} \tag{4.12}
\end{align*}
$$

here and below $C$ is a constant depending on $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d, C_{\kappa}$ is a constant depending on $\kappa, \nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$. Applying (2.28) with $s=\sigma_{n}$ and $T=\sigma_{n+1}$, also with the help of (4.12) and the expression of $\rho_{\sigma_{n+2}}^{(1)}$, we have

$$
\begin{align*}
& \left\|\rho_{\sigma_{n+2}}^{(1)}\right\|^{40} \leq\left\|J_{\sigma_{n+1}, \sigma_{n+2}} Q_{N}\right\|^{40}\left\|J_{\sigma_{n}, \sigma_{n+1}}\right\|^{40}\left\|\rho_{\sigma_{n}}\right\|^{40} \\
& \leq C \exp \left\{\frac{\nu \kappa}{6} \int_{\sigma_{n}}^{\sigma_{n+1}}\left\|w_{s}\right\|_{1}^{2} e^{-\nu\left(\sigma_{n+1}-s\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma_{n+1}}-\ell_{s}\right)} \mathrm{d} s-\frac{\kappa}{6} \mathfrak{B}_{0}\left(\ell_{\sigma_{n+1}}-\ell_{\sigma_{n}}\right)\right\} \\
& \times \exp \left\{\frac{\nu \kappa}{6} \int_{\sigma_{n+1}}^{\sigma_{n+2}}\left\|w_{s}\right\|_{1}^{2} e^{-\nu\left(\sigma_{n+2}-s\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma_{n+2}}-\ell_{s}\right)} \mathrm{d} s-\frac{\kappa}{6} \mathfrak{B}_{0}\left(\ell_{\sigma_{n+2}}-\ell_{\sigma_{n+1}}\right)\right\} \\
& \times \sup _{s \in\left[\sigma_{n+1}, \sigma_{n+2}\right]}\left(\left\|w_{s}\right\|^{50}+1\right) \\
& \times\left(\exp \left\{-20 \nu N^{2}\left(\sigma_{n+2}-\sigma_{n+1}\right)\right\}+\frac{\left(1+\sigma_{n+2}-\sigma_{n+1}\right)^{20}}{N^{5}}\right) \\
& \times \exp \left\{\frac{\kappa}{6} \mathfrak{B}_{0}\left(\ell_{\sigma_{n+1}}-\ell_{\sigma_{n}}\right)+\frac{\kappa}{6} \mathfrak{B}_{0}\left(\ell_{\sigma_{n+2}}-\ell_{\sigma_{n+1}}\right)\right\} \\
& \times \exp \left\{C_{\kappa} \int_{\sigma_{n}}^{\sigma_{n+1}} e^{2 \nu\left(\sigma_{n+1}-s\right)-16 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma_{n+1}}-\ell_{s}\right)} \mathrm{d} s+C_{\kappa} \int_{\sigma_{n+1}}^{\sigma_{n+2}} e^{2 \nu\left(\sigma_{n+2}-s\right)-16 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma_{n+2}}-\ell_{s}\right)} \mathrm{d} s\right\} \\
& \times\left\|\rho_{\sigma_{n}}\right\|^{40} \\
& :=C U_{n} U_{n+1} V_{n} R_{n} e^{C_{\kappa}\left(Y_{n}+Y_{n+1}\right)} e^{C_{\kappa}\left(X_{n}+X_{n+1}\right)}\left\|\rho_{\sigma_{n}}\right\|^{40}, \tag{4.13}
\end{align*}
$$

where $X_{n}, Y_{n}$ are defined in (4.2), and

$$
\begin{aligned}
U_{n} & =\exp \left\{\frac{\nu \kappa}{6} \int_{\sigma_{n}}^{\sigma_{n+1}}\left\|w_{s}\right\|_{1}^{2} e^{-\nu\left(\sigma_{n+1}-s\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma_{n+1}}-\ell_{s}\right)} \mathrm{d} s-\frac{\kappa}{6} \mathfrak{B}_{0}\left(\ell_{\sigma_{n+1}}-\ell_{\sigma_{n}}\right)\right\} \\
V_{n} & =\sup _{s \in\left[\sigma_{n+1}, \sigma_{n+2}\right]}\left(\left\|w_{s}\right\|^{50}+1\right), \\
R_{n} & =\exp \left\{-20 \nu N^{2}\left(\sigma_{n+2}-\sigma_{n+1}\right)\right\}+\frac{\left(1+\sigma_{n+2}-\sigma_{n+1}\right)^{20}}{N^{5}} .
\end{aligned}
$$

For the second term $\rho_{\sigma_{n+2}}^{(2)}$, using (2.28) twice, with the similar arguments as that for deducing (4.13), we obtain

$$
\begin{aligned}
& \left\|\rho_{\sigma_{n+2}}^{(2)}\right\|^{40} \\
& \leq C U_{n} U_{n+1}\left\|P_{N} \mathcal{R}_{\sigma_{n}, \sigma_{n+1}}^{\beta}\right\|^{40} e^{C_{\kappa}\left(Y_{n}+Y_{n+1}\right)+C_{\kappa}\left(X_{n}+X_{n+1}\right)}\left\|\rho_{\sigma_{n}}\right\|^{40} . \text { (4.14) }
\end{aligned}
$$

Combining (4.11) with (4.13)(4.14), for any $n \in 2 \mathbb{N}$, one arrives at

$$
\left\|\rho_{\sigma_{n+2}}\right\|^{40} \leq C \theta_{n} e^{C_{\kappa}\left(Y_{n}+Y_{n+1}\right)+C_{\kappa}\left(X_{n}+X_{n+1}\right)}\left\|\rho_{\sigma_{n}}\right\|^{40}
$$

where $\theta_{n}=U_{n} U_{n+1} V_{n} R_{n}+U_{n} U_{n+1}\left\|P_{N} \mathcal{R}_{\sigma_{n}, \sigma_{n+1}}^{\beta}\right\|^{40}$. Assume that $\|\xi\|=1$, then $\left\|\rho_{0}\right\|=1$. By recursion, for any $n \in \mathbb{N}$, we have

$$
\begin{equation*}
\left\|\rho_{\sigma_{2 n+2}}\right\|^{40} \leq C^{n+1}\left(\prod_{i=0}^{n} \theta_{2 i}\right) e^{C_{\kappa} \sum_{i=0}^{2 n+1}\left(X_{i}+Y_{i}\right)} \tag{4.15}
\end{equation*}
$$

Recall that $\Theta=\sup _{n \geq 1} \frac{\sum_{i=0}^{n-1} X_{i}}{n}+\sup _{n \geq 1} \frac{\sum_{i=0}^{n-1} Y_{i}}{n}$. Thus, on the event

$$
\{\ell: \Theta(\ell) \leq M\}
$$

it holds that

$$
\begin{equation*}
\left\|\rho_{\sigma_{2 n+2}}\right\|^{40} \leq C^{n+1} e^{C_{\kappa} M(2 n+2)} \prod_{i=0}^{n} \theta_{2 i}, \quad \forall n \in \mathbb{N} \tag{4.16}
\end{equation*}
$$

Notice that $\theta_{i}$ depends on the parameters $N, \beta$. We have the following estimates for $\theta_{i}, i \in \mathbb{N}$.

30-6 Lemma 4.4. For any $\kappa \in\left(0, \kappa_{0}\right]$ and $\delta \in(0,1)$, there exist constants $\beta>0$ and $N \in \mathbb{N}$ which depend on $\kappa, \delta, \nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$ such that

$$
\mathbb{E}\left[\theta_{i} \mid \mathcal{F}_{\sigma_{i}}\right] \leq \delta \exp \left\{\kappa\left\|w_{\sigma_{i}}\right\|^{2}\right\}, \quad \forall i \in \mathbb{N}
$$

Proof. Obviously, for any $\kappa>0, \delta^{\prime}>0$, by Lemma 2.2, there exists a $N=$ $N\left(\kappa, \delta^{\prime}, \nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d\right)>0$ such that

$$
\left(\mathbb{E}\left|R_{i}\right|^{3} \mid \mathcal{F}_{\sigma_{i}}\right)^{1 / 3} \leq \frac{\delta^{\prime}}{4}, \quad \forall i \in \mathbb{N}
$$

Hence, also with the help of Lemma 2.2, it holds that

$$
\begin{align*}
& \mathbb{E}\left[U_{i} U_{i+1} V_{i} R_{i} \mid \mathcal{F}_{\sigma_{i}}\right] \\
& \leq\left(\mathbb{E}\left|U_{i}\right|^{6} \mid \mathcal{F}_{\sigma_{i}}\right)^{1 / 6}\left(\mathbb{E}\left|U_{i+1}\right|^{6} \mid \mathcal{F}_{\sigma_{i}}\right)^{1 / 6}\left(\mathbb{E}\left|V_{i}\right|^{3} \mid \mathcal{F}_{\sigma_{i}}\right)^{1 / 3}\left(\mathbb{E}\left|R_{i}\right|^{3} \mid \mathcal{F}_{\sigma_{i}}\right)^{1 / 3} \\
& \leq C_{\kappa} \frac{\delta^{\prime}}{4} e^{\kappa\left\|w_{\sigma_{i}}\right\|^{2}} \tag{4.17}
\end{align*}
$$

By Lemma 4.3, for any $\delta^{\prime}>0, \kappa>0$ and the $N=N\left(\kappa, \delta^{\prime}, \nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d\right)$ given above, there exists a $\beta=\beta\left(\kappa, \delta^{\prime}, N\right)>0$ such that

$$
\left(\mathbb{E}\left\|P_{N} \mathcal{R}_{\sigma_{i}, \sigma_{i+1}}^{\beta}\right\|^{60} \mid \mathcal{F}_{\sigma_{i}}\right)^{2 / 3} \leq \frac{\delta^{\prime}}{4} \exp \left\{\frac{\kappa}{3}\left\|w_{\sigma_{i}}\right\|^{2}\right\}, \quad \forall i \in \mathbb{N} .
$$

Hence,

$$
\begin{aligned}
& \mathbb{E}\left(U_{i} U_{i+1}\left\|P_{N} \mathcal{R}_{\sigma_{i}, \sigma_{i+1}}^{\beta}\right\|^{40} \mid \mathcal{F}_{\sigma_{i}}\right) \\
& \leq\left(\mathbb{E}\left|U_{i}\right|^{6} \mid \mathcal{F}_{\sigma_{i}}\right)^{1 / 6}\left(\mathbb{E}\left|U_{i+1}\right|^{6} \mid \mathcal{F}_{\sigma_{i}}\right)^{1 / 6}\left(\mathbb{E}\left\|P_{N} \mathcal{R}_{\sigma_{i}, \sigma_{i+1}}^{\beta}\right\|^{60} \mid \mathcal{F}_{\sigma_{i}}\right)^{2 / 3} \\
& \leq C_{\kappa} \frac{\delta^{\prime}}{4} e^{\kappa\left\|w_{\sigma_{i}}\right\|^{2}} .
\end{aligned}
$$

Combining this with (4.17) and setting $\delta^{\prime}=\frac{2 \delta}{C_{\kappa}}$, we complete the proof.
For any $\kappa \in\left(0, \kappa_{0}\right], M>0$ and $\gamma_{0}>0$, the constant $\beta$ in (4.7) is decided through the following Lemma. Recall that, $\mathcal{C}_{0}=\mathcal{C}_{0}(\nu)$ is the constant introduced in Lemma 2.4.

30-1 Lemma 4.5. For any $\kappa \in\left(0, \kappa_{0}\right], M>0, \gamma_{0}>0$, there exists a positive constant $\beta=\beta\left(\kappa, M, \gamma_{0}, \nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d\right)$ such that if we define the direction $v$ according to (4.7), then the following holds

$$
\begin{align*}
& \mathbb{E}_{w_{0}}\left[\left(1+\sigma_{2 n+2}-\sigma_{2 n}\right)^{8} \exp \left\{8 \mathcal{C}_{0} \int_{\sigma_{2 n}}^{\sigma_{2 n+2}}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\}\left\|\rho_{\sigma_{2 n}}\right\|^{4} I_{\{\Theta \leq M\}}\right] \\
& \leq C_{\kappa, M, \gamma_{0}} \exp \left\{4 \kappa a\left\|w_{0}\right\|^{2}-n \gamma_{0}\right\} \tag{4.18}
\end{align*}
$$

for every $n \in \mathbb{N}$ and $w_{0} \in H$, where $a=\frac{1}{1-e^{-1}}, C_{\kappa, M, \gamma_{0}}$ is a constant depending on $\kappa, M, \gamma_{0}$ and $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$.

Proof. As in other places of this paper, $C$ denotes a constant that may depend on $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d ; C_{\kappa}$ denotes a constant that may depend on $\kappa, \nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$; First, we have the following

Claim. For any $\delta \in(0,1)$ and $\kappa \in\left(0, \kappa_{0}\right]$, there exists constants $\beta=$ $\beta\left(\kappa, \delta, \nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d\right)$ and $N=N\left(\kappa, \delta, \nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d\right)$ such that

$$
\begin{equation*}
\mathbb{E} \prod_{i=0}^{n} \theta_{2 i}^{1 / 2} \leq \delta^{(n+1) / 2} e^{2 a \kappa\left\|w_{0}\right\|^{2}+C_{\kappa} n}, \quad \forall n \in \mathbb{N} \tag{4.19}
\end{equation*}
$$

where $a=\frac{1}{1-e^{-1}}, \theta_{i}$ is defined as in Lemma 4.4. Indeed, by Lemma 4.4 and (2.14), we can choose a $\beta>0$ and a $N \in \mathbb{N}$ such that

$$
\begin{aligned}
& \mathbb{E} \prod_{i=0}^{n} \theta_{2 i}^{1 / 2}=\mathbb{E}\left[\prod_{i=0}^{n} \theta_{2 i}^{1 / 2} e^{-\frac{\kappa}{2} \sum_{i=0}^{n}\left\|w_{\sigma_{2 i}}\right\|^{2}} e^{\frac{\kappa}{2} \sum_{i=0}^{n}\left\|w_{\sigma_{2 i}}\right\|^{2}}\right] \\
& \leq\left(\mathbb{E} \prod_{i=0}^{n} \theta_{2 i} e^{-\kappa \sum_{i=0}^{n}\left\|w_{\sigma_{2 i}}\right\|^{2}}\right)^{1 / 2}\left(\mathbb{E} e^{\kappa \sum_{i=0}^{n}\left\|w_{\sigma_{2 i}}\right\|^{2}}\right)^{1 / 2} \\
& \leq \delta^{(n+1) / 2} e^{(a+1) \kappa\left\|w_{0}\right\|^{2}+C_{\kappa} n},
\end{aligned}
$$

which yields (4.19).
Now we are in a position to prove (4.18). By Young's inequality and (4.15), we have

$$
\begin{align*}
& \left(1+\sigma_{2 n+2}-\sigma_{2 n}\right)^{8} \exp \left\{8 \mathcal{C}_{0} \int_{\sigma_{2 n}}^{\sigma_{2 n+2}}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\}\left\|\rho_{\sigma_{2 n}}\right\|^{4} \\
& \leq\left(1+\sigma_{2 n+2}-\sigma_{2 n}\right)^{8} \\
& \times \exp \left\{\frac{\nu \kappa}{6} \int_{\sigma_{2 n}}^{\sigma_{2 n+1}}\left\|w_{s}\right\|_{1}^{2} e^{-\nu\left(\sigma_{2 n+1}-s\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma_{2 n+1}}-\ell_{s}\right)} \mathrm{d} s-\frac{\kappa}{6} \mathfrak{B}_{0}\left(\ell_{\sigma_{2 n+1}}-\ell_{\sigma_{2 n}}\right)\right\} \\
& \times \exp \left\{\frac{\nu \kappa}{6} \int_{\sigma_{2 n+1}}^{\sigma_{2 n+2}}\left\|w_{s}\right\|_{1}^{2} e^{-\nu\left(\sigma_{2 n+2}-s\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma_{2 n+2}}-\ell_{s}\right)} \mathrm{d} s-\frac{\kappa}{6} \mathfrak{B}_{0}\left(\ell_{\sigma_{2 n+2}}-\ell_{\sigma_{2 n+1}}\right)\right\} \\
& \times \exp \left\{C_{\kappa} X_{2 n}+C_{\kappa} X_{2 n+1}+\frac{\kappa}{6} \mathfrak{B}_{0}\left(Y_{2 n}+Y_{2 n+1}\right)\right\} \\
& \times C^{n / 10}\left(\prod_{i=0}^{n-1} \theta_{2 i}^{1 / 10}\right) e^{\sum_{i=0}^{2 n-1} C_{\kappa}\left(X_{i}+Y_{i}\right)} \tag{4.20}
\end{align*}
$$

where $X_{i}=\int_{\sigma_{i}}^{\sigma_{i+1}} e^{2 \nu\left(\sigma_{i+1}-s\right)-16 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma_{i+1}}-\ell_{s}\right)} \mathrm{d} s, Y_{i}=\ell_{\sigma_{i+1}}-\ell_{\sigma_{i}}$. Notice that the right hand of (4.20) depends on $\beta$ and $N$ through $\theta_{i}$. We will determine the values of $\beta$ and $N$ such that (4.18) hold.

For $i \in\{2 n, 2 n+1\}$, set
$\zeta_{i}=\mathbb{E} \exp \left\{\frac{\nu \kappa}{6} \int_{\sigma_{i}}^{\sigma_{i+1}}\left\|w_{s}\right\|_{1}^{2} e^{-\nu\left(\sigma_{i+1}-s\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma_{i+1}}-\ell_{s}\right)} \mathrm{d} s-\frac{\kappa}{6} \mathfrak{B}_{0}\left(\ell_{\sigma_{i+1}}-\ell_{\sigma_{i}}\right)\right\}$.
By Lemma 2.2, we have

$$
\begin{align*}
& \left(\mathbb{E}\left(1+\sigma_{2 n+2}-\sigma_{2 n}\right)^{\frac{120}{7}}\right)^{7 / 15} \leq C_{\kappa},  \tag{4.21}\\
& \left(\mathbb{E} \zeta_{i}^{6}\right)^{1 / 6} \leq C_{\kappa} \mathbb{E} \exp \left\{\kappa e^{-1}\left\|w_{\sigma_{i}}\right\|^{2} / 6\right\}, \quad i \in\{2 n, 2 n+1\} \tag{4.22}
\end{align*}
$$

By (4.19), for any $\delta \in(0,1)$ and $\kappa \in\left(0, \kappa_{0}\right]$, there exist constants $\beta$ and $N$ such that

$$
\begin{equation*}
\left(\mathbb{E} \prod_{i=0}^{n-1} \theta_{2 i}^{1 / 2}\right)^{1 / 5} \leq \delta^{n / 10} e^{2 a \kappa\left\|w_{0}\right\|^{2} / 5+C_{\kappa}(n-1) / 5} \tag{4.23}
\end{equation*}
$$

Notice that on the event $\{\ell: \Theta(\ell) \leq M\}$, it holds that

$$
\begin{equation*}
\sum_{i=0}^{2 n+1}\left(X_{i}+Y_{i}\right) \leq M(2 n+2) \tag{4.24}
\end{equation*}
$$

Hence, by Hölder's inequality, using (4.20)-(4.24) and (2.14), we have

$$
\begin{aligned}
& \mathbb{E}\left[\left(1+\sigma_{2 n+2}-\sigma_{2 n}\right)^{8} \exp \left\{\mathcal{C}_{0} \int_{\sigma_{2 n}}^{\sigma_{2 n+2}}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\}\left\|\rho_{\sigma_{2 n}}\right\|^{4} I_{\{\theta \leq M\}}\right] \\
& \leq \mathbb{E}\left[\left(1+\sigma_{2 n+2}-\sigma_{2 n}\right)^{8} \zeta_{2 n} \zeta_{2 n+1}\left(\prod_{i=0}^{n-1} \theta_{2 i}^{1 / 10}\right) e^{C_{\kappa} M(2 n+2)} C^{n / 10}\right] \\
& \leq C_{\kappa, M}^{n}\left(\mathbb{E}\left(1+\sigma_{2 n+2}-\sigma_{2 n}\right)^{\frac{120}{7}}\right)^{7 / 15}\left(\mathbb{E} \zeta_{2 n}^{6}\right)^{1 / 6}\left(\mathbb{E} \zeta_{2 n+1}^{6}\right)^{1 / 6}\left(\mathbb{E} \prod_{i=0}^{n-1} \theta_{2 i}^{1 / 2}\right)^{1 / 5} \\
& \leq C_{\kappa, M}^{n} \delta^{n / 10} e^{a \kappa\left\|w_{0}\right\|^{2}} \mathbb{E} \exp \left\{\kappa e^{-1}\left\|w_{\sigma_{2 n}}\right\|^{2} / 6\right\} \mathbb{E} \exp \left\{\kappa e^{-1}\left\|w_{\sigma_{2 n+1}}\right\|^{2} / 6\right\} \\
& \leq C_{\kappa, M}^{n} \delta^{n / 10} e^{4 a \kappa\left\|w_{0}\right\|^{2}}, \quad \forall n \in \mathbb{N}
\end{aligned}
$$

where $C_{\kappa, M} \geq 1$ is a constant depending on $\kappa, M$ and $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$. Choose $\delta=\delta\left(\kappa, M, \gamma_{0}, \nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d\right)$ sufficiently small so that

$$
C_{\kappa, M}^{n} \delta^{n / 10} \leq e^{-n \gamma_{0}}, \quad \forall n \in \mathbb{N} .
$$

Then, we adjust the values of $\beta$ and $N$ according to (4.23). The proof of (4.18) is complete.

### 4.3 The control of $\int_{0}^{\ell_{t}} v(s) \mathrm{d} W(s)$.

For any $M, t>0$ and $n \in \mathbb{N}$, the aim of this subsection is to give an estimate for the moment of the stochastic integral:

$$
\mathbb{E}\left[\left|\int_{\ell_{\sigma_{2 n}}}^{\left(\ell_{\sigma_{2 n+1}} \wedge \ell_{t}\right) \vee \ell_{\sigma_{2 n}}} v(s) \mathrm{d} W(s)\right|^{2} I_{\{\Theta \leq M\}}\right]
$$

We start with an estimate on the moments of $\rho_{t}$.
L:3.3 Lemma 4.6. For any $\kappa \in\left(0, \kappa_{0}\right], M>0, \gamma_{0}>0$, let $\beta$ be the constant chosen according to Lemma 4.5. Then, for any $w_{0} \in H, n \in \mathbb{N}$ and $t \geq 0$, one has

$$
\begin{equation*}
\mathbb{E}_{w_{0}}\left[\left\|\rho_{t}\right\|^{4} I_{\{\Theta \leq M\}} I_{\left\{t \in\left[\sigma_{2 n}, \sigma_{2 n+2}\right)\right\}}\right] \leq C_{\kappa, M, \gamma_{0}} \exp \left\{4 a \kappa\left\|w_{0}\right\|^{2}-n \gamma_{0}\right\} \tag{4.25}
\end{equation*}
$$

3.6
where $C_{\kappa, M, \gamma_{0}}$ is a constant depending on $\kappa, M, \gamma_{0}$ and $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$.
Proof. From the construction we have

$$
\rho_{t}= \begin{cases}J_{\sigma_{2 n}, t} \rho_{\sigma_{2 n}}-\mathcal{A}_{\sigma_{2 n}, t} v_{\sigma_{2 n}, t}, & \text { for } t \in\left[\sigma_{2 n}, \sigma_{2 n+1}\right] \\ J_{\sigma_{2 n+1}, t} \rho_{\sigma_{2 n+1}}, & \text { for } t \in\left[\sigma_{2 n+1}, \sigma_{2 n+2}\right]\end{cases}
$$

for any $n \geq 0$. Using (4.7) and inequalities (2.31)(2.33), we get

$$
\begin{equation*}
\left\|v_{\sigma_{2 n}, \sigma_{2 n+1}}\right\|_{L^{2}\left(\left[\ell_{\sigma_{2 n}}, \ell_{\sigma_{2 n+1}}\right] ; \mathbb{R}^{d}\right)} \leq \beta^{-1 / 2}\left\|J_{\sigma_{2 n}, \sigma_{2 n+1}} \rho_{\sigma_{2 n}}\right\| \tag{4.26}
\end{equation*}
$$

$\square$
Hence, by Lemma 2.4, (2.30) and the definition of $\sigma_{2 n+1}$, for any $t \in\left[\sigma_{2 n}, \sigma_{2 n+1}\right]$,

$$
\begin{aligned}
\left\|\rho_{t}\right\| & \leq\left\|J_{\sigma_{2 n}, t} \rho_{\sigma_{2 n}}\right\|+\left\|\mathcal{A}_{\sigma_{2 n}, t} v_{\sigma_{2 n}, t}\right\| \\
& \leq\left\|J_{\sigma_{2 n}, t} \rho_{\sigma_{2 n}}\right\|+\left\|\mathcal{A}_{\sigma_{2 n}, t}\right\|_{\mathcal{L}\left(L^{2}\left(\left[\ell_{\sigma_{2 n}}, \ell_{t}\right] ; \mathbb{R}^{d}\right), H\right)}\left\|v_{\sigma_{2 n}, t}\right\|_{L^{2}\left(\left[\ell_{\sigma_{2 n}}, \ell_{\sigma_{2 n+1}}\right] ; \mathbb{R}^{d}\right)} \\
& \leq\left\|J_{\sigma_{2 n}, t} \rho_{\sigma_{2 n}}\right\|+\left\|\mathcal{A}_{\sigma_{2 n}, t}\right\|_{\mathcal{L}\left(L^{2}\left(\left[\ell_{\sigma_{2 n}}, \ell_{t}\right] ; \mathbb{R}^{d}\right), H\right)}\left\|v_{\sigma_{2 n}, \sigma_{2 n+1}}\right\|_{L^{2}\left(\left[\ell_{\sigma_{2 n}}, \ell_{\sigma_{2 n}+1}\right] ; \mathbb{R}^{d}\right)} \\
& \leq\left\|J_{\sigma_{2 n}, t} \rho_{\sigma_{2 n}}\right\|+C \beta^{-1 / 2}\left(\ell_{\sigma_{2 n+1}}-\ell_{\sigma_{2 n}}\right)^{1 / 2}\left\|J_{\sigma_{2 n}, \sigma_{2 n+1}} \rho_{\sigma_{2 n}}\right\| \sup _{s \in\left[\sigma_{2 n}, t\right]}\left\|J_{s, t}\right\|_{\mathcal{L}(H, H)} \\
& \leq C_{\kappa, \beta}\left(1+\sigma_{2 n+1}-\sigma_{2 n}\right) \exp \left\{\mathcal{C}_{0} \int_{\sigma_{2 n}}^{\sigma_{2 n+1}}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\}\left\|\rho_{\sigma_{2 n}}\right\|,
\end{aligned}
$$

where $C_{\kappa, \beta}$ is a constant depending on $\kappa, \beta, \nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$. For any $t \in$ $\left[\sigma_{2 n+1}, \sigma_{2 n+2}\right]$, also by Lemma 2.4, it holds that
$\left\|\rho_{t}\right\| \leq \sup _{t \in\left[\sigma_{2 n+1}, \sigma_{2 n+2}\right]}\left\|J_{\sigma_{2 n+1}, t} \rho_{\sigma_{2 n+1}}\right\| \leq \mathcal{C}_{0} \exp \left\{\mathcal{C}_{0} \int_{\sigma_{2 n+1}}^{\sigma_{2 n+2}}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\}\left\|\rho_{\sigma_{2 n+1}}\right\|(.4 .28)$
Combining (4.27),(4.28) with (4.18), we complete the proof.

L:3.4 Lemma 4.7. For any $M>0, \gamma_{0}>0, \kappa \in\left(0, \kappa_{0}\right]$, let $\beta$ be the constant chosen according to Lemma 4.5. Then,

$$
\begin{align*}
& \mathbb{E}_{w_{0}}\left[\left|\int_{\ell_{\sigma_{2 n}}}^{\ell_{\sigma_{2 n+1}}} v(s) \mathrm{d} W(s)\right|^{2} I_{\{\Theta \leq M\}}\right]  \tag{4.29}\\
& \leq C_{\kappa, M, \gamma_{0}} \exp \left\{2 \kappa a\left\|w_{0}\right\|^{2}-\gamma_{0} n / 2\right\}
\end{align*}
$$

and

$$
\begin{align*}
& \mathbb{E}_{w_{0}}\left[\left|\int_{\ell_{\sigma_{2 n}}}^{\left(\ell_{\sigma_{2 n+1}} \wedge \ell_{t}\right) \vee \ell_{\sigma_{2 n}}} v(s) \mathrm{d} W(s)\right|^{2} I_{\{\Theta \leq M\}}\right]  \tag{4.30}\\
& \leq C_{\kappa, M, \gamma_{0}} \exp \left\{2 \kappa a\left\|w_{0}\right\|^{2}-\gamma_{0} n / 2\right\}
\end{align*}
$$

$\square$
for $n \geq 0, t \geq 0$ and $w_{0} \in H$, here $C_{\kappa, M, \gamma_{0}}$ is a constant depending on $\kappa, M, \gamma_{0}$ and $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$.

Proof. We only prove (4.29); the estimate (4.30) is treated in a similar way. Using the generalised Itô isometry (see Section 1.3 in [Nua06]) and the fact that $v(t)=0$ for $t \in\left[\ell_{\sigma_{2 n+1}}, \ell_{\sigma_{2 n+2}}\right]$, we have

$$
\begin{align*}
& \mathbb{E}_{w_{0}}\left[\left|\int_{\ell_{\sigma_{2 n}}}^{\ell_{\sigma_{2 n+1}}} v(s) \mathrm{d} W(s)\right|^{2} I_{\{\Theta \leq M\}}\right] \\
& =\mathbb{E}_{w_{0}}\left[\int_{\ell_{\sigma_{2 n}}}^{\ell_{\sigma_{2 n+1}}}|v(s)|_{\mathbb{R}^{d}}^{2} \mathrm{~d} s I_{\{\Theta \leq M\}}\right] \\
& \quad+\mathbb{E}_{w_{0}}\left[\int_{\ell_{\sigma_{2 n}}}^{\ell_{\sigma_{2 n+1}}} \int_{\ell_{\sigma_{2 n}}}^{\ell_{\sigma_{2 n+1}}} \operatorname{Tr}\left(\mathcal{D}_{s} v(r) \mathcal{D}_{r} v(s)\right) \mathrm{d} s \mathrm{~d} r I_{\{\Theta \leq M\}}\right] \\
& \leq \mathbb{E}_{w_{0}}\left[\int_{\ell_{\sigma_{2 n}}}^{\ell_{\sigma_{2 n+1}}}|v(s)|_{\mathbb{R}^{d}} \mathrm{~d} s I_{\{\Theta \leq M\}}\right] \\
& \quad+\mathbb{E}_{w_{0}}\left[\int_{\ell_{\sigma_{2 n}}}^{\ell_{\sigma_{2 n+1}}} \int_{\ell_{\sigma_{2 n}}}^{\ell_{\sigma_{2 n+1}}}\left|\mathcal{D}_{r} v_{\sigma_{2 n}, \sigma_{2 n+1}}(s)\right|_{\mathbb{R}^{d} \times \mathbb{R}^{d}}^{2} \mathrm{~d} s \mathrm{~d} r I_{\{\Theta \leq M\}}\right] \\
& =L_{1}+L_{2} \tag{4.31}
\end{align*}
$$

3.11

Using (4.7), (4.26), Lemma 2.4 and Lemma 4.5, we have

$$
\begin{align*}
L_{1} & =\mathbb{E}_{w_{0}} \int_{\ell_{\sigma_{2 n}}}^{\ell_{\sigma_{2 n+1}}}|v(s)|_{\mathbb{R}^{d}}^{2} I_{\{\Theta \leq M\}} \mathrm{d} s \leq \beta^{-1} \mathbb{E}_{w_{0}}\left\|J_{\sigma_{2 n}, \sigma_{2 n+1}} \rho_{\sigma_{2 n}}\right\|^{2} I_{\{\Theta \leq M\}} \\
& \leq \beta^{-1} \mathcal{C}_{0} \mathbb{E}\left[e^{\mathcal{C}_{0} \int_{\sigma_{2 n}}^{\sigma_{2 n+1}}\left\|w_{r}\right\|_{1}^{4 / 3} \mathrm{~d} r}\left\|\rho_{\sigma_{2 n}}\right\|^{2} I_{\{\Theta \leq M\}}\right] \\
& \leq C_{\kappa, M, \gamma_{0}} \exp \left\{2 \kappa a\left\|w_{0}\right\|^{2}-n \gamma_{0} / 2\right\} \tag{4.32}
\end{align*}
$$

where $\kappa \in\left(0, \kappa_{0}\right]$ and $C_{\kappa, M, \gamma_{0}}$ is constant depending on $\kappa, M, \gamma_{0}$ and $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$ that may change from line to line. To estimate $L_{2}$, we use the explicit form of $\mathcal{D}_{r} v$. Notice that, for any $r \in\left[\ell_{\sigma_{2 n}}, \ell_{\sigma_{2 n+1}}\right]$ and $i=1, \ldots, d$,

$$
\mathcal{D}_{r}^{i} v_{\sigma_{2 n}, \sigma_{2 n+1}}=\mathcal{D}_{r}^{i}\left(\mathcal{A}_{\sigma_{2 n}, \sigma_{2 n+1}}^{*}\right)\left(\mathcal{M}_{\sigma_{2 n}, \sigma_{2 n+1}}+\beta \mathbb{I}\right)^{-1} J_{\sigma_{2 n}, \sigma_{2 n+1}} \rho_{\sigma_{2 n}}
$$

$$
\begin{aligned}
+ & \mathcal{A}_{\sigma_{2 n}, \sigma_{2 n+1}}^{*}\left(\mathcal{M}_{\sigma_{2 n}, \sigma_{2 n+1}}+\beta \mathbb{I}\right)^{-1} \\
& \times\left(\mathcal{D}_{r}^{i}\left(\mathcal{A}_{\sigma_{2 n}, \sigma_{2 n+1}}\right) \mathcal{A}_{\sigma_{2 n}, \sigma_{2 n+1}}^{*}+\mathcal{A}_{\sigma_{2 n}, \sigma_{2 n+1}} \mathcal{D}_{r}^{i}\left(\mathcal{A}_{\sigma_{2 n}, \sigma_{2 n+1}}^{*}\right)\right) \\
& \times\left(\mathcal{M}_{\sigma_{2 n}, \sigma_{2 n+1}}+\beta \mathbb{I}\right)^{-1} J_{\sigma_{2 n}, \sigma_{2 n+1}} \rho_{\sigma_{2 n}} \\
+ & \mathcal{A}_{\sigma_{2 n}, \sigma_{2 n+1}}^{*}\left(\mathcal{M}_{\sigma_{2 n}, \sigma_{2 n+1}}+\beta \mathbb{I}\right)^{-1} \mathcal{D}_{r}^{i}\left(J_{\sigma_{2 n}, \sigma_{2 n+1}}\right) \rho_{\sigma_{2 n}}
\end{aligned}
$$

By inequalities (2.31)-(2.33), Lemma 2.4 and Lemma 2.10, we have

$$
\begin{aligned}
\| & \mathcal{D}_{r}^{i} v_{\sigma_{2 n}, \sigma_{2 n+1}} \|_{L^{2}\left(\left[\ell_{\sigma_{2 n}}, \ell_{\sigma_{2 n+1}}\right] ; \mathbb{R}^{d}\right)} \\
\leq & \beta^{-1}\left\|\mathcal{D}_{r}^{i}\left(\mathcal{A}_{\sigma_{2 n}, \sigma_{2 n+1}}\right)\right\|_{\mathcal{L}\left(L^{2}\left(\left[\ell_{\sigma_{2 n}}, \ell_{\sigma_{2 n+1}}\right] ; \mathbb{R}^{d}\right), H\right)}\left\|J_{\sigma_{2 n}, \sigma_{2 n+1}} \rho_{\sigma_{2 n}}\right\| \\
& +2 \beta^{-1}\left\|\mathcal{D}_{r}^{i}\left(\mathcal{A}_{\sigma_{2 n}, \sigma_{2 n+1}}^{*}\right)\right\|_{\mathcal{L}\left(H, L^{2}\left(\left[\ell_{\sigma_{2 n}}, \ell_{\sigma_{2 n+1}}\right] ; \mathbb{R}^{d}\right)\right)}\left\|J_{\sigma_{2 n}, \sigma_{2 n+1}} \rho_{\sigma_{2 n}}\right\| \\
& +\beta^{-1 / 2}\left\|\mathcal{D}_{r}^{i}\left(J_{\sigma_{2 n}, \sigma_{2 n+1}}\right) \rho_{\sigma_{2 n}}\right\| \\
\leq & C_{\kappa, \beta}\left(1+\sigma_{2 n+1}-\sigma_{2 n}\right) \exp \left\{2 \mathcal{C}_{0} \int_{\sigma_{2 n}}^{\sigma_{2 n+1}}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\}\left\|\rho_{\sigma_{2 n}}\right\|
\end{aligned}
$$

where $C_{\kappa, \beta}$ is a constant depending on $\kappa, \beta$ and $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$. By Lemma 4.5 and the fact that $\ell_{\sigma_{2 n+1}}-\ell_{\sigma_{2 n}} \leq \frac{\nu}{8 \mathfrak{B}_{0} \kappa}\left(\sigma_{2 n+1}-\sigma_{2 n}\right)$, it follows that

$$
\begin{aligned}
& \mathbb{E}_{w_{0}} \int_{\ell_{\sigma_{2 n}}}^{\ell_{\sigma_{2 n+1}}} \int_{\ell_{\sigma_{2 n}}}^{\ell_{\sigma_{2 n+1}}}\left|\mathcal{D}_{r} v_{\sigma_{2 n}, \sigma_{2 n+1}}(s)\right|_{\mathbb{R}^{d} \times \mathbb{R}^{d}}^{2} I_{\{\Theta \leq M\}} \mathrm{d} s \mathrm{~d} r \\
& \leq \mathbb{E}\left[\left(\ell_{\sigma_{2 n+1}}-\ell_{\sigma_{2 n}}\right) C_{\kappa, \beta}\left(1+\sigma_{2 n+1}-\sigma_{2 n}\right)^{2} \exp \left\{4 \mathcal{C}_{0} \int_{\sigma_{2 n}}^{\sigma_{2 n+1}}\left\|w_{s}\right\|_{1}^{4 / 3} \mathrm{~d} s\right\}\left\|\rho_{\sigma_{2 n}}\right\|^{2}\right] \\
& \leq C_{\kappa, M, \gamma_{0}} \exp \left\{2 \kappa a\left\|w_{0}\right\|^{2}-n \gamma_{0} / 2\right\}
\end{aligned}
$$

Combining the above estimate with (4.31) and (4.32), we complete the proof.

### 4.4 Proof of Proposition 1.4(continued).

For $\xi \in H$, let $v$ be the process chosen as in (4.7). In order to treat the term $I_{1}$ in (4.4), we observe that

$$
\begin{aligned}
& \left|\nabla_{\xi} P_{t}^{M} f\left(w_{0}\right)\right|=\left|\mathbb{E} \nabla f\left(w_{t}\right) J_{0, t} \xi I_{\{\Theta \leq M\}}\right| \\
& =\left|\mathbb{E}\left[\nabla f\left(w_{t}\right) \mathcal{D}^{v} w_{t} I_{\{\Theta \leq M\}}\right]+\mathbb{E}\left[\nabla f\left(w_{t}\right) \rho_{t} I_{\{\Theta \leq M\}}\right]\right| \\
& =\mathbb{E}\left[f\left(w_{t}\right) \int_{0}^{\ell_{t}} v(s) \mathrm{d} W(s) I_{\{\Theta \leq M\}}\right]+\mathbb{E}\left[\nabla f\left(w_{t}\right) \rho_{t} I_{\{\Theta \leq M\}}\right] \\
& :=I_{11}+I_{12} .
\end{aligned}
$$

For any $M>0, \kappa \in\left(0, \kappa_{0}\right]$ and $\gamma_{0}>0$, we set the value of $\beta$ in (4.7) according to Lemma 4.5. For the term $I_{11}$, by Lemma 4.7 in subsection 4.3, we have

$$
I_{11} \leq\|f\|_{\infty} \sum_{n=0}^{\infty} \mathbb{E}\left|\int_{\ell_{\sigma_{2 n}}}^{\left(\ell_{\sigma_{2 n+1}} \wedge \ell_{t}\right) \vee \ell_{\sigma_{2 n}}} v(s) \mathrm{d} W_{s} I_{\{\Theta \leq M\}}\right|
$$

$$
\begin{aligned}
& \leq\|f\|_{\infty} \sum_{n=0}^{\infty}\left(\mathbb{E}\left|\int_{\ell_{\sigma_{2 n}}}^{\left(\ell_{\sigma_{2 n+1}} \wedge \ell_{t}\right) \vee \ell_{\sigma_{2 n}}} v(s) \mathrm{d} W_{s}\right|^{2} I_{\{\Theta \leq M\}}\right)^{1 / 2} \\
& \leq\|f\|_{\infty} \sum_{n=0}^{\infty} C_{\kappa, M, \gamma_{0}} \exp \left\{\kappa a\left\|w_{0}\right\|^{2}-\gamma_{0} n / 4\right\}
\end{aligned}
$$

Now consider the term $I_{12}$. By Lemma 4.6 in subsection 4.3, we have

$$
\begin{aligned}
I_{12} & \leq \sum_{n=0}^{\infty}\|\nabla f\|_{\infty} \mathbb{E}_{w_{0}}\left[\left\|\rho_{t}\right\| I_{\{\Theta \leq M\}} I_{\left\{t \in\left[\sigma_{2 n}, \sigma_{2 n+2}\right)\right\}}\right] \\
& \leq \sum_{n=0}^{\infty}\|\nabla f\|_{\infty} C_{\kappa, M, \gamma_{0}} \exp \left\{a \kappa\left\|w_{0}\right\|^{2}-\gamma_{0} n / 4\right\}
\end{aligned}
$$

Combining the estimates of $I_{11}, I_{12}$, for any $\xi$ with $\|\xi\|=1$, we conclude that

$$
\begin{equation*}
\left|\nabla_{\xi} P_{t}^{M} f\left(w_{0}\right)\right| \leq C_{\kappa, M, \gamma_{0}}\left(\|f\|_{\infty}+\|\nabla f\|_{\infty} \exp \left\{a \kappa\left\|w_{0}\right\|^{2}\right\}\right) \tag{4.33}
\end{equation*}
$$

where $C_{\kappa, M, \gamma_{0}}$ is a constant independent of $t$. Let $\gamma(s)=s w_{0}+(1-s) w_{0}^{\prime}$. Then, by (4.33),

$$
\begin{align*}
& \mathbb{E} f\left(w_{t}^{w_{0}}\right) I_{\{\Theta \leq M\}}-\mathbb{E} f\left(w_{t}^{w_{0}^{\prime}}\right) I_{\{\Theta \leq M\}}=\int_{0}^{1}\left\langle\nabla P_{t}^{M} f(\gamma(s)), w_{0}-w_{0}^{\prime}\right\rangle \mathrm{d} s \\
& \leq C_{\kappa, M, \gamma_{0}}\left\|w_{0}-w_{0}^{\prime}\right\|\left(\|f\|_{\infty}+\|\nabla f\|_{\infty} \sup _{s \in[0,1]} \exp \left\{a \kappa\|\gamma(s)\|^{2}\right\}\right) \tag{4.34}
\end{align*}
$$

Combining the estimates (4.34) with (4.4), for any $\kappa \in\left(0, \kappa_{0}\right]$, $w_{0}, w_{0}^{\prime} \in$ $B_{H}(\Upsilon), f \in C_{b}^{1}(H)$ and $M \geq 1, t>0$, we obtain that

$$
\begin{aligned}
& \left|P_{t} f\left(w_{0}\right)-P_{t} f\left(w_{0}^{\prime}\right)\right| \\
& \leq C_{\kappa, M, \gamma_{0}}\left\|w_{0}-w_{0}^{\prime}\right\|\left(\|f\|_{\infty}+\|\nabla f\|_{\infty} \exp \left\{a \kappa \Upsilon^{2}\right\}\right)+2\|f\|_{\infty} \mathbb{P}(\Theta \geq M)
\end{aligned}
$$

For any bounded and Lipschitz continuous function $f$ on $H$, by the arguments in [KPS10, Page 1431], there exists a sequence $\left(f_{n}\right)$ satisfies $\left(f_{n}\right) \subseteq C_{b}^{1}(H)$ and $\lim _{n \rightarrow \infty} f_{n}(x)=f(x)$ pointwise. In addition, $\left\|f_{n}\right\|_{\infty} \leq\|f\|_{\infty}$ and $\left\|\nabla f_{n}\right\|_{\infty} \leq$ $\operatorname{Lip}(f)$, where $\operatorname{Lip}(f)=\sup _{x \neq y} \frac{|f(x)-f(y)|}{\|x-y\|}$. Therefore, for any $t \geq 0$, one has

$$
\begin{align*}
& \left|P_{t} f\left(w_{0}\right)-P_{t} f\left(w_{0}^{\prime}\right)\right|=\lim _{n \rightarrow \infty}\left|P_{t} f_{n}\left(w_{0}\right)-P_{t} f_{n}\left(w_{0}^{\prime}\right)\right| \\
& \leq \lim _{n \rightarrow \infty}\left[C_{\kappa, M, \gamma_{0}}\left\|w_{0}-w_{0}^{\prime}\right\|\left(\left\|f_{n}\right\|_{\infty}+\left\|\nabla f_{n}\right\|_{\infty} \exp \left\{a \kappa \Upsilon^{2}\right\}\right)+2\left\|f_{n}\right\|_{\infty} \mathbb{P}(\Theta \geq M)\right] \\
& \leq C_{\kappa, M, \gamma_{0}}\left\|w_{0}-w_{0}^{\prime}\right\|\left(\|f\|_{\infty}+\operatorname{Lip}(f) \exp \left\{a \kappa \Upsilon^{2}\right\}+2\|f\|_{\infty} \mathbb{P}(\Theta \geq M)\right. \\
& :=J_{1}+J_{2} \tag{4.35}
\end{align*}
$$

For any $\varepsilon>0$, by (4.3), we can set a $M>0$ such that $J_{2}<\frac{\varepsilon}{2}$. Obviously, there exists a $\delta>0$ such that for any $w_{0}, w_{0}^{\prime} \in B_{H}(\Upsilon)$ with $\left\|w_{0}-w_{0}^{\prime}\right\|<\delta, J_{1}<\frac{\varepsilon}{2}$. Combining these with (4.35), one arrives at that for any positive constants $\varepsilon, \Upsilon$, there exists a $\delta>0$, such that
$\left|P_{t} f\left(w_{0}\right)-P_{t} f\left(w_{0}^{\prime}\right)\right|<\varepsilon, \quad \forall t \geq 0$ and $w_{0}, w_{0}^{\prime} \in B_{H}(\Upsilon)$ with $\left\|w_{0}-w_{0}^{\prime}\right\|<\delta$.
This completes the proof of Proposition 1.4.

## 5 Proof of weak irreducibility

We start with the following lemma.
16-1 Lemma 5.1. For any $T>0, \varepsilon>0$ and non-zero reals numbers $b_{i}, i \in \mathcal{Z}_{0}$, one has

$$
\mathbb{P}\left(\sup _{t \in[0, T]}\left\|\sum_{i \in \mathcal{Z}_{0}} b_{i} W_{S_{t}}^{i} e_{i}\right\|<\varepsilon\right) \geq p_{0}>0
$$

where $p_{0}=p_{0}\left(T, \varepsilon,\left\{b_{i}\right\}_{i \in \mathcal{Z}_{0}}\right)$ is a constant.
Proof. For sufficiently big constant $M>0$, one sees that

$$
\begin{aligned}
& \mathbb{P}\left(\sup _{t \in[0, T]}\left\|\sum_{i \in \mathcal{Z}_{0}} b_{i} W_{S_{t}}^{i} e_{i}\right\|<\varepsilon\right) \\
& \geq \mathbb{P}^{\mu_{\mathbb{S}}}\left(S_{T} \leq M\right) \mathbb{P}^{\mu_{\mathbb{W}}}\left(\sup _{t \in[0, M]}\left\|\sum_{i \in \mathcal{Z}_{0}} b_{i} W_{t}^{i} e_{i}\right\|<\varepsilon\right)>0
\end{aligned}
$$

which completes the proof.

Now we are in position to give a proof of Proposition 1.5.
Proof. The proof of this proposition is the same as that in [EM01, Lemma 3.1]. For the reader's convenience, we provide the proof here. Define $v_{t}=w_{t}-\eta_{t}$, where $\eta_{t}=\sum_{i \in \mathcal{Z}_{0}} b_{i} W_{S_{t}}^{i} e_{i}, w_{t}$ is the solution to (1.2) at time $t$. Then, $v_{t}$ satisfies

$$
\frac{\partial v_{t}}{\partial t}=\nu \Delta\left(v_{t}+\eta_{t}\right)+B\left(\mathcal{K} w_{t}, w_{t}\right)=\nu \Delta\left(v_{t}+\eta_{t}\right)+B\left(\mathcal{K} w_{t}, v_{t}+\eta_{t}\right)
$$

Taking the $L^{2}$-inner product of this equation with $v_{t}$ produces

$$
\begin{aligned}
\frac{1}{2} \frac{d}{d t}\left\|v_{t}\right\|^{2} & =-\nu\left\|\nabla v_{t}\right\|^{2}+\left\langle\nu \Delta \eta_{t}, v_{t}\right\rangle+\left\langle B\left(\mathcal{K} w_{t}, \eta_{t}\right), v_{t}\right\rangle \\
& \leq-\nu\left\|\nabla v_{t}\right\|^{2}+C_{1}\left\|v_{t}\right\|\left\|\Delta \eta_{t}\right\|+C_{1}\left\|v_{t}\right\|\left\|\Delta \eta_{t}\right\|\left\|\mathcal{K} w_{t}\right\|_{1} \\
& =-\nu\left\|\nabla v_{t}\right\|^{2}+C_{1}\left\|v_{t}\right\|\left\|\Delta \eta_{t}\right\|+C_{1}\left\|v_{t}\right\|\left\|\Delta \eta_{t}\right\|\left\|\left(v_{t}+\eta_{t}\right)\right\| \\
& \leq-\nu\left\|\nabla v_{t}\right\|^{2}+C_{1}\left\|v_{t}\right\|\left\|\Delta \eta_{t}\right\|+C_{1}\left\|v_{t}\right\|\left\|\Delta \eta_{t}\right\|\left\|v_{t}\right\|+C_{1}\left\|v_{t}\right\|\left\|\Delta \eta_{t}\right\|\left\|\eta_{t}\right\| \\
& \leq-\frac{\nu}{2}\left\|\nabla v_{t}\right\|^{2}+\frac{4 C_{1}^{2}}{\nu}\left\|\Delta \eta_{t}\right\|^{2}+C_{1}\left\|v_{t}\right\|^{2}\left\|\Delta \eta_{t}\right\|+\frac{4 C_{1}^{2}}{\nu}\left\|\Delta \eta_{t}\right\|^{2}\left\|\eta_{t}\right\|^{2}
\end{aligned}
$$

where $C_{1}=C_{1}(\nu)$ is a constant. For any $T, \delta>0$, we define,

$$
\Omega^{\prime}(\delta, T)=\left\{g=\left(g_{s}\right)_{s \in[0, T]} \in D([0, T] ; H): \sup _{s \in[0, T]}\left\|\Delta g_{s}\right\| \leq \min \left\{\delta, \frac{\nu}{4 C_{1}}\right\}\right\}
$$

where $\Delta$ stands for the Laplacian operator. If $\eta \in \Omega^{\prime}(\delta, T)$, one has

$$
\left\|v_{t}\right\|^{2} \leq\left\|v_{0}\right\|^{2} e^{-\frac{\nu}{2} t}+\frac{4 C_{1}^{2}}{\nu} \cdot \frac{2}{\nu} \cdot\left[\min \left(\delta, \frac{\nu}{4 C_{1}}\right)^{4}+\min \left(\delta, \frac{\nu}{4 C_{1}}\right)^{2}\right]
$$

Let $\mathcal{C}$ and $\gamma$ be given as in the statement of Proposition 1.5. As $\left\|w_{0}\right\| \leq \mathcal{C}$, there exists a $T$ and a $\delta$ such that

$$
\left\|v_{T}\right\| \leq \frac{\gamma}{2} \text { and } \delta \leq \frac{\gamma}{2}
$$

Putting everything together, one has

$$
\left\|w_{0}\right\| \leq \mathcal{C} \text { and } \eta \in \Omega^{\prime}(\delta, T) \Rightarrow\left\|w_{T}\right\| \leq\left\|v_{T}\right\|+\left\|\eta_{T}\right\| \leq \gamma
$$

Combining this fact with Lemma 5.1, we complete the proof.

## A Proof of Lemma 2.2.

This section is organized as follows. In the subsection A.1, we make some preparations. Then, we provide the proofs of (2.9)-(2.14) in subsection A.2, the proof of (2.15) is given in subsection A.3.

## A. 1 Preparations

For $\kappa>0, \varepsilon \in(0,1]$ and $\ell \in \mathbb{S}$, set

$$
\ell_{t}^{\varepsilon}=\frac{1}{\varepsilon} \int_{t}^{t+\varepsilon} \ell_{s} \mathrm{~d} s+\varepsilon t
$$

and

$$
\sigma^{\varepsilon}=\sigma^{\varepsilon}(\ell):=\inf \left\{t \geq 0: \nu t-8 \mathfrak{B}_{0} \kappa \ell_{t}^{\varepsilon}>1\right\} .
$$

Keeping in mind that $\ell$ is a càdlàg increasing function from $\mathbb{R}^{+}$to $\mathbb{R}^{+}$with $\ell_{0}=0$, it is easy to see that the following lemma is valid.

Lemma A.1. For $\ell \in \mathbb{S}$,
(i) $\ell^{\varepsilon}:[0, \infty) \rightarrow[0, \infty)$ is continuous and strictly increasing;
(ii) for any $t \geq 0, \ell_{t}^{\varepsilon}$ strictly decreases to $\ell_{t}$ as $\varepsilon$ decreases to 0 .

With regard to stopping times $\sigma^{\varepsilon}$ and $\sigma$, the following moment estimates hold.

Lemma A2
Lemma A.2. There exists a constant $\widetilde{\kappa}_{0}>0$ such that, for any $\kappa \in\left(0, \widetilde{\kappa}_{0}\right]$,

$$
\begin{align*}
\sup _{\varepsilon \in(0,1]} \mathbb{E}^{\mu_{s}} \exp \left\{10 \nu \sigma^{\varepsilon}\right\} & \leq C_{\kappa}  \tag{A.1}\\
\mathbb{E}^{\mu_{s}} e^{10 \nu \sigma} & \leq C_{\kappa} \tag{A.2}
\end{align*}
$$

where $C_{\kappa}$ is a constant depending on $\kappa, \nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$.

Proof. We only give a proof for(A.1); the proof of (A.2) is similar. Let $c_{\kappa}=$ $\int_{0}^{\infty}\left(e^{160 \mathfrak{B}_{0} \kappa u}-1\right) \nu_{S}(\mathrm{~d} u)$. Then $c_{\kappa}<\infty$ for sufficiently small constant $\kappa$. For any $n \in \mathbb{N}$ and $\varepsilon \in(0,1)$, by the fact that $e^{160 \mathfrak{B}_{0} \kappa S_{t}(\ell)-c_{\kappa} t}=e^{160 \mathfrak{B}_{0} \kappa \ell_{t}-c_{\kappa} t}$ is a local martingale(c.f.[App09, Corollary 5.2.2]) and that $\ell_{n}^{\varepsilon} \leq \ell_{n+\varepsilon}+\varepsilon n$, we have

$$
\begin{align*}
& \mathbb{P}^{\mu_{\mathrm{s}}}\left(\ell: \sigma^{\varepsilon}(\ell)>n\right) \leq \mathbb{P}^{\mu_{\mathrm{s}}}\left(\nu n-8 \mathfrak{B}_{0} \kappa \ell_{n}^{\varepsilon} \leq 1\right) \\
& =\mathbb{P}^{\mu_{\mathrm{S}}}\left(160 \mathfrak{B}_{0} \kappa \ell_{n}^{\varepsilon} \geq 20 \nu n-20\right) \\
& \leq \mathbb{P}^{\mu_{\mathrm{S}}}\left(160 \mathfrak{B}_{0} \kappa \ell_{n+\varepsilon} \geq\left(20 \nu-160 \mathfrak{B}_{0} \kappa \varepsilon\right) n-20\right) \\
& =\mathbb{P}^{\mu_{\mathrm{S}}}\left(160 \mathfrak{B}_{0} \kappa \ell_{n+\varepsilon}-c_{\kappa}(n+\varepsilon) \geq\left(20 \nu-160 \mathfrak{B}_{0} \kappa \varepsilon-c_{\kappa}\right) n-20-c_{\kappa} \varepsilon\right) \\
& \leq \exp \left\{-\left(20 \nu-160 \mathfrak{B}_{0} \kappa \varepsilon-c_{\kappa}\right) n+20+c_{\kappa} \varepsilon\right\} . \tag{A.3}
\end{align*}
$$

By the Condition 1.2, one has $\lim _{\kappa \rightarrow 0} c_{\kappa}=0$. Therefore, there exists a constant $\widetilde{\kappa}_{0}>0$ such that

$$
160 \mathfrak{B}_{0} \kappa+c_{\kappa}<10 \nu, \quad \forall \kappa \in\left(0, \widetilde{\kappa}_{0}\right] .
$$

For any $\kappa \in\left(0, \widetilde{\kappa}_{0}\right]$ and $\varepsilon \in(0,1]$, by (A.3) and the above inequality, we conclude that

$$
\begin{aligned}
& \mathbb{E}^{\mu_{s}} e^{10 \nu \sigma^{\varepsilon}} \leq 1+\sum_{n=0}^{\infty} e^{10 \nu(n+1)} \mathbb{P}\left(\sigma^{\varepsilon} \in(n, n+1]\right) \\
& \leq 1+\sum_{n=0}^{\infty} e^{10 \nu(n+1)} \exp \left\{-\left(20 \nu-160 \mathfrak{B}_{0} \kappa \varepsilon-c_{\kappa}\right) n+20+c_{\kappa} \varepsilon\right\} \\
& \leq 1+\frac{\exp \left\{10 \nu+20+c_{\kappa}\right\}}{1-\exp \left\{-10 \nu+160 \mathfrak{B}_{0} \kappa+c_{\kappa}\right\}}<\infty
\end{aligned}
$$

The proof is complete.
For any $\kappa \in\left(0, \widetilde{\kappa}_{0}\right]$, set

$$
\begin{equation*}
\mathbb{S}_{1}=\left\{\ell \in \mathbb{S}: \sigma^{1}(\ell)<\infty\right\} \tag{A.4}
\end{equation*}
$$

We have the following lemma.
lemma A3
Lemma A.3. $\mathbb{P}^{\mu_{s}}\left(\mathbb{S}_{1}\right)=1$. And for any $\ell \in \mathbb{S}_{1}$, the following statements hold.
(1) For any $\varepsilon \in(0,1), \sigma^{\varepsilon}<\sigma^{1}<\infty$ and $\nu \sigma^{\varepsilon}-8 \mathfrak{B}_{0} \kappa \ell_{\sigma^{\varepsilon}}^{\varepsilon}=1$;
(2) $\sigma^{\varepsilon}$ strictly decreases to $\sigma$ as $\varepsilon$ decreases to 0 ;
(3) $\ell_{\sigma^{\varepsilon}}^{\varepsilon}$ strictly decreases to $\ell_{\sigma}$ as $\varepsilon$ decreases to 0 ;
(4) $\nu \sigma-8 \mathfrak{B}_{0} \kappa \ell_{\sigma}=1$;
(5) $\lim \sup _{\varepsilon \rightarrow 0} \int_{0}^{\sigma^{\varepsilon}} e^{-\nu\left(\sigma^{\varepsilon}-s\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma^{\varepsilon}}^{\varepsilon}-\ell_{s}^{\varepsilon}\right)} \mathrm{d} \ell_{s}^{\varepsilon} \leq \ell_{\sigma}$.

Proof. Lemma A. 2 implies that $\mathbb{P}^{\mu_{s}}\left(\mathbb{S}_{1}\right)=1$.
By Lemma A. 1 and the definition of $\sigma^{\varepsilon}$, it is easy to see that (1) holds. Moreover, for any $\ell \in \mathbb{S}_{1}, \sigma^{\varepsilon}$ strictly decreases to a constant $a$ as $\varepsilon$ decreases to 0 . On the other hand, for any $\varepsilon \in(0,1]$,

$$
\begin{equation*}
\sigma^{\varepsilon}>a \geq \sigma \tag{A.5}
\end{equation*}
$$

For any $s<a$, by the definition of $\sigma^{\varepsilon}$,

$$
\nu s-8 \mathfrak{B}_{0} \kappa \ell_{s}^{\varepsilon} \leq 1, \quad \forall \varepsilon \in(0,1]
$$

Letting $\varepsilon \rightarrow 0$, by Lemma A.1, we get

$$
\nu s-8 \mathfrak{B}_{0} \kappa \ell_{s} \leq 1
$$

Hence $\sigma \geq a$. Combining this with (A.5), we get (2).
Using Lemma A.1, (2), the definition of $\ell^{\varepsilon}$, and the fact that $\ell$ is increasing yields, it follows that for any $\varepsilon \in(0,1]$,

$$
\ell_{\sigma}<\ell_{\sigma}^{\varepsilon} \leq \ell_{\sigma^{\varepsilon}}^{\varepsilon} \leq \ell_{\sigma^{\varepsilon}+\varepsilon}+\varepsilon \sigma^{\varepsilon}
$$

Letting $\varepsilon \rightarrow 0$, by (2) and the right continuity of $\ell$, we get (3). Combining (1), (2) and (3), one gets (4).

Combining (1) with the fact that for any $s \leq \sigma^{\varepsilon}, \nu s-8 \mathfrak{B}_{0} \kappa \ell_{s}^{\varepsilon} \leq 1$, one arrives at

$$
\begin{gathered}
\limsup _{\varepsilon \rightarrow 0} \int_{0}^{\sigma^{\varepsilon}} e^{-\nu\left(\sigma^{\varepsilon}-s\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma}^{\varepsilon}-\ell_{s}^{\varepsilon}\right)} \mathrm{d} \ell_{s}^{\varepsilon} \\
\leq \\
\limsup _{\varepsilon \rightarrow 0} e^{-1+1} \int_{0}^{\sigma^{\varepsilon}} \mathrm{d} \ell_{s}^{\varepsilon} \leq \limsup _{\varepsilon \rightarrow 0} \ell_{\sigma^{\varepsilon}}^{\varepsilon}=\ell_{\sigma}
\end{gathered}
$$

completing the proof of (5).
The proof of Lemma A. 3 is complete.
Let $\mathcal{H}_{0}=\operatorname{span}\left\{e_{k}: k \in \mathcal{Z}_{0}\right\}$ and $D\left([0, \infty) ; \mathcal{H}_{0}\right)$ be the space of all càdlàg functions taking values in $\mathcal{H}_{0}$. Keeping in mind that $d=\left|\mathcal{Z}_{0}\right|<\infty$, it is well-known that, for any $w_{0} \in H$ and $g \in D\left([0, \infty) ; \mathcal{H}_{0}\right)$, there exists a unique solution $\Psi\left(w_{0}, g\right) \in C([0, \infty) ; H) \cap L_{l o c}^{2}([0, \infty) ; V)$ to the following PDE:

$$
\begin{aligned}
\Psi\left(w_{0}, g\right)(t)= & w_{0}+\nu \int_{0}^{t} \Delta\left(\Psi\left(w_{0}, g\right)(s)+g_{s}\right) \mathrm{d} s \\
& +\int_{0}^{t} B\left(\mathcal{K} \Psi\left(w_{0}, g\right)(s)+\mathcal{K} g_{s}, \Psi\left(w_{0}, g\right)(s)+g_{s}\right) \mathrm{d} s
\end{aligned}
$$

Here $V=\left\{h \in H:\|h\|_{1}<\infty\right\}$.
We denote $\eta_{t}^{\varepsilon}=Q\left(W_{\ell_{t}^{\varepsilon}}-W_{\ell_{0}^{\varepsilon}}\right), \eta_{t}=Q W_{\ell_{t}}, v_{t}^{\varepsilon}=\Psi\left(w_{0}, \eta^{\varepsilon}\right)(t)$, and $v_{t}=$ $\Psi\left(w_{0}, \eta\right)(t)$. It is easy to see that $v_{t}+\eta_{t}$ is the unique solution $w_{t}$ to (1.2), i.e.,
$w_{t}=v_{t}+\eta_{t}$, and for any $\ell \in \mathbb{S}$ and $\varepsilon \in(0,1], w_{t}^{\varepsilon}:=v_{t}^{\varepsilon}+\eta_{t}^{\varepsilon}$ is the solution of the following PDE:

$$
w_{t}^{\varepsilon}=w_{0}+\int_{0}^{t}\left[\nu \Delta w_{s}^{\varepsilon}+B\left(\mathcal{K} w_{s}^{\varepsilon}, w_{s}^{\varepsilon}\right)\right] \mathrm{d} s+Q\left(W_{\ell_{t}^{\varepsilon}}-W_{\ell_{0}^{\varepsilon}}\right)
$$

Recall $\mathbb{S}_{1}$ introduced in (A.4). We have

## lem A4

Lemma A.4. For any $\ell \in \mathbb{S}_{1}$ and $\mathrm{w} \in \mathbb{W}$, the following statements hold:

$$
\begin{align*}
& \lim _{\varepsilon \rightarrow 0} \int_{0}^{\sigma^{\varepsilon}} e^{-\nu\left(\sigma^{\varepsilon}-s\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma}^{\varepsilon}-\ell_{s}^{\varepsilon}\right)}\left\|w_{s}^{\varepsilon}\right\|_{1}^{2} \mathrm{~d} s  \tag{A.6}\\
& =\int_{0}^{\sigma} e^{-\nu(\sigma-s)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma}-\ell_{s}\right)}\left\|w_{s}\right\|_{1}^{2} \mathrm{~d} s
\end{align*}
$$

and

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0}\left\|w_{\sigma^{\varepsilon}}^{\varepsilon}-w_{\sigma}\right\|^{2}=0 \tag{A.7}
\end{equation*}
$$

Proof. To prove this lemma, we first need some a priori estimates for $\Psi$.
By the chain rule and (2.2), there exists a constant $C=C(\nu)>0$ such that, for any $w_{0} \in H, g \in D\left([0, \infty) ; \mathcal{H}_{0}\right)$ and $t \geq 0$,

$$
\begin{aligned}
& \left\|\Psi\left(w_{0}, g\right)(t)\right\|^{2}+\nu \int_{0}^{t}\left\|\Psi\left(w_{0}, g\right)(s)\right\|_{1}^{2} d s \\
\leq & \left\|w_{0}\right\|^{2}+\nu \int_{0}^{t}\left\|g_{s}\right\|_{1}^{2} \mathrm{~d} s+C \int_{0}^{t}\left\|g_{s}\right\|_{2}\left\|\Psi\left(w_{0}, g\right)(s)\right\|^{2} d s+C \int_{0}^{t}\left\|g_{s}\right\|_{1}^{2}\left\|\Psi\left(w_{0}, g\right)(s)\right\| d s
\end{aligned}
$$

Applying the Gronwall lemma and using the fact that, for any $\alpha>0$, there exists a constant $C_{\alpha}$ such that $\|h\|_{\alpha} \leq C_{\alpha}\|h\|, \forall h \in \mathcal{H}_{0}$, there exists a constant $C>0$ such that, for any $T>0$,

$$
\begin{align*}
& \sup _{t \in[0, T]}\left\|\Psi\left(w_{0}, g\right)(t)\right\|^{2}+\nu \int_{0}^{T}\left\|\Psi\left(w_{0}, g\right)(s)\right\|_{1}^{2} d s  \tag{A.8}\\
& \leq C\left(\left\|w_{0}\right\|^{2}+\int_{0}^{T}\left(1+\left\|g_{s}\right\|^{2}\right) d s\right) e^{C \int_{0}^{T}\left(1+\left\|g_{s}\right\|^{2}\right) d s}
\end{align*}
$$

For any $g^{1}, g^{2} \in D\left([0, \infty) ; \mathcal{H}_{0}\right)$, put $\Psi^{1}(t)=\Psi\left(w_{0}, g^{1}\right)(t)$ and $\Psi^{2}(t)=$ $\Psi\left(w_{0}, g^{2}\right)(t)$, simplifying the notation. Using similar arguments as above,

$$
\begin{aligned}
& \left\|\Psi^{1}(t)-\Psi^{2}(t)\right\|^{2}+\nu \int_{0}^{t}\left\|\Psi^{1}(s)-\Psi^{2}(s)\right\|_{1}^{2} d s \\
\leq \quad & \nu \int_{0}^{t}\left\|g^{1}(s)-g^{2}(s)\right\|_{1}^{2} d s \\
& +2 \int_{0}^{t}\left\langle B\left(\mathcal{K} \Psi^{1}(s)+\mathcal{K} g_{s}^{1}, \Psi^{1}(s)+g_{s}^{1}\right)-B\left(\mathcal{K} \Psi^{2}(s)+\mathcal{K} g_{s}^{2}, \Psi^{2}(s)+g_{s}^{2}\right), \Psi^{1}(s)-\Psi^{2}(s)\right\rangle d s
\end{aligned}
$$

$$
\begin{aligned}
= & \nu \int_{0}^{t}\left\|g^{1}(s)-g^{2}(s)\right\|_{1}^{2} d s \\
& +2 \int_{0}^{t}\left\langle B\left(\mathcal{K} \Psi^{1}(s)+\mathcal{K} g_{s}^{1}, \Psi^{1}(s)+g_{s}^{1}\right)-B\left(\mathcal{K} \Psi^{1}(s)+\mathcal{K} g_{s}^{1}, \Psi^{2}(s)+g_{s}^{2}\right), \Psi^{1}(s)-\Psi^{2}(s)\right\rangle d s \\
& +2 \int_{0}^{t}\left\langle B\left(\mathcal{K} \Psi^{1}(s)+\mathcal{K} g_{s}^{1}, \Psi^{2}(s)+g_{s}^{2}\right)-B\left(\mathcal{K} \Psi^{2}(s)+\mathcal{K} g_{s}^{2}, \Psi^{2}(s)+g_{s}^{2}\right), \Psi^{1}(s)-\Psi^{2}(s)\right\rangle d s \\
\leq & \nu \int_{0}^{t}\left\|g^{1}(s)-g^{2}(s)\right\|_{1}^{2} d s+C \int_{0}^{t}\left\|\Psi^{1}(s)-\Psi^{2}(s)\right\|\left\|\Psi^{1}(s)+g_{s}^{1}\right\|\left\|g^{1}(s)-g^{2}(s)\right\|_{2} d s \\
& +C \int_{0}^{t}\left\|\Psi^{1}(s)-\Psi^{2}(s)\right\|_{1}\left\|\Psi^{2}(s)+g_{s}^{2}\right\|_{1}\left\|\Psi^{1}(s)-\Psi^{2}(s)\right\| d s \\
& +C \int_{0}^{t}\left\|g^{1}(s)-g^{2}(s)\right\|_{1}\left\|\Psi^{2}(s)+g_{s}^{2}\right\|_{1}\left\|\Psi^{1}(s)-\Psi^{2}(s)\right\| d s \\
\leq & C\left(1+\sup _{s \in[0, t]}\left\|\Psi^{1}(s)+g_{s}^{1}\right\|^{2}\right) \int_{0}^{t}\left\|g^{1}(s)-g^{2}(s)\right\|^{2} d s \\
& +C \int_{0}^{t}\left(1+\left\|\Psi^{2}(s)+g_{s}^{2}\right\|_{1}^{2}\right)\left\|\Psi^{1}(s)-\Psi^{2}(s)\right\|^{2} d s+\frac{\nu}{2} \int_{0}^{t}\left\|\Psi^{1}(s)-\Psi^{2}(s)\right\|_{1}^{2} d s .
\end{aligned}
$$

Rearranging terms and using the Gronwall lemma, we arrive at, for any $T \geq 0$,

$$
\begin{align*}
& \sup _{t \in[0, T]}\left\|\Psi^{1}(t)-\Psi^{2}(t)\right\|^{2}+\frac{\nu}{2} \int_{0}^{T}\left\|\Psi^{1}(s)-\Psi^{2}(s)\right\|_{1}^{2} d s \\
& \leq C\left(1+\sup _{s \in[0, T]}\left\|\Psi^{1}(s)+g_{s}^{1}\right\|^{2}\right) \int_{0}^{T}\left\|g^{1}(s)-g^{2}(s)\right\|^{2} d s \exp \left\{C \int_{0}^{T}\left(1+\left\|\Psi^{2}(s)+g_{s}^{2}\right\|_{1}^{2}\right) d s\right\} \tag{A.9}
\end{align*}
$$

For any $(\mathrm{w}, \ell) \in \mathbb{W} \times \mathbb{S}$, from the definitions of $\eta_{t}^{\varepsilon}$ and $\eta_{t}$, it is easy to see that, for any $T \geq 0$,

$$
\begin{equation*}
\sup _{\varepsilon \in(0,1]} \sup _{t \in[0, T]}\left(\left\|\eta_{t}^{\varepsilon}(\mathrm{w}, \ell)\right\|+\left\|\eta_{t}(\mathrm{w}, \ell)\right\|\right) \leq C \sup _{t \in\left[0, \ell_{T+1}+T\right]}\left\|\mathrm{w}_{t}\right\|<\infty \tag{A.10}
\end{equation*}
$$

and

$$
\lim _{\varepsilon \rightarrow 0} \int_{0}^{T}\left\|\eta_{t}^{\varepsilon}(\mathrm{w}, \ell)-\eta_{t}(\mathrm{w}, \ell)\right\|^{2} d t=0
$$

Combining the above two estimates with (A.8) and (A.9), there exists a constant $C$ dependent on $\left\|w_{0}\right\|, T, \sup _{t \in\left[0, \ell_{T+1}+T\right]}\left\|\mathrm{w}_{t}\right\|$ such that

$$
\begin{align*}
& \sup _{\varepsilon \in(0,1]}\left(\sup _{t \in[0, T]}\left\|w_{t}^{\varepsilon}\right\|^{2}+\int_{0}^{T}\left\|w_{t}^{\varepsilon}\right\|_{1}^{2} d t\right)(\mathrm{w}, \ell) \\
& \quad+\left(\sup _{t \in[0, T]}\left\|w_{t}\right\|^{2}+\int_{0}^{T}\left\|w_{t}\right\|_{1}^{2} d t\right)(\mathrm{w}, \ell) \leq C \tag{A.11}
\end{align*}
$$

and

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0}\left(\sup _{t \in[0, T]}\left\|v_{t}^{\varepsilon}-v_{t}\right\|^{2}+\int_{0}^{T}\left\|w_{t}^{\varepsilon}-w_{t}\right\|_{1}^{2} d t\right)(\mathrm{w}, \ell)=0 \tag{A.12}
\end{equation*}
$$

Notice that, for any $\ell \in \mathbb{S}_{1}$ and $w \in \mathbb{W}$,

$$
\begin{align*}
& \left\|w_{\sigma^{\varepsilon}}^{\varepsilon}-w_{\sigma}\right\| \leq\left\|v_{\sigma^{\varepsilon}}^{\varepsilon}-v_{\sigma}\right\|+\left\|\eta_{\sigma^{\varepsilon}}^{\varepsilon}-\eta_{\sigma}\right\| \\
& \leq\left\|v_{\sigma^{\varepsilon}}^{\varepsilon}-v_{\sigma^{\varepsilon}}\right\|+\left\|v_{\sigma^{\varepsilon}}-v_{\sigma}\right\|+\left\|\eta_{\sigma^{\varepsilon}}^{\varepsilon}-\eta_{\sigma}\right\| \\
& \leq \sup _{t \in\left[0, \sigma^{1}\right]}\left\|v_{t}^{\varepsilon}-v_{t}\right\|+\left\|v_{\sigma^{\varepsilon}}-v_{\sigma}\right\|+\left\|Q\left(W_{\ell_{\sigma^{\varepsilon}}}-W_{\ell_{0}^{\varepsilon}}\right)-Q W_{\ell_{\sigma}}\right\| . \tag{A.13}
\end{align*}
$$

Applying Lemmas A. 1 and A.3, (A.11)-(A.13), and the fact that $v_{t}$ is continuous in $H$, we deduce (A.6) and (A.7), completing the proof of Lemma A.4.

We also have the following estimate on $w_{t}^{\varepsilon}$.
qu-1 Lemma A.5. There exists a positive constant $C$ which only depends on $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}$ and $d=\left|\mathcal{Z}_{0}\right|$ such that, for any $\kappa \in\left(0, \widetilde{\kappa}_{0}\right], \varepsilon \in(0,1]$ and $\ell \in \mathbb{S}_{1}($ see (A.4)),

$$
\begin{align*}
\mathbb{E}^{\mu_{\mathbb{W}}} \exp \{ & \kappa\left\|w_{\sigma^{\varepsilon}}^{\varepsilon}\right\|-\kappa\left\|w_{0}\right\|^{2} e^{-\nu \sigma^{\varepsilon}+8 \mathfrak{B}_{0} \kappa \ell_{\sigma^{\varepsilon}}^{\varepsilon}} \\
& +\nu \kappa \int_{0}^{\sigma^{\varepsilon}} e^{-\nu\left(\sigma^{\varepsilon}-s\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma^{\varepsilon} \varepsilon}^{\varepsilon}-\ell_{s}^{\varepsilon}\right)}\left\|w_{s}^{\varepsilon}\right\|_{1}^{2} \mathrm{~d} s  \tag{A.14}\\
& \left.-\kappa \mathfrak{B}_{0} \int_{0}^{\sigma^{\varepsilon}} e^{-\nu\left(\sigma^{\varepsilon}-s\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma^{\varepsilon}}^{\varepsilon}-\ell_{s}^{\varepsilon}\right)} \mathrm{d} \ell_{s}^{\varepsilon}\right\} \leq C .
\end{align*}
$$

Here $\widetilde{\kappa}_{0}$ is a constant appeared in Lemma A.2.
Proof. Now we fix $\kappa \in\left(0, \widetilde{\kappa}_{0}\right], \varepsilon \in(0,1]$ and $\ell \in \mathbb{S}_{1}$.
Let $\gamma^{\varepsilon}$ be the inverse function of $\ell^{\varepsilon}$. By a change of variable, for $t \geq \ell_{0}^{\varepsilon}$, $Y_{t}^{\varepsilon}:=w_{\gamma_{t}^{\varepsilon}}^{\varepsilon}, t \in\left[\ell_{0}^{\varepsilon}, \infty\right)$ satisfies the following stochastic equation

$$
\begin{equation*}
Y_{t}^{\varepsilon}=w_{0}+\int_{\ell_{0}^{\varepsilon}}^{t}\left[\nu \Delta Y_{s}^{\varepsilon}+B\left(\mathcal{K} Y_{s}^{\varepsilon}, Y_{s}^{\varepsilon}\right)\right] \dot{\gamma}_{s}^{\varepsilon} \mathrm{d} s+Q\left(W_{t}-W_{\ell_{0}^{\varepsilon}}\right) \tag{A.15}
\end{equation*}
$$

By Itô's formula we have

$$
\mathrm{d}\left\|Y_{t}^{\varepsilon}\right\|^{2}=-2 \nu\left\|Y_{t}^{\varepsilon}\right\|_{1}^{2} \dot{\gamma}_{t}^{\varepsilon} \mathrm{d} t+2\left\langle Y_{t}^{\varepsilon}, Q \mathrm{~d} W_{t}\right\rangle+\mathfrak{B}_{0} \mathrm{~d} t
$$

and

$$
\begin{aligned}
& \mathrm{d} \kappa\left\|Y_{t}^{\varepsilon}\right\|^{2} e^{\nu \gamma_{t}^{\varepsilon}-8 \mathfrak{B}_{0} \kappa t} \\
= & e^{\nu \gamma_{t}^{\varepsilon}-8 \mathfrak{B}_{0} \kappa t}\left[-2 \nu \kappa\left\|Y_{t}^{\varepsilon}\right\|_{1}^{2} \dot{\gamma}_{t}^{\varepsilon} \mathrm{d} t+2 \kappa\left\langle Y_{t}^{\varepsilon}, Q \mathrm{~d} W_{t}\right\rangle+\kappa \mathfrak{B}_{0} \mathrm{~d} t\right] \\
& \quad+\kappa\left\|Y_{t}^{\varepsilon}\right\|^{2} e^{\nu \gamma_{t}^{\varepsilon}-8 \mathfrak{B}_{0} \kappa t}\left(\nu \dot{\gamma}_{t}^{\varepsilon}-8 \mathfrak{B}_{0} \kappa\right) \mathrm{d} t \\
\leq & -\nu \kappa e^{\nu \gamma_{t}^{\varepsilon}-8 \mathfrak{B}_{0} \kappa t}\left\|Y_{t}^{\varepsilon}\right\|_{1}^{2} \dot{\gamma}_{t}^{\varepsilon} \mathrm{d} t+\kappa \mathfrak{B}_{0} e^{\nu \gamma_{t}^{\varepsilon}-8 \mathfrak{B}_{0} \kappa t} \mathrm{~d} t+2 \kappa e^{\nu \gamma_{t}^{\varepsilon}-8 \mathfrak{B}_{0} \kappa t}\left\langle Y_{t}^{\varepsilon}, Q \mathrm{~d} W_{t}\right\rangle \\
& -8 \mathfrak{B}_{0} \kappa^{2}\left\|Y_{t}^{\varepsilon}\right\|^{2} e^{\nu \gamma_{t}^{\varepsilon}-8 \mathfrak{B}_{0} \kappa t} \mathrm{~d} t .
\end{aligned}
$$

Here we have used the inequality: $\|h\|_{1} \geq\|h\|, \forall h \in H$. Hence,

$$
\begin{align*}
& \kappa\left\|Y_{t}^{\varepsilon}\right\|^{2}+\nu \kappa \int_{\ell_{0}^{\varepsilon}}^{t} e^{-\nu\left(\gamma_{t}^{\varepsilon}-\gamma_{s}^{\varepsilon}\right)+8 \mathfrak{B}_{0} \kappa(t-s)}\left\|Y_{s}^{\varepsilon}\right\|_{1}^{2} \dot{\gamma}_{s}^{\varepsilon} \mathrm{d} s \\
& \leq \kappa\left\|w_{0}\right\|^{2} e^{-\nu \gamma_{t}^{\varepsilon}+8 \mathfrak{B}_{0} \kappa t}+\kappa \mathfrak{B}_{0} \int_{\ell_{0}^{\varepsilon}}^{t} e^{-\nu\left(\gamma_{t}^{\varepsilon}-\gamma_{s}^{\varepsilon}\right)+8 \mathfrak{B}_{0} \kappa(t-s)} \mathrm{d} s+\tilde{M}_{t} \tag{A.16}
\end{align*}
$$

where

$$
\begin{aligned}
\tilde{M}_{t}=\tilde{M}_{t}^{\kappa, \varepsilon}= & 2 \kappa \int_{\ell_{0}^{\varepsilon}}^{t} e^{-\nu\left(\gamma_{t}^{\varepsilon}-\gamma_{s}^{\varepsilon}\right)+8 \mathfrak{B}_{0} \kappa(t-s)}\left\langle Y_{s}^{\varepsilon}, Q \mathrm{~d} W_{s}\right\rangle \\
& -8 \mathfrak{B}_{0} \kappa^{2} \int_{\ell_{0}^{\varepsilon}}^{t}\left\|Y_{s}^{\varepsilon}\right\|^{2} e^{-\nu\left(\gamma_{t}^{\varepsilon}-\gamma_{s}^{\varepsilon}\right)+8 \mathfrak{B}_{0} \kappa(t-s)} \mathrm{d} s .
\end{aligned}
$$

Next we prove that

$$
\begin{equation*}
\mathbb{E}^{\mu_{\mathbb{W}}} \exp \left\{\tilde{M}_{\ell_{\sigma^{\varepsilon}}}\right\} \leq C \tag{A.17}
\end{equation*}
$$

Denote

$$
\begin{aligned}
M_{t} & =2 \kappa \int_{\ell_{0}^{\varepsilon}}^{t}\left\langle Y_{s}^{\varepsilon}, Q \mathrm{~d} W_{s}\right\rangle, \\
N(s) & =M(s)-2[M, M](s),
\end{aligned} \quad g(t, s)=e^{-\nu\left(\gamma_{t}^{\varepsilon}-\gamma_{s}^{\varepsilon}\right)+8 \mathfrak{B}_{0} \kappa(t-s)} .
$$

With these notations, one has $\tilde{M}_{t}=\int_{\ell_{0}^{\varepsilon}}^{t} g(t, s) \mathrm{d} N(s)$. For any $K>0$, by the definition of $\sigma^{\varepsilon}$ and the fact that $\nu \sigma^{\varepsilon}-8 \mathfrak{B}_{0} \kappa \ell_{\sigma^{\varepsilon}}^{\varepsilon}=1$ (see (1) of Lemma A.3),

$$
\begin{align*}
& \mathbb{P}^{\mu_{\mathbb{W}}}\left(\tilde{M}_{\ell_{\sigma}^{\varepsilon}}>K\right)=\mathbb{P}^{\mu_{\mathbb{W}}}\left(\int_{\ell_{0}^{\varepsilon}}^{\ell_{\sigma}^{\varepsilon}} g\left(\ell_{\sigma^{\varepsilon}}^{\varepsilon}, s\right) \mathrm{d} N(s)>K\right) \\
& =\mathbb{P}^{\mu_{\mathbb{W}}}\left(e^{-\nu \sigma^{\varepsilon}+8 \mathfrak{B}_{0} \kappa \ell_{\sigma^{\varepsilon}}^{\varepsilon}} \int_{\ell_{0}^{\varepsilon}}^{\ell_{\sigma}^{\varepsilon} \varepsilon} e^{\nu \gamma_{s}^{\varepsilon}-8 \mathfrak{B}_{0} \kappa s} \mathrm{~d} N(s)>K\right) \\
& =\mathbb{P}^{\mu_{\mathbb{W}}}\left(\int_{\ell_{0}^{\varepsilon}}^{\ell_{\sigma}^{\varepsilon}} e^{\nu \gamma_{s}^{\varepsilon}-8 \mathfrak{B}_{0} \kappa s} \mathrm{~d} N(s)>e K\right) \\
& =\mathbb{P}^{\mu_{\mathbb{W}}}\left(\int_{\ell_{0}^{\varepsilon}}^{\ell_{\sigma}^{\varepsilon} \varepsilon} e^{\nu \gamma_{s}^{\varepsilon}-8 \mathfrak{B}_{0} \kappa s} \mathrm{~d} M(s)\right. \\
& \left.\quad-\int_{\ell_{0}^{\varepsilon}}^{\ell_{\sigma}^{\varepsilon} \varepsilon} e^{2 \nu \gamma_{s}^{\varepsilon}-16 \mathfrak{B}_{0} \kappa s} 2 e^{-\nu \gamma_{s}^{\varepsilon}+8 \mathfrak{B}_{0} \kappa s} \mathrm{~d}[M, M](s)>e K\right) \\
& \leq \mathbb{P}^{\mu_{\mathbb{W}}}\left(\int_{\ell_{0}^{\varepsilon}}^{\ell_{\sigma}^{\varepsilon}} e^{\nu \gamma_{s}^{\varepsilon}-8 \mathfrak{B}_{0} \kappa s} \mathrm{~d} M(s)-\int_{\ell_{0}^{\varepsilon}}^{\ell_{\sigma}^{\varepsilon} \varepsilon}\right. \\
& \leq \exp \left\{-4 e^{-1} e K\right\}=\gamma_{s}^{\varepsilon}-16 \mathfrak{B}_{0} \kappa s  \tag{A.18}\\
& \left.\leq e^{-1} \mathrm{~d}[M, M](s)>e K\right)
\end{align*}
$$

In the first inequality, we have used the fact: for any $s=\ell_{r}^{\varepsilon}\left(r \leq \sigma^{\varepsilon}\right),-\nu \gamma_{s}^{\varepsilon}+$ $8 \mathfrak{B}_{0} \kappa s=-\nu r+8 \mathfrak{B}_{0} \kappa \ell_{r}^{\varepsilon} \geq-1$. For the last inequality we have used the following
fact (c.f. [Ap-2009, Theorem 5.2.9]):

$$
\mathbb{P}^{\mu_{\mathbb{W}}}\left(\int_{\ell_{0}^{\varepsilon}}^{\ell_{\sigma}^{\varepsilon} \varepsilon} e^{\nu \gamma_{s}^{\varepsilon}-8 \mathfrak{B}_{0} \kappa s} \mathrm{~d} M(s)-\frac{\alpha}{2} \int_{\ell_{0}^{\varepsilon}}^{\ell_{\sigma}^{\varepsilon} \varepsilon} e^{2 \nu \gamma_{s}^{\varepsilon}-16 \mathfrak{B}_{0} \kappa s} \mathrm{~d}[M, M](s) \geq \beta\right) \leq e^{-\alpha \beta}
$$

The desired result (A.17) follows immediately from (A.18). The proof of (A.17) is complete.

Now replacing the $t$ in (A.16) by $\ell_{\sigma^{\varepsilon}}^{\varepsilon}$, we obtain

$$
\begin{aligned}
& \mathbb{E}^{\mu_{W}}\left[\operatorname { e x p } \left\{\kappa\left\|Y_{\ell_{\sigma^{\varepsilon}}^{\varepsilon}}^{\varepsilon}\right\|^{2}+\nu \kappa \int_{\ell_{0}^{\varepsilon}}^{\ell_{\sigma^{\varepsilon} \varepsilon}^{\varepsilon}} e^{-\nu\left(\sigma^{\varepsilon}-\gamma_{s}^{\varepsilon}\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma}^{\varepsilon}-s\right)}\left\|Y_{s}^{\varepsilon}\right\|_{1}^{2} \dot{\gamma}_{s}^{\varepsilon} \mathrm{d} s\right.\right. \\
& \left.\left.-\kappa\left\|w_{0}\right\|^{2} e^{-\nu \sigma^{\varepsilon}+8 \mathfrak{B}_{0} \kappa \ell_{\sigma}^{\varepsilon} \varepsilon}-\kappa \mathfrak{B}_{0} \int_{\ell_{0}^{\varepsilon}}^{\ell_{\sigma}^{\varepsilon} \varepsilon} e^{-\nu\left(\sigma^{\varepsilon}-\gamma_{s}^{\varepsilon}\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma}^{\varepsilon}-s\right)} \mathrm{d} s\right\}\right] \\
& \leq \mathbb{E}^{\mu_{\mathbb{W}}}\left[\exp \left\{\tilde{M}_{\ell_{\sigma \varepsilon}}\right\}\right] .
\end{aligned}
$$

Combining the above inequality with $Y_{\ell_{\sigma}^{\varepsilon} \varepsilon}^{\varepsilon}=w_{\sigma^{\varepsilon}}^{\varepsilon},\left.\gamma_{s}^{\varepsilon}\right|_{s=\ell_{r}^{\varepsilon}}=r$, (A.17), and

$$
\begin{aligned}
& \int_{\ell_{0}^{\varepsilon}}^{\ell_{\sigma}^{\varepsilon}} e^{-\nu\left(\sigma^{\varepsilon}-\gamma_{s}^{\varepsilon}\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma}^{\varepsilon}-s\right)}\left\|Y_{s}^{\varepsilon}\right\|_{1}^{2} \dot{\gamma}_{s}^{\varepsilon} \mathrm{d} s=\int_{0}^{\sigma^{\varepsilon}} e^{-\nu\left(\sigma^{\varepsilon}-r\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma^{\varepsilon} \varepsilon}^{\varepsilon}-\ell_{r}^{\varepsilon}\right)}\left\|w_{r}^{\varepsilon}\right\|_{1}^{2} \mathrm{~d} r, \\
& \int_{\ell_{0}^{\varepsilon}}^{\ell_{\sigma}^{\varepsilon} \varepsilon} e^{-\nu\left(\sigma^{\varepsilon}-\gamma_{s}^{\varepsilon}\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma}^{\varepsilon} \varepsilon-s\right)} \mathrm{d} s=\int_{0}^{\sigma^{\varepsilon}} e^{-\nu\left(\sigma^{\varepsilon}-r\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma}^{\varepsilon}--\ell_{r}^{\varepsilon}\right)} \mathrm{d} \ell_{r}^{\varepsilon},
\end{aligned}
$$

we obtain the desired result (A.14).
The proof of Lemma A. 5 is complete.

## A. 2 Proof of (2.9)-(2.14).

(2.9) was already proved in Lemma A.2, and it obviously implies (2.10). We now prove the inequalities (2.11) and (2.12) with $k=1$, and it is straightforward to extend it to the general case. (A.14), $\ell_{0}=0$, Fatou's Lemma and Lemmas A. 3 and A. 4 imply that

$$
\begin{aligned}
& C \geq \mathbb{E}^{\mu_{\mathbb{W}}} \liminf _{\varepsilon \rightarrow 0} \exp \left\{\kappa\left\|w_{\sigma^{\varepsilon}}^{\varepsilon}\right\|^{2}-\kappa\left\|w_{0}\right\|^{2} e^{-\nu \sigma^{\varepsilon}+8 \mathfrak{B}_{0} \kappa \ell_{\sigma^{\varepsilon} \varepsilon}^{\varepsilon}}\right. \\
&+\nu \kappa \int_{0}^{\sigma^{\varepsilon}} e^{-\nu\left(\sigma^{\varepsilon}-s\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma}^{\varepsilon}-\ell_{s}^{\varepsilon}\right)}\left\|w_{s}^{\varepsilon}\right\|_{1}^{2} \mathrm{~d} s \\
& \geq \mathbb{E}^{\mu \mathbb{W}} \exp \left\{\liminf _{\varepsilon \rightarrow 0} \kappa\left\|w_{\sigma^{\varepsilon}}^{\varepsilon}\right\|^{2}-\lim _{\varepsilon \rightarrow 0} \kappa\left\|w_{0}\right\|^{2} e^{-\nu \sigma^{\varepsilon}+8 \mathfrak{B}_{0} \kappa \ell_{\sigma}^{\varepsilon} \varepsilon}\right. \\
&\left.+\liminf _{\varepsilon \rightarrow 0} \nu \kappa \int_{0}^{\sigma^{\varepsilon}} e^{-\nu\left(\sigma^{\varepsilon}-s\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma^{\varepsilon}}^{\varepsilon}-\ell_{s}^{\varepsilon}\right)} \mathrm{d} \ell_{s}^{\varepsilon}\right\} \\
& e^{-\nu\left(\sigma^{\varepsilon}-s\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma}^{\varepsilon}-\ell_{s}^{\varepsilon}\right)}\left\|w_{s}^{\varepsilon}\right\|_{1}^{2} \mathrm{~d} s
\end{aligned}
$$

$$
\begin{aligned}
&\left.-\limsup _{\varepsilon \rightarrow 0} \kappa \mathfrak{B}_{0} \int_{0}^{\sigma^{\varepsilon}} e^{-\nu\left(\sigma^{\varepsilon}-s\right)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma}^{\varepsilon}-\ell_{s}^{\varepsilon}\right)} \mathrm{d} \ell_{s}^{\varepsilon}\right\} \\
& \geq \mathbb{E}^{\mu_{\mathbb{W}}} \exp \left\{\kappa\left\|w_{\sigma}\right\|^{2}-\kappa\left\|w_{0}\right\|^{2} e^{-\nu \sigma+8 \mathfrak{B}_{0} \kappa \ell_{\sigma}}\right. \\
&\left.+\nu \kappa \int_{0}^{\sigma} e^{-\nu(\sigma-s)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma}-\ell_{s}\right)}\left\|w_{s}\right\|_{1}^{2} \mathrm{~d} s-\kappa \mathfrak{B}_{0} \ell_{\sigma}\right\} \\
&=\mathbb{E}^{\mu_{\mathbb{W}}} \exp \left\{\kappa\left\|w_{\sigma}\right\|^{2}-\kappa\left\|w_{0}\right\|^{2} e^{-1}\right. \\
&\left.+\nu \kappa \int_{0}^{\sigma} e^{-\nu(\sigma-s)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma}-\ell_{s}\right)}\left\|w_{s}\right\|_{1}^{2} \mathrm{~d} s-\kappa \mathfrak{B}_{0} \ell_{\sigma}\right\}
\end{aligned}
$$

which gives the desired result (2.11).
In the above inequality, taking expectation under the probability measure $\mathbb{P}^{\mu_{\mathrm{s}}}$, we get

$$
\begin{aligned}
\mathbb{E}[\exp \{ & \kappa\left\|w_{\sigma}\right\|^{2}-\kappa\left\|w_{0}\right\|^{2} e^{-1} \\
& \left.\left.+\nu \kappa \int_{0}^{\sigma} e^{-\nu(\sigma-s)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma}-\ell_{s}\right)}\left\|w_{s}\right\|_{1}^{2} \mathrm{~d} s-\kappa \mathfrak{B}_{0}\left(\ell_{\sigma}\right)\right\}\right] \leq C
\end{aligned}
$$

which is (2.12).
We consider now (2.13) and only prove it for $k=0$. Noticing that $\nu \sigma-$ $8 \mathfrak{B}_{0} \kappa \ell_{\sigma}=1$ (see (4) in Lemma A.3), using (2.12) with $k=1$, there exists a $\kappa_{0}>0$ such that

$$
\mathbb{E} \exp \left\{2 \kappa\left\|w_{\sigma}\right\|^{2}-2 \kappa \mathfrak{B}_{0} \ell_{\sigma}\right\} \leq C \exp \left\{2 \kappa e^{-1}\left\|w_{0}\right\|^{2}\right\}, \forall \kappa \in\left(0, \kappa_{0}\right]
$$

Thus, combining (2.10), we have

$$
\begin{aligned}
& \mathbb{E} \exp \left\{\kappa\left\|w_{\sigma}\right\|^{2}\right\} \leq\left(\mathbb{E} \exp \left\{2 \kappa\left\|w_{\sigma}\right\|^{2}-2 \kappa \mathfrak{B}_{0} \ell_{\sigma}\right\}\right)^{1 / 2}\left(\mathbb{E} \exp \left\{2 \kappa \mathfrak{B}_{0} \ell_{\sigma}\right\}\right)^{1 / 2} \\
& \leq C \exp \left\{\kappa e^{-1}\left\|w_{0}\right\|^{2}\right\}(\mathbb{E} \exp \{\nu \sigma\})^{1 / 2} \leq C_{\kappa} \exp \left\{\kappa e^{-1}\left\|w_{0}\right\|^{2}\right\}
\end{aligned}
$$

which yields the desired result (2.13) for the case $k=0$.
Following the arguments in the proof of [HM06, (4.7)], also with the help of (2.13), we arrive at (2.14).

## A. 3 Proof of (2.15).

Recall $\mathbb{S}_{1}$ introduced in (A.4). Fix any $\ell \in \mathbb{S}_{1}$ and recall $Y_{t}^{\varepsilon}$ introduced in (A.15). By Itô's formula,

$$
\begin{align*}
& \left\|Y_{s}^{\varepsilon}\right\|^{2 n}+2 \nu n \int_{\ell_{0}^{\varepsilon}}^{s}\left\|Y_{u}^{\varepsilon}\right\|^{2 n-2}\left\|Y_{u}^{\varepsilon}\right\|_{1}^{2} \dot{\gamma}_{u}^{\varepsilon} d u  \tag{A.19}\\
= & \left\|w_{0}\right\|^{2 n}+2 n \int_{\ell_{0}^{\varepsilon}}^{s}\left\|Y_{u}^{\varepsilon}\right\|^{2 n-2}\left\langle Y_{u}^{\varepsilon}, Q d W_{u}\right\rangle+n \int_{\ell_{0}^{\varepsilon}}^{s}\left\|Y_{u}^{\varepsilon}\right\|^{2 n-2} \mathfrak{B}_{0} d u
\end{align*}
$$

$$
+2 n(n-1) \int_{\ell_{0}^{\varepsilon}}^{s}\left\|Y_{u}^{\varepsilon}\right\|^{2 n-4} \sum_{j \in \mathcal{Z}_{0}}\left\langle Y_{u}^{\varepsilon}, e_{j}\right\rangle^{2} b_{j}^{2} \mathrm{~d} u, \quad s \geq \ell_{0}^{\varepsilon}
$$

Taking expectations with respect to $\mathbb{P}^{\mu_{W}}$ yields

$$
\begin{aligned}
& \mathbb{E}^{\mu_{\mathbb{W}}}\left\|Y_{s}^{\varepsilon}\right\|^{2 n} \leq\left\|w_{0}\right\|^{2 n}+C_{n} \int_{\ell_{0}^{\varepsilon}}^{s} \mathbb{E}^{\mu_{\mathbb{W}}}\left\|Y_{u}^{\varepsilon}\right\|^{2 n-2} d u \\
& \leq\left\|w_{0}\right\|^{2 n}+\frac{1}{2} \sup _{u \in\left[\ell_{0}^{\varepsilon}, t\right]} \mathbb{E}^{\mu_{\mathbb{W}}}\left\|Y_{u}^{\varepsilon}\right\|^{2 n}+C_{n} t^{n}, \quad \forall s \in\left[\ell_{0}^{\varepsilon}, t\right]
\end{aligned}
$$

Rearranging terms, we arrive at, for any $t \geq \ell_{0}^{\varepsilon}$,

$$
\sup _{s \in\left[\ell_{0}^{\varepsilon}, t\right]} \mathbb{E}^{\mu_{\mathbb{W}}}\left\|Y_{t}^{\varepsilon}\right\|^{2 n} \leq 2\left\|w_{0}\right\|^{2 n}+C_{n} t^{n}
$$

Using (A.19) and the BDG inequality, we have

$$
\begin{aligned}
& \mathbb{E}^{\mu_{\mathbb{W}}}\left(\sup _{s \in\left[\ell_{0}^{\varepsilon}, t\right]}\left\|Y_{s}^{\varepsilon}\right\|^{2 n}\right) \\
\leq & \left\|w_{0}\right\|^{2 n}+2 n \mathbb{E}^{\mu_{\mathbb{W}}} \sup _{s \in\left[\ell_{0}^{\varepsilon}, t\right]}\left|\int_{\ell_{0}^{\varepsilon}}^{s}\left\|Y_{r}^{\varepsilon}\right\|^{2 n-2}\left\langle Y_{r}^{\varepsilon}, Q d W_{r}\right\rangle\right|+C_{n} \mathbb{E}^{\mu_{\mathbb{W}}} \int_{\ell_{0}^{\varepsilon}}^{t}\left\|Y_{r}^{\varepsilon}\right\|^{2 n-2} d r \\
\leq & \left\|w_{0}\right\|^{2 n}+C_{n}\left(\mathbb{E}^{\mu_{\mathbb{W}}} \sup _{s \in\left[\ell_{0}^{\varepsilon}, t\right]}\left|\int_{\ell_{0}^{\varepsilon}}^{s}\left\|Y_{r}^{\varepsilon}\right\|^{2 n-2}\left\langle Y_{r}^{\varepsilon}, Q d W_{r}\right\rangle\right|^{2}\right)^{1 / 2}+C_{n} \mathbb{E}^{\mu_{\mathbb{W}}} \int_{\ell_{0}^{\varepsilon}}^{t}\left\|Y_{r}^{\varepsilon}\right\|^{2 n-2} d r \\
\leq & \left\|w_{0}\right\|^{2 n}+C_{n}\left(\mathbb{E}^{\mu_{\mathbb{W}}} \int_{\ell_{0}^{\varepsilon}}^{t}\left\|Y_{r}^{\varepsilon}\right\|^{4 n-2} d r\right)^{1 / 2}+C_{n} \mathbb{E}^{\mu_{\mathbb{W}}} \int_{\ell_{0}^{\varepsilon}}^{t}\left\|Y_{r}^{\varepsilon}\right\|^{2 n-2} d r \\
\leq & \left\|w_{0}\right\|^{2 n}+C_{n}\left(\left\|w_{0}\right\|^{4 n-2} t+t^{2 n}\right)^{1 / 2}+C_{n}\left(\left\|w_{0}\right\|^{2 n-2} t+t^{n}\right) \\
\leq & C_{n}(1+t)\left\|w_{0}\right\|^{2 n}+C_{n}\left(1+t^{n}\right) .
\end{aligned}
$$

Using the fact that $w_{t}^{\varepsilon}=Y_{\ell_{t}^{\varepsilon}}^{\varepsilon}$, we arrive at

$$
\mathbb{E}^{\mu_{\mathbb{W}}} \sup _{s \in[0, t]}\left\|w_{s}^{\varepsilon}\right\|^{2 n}=\mathbb{E}^{\mu_{\mathbb{W}}} \sup _{s \in\left[\ell_{0}^{\varepsilon}, \ell_{t}^{\varepsilon}\right]}\left\|Y_{s}^{\varepsilon}\right\|^{2 n} \leq C_{n}\left(1+\ell_{t}^{\varepsilon}\right)\left\|w_{0}\right\|^{2 n}+C_{n}\left(1+\left(\ell_{t}^{\varepsilon}\right)^{n}\right)
$$

By (A.12), Fatou's lemma, (ii) of Lemma A. 1 and
$w_{s}=\left(w_{s}-\eta_{s}\right)-\left(w_{s}^{\varepsilon}-\eta_{s}^{\varepsilon}\right)+w_{s}^{\varepsilon}+\left(\eta_{s}-\eta_{s}^{\varepsilon}\right)=v_{s}-v_{s}^{\varepsilon}+w_{s}^{\varepsilon}+\left(\eta_{s}-\eta_{s}^{\varepsilon}\right), \forall \varepsilon>0$,
one has

$$
\begin{aligned}
& \mathbb{E}^{\mu_{\mathbb{W}}} \sup _{s \in[0, t]}\left\|w_{s}\right\|^{2 n} \\
& \leq C_{n} \mathbb{E}^{\mu_{\mathbb{W}}}\left[\liminf _{\varepsilon \rightarrow 0} \sup _{s \in[0, t]}\left\|w_{s}^{\varepsilon}\right\|^{2 n}+\liminf _{\varepsilon \rightarrow 0} \sup _{s \in[0, t]}\left\|v_{s}-v_{s}^{\varepsilon}\right\|^{2 n}+\liminf _{\varepsilon \rightarrow 0} \sup _{s \in[0, t]}\left|\eta_{s}-\eta_{s}^{\varepsilon}\right|^{2 n}\right] \\
& \leq C_{n} \liminf _{\varepsilon \rightarrow 0} \mathbb{E}^{\mu_{\mathbb{W}}} \sup _{s \in[0, t]}\left\|w_{s}^{\varepsilon}\right\|^{2 n}+C_{n} \liminf _{\varepsilon \rightarrow 0} \mathbb{E}^{\mu_{\mathbb{W}}} \sup _{s \in[0, t]}\left(\left|\eta_{s}\right|^{2 n}+\left|\eta_{s}^{\varepsilon}\right|^{2 n}\right)
\end{aligned}
$$

$$
\leq C_{n}\left(1+\ell_{t}\right)\left\|w_{0}\right\|^{2 n}+C_{n}\left(1+\left(\ell_{t}\right)^{n}\right)
$$

Since the above estimate holds for any fixed $\ell \in \mathbb{S}_{1}, \mu_{\mathbb{S}}\left(\mathbb{S}_{1}\right)=1$ (see Lemma A.3) and since $\sigma$ only depends on $\ell \in \mathbb{S}$, we first replace the $t$ by $\sigma$, and then take expectations with respect to $\mu_{\mathbb{S}}$, to obtain

$$
\begin{aligned}
\mathbb{E} \sup _{s \in[0, \sigma]}\left\|w_{s}\right\|^{2 n} & \leq C_{n}\left(1+\mathbb{E}^{\mu_{\mathrm{s}}} \ell_{\sigma}\right)\left\|w_{0}\right\|^{2 n}+C_{n}\left(1+\mathbb{E}^{\mu_{\mathbb{S}}}\left(\ell_{\sigma}\right)^{n}\right) \\
& \leq C_{n}\left(1+\mathbb{E}^{\mu_{\mathbb{S}}}\left(\frac{\nu \sigma}{8 \kappa \mathfrak{B}_{0}}\right)\right)\left\|w_{0}\right\|^{2 n}+C_{n}\left(1+\mathbb{E}^{\mu_{\mathbb{S}}}\left(\frac{\nu \sigma}{8 \kappa \mathfrak{B}_{0}}\right)^{n}\right) \\
& \leq C_{n, \kappa}\left(1+\left\|w_{0}\right\|^{2 n}\right)
\end{aligned}
$$

Here we have used the fact that $\ell_{\sigma} \leq \frac{\nu \sigma}{8 \kappa \mathfrak{B}_{0}}$ (see (4) of Lemma A.3) and (2.9), $C_{n, \kappa}$ is a constant depending on $n, \kappa$ and $\nu, \mathfrak{B}_{0}$.

The proof of (2.15) is complete.

## B Proof of Proposition 3.5

We will prove Proposition 3.5 by contradiction.
Suppose that Proposition 3.5 were not true, then there exist sequences $\left\{w_{0}^{(k)}\right\} \subseteq B_{H}(\mathfrak{R}),\left\{\varepsilon_{k}\right\} \subseteq(0,1)$ and a positive number $\delta_{0}$ such that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \mathbb{P}\left(X^{w_{0}^{(k)}, \alpha, N}<\varepsilon_{k}\right) \geq \delta_{0}>0 \text { and } \lim _{k \rightarrow \infty} \varepsilon_{k}=0 \tag{B.1}
\end{equation*}
$$

Our aim is to find something which contradicts (B.1).
Since $H$ is a Hilbert space, there exists a subsequence $\left\{w_{0}^{\left(n_{k}\right)}, k \geq 1\right\}$ of $\left\{w_{0}^{(k)}, k \geq 1\right\}$ such that $w_{0}^{\left(n_{k}\right)}$ converges weakly to some $w_{0}^{(0)} \in H$. We still denote this subsequence by $\left\{w_{0}^{(k)}, k \geq 1\right\}$. Let $w_{t}^{(k)}$ denote the solution of equation (1.2) with $\left.w_{t}\right|_{t=0}=w_{0}^{(k)}(k \geq 0)$. In the following equation

$$
\begin{align*}
\partial_{t} J_{s, t} \xi-\nu \Delta J_{s, t} \xi-\tilde{B}\left(w_{t}, J_{s, t} \xi\right) & =0  \tag{B.2}\\
J_{s, s} \xi & =\xi \tag{tabular}
\end{align*}
$$

when $w_{t}$ is replaced by $w_{t}^{(k)}$, we denote its solution by $J_{s, t}^{(k)} \xi$. We denote the adjoint of $J_{s, t}^{(k)}$ by $K_{s, t}^{(k)}$.

Recall that for any $M \in \mathbb{N}, H_{M}=\left\{e_{j}: j \in \mathbb{Z}_{*}^{2}\right.$ and $\left.|j| \leq M\right\}, P_{M}$ denotes the orthogonal projections from $H$ onto $H_{M}$ and $Q_{M} u:=u-P_{M} u, \forall u \in H$. As before, $C$ denotes a constant depending $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d, C_{\Re}$ denotes a constant depending on $\mathfrak{R}$ and $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$. The values of the constants may change from line to line.

In what follows, we will give some estimates for $\left\|P_{M} w_{t}^{(k)}-P_{M} w_{t}^{(0)}\right\|$ and $\left\|Q_{M} w_{t}^{(k)}\right\|$ in Lemma B.1, and for $\left\|J_{s, t}^{(k)} \xi-J_{s, t}^{(0)} \xi\right\|, s, t \in(0, \sigma]$ in Lemma B.2. We finish the proof of Proposition 3.5 at the end of the section.

E-5 Lemma B.1. For any $t \geq 0, k \in \mathbb{N}$ and $M>\max \left\{|j|: j \in \mathcal{Z}_{0}\right\}$, one has

$$
\begin{align*}
\left\|Q_{M} w_{t}^{(k)}\right\|^{2} & \leq e^{-\nu M^{2} t}\left\|Q_{M} w_{0}^{(k)}\right\|^{2}+ \\
& \frac{C}{M^{1 / 2}}\left(\int_{0}^{t}\left\|w_{s}^{(k)}\right\|_{1}^{4 / 3} \mathrm{~d} s\right)^{3 / 4} \sup _{s \in[0, t]}\left\|w_{s}^{(k)}\right\|^{3} \tag{B.3}
\end{align*}
$$

and

$$
\begin{align*}
& \left\|P_{M} w_{t}^{(k)}-P_{M} w_{t}^{(0)}\right\|^{2} \leq C_{\Re} \exp \left\{C \int_{0}^{t}\left(\left\|w_{s}^{(k)}\right\|_{1}^{4 / 3}+\left\|w_{s}^{(0)}\right\|_{1}^{4 / 3}\right) \mathrm{d} s\right\} \\
& \quad \times \sup _{r \in[0, t]}\left(1+\left\|w_{r}^{(k)}\right\|^{4}+\left\|w_{r}^{(0)}\right\|^{4}\right)  \tag{B.4}\\
& \quad \times\left[\left\|P_{M} w_{0}^{(k)}-P_{M} w_{0}^{(0)}\right\|^{2}+\frac{1+t}{M^{1 / 2}}\right]
\end{align*}
$$

Proof. First, we give a proof of (B.3). By (2.2), one has

$$
\begin{aligned}
\left\langle B\left(\mathcal{K} w_{t}^{(k)}, w_{t}^{(k)}\right), Q_{M} w_{t}^{(k)}\right\rangle & \leq C\left\|Q_{M} w_{t}^{(k)}\right\|_{1}\left\|w_{t}^{(k)}\right\|_{1 / 2}\left\|w_{t}^{(k)}\right\| \\
& \leq \frac{\nu}{4}\left\|Q_{M} w_{t}^{(k)}\right\|_{1}^{2}+C\left\|w_{t}^{(k)}\right\|\left\|_{1}\right\| w_{t}^{(k)} \|^{3}
\end{aligned}
$$

Thus, in view of the equation (1.2), we obtain

$$
\begin{aligned}
\mathrm{d}\left\|Q_{M} w_{t}^{(k)}\right\|^{2} & =-2 \nu\left\|Q_{M} w_{t}^{(k)}\right\|_{1}^{2}+\left\langle B\left(\mathcal{K} w_{t}^{(k)}, w_{t}^{(k)}\right), Q_{M} w_{t}^{(k)}\right\rangle \\
& \leq-\nu M^{2}\left\|Q_{M} w_{t}^{(k)}\right\|^{2} \mathrm{~d} t+C\left\|w_{t}^{(k)}\right\|_{1}\left\|w_{t}^{(k)}\right\|^{3} \mathrm{~d} t
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \left\|Q_{M} w_{t}^{(k)}\right\|^{2} \leq e^{-\nu M^{2} t}\left\|Q_{M} w_{0}^{(k)}\right\|^{2}+C \int_{0}^{t} e^{-\nu M^{2}(t-s)}\left\|w_{s}^{(k)}\right\|_{1}\left\|w_{s}^{(k)}\right\|^{3} \mathrm{~d} s \\
& \leq e^{-\nu M^{2} t}\left\|Q_{M} w_{0}^{(k)}\right\|^{2} \\
& \quad+C\left(\int_{0}^{t} e^{-4 \nu M^{2}(t-s)} \mathrm{d} s\right)^{1 / 4}\left(\int_{0}^{t}\left\|w_{s}^{(k)}\right\|_{1}^{4 / 3} \mathrm{~d} s\right)^{3 / 4} \sup _{s \in[0, t]}\left\|w_{s}^{(k)}\right\|^{3}
\end{aligned}
$$

This completes the proof of (B.3).
Next, we will prove(B.4). One easily sees that

$$
\begin{align*}
& \mathrm{d} \| P_{M} w_{t}^{(k)}-P_{M} w_{t}^{(0)} \|^{2} \\
&=-\nu\left\|P_{M} w_{t}^{(k)}-P_{M} w_{t}^{(0)}\right\|_{1}^{2} \mathrm{~d} t  \tag{B.5}\\
& \quad+\left\langle P_{M} w_{t}^{(k)}-P_{M} w_{t}^{(0)}, B\left(\mathcal{K} w_{t}^{(k)}, w_{t}^{(k)}\right)-B\left(\mathcal{K} w_{t}^{(0)}, w_{t}^{(0)}\right)\right\rangle
\end{align*}
$$

Clearly, we have

$$
\begin{aligned}
& \left\langle P_{M} w_{t}^{(k)}-P_{M} w_{t}^{(0)}, B\left(\mathcal{K} w_{t}^{(k)}, w_{t}^{(k)}\right)-B\left(\mathcal{K} w_{t}^{(0)}, w_{t}^{(0)}\right)\right\rangle \\
& =\left\langle P_{M} w_{t}^{(k)}-P_{M} w_{t}^{(0)}, B\left(\mathcal{K} w_{t}^{(k)}-\mathcal{K} w_{t}^{(0)}, w_{t}^{(k)}\right)\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
& +\left\langle P_{M} w_{t}^{(k)}-P_{M} w_{t}^{(0)}, B\left(\mathcal{K} w_{t}^{(0)}, w_{t}^{(k)}-w_{t}^{(0)}\right)\right\rangle \\
:= & I_{1}+I_{2}
\end{aligned}
$$

For the term $I_{1}$, we have

$$
\begin{aligned}
& I_{1} \leq C\left\|P_{M} w_{t}^{(k)}-P_{M} w_{t}^{(0)}\right\|_{1 / 2}\left\|P_{M} w_{t}^{(k)}-P_{M} w_{t}^{(0)}\right\|\left\|w_{t}^{(k)}\right\|_{1} \\
& \quad+C\left\|P_{M} w_{t}^{(k)}-P_{M} w_{t}^{(0)}\right\|_{1}\left\|Q_{M} w_{t}^{(k)}-Q_{M} w_{t}^{(0)}\right\|\left\|w_{t}^{(k)}\right\|_{1 / 2} \\
& \leq \frac{\nu}{6}\left\|P_{M} w_{t}^{(k)}-P_{M} w_{t}^{(0)}\right\|_{1}^{2}+C\left\|P_{M} w_{t}^{(k)}-P_{M} w_{t}^{(0)}\right\|^{2}\| \| w_{t}^{(k)} \|_{1}^{4 / 3} \\
& \\
& \quad+C\left\|Q_{M} w_{t}^{(k)}-Q_{M} w_{t}^{(0)}\right\|^{2}\left\|w_{t}^{(k)}\right\|_{1 / 2}^{2} .
\end{aligned}
$$

Obviously,

$$
\begin{aligned}
I_{2} & \leq C\left\|P_{M} w_{t}^{(k)}-P_{M} w_{t}^{(0)}\right\|_{1}\left\|Q_{M} w_{t}^{(k)}-Q_{M} w_{t}^{(0)}\right\|\left\|w_{t}^{(0)}\right\|_{1 / 2} \\
& \leq \frac{\nu}{6}\left\|P_{M} w_{t}^{(k)}-P_{M} w_{t}^{(0)}\right\|_{1}^{2}+C\left\|Q_{M} w_{t}^{(k)}-Q_{M} w_{t}^{(0)}\right\|^{2}\left\|w_{t}^{(0)}\right\|_{1 / 2}^{2}
\end{aligned}
$$

Combining the estimates of $I_{1}, I_{2}$ with (B.5),(B.3), taking into account the fact that $\left\|Q_{M} w_{0}^{(k)}\right\| \leq \mathfrak{R}$, we obtain

$$
\begin{aligned}
& \left\|P_{M} w_{t}^{(k)}-P_{M} w_{t}^{(0)}\right\|^{2} \leq\left\|P_{M} w_{0}^{(k)}-P_{M} w_{0}^{(0)}\right\|^{2} e^{C \int_{0}^{t}\left\|w_{s}^{(k)}\right\|_{1}^{4 / 3} \mathrm{~d} s} \\
& \quad+C e^{C \int_{0}^{t}\left\|w_{s}^{(k)}\right\|_{1}^{4 / 3} \mathrm{~d} s} \int_{0}^{t}\left\|Q_{M} w_{s}^{(k)}-Q_{M} w_{s}^{(0)}\right\|^{2}\left(\left\|w_{s}^{(k)}\right\|_{1 / 2}^{2}+\left\|w_{s}^{(0)}\right\|_{1 / 2}^{2}\right) \mathrm{d} s \\
& \leq\left\|P_{M} w_{0}^{(k)}-P_{M} w_{0}^{(0)}\right\|^{2} e^{C \int_{0}^{t}\left\|w_{s}^{(k)}\right\|_{1}^{4 / 3} \mathrm{~d} s} \\
& \quad+C e^{C \int_{0}^{t}\left\|w_{s}^{(k)}\right\|_{1}^{4 / 3} \mathrm{~d} s}\left(\int_{0}^{t}\left\|Q_{M} w_{s}^{(k)}-Q_{M} w_{s}^{(0)}\right\|^{8} \mathrm{~d} s\right)^{1 / 4} \\
& \left.\quad \times\left(\left(\int_{0}^{t}\left\|w_{s}^{(k)}\right\|_{1}^{4 / 3} \mathrm{~d} s\right)^{3 / 4}+\left(\int_{0}^{t}\left\|w_{s}^{(0)}\right\|_{1}^{4 / 3}\right) \mathrm{d} s\right)^{3 / 4}\right) \sup _{s \in[0, t]}\left(\left\|w_{s}^{(k)}\right\|+\left\|w_{s}^{(0)}\right\|\right) \\
& \leq e^{C \int_{0}^{t}\left\|w_{s}^{(k)}\right\|_{1}^{4 / 3} \mathrm{~d} s}\left\|P_{M} w_{0}^{(k)}-P_{M} w_{0}^{(0)}\right\|^{2} \\
& \quad+C_{\mathfrak{R}} e^{C \int_{0}^{t}\left\|w_{s}^{(k)}\right\|_{1}^{4 / 3} \mathrm{~d} s+C \int_{0}^{t}\left\|w_{s}^{(0)}\right\|_{1}^{4 / 3} \mathrm{~d} s} \sup _{s \in[0, t]}\left(\left\|w_{s}^{(k)}\right\|+\left\|w_{s}^{(0)}\right\|\right) \\
& \times\left(\int _ { 0 } ^ { t } \left[e^{-4 \nu M^{2} s}+\frac{1}{M^{2}}\left(\int_{0}^{s}\left\|w_{r}^{(k)}\right\|_{1}^{4 / 3} \mathrm{~d} r\right)^{3} \sup _{s \in[0, t]}\left\|w_{s}^{(k)}\right\|^{12}\right.\right. \\
& \left.\left.\quad+\frac{1}{M^{2}}\left(\int_{0}^{s}\left\|w_{r}^{(0)}\right\|_{1}^{4 / 3} \mathrm{~d} r\right)^{3} \sup _{s \in[0, t]}\left\|w_{s}^{(0)}\right\|^{12}\right] \mathrm{~d} s\right)^{1 / 4},
\end{aligned}
$$

which implies the desired result (B.4).
9-3 Lemma B.2. For any $0 \leq s \leq t, k \in \mathbb{N}$ and $\xi \in H$ with $\|\xi\|=1$, one has

$$
\left\|J_{s, t}^{(k)} \xi-J_{s, t}^{(0)} \xi\right\|^{2} \leq C \sup _{r \in[s, t]}\left\|w_{r}^{(k)}-w_{r}^{(0)}\right\|^{2} \cdot e^{C \int_{s}^{t}\left(\left\|w_{r}^{(k)}\right\|_{1}^{4 / 3}+\left\|w_{r}^{(0)}\right\|_{1}^{4 / 3}\right) \mathrm{d} r}
$$

where $C$ is a constant depending on $\nu,\left\{b_{j}\right\}_{j \in \mathcal{Z}_{0}}, \nu_{S}, d$.

Proof. By the equation (B.2), one easily sees that

$$
\begin{align*}
& \mathrm{d}\left\|J_{s, t}^{(k)} \xi-J_{s, t}^{(0)} \xi\right\|^{2} \leq-\nu\left\|J_{s, t}^{(k)} \xi-J_{s, t}^{(0)} \xi\right\|_{1}^{2} \mathrm{~d} t \\
& \quad+\left\langle J_{s, t}^{(k)} \xi-J_{s, t}^{(0)} \xi, B\left(\mathcal{K} w_{t}^{(k)}, J_{s, t}^{(k)} \xi\right)-B\left(\mathcal{K} w_{t}^{(0)}, J_{s, t}^{(0)} \xi\right)\right\rangle \mathrm{d} t \\
& \quad+\left\langle J_{s, t}^{(k)} \xi-J_{s, t}^{(0)} \xi, B\left(\mathcal{K} J_{s, t}^{(k)} \xi, w_{t}^{(k)}\right)-B\left(\mathcal{K} J_{s, t}^{(0)} \xi, w_{t}^{(0)}\right)\right\rangle \mathrm{d} t \\
& :=  \tag{B.6}\\
& I_{1}(t) \mathrm{d} t+I_{2}(t) \mathrm{d} t+I_{3}(t) \mathrm{d} t .
\end{align*}
$$

For the terms $I_{2}(t), I_{3}(t)$, we have

$$
\begin{aligned}
I_{2}(t) & =\left\langle J_{s, t}^{(k)} \xi-J_{s, t}^{(0)} \xi, B\left(\mathcal{K} w_{t}^{(k)}-\mathcal{K} w_{t}^{(0)}, J_{s, t}^{(k)} \xi\right)\right\rangle \\
& \leq C\left\|w_{t}^{(k)}-w_{t}^{(0)}\right\|\left\|J_{s, t}^{(k)} \xi-J_{s, t}^{(0)} \xi\right\|_{1}\left\|J_{s, t}^{(k)} \xi\right\|_{1 / 2} \\
& \leq C\left\|w_{t}^{(k)}-w_{t}^{(0)}\right\|^{2}\left\|J_{s, t}^{(k)} \xi\right\|_{1 / 2}^{2}+\frac{\nu}{6}\left\|J_{s, t}^{(k)} \xi-J_{s, t}^{(0)} \xi\right\|_{1}^{2}
\end{aligned}
$$

and

$$
\begin{aligned}
I_{3}(t)= & \left\langle J_{s, t}^{(k)} \xi-J_{s, t}^{(0)} \xi, B\left(\mathcal{K} J_{s, t}^{(k)} \xi-\mathcal{K} J_{s, t}^{(0)} \xi, w_{t}^{(k)}\right)+B\left(\mathcal{K} J_{s, t}^{(0)} \xi, w_{t}^{(k)}-w_{t}^{(0)}\right)\right\rangle \\
\leq & C\left\|w_{t}^{(k)}\right\|_{1}\left\|J_{s, t}^{(k)} \xi-J_{s, t}^{(0)} \xi\right\|^{3 / 2}\left\|J_{s, t}^{(k)} \xi-J_{s, t}^{(0)} \xi\right\|_{1}^{1 / 2} \\
& +C\left\|w_{t}^{(k)}-w_{t}^{(0)}\right\|\left\|J_{s, t}^{(k)} \xi-J_{s, t}^{(0)} \xi\right\|_{1}\left\|J_{s, t}^{(0)} \xi\right\|_{1 / 2} \\
\leq & \frac{\nu}{6}\left\|J_{s, t}^{(k)} \xi-J_{s, t}^{(0)} \xi\right\|_{1}^{2}+C\left\|J_{s, t}^{(k)} \xi-J_{s, t}^{(0)} \xi\right\|^{2}\left\|w_{t}^{(k)}\right\|_{1}^{4 / 3} \\
& +C\left\|w_{t}^{(k)}-w_{t}^{(0)}\right\|^{2}\left\|J_{s, t}^{(0)} \xi\right\|_{1 / 2}^{2} .
\end{aligned}
$$

Combining the above estimates of $I_{2}, I_{3}$ with (B.6), we obtain

$$
\begin{aligned}
& \left\|J_{s, t}^{(k)} \xi-J_{s, t}^{(0)} \xi\right\|^{2} \\
& \leq C \sup _{r \in[s, t]}\left\|w_{r}^{(k)}-w_{r}^{(0)}\right\|^{2} \cdot e^{C \int_{s}^{t}\left\|w_{r}^{(k)}\right\|_{1}^{4 / 3} \mathrm{~d} r} \int_{s}^{t}\left[\left\|J_{s, r}^{(k)} \xi\right\|_{1 / 2}^{2}+\left\|J_{s, r}^{(0)} \xi\right\|_{1 / 2}^{2}\right] \mathrm{d} r .
\end{aligned}
$$

By Lemma 2.4, Hölder's inequality and using the fact that $\|a\|_{1 / 2}^{2} \leq\|a\|\|a\|_{1}$, we obtain the desired result.

## Now we are in a position to complete the proof of Proposition 3.5.

With the help of Lemmas B.1, B.2, for any $\kappa>0, r \in\left[\frac{\sigma}{2}, \sigma\right]$ and $\phi \in H$ with $\|\phi\|=1$, we have

$$
\begin{aligned}
& \left\|J_{r, \sigma}^{(k)} \phi-J_{r, \sigma}^{(0)} \phi\right\| \leq C e^{C \int_{\sigma / 2}^{\sigma}\left(\left\|w_{r}^{(k)}\right\|_{1}^{4 / 3}+\left\|w_{r}^{(0)}\right\|_{1}^{4 / 3}\right) \mathrm{d} r} \cdot \sup _{r \in\left[\frac{\sigma}{2}, \sigma\right]}\left\|w_{r}^{(k)}-w_{r}^{(0)}\right\|^{2} \\
& \leq C_{\Re} \exp \left\{C \int_{0}^{\sigma}\left(\left\|w_{s}^{(k)}\right\|_{1}^{4 / 3}+\left\|w_{s}^{(0)}\right\|_{1}^{4 / 3}\right) \mathrm{d} s\right\} \sup _{r \in[0, \sigma]}\left(1+\left\|w_{r}^{(k)}\right\|^{4}+\left\|w_{r}^{(0)}\right\|^{4}\right) \\
& \quad \times\left[e^{-\nu M^{2} \sigma / 2}+\left\|P_{M} w_{0}^{(k)}-P_{M} w_{0}^{(0)}\right\|^{2}+\frac{1+\sigma}{M^{1 / 2}}\right]
\end{aligned}
$$

$$
\begin{align*}
\leq & {\left[C_{\mathfrak{R}} \exp \left\{\frac{\nu \kappa}{6} \int_{0}^{\sigma}\left(\left\|w_{s}^{(k)}\right\|_{1}^{2}+\left\|w_{s}^{(0)}\right\|_{1}^{2}\right) e^{-\nu(\sigma-s)+8 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma}-\ell_{s}\right)} \mathrm{d} s\right\}\right.} \\
& \left.\cdot \sup _{r \in[0, \sigma]}\left(1+\left\|w_{r}^{(k)}\right\|^{4}+\left\|w_{r}^{(0)}\right\|^{4}\right)\right] \\
& \times \exp \left\{C_{\kappa} \int_{0}^{\sigma} e^{2 \nu(\sigma-s)-16 \mathfrak{B}_{0} \kappa\left(\ell_{\sigma}-\ell_{s}\right)} \mathrm{d} s\right\} \\
& \times\left[e^{-\nu M^{2} \sigma / 2}+\left\|P_{M} w_{0}^{(k)}-P_{M} w_{0}^{(0)}\right\|^{2}+\frac{1+\sigma}{M^{1 / 2}}\right] \\
:= & \Xi(k) \cdot \tilde{X} \cdot \Upsilon(k, M), \quad \forall r \in\left[\frac{\sigma}{2}, \sigma\right] . \tag{B.7}
\end{align*}
$$

Note that

$$
\begin{aligned}
& \left\langle K_{r, \sigma}^{(k)} \phi, e_{j}\right\rangle^{2}=\left(\left\langle K_{r, \sigma}^{(0)} \phi, e_{j}\right\rangle+\left\langle K_{r, \sigma}^{(k)} \phi-K_{r, \sigma}^{(0)} \phi, e_{j}\right\rangle\right)^{2} \\
& \geq \frac{1}{2}\left\langle K_{r, \sigma}^{(0)} \phi, e_{j}\right\rangle^{2}-3\left\langle K_{r, \sigma}^{(k)} \phi-K_{r, \sigma}^{(0)} \phi, e_{j}\right\rangle^{2} \geq \frac{1}{2}\left\langle K_{r, \sigma}^{(0)} \phi, e_{j}\right\rangle^{2}-3\left\|K_{r, \sigma}^{(k)} \phi-K_{r, \sigma}^{(0)} \phi\right\|^{2}
\end{aligned}
$$

and recall that $K_{r, \sigma}$ is the adjoint of $J_{r, \sigma}$. It follows from (B.7) that

$$
\begin{aligned}
& \mathbb{P}\left(\inf _{\phi \in \mathcal{S}_{\alpha, N}} \sum_{j \in \mathcal{Z}_{0}} \int_{0}^{\sigma}\left\langle K_{r, \sigma}^{(k)} \phi, e_{j}\right\rangle^{2} \mathrm{~d} S_{r}<\varepsilon_{k}\right) \\
& \leq \mathbb{P}\left(\frac{1}{2} \inf _{\phi \in \mathcal{S}_{\alpha, N}} \sum_{j \in \mathcal{Z}_{0}} \int_{\sigma / 2}^{\sigma}\left\langle K_{r, \sigma}^{(0)} \phi, e_{j}\right\rangle^{2} \mathrm{~d} S_{r}<\varepsilon_{k}\right. \\
& \left.\quad+3 d \sup _{\phi \in S_{\alpha, N}} \sup _{r \in[\sigma / 2, \sigma]}\left\|K_{r, \sigma}^{(k)} \phi-K_{r, \sigma}^{(0)} \phi\right\|^{2} S_{\sigma}\right) \\
& \leq \mathbb{P}\left(\inf _{\phi \in \mathcal{S}_{\alpha, N}} \sum_{j \in \mathcal{Z}_{0}} \int_{\sigma / 2}^{\sigma}\left\langle K_{r, \sigma}^{(0)} \phi, e_{j}\right\rangle^{2} \mathrm{~d} S_{r}<2 \varepsilon_{k}+\Xi(k) \cdot\left(6 d S_{\sigma} \tilde{X}\right) \cdot \Upsilon(k, M)\right)
\end{aligned}
$$

Therefore, for any $\mathcal{C}>0$, we deduce that

$$
\begin{align*}
& \mathbb{P}\left(\inf _{\phi \in \mathcal{S}_{\alpha, N}} \sum_{j \in \mathcal{Z}_{0}} \int_{0}^{\sigma}\left\langle K_{r, \sigma}^{(k)} \phi, e_{j}\right\rangle^{2} \mathrm{~d} S_{r}<\varepsilon_{k}\right) \\
& \leq \mathbb{P}\left(\inf _{\phi \in \mathcal{S}_{\alpha, N}} \sum_{j \in \mathcal{Z}_{0}} \int_{\sigma / 2}^{\sigma}\left\langle K_{r, \sigma}^{(0)} \phi, e_{j}\right\rangle^{2} \mathrm{~d} S_{r}<2 \varepsilon_{k}+\mathcal{C}^{2} \Upsilon(k, M)\right) \\
& +\mathbb{P}(\Xi(k) \geq \mathcal{C})+\mathbb{P}\left(6 d S_{\sigma} \tilde{X}>\mathcal{C}\right) \tag{B.8}
\end{align*}
$$

Letting $k \rightarrow \infty$ in (B.8), by (B.1), one sees that

$$
\delta_{0} \leq \mathbb{P}\left(\inf _{\phi \in \mathcal{S}_{\alpha, N}} \sum_{j \in \mathcal{Z}_{0}} \int_{\sigma / 2}^{\sigma}\left\langle K_{r, \sigma}^{(0)} \phi, e_{j}\right\rangle^{2} \mathrm{~d} S_{r} \leq \mathcal{C}^{2}\left(e^{-\nu M^{2} \sigma / 2}+\frac{1+\sigma}{M^{1 / 2}}\right)\right)
$$

$$
\begin{equation*}
+\frac{\sup _{k \geq 0} \mathbb{E} \Xi(k)}{\mathcal{C}}+\mathbb{P}\left(6 d S_{\sigma} \tilde{X}>\mathcal{C}\right) \tag{B.9}
\end{equation*}
$$

By Lemma 2.2 and the fact that $\sup _{k \in \mathbb{N}}\left\|w_{0}^{(k)}\right\| \leq \mathfrak{R}$, for any $\kappa \in\left(0, \kappa_{0}\right]$, one has $\sup _{k} \mathbb{E} \Xi(k)<\infty$. In (B.9), first letting $M \rightarrow \infty$ and then letting $\mathcal{C} \rightarrow \infty$, we conclude that

$$
\begin{equation*}
\delta_{0} \leq \mathbb{P}\left(\inf _{\phi \in \mathcal{S}_{\alpha, N}} \sum_{j \in \mathcal{Z}_{0}} \int_{\sigma / 2}^{\sigma}\left\langle K_{r, \sigma}^{(0)} \phi, e_{j}\right\rangle^{2} \mathrm{~d} S_{r}=0\right) \tag{B.10}
\end{equation*}
$$

On the other hand, since $K_{r, t}^{(0)}$ is the solution of equation (2.17) with $w_{t}$ issued from $\left.w_{t}\right|_{t=0}=w_{0}^{(0)},(3.5)$ implies that

$$
\mathbb{P}\left(\inf _{\phi \in \mathcal{S}_{\alpha, N}} \sum_{j \in \mathcal{Z}_{0}} \int_{\sigma / 2}^{\sigma}\left\langle K_{r, \sigma}^{(0)} \phi, e_{j}\right\rangle^{2} \mathrm{~d} S_{r}=0\right)=0
$$

This is in conflict with (B.10) and the proof is complete.

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[^1]:    ${ }^{2}$ Note that $\mathcal{M}_{0, t}$ is the Malliavin matrix of $w_{t}$, the solution of equation (1.2) at time $t$ with initial value $w_{0}$. Therefore, $\mathcal{M}_{0, \sigma}$ also depends on $w_{0}$.

[^2]:    ${ }^{3}$ Actually, by Lemma 2.2, $\mathbb{E} X_{n}^{2} \leq \mathbb{E}\left[\left(\sigma_{n+1}-\sigma_{n}\right)^{2} e^{4 \nu\left(\sigma_{n+1}-\sigma_{n}\right)}\right]<\infty$.

