ADDITIVE ACTIONS ON PROJECTIVE HYPERSURFACES WITH A FINITE NUMBER OF ORBITS

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ABSTRACT. An induced additive action on a projective variety $X \subseteq \mathbb{P}^n$ is a regular action of the group \mathbb{G}_a^m on X with an open orbit, which can be extended to a regular action on the ambient projective space \mathbb{P}^n . In this work, we classify all projective hypersurfaces admitting an induced additive action with a finite number of orbits.

1. INTRODUCTION

Let \mathbb{K} be an algebraically closed field of zero characteristic. By a variety or an algebraic group we always mean an algebraic variety or an algebraic group over \mathbb{K} . By open and closed subsets of algebraic varieties we always mean open and closed subsets in Zariski topology. We denote by \mathbb{G}_a the additive group of a field $(\mathbb{K}, +)$ and by \mathbb{G}_a^m the group

$$\mathbb{G}_a^m = \underbrace{\mathbb{G}_a \times \cdots \times \mathbb{G}_a}_{m \text{ times}}.$$

Definition 1. An *additive action* on an algebraic variety X is a regular effective action of \mathbb{G}_a^m on X with an open orbit. By an *induced additive action* on an embedded projective algebraic variety $X \subseteq \mathbb{P}^n$ we mean a regular effective action of \mathbb{G}_a^m on \mathbb{P}^n such that the variety X is the closure of the open orbit of \mathbb{G}_a^m .

Not every additive action on a projective variety is induced. An example can be found in [3, Example 1]. However, when the projective variety $X \subseteq \mathbb{P}^n$ is normal and linearly normal, then this holds and every additive action of \mathbb{G}_a^m on X is induced and lifts to the regular effective action of \mathbb{G}_a^m on the projective space \mathbb{P}^n .

In [16] a remarkable correspondence between additive actions on the projective space \mathbb{P}^n and local algebras of dimension n + 1 was obtained. By a *local algebra* we mean a commutative associative algebra over \mathbb{K} with a unit and a unique maximal ideal. We will recall this correspondence in Section 2. A more general correspondence between actions of commutative algebraic groups on \mathbb{P}^n with an open orbit and associative commutative algebras with a unit element of dimension n + 1 was established in [17].

The systematic study of additive actions on projective and complete varieties was initiated in [3, 5, 20]. There are several results on additive actions on projective hypersurfaces. For example, it was proven in [20] that there is a unique additive action on a non-degenerate quadric. This result was generalized in [10], where actions of arbitrary algebraic commutative groups on non-degenerate quadrics with an open orbit were described. In [3] and [5]induced actions on projective hypersurfaces were studied. It was proven in [6] that a non-degenerate hypersurface (see Definition 2) admits at most one additive action. And

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when a degenerate hypersurface admits an additive action, then there are at least two nonisomorphic additive actions on it, see [9]. For additive actions on degenerate hypersurfaces we refer also to [18].

Flag varieties admitting an additive action were classified in [1] and all additive actions on flag varieties were classified in [13]. Additive actions on toric varieties were studied in [2,4,14,15,19,21,22]. There are results on additive actions on Fano varieties in [7,8,11,12,23]. For a detailed review of the results on additive actions we refer to [6].

In this paper we find all projective hypersurfaces admitting an induced additive action with a finite number of orbits. We use the technique developed in [3, 5, 6, 20], generalizing the correspondence from [16, 17]. Each hypersurface admitting an induced additive action corresponds to a pair (A, U), where A is a local algebra with the maximal ideal \mathfrak{m} and U is a subspace in \mathfrak{m} of codimension 1 generating A as an algebra with unit. We classify all such pairs (A, U) that correspond to hypersurfaces admitting an induced additive action with a finite number of orbits, see Theorem 3. By a pair (A, U), one can find an equation defining the hypersurface using [6, Theorem 2.14]. Our final result is formulated in Corollary 4.

2. Additive actions on projective varieties

In this section we recall some of the facts on additive actions on projective varieties. We say that two induced additive actions on a projective variety $X \subseteq \mathbb{P}^n$ are *equivalent* if one is obtained from the other via an automorphism of \mathbb{P}^n preserving X.

Proposition 1. [16, Proposition 2.15] There is a one-to-one correspondence between

- (1) equivalence classes of additive actions on \mathbb{P}^n ;
- (2) isomorphism classes of local algebras of dimension n + 1.

We now recall how to construct an additive action on \mathbb{P}^n by an (n+1)-dimensional local algebra A. Let \mathfrak{m} be the maximal ideal in A. Then $A = \mathbb{K} \oplus \mathfrak{m}$ (a direct sum of vector spaces) and all elements in the ideal \mathfrak{m} are nilpotent. This is a well-known fact, for the proof we refer to [6, Lemma 1.2]. Consider an exponential map on \mathfrak{m} :

$$m \mapsto \exp(m) = \sum_{i \ge 0} \frac{m^i}{i!}, \text{ for } m \in \mathfrak{m}.$$

This map is well-defined on \mathfrak{m} . The additive group of \mathfrak{m} is isomorphic to \mathbb{G}_a^n and \mathfrak{m} acts on the algebra A by the following rule: $m \circ a = \exp(m) \cdot a$. This is an algebraic action. The stabilizer of a unit is trivial, so we have the following isomorphisms of algebraic varieties

$$\mathbb{A}^n \simeq \mathbb{G}^n \simeq \exp(\mathfrak{m}) \cdot 1 = 1 + \mathfrak{m},$$

where the last equality is satisfied since the map

$$1 + m \mapsto \ln(1 + m) = \sum_{i>0} (-1)^{i-1} \frac{m^i}{i}, \text{ for } m \in \mathfrak{m},$$

is well-defined on $1 + \mathfrak{m}$ and $\exp(\ln(1+m)) = 1 + m$. The action of \mathfrak{m} on the algebra A defines an algebraic action of \mathbb{G}_a^n on the projective space $\mathbb{P}^n = \mathbb{P}(A)$ by the rule

$$m \circ \pi(a) = \pi(\exp(m) \cdot a),$$

where the map $\pi: A \setminus \{0\} \to \mathbb{P}(A)$ is the canonical projection. The orbit of $\pi(1)$ is the open orbit, so this defines an additive action on \mathbb{P}^n . See [6, Example 1.50] for further examples of this construction.

Proposition 2. [3, Proposition 3]

Let X be a projective hypersurface in \mathbb{P}^n of degree at least 2. Then there is a one-to-one correspondence between

- (1) equivalence classes of induced additive actions on X;
- (2) isomorphism classes of pairs (A, U), where A is a local (n + 1)-dimensional algebra with the maximal ideal \mathfrak{m} and U is an (n 1)-dimensional subspace in \mathfrak{m} that generates A as an algebra with a unit.

The pairs (A, U) from Proposition 2 are called *H*-pairs. We say that two *H*-pairs (A_1, U_1) and (A_2, U_2) are isomorphic if there is an isomorphism of local algebras $\varphi \colon A_1 \to A_2$ such that $\varphi(U_1) = U_2$.

We can construct an additive action on a projective hypersurface $X \subseteq \mathbb{P}^n$ similarly to one from Proposition 1. From now on until the end of the section, we fix an *H*-pair (A, U). Let \mathfrak{m} be the maximal ideal of A. We define the action of \mathfrak{m} on $\mathbb{P}(A)$ in the same way as in Proposition 1. We then restrict this action on the subgroup $U \simeq \mathbb{G}_a^{n-1}$ and consider the subvariety

$$X = \pi(\overline{\exp(U) \cdot 1}).$$

Then X is a hypersurface in $\mathbb{P}(A) = \mathbb{P}^n$ and the group U acts on X with an open orbit.

The following results illustrate how to find the defining equation and degree of X.

Theorem 1. [5, Theorem 5.1] The degree of the hypersurface X is equal to the largest number $d \in \mathbb{N}$ such that $\mathfrak{m}^d \nsubseteq U$, where \mathfrak{m} is the maximal ideal in the corresponding local algebra A.

Theorem 2. [6, Theorem 2.14] The hypersurface X is given in $\mathbb{P}(A)$ by the following homogeneous equation:

$$z_0^d \pi \left(\ln \left(1 + \frac{z}{z_0} \right) \right) = 0,$$

where $z_0 \in \mathbb{K}$, $z \in \mathfrak{m}$ and $\pi \colon \mathfrak{m} \to \mathfrak{m}/U \simeq \mathbb{K}$ is the canonical projection.

It is also possible to describe elements $a \in A$ such that $\pi(a) \in X$.

Proposition 3. [6, Corollary 2.18] The complement of the open U-orbit in X is the set

$$\{ \pi(z) \mid z \in \mathfrak{m} \text{ such that } z^d \in U \},\$$

where $\pi: A \setminus \{0\} \to \mathbb{P}(A)$ is the canonical projection and d is the degree of X.

Corollary 1. Suppose that the point $x \in X$ belongs to the complement of the open orbit of the group U. Then the m-orbit of x is contained in X.

Proof. Let us take $z \in \mathfrak{m}$ such that $\pi(z) = x$ lies in X. Then $z^d \in \mathfrak{m}^d \cap U$. The \mathfrak{m} -orbit of the element z is $z + z \cdot \mathfrak{m}$. But then $(z + z \cdot \mathfrak{m})^d \subseteq z^d + \mathfrak{m}^{d+1} \subseteq U$. So $\pi(z + z \cdot \mathfrak{m}) \subseteq X$. \Box

We recall that a *socle* of a local algebra A is the ideal $Soc(A) := \{z \in A \mid z \cdot \mathfrak{m} = 0\}.$

Corollary 2. The set $\{\pi(z) \mid z \in \text{Soc}(A) \setminus \{0\}\}$ is contained in X.

Proof. For all $z \in \text{Soc}(A)$ we have $z^d = 0$ is in the group U.

Corollary 3. When $\dim(\operatorname{Soc}(A)) > 1$ then there are infinitely many U-orbits on X.

Proof. If $z \in \text{Soc}(A)$ then $\exp(U) \cdot z = \{z\}$. So the set $\{\pi(z) \mid z \in \text{Soc}(A) \setminus \{0\}\} \subseteq X$ consists of the U-fixed points and has dimension at least 1. \Box

It is also possible to describe the relationship between \mathfrak{m} -orbits on X and U-orbits. For an element $z \in A$ we denote by Ann(z) the ideal $\{a \in A \mid az = 0\}$.

Proposition 4. Let $z \in \mathfrak{m} \setminus \{0\}$ be an element with $\pi(z) \in X$.

- (1) If $\operatorname{Ann}(z) + U = \mathfrak{m}$, then the \mathfrak{m} -orbit of $\pi(z)$ coincides with the U-orbit.
- (2) Otherwise, the \mathfrak{m} -orbit of $\pi(z)$ is the union of an infinite number of U-orbits.

Proof. We will show that Ann(z) coincides with the stabilizer $St_{\mathfrak{m}}(\pi(z))$ with respect to the m-action. The inclusion $\operatorname{Ann}(z) \subseteq \operatorname{St}_{\mathfrak{m}}(\pi(z))$ is clear. Indeed, if $a \in \operatorname{Ann}(z)$ then we have az = 0 and $\exp(a) \cdot z = z$. We now show the reverse inclusion. If $a \in St_m(\pi(z))$ then

$$\exp(a) \cdot \pi(z) = \pi(z + az + \frac{a^2}{2}z + \frac{a^3}{6}z + \ldots) = \pi(z),$$

which implies

$$az + \frac{a^2}{2}z + \frac{a^3}{6}z + \ldots = 0.$$

Assume $az \neq 0$. Then there is a number $k \in \mathbb{N}$ such that $az \in \mathfrak{m}^k \setminus \mathfrak{m}^{k+1}$. However, then the element $\frac{a^2}{2}z + \frac{a^3}{6}z + \ldots$ lies in the ideal \mathfrak{m}^{k+1} . So az = 0 and $a \in \operatorname{Ann}(z)$. Thus, the \mathfrak{m} -orbit of $\pi(z)$ is isomorphic to $\mathfrak{m}/\operatorname{Ann}(z)$ and U-orbit of $\pi(z)$ is isomorphic to

$$U/(\operatorname{Ann}(z) \cap U) \simeq (U + \operatorname{Ann}(z))/\operatorname{Ann}(z).$$

Hence, if $U + \operatorname{Ann}(z) = \mathfrak{m}$ then the \mathfrak{m} -orbit of $\pi(z)$ coincides with the U-orbit, and if $U + \operatorname{Ann}(z) \neq \mathfrak{m}$ then the action of U on $\mathfrak{m}/\operatorname{Ann}(z)$ has infinitely many orbits.

3. Main result

In this section we state our main result. Recall that for a local algebra A with the maximal ideal \mathfrak{m} the following sequence of numbers

$$(\dim_{\mathbb{K}} A/\mathfrak{m}, \dim_{\mathbb{K}} \mathfrak{m}/\mathfrak{m}^2, \dim_{\mathbb{K}} \mathfrak{m}^2/\mathfrak{m}^3, \ldots)$$

is called a *Hilbert-Samuel sequence*.

Proposition 5. Let (A, U) be an *H*-pair and *X* be the corresponding hypersurface in $\mathbb{P}(A)$. Suppose that there are finitely many U-orbits in X. Then the Hilbert-Samuel sequence of Ais either $(1, 1, 1, \dots, 1)$ or $(1, 2, 1, \dots, 1)$.

Proof. Since \mathbb{K} is algebraically closed and A/\mathfrak{m} is a finite dimensional field over \mathbb{K} we have

$$\dim_{\mathbb{K}} A/\mathfrak{m} = 1.$$

Suppose there is a number $k \geq 2$ with $\dim_{\mathbb{K}} \mathfrak{m}^k/\mathfrak{m}^{k+1} > 1$. For all $z \in \mathfrak{m}^k \setminus \{0\}$ we have

$$z^d \in \mathfrak{m}^{kd} \subseteq \mathfrak{m}^{d+1} \subseteq U_t$$

where d is the degree of X. Then $\pi(z)$ lies in X for all $z \in \mathfrak{m}^k \setminus \{0\}$, where $\pi: A \setminus \{0\} \to \mathbb{P}(A)$ is the canonical projection. The m-orbit of $\pi(z)$ is $\pi(z + z \cdot \mathfrak{m}) \subseteq \pi(z + \mathfrak{m}^{k+1})$. Thus, if the images of elements z_1 and z_2 from \mathfrak{m}^k in $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ are not proportional, then the \mathfrak{m} orbits of $\pi(z_1)$ and $\pi(z_2)$ do not coincide, so their U-orbits are also different. Therefore, if $\dim_{\mathbb{K}} \mathfrak{m}^k/\mathfrak{m}^{k+1} > 1$, there are infinitely many U-orbits on X, this contradicts our assumption.

It implies that the Hilbert-Samuel sequence has the following form: (1, r, 1, ..., 1). Now suppose that $r \ge 3$ and consider a map:

$$\begin{split} \varphi \colon \mathfrak{m}/\mathfrak{m}^2 &\to \mathfrak{m}^d/\mathfrak{m}^{d+1}, \\ z &+ \mathfrak{m}^2 \mapsto z^d + \mathfrak{m}^{d+1}. \end{split}$$

The map φ is a morphism between algebraic varieties $\mathfrak{m}/\mathfrak{m}^2 \simeq \mathbb{A}^r$ and $\mathfrak{m}^d/\mathfrak{m}^{d+1} \simeq \mathbb{A}^1$. The set $Z := \varphi^{-1}(0 + \mathfrak{m}^{d+1})$ is non-empty, so dim $(Z) \ge r - 1 \ge 2$. For all elements $z \in \mathfrak{m} \setminus \{0\}$ with $z + \mathfrak{m}^2 \in Z$ we have $\pi(z)$ is lying in X. As previously, when elements $z_1 + \mathfrak{m}^2 \in Z$ and $z_2 + \mathfrak{m}^2 \in Z$ are not proportional then the U-orbits of $\pi(z_1)$ and $\pi(z_2)$ are different.

Since dim $(Z) \ge 2$ there are infinitely many U-orbits on X. This contradicts our assumption, thus $r \le 2$ and the Hilbert-Samuel sequence of A equals $(1, \ldots, 1)$ or $(1, 2, 1, \ldots, 1)$. \Box

Proposition 6. Let (A, U) be an *H*-pair and *X* be the corresponding hypersurface in $\mathbb{P}(A)$. Suppose that there are finitely many *U*-orbits in *X*. Then for $n \ge 1$ we have

$$A\simeq \mathbb{K}[x]/(x^{n+1}) \quad or \quad A\simeq \mathbb{K}[x,y]/(xy,x^3,y^2-x^2)$$

Proof. First suppose the Hilbert-Samuel sequence of A equals to (1, 1, ..., 1). Then A is generated by one nilpotent element, so A is isomorphic to $\mathbb{K}[x]/(x^{n+1})$.

Now consider the case when the Hilbert-Samuel sequence of A is (1, 2, ..., 1). Denote by r the maximal number such that $\mathfrak{m}^r \neq 0$, where \mathfrak{m} is the maximal ideal in A. If r = 1then $A \simeq \mathbb{K}[x, y]/(x^2, xy, y^2)$. In this case $\operatorname{Soc}(A) = \langle x, y \rangle$, this contradicts Corollary 3.

Now consider the case r > 1. Then there is an element $x \in \mathfrak{m}$ such that $\langle x^r \rangle = \mathfrak{m}^r$, see [6, Lemma 2.13]. Hence, $\mathfrak{m} = \langle x, x^2, \dots, x^r, y \rangle$ where $y \in \mathfrak{m} \setminus \mathfrak{m}^2$ and images of x and y are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$. Thus, xy lies in \mathfrak{m}^2 , so xy = f(x), where f(x) is a polynomial divisible by x^2 . We replace y with $y - \frac{f(x)}{x}$ to obtain xy = 0. The element y^2 belongs to \mathfrak{m}^2 . Thus, $y^2 = g(x)$, where g(x) is a polynomial divisible by

The element y^2 belongs to \mathfrak{m}^2 . Thus, $y^2 = g(x)$, where g(x) is a polynomial divisible by x^2 . Assume that $y^2 = 0$, then $\operatorname{Soc}(A) = \langle x^r, y \rangle$ which contradicts Corollary 3. On the other hand, $xy^2 = (xy)y = 0 = xg(x)$. It implies that $g(x) = \lambda x^r$, where $\lambda \in \mathbb{K} \setminus \{0\}$. We replace y with $\sqrt{\lambda}y$ to get $y^2 = x^r$. Then A is isomorphic to the algebra $\mathbb{K}[x, y]/(xy, x^{r+1}, y^2 - x^r)$.

To complete the proof we should show that $r \leq 2$. Assume the opposite, i.e., r > 2. If we denote by $d \geq 2$ the degree of the hypersurface X, then $\mathfrak{m}^{d+1} \subseteq U$. We have

$$(y + \alpha x^2)^d = y^d + \alpha^d x^{2d} \in \mathfrak{m}^{d+1}$$
 for all $\alpha \in \mathbb{K}$.

Here we use that $y^2 = x^r$ and $y^3 = 0$. Therefore, $\pi(y + \alpha x^2) \in X$ for all $\alpha \in \mathbb{K}$, where, as before, $\pi: A \to \mathbb{P}(A)$ is the canonical projection. The **m**-orbit of $(y + \alpha x^2)$ is the set

$$y + \alpha x^2 + (y + \alpha x^2)\mathfrak{m} \subseteq y + \alpha x^2 + \mathfrak{m}^3.$$

That is, the **m**-orbits of the points $\pi(y + \alpha x^2)$ do not coincide. Hence, if r > 2 there are infinitely many U-orbits on X, which leads to a contradiction.

Remark 1. Note that the algebra $\mathbb{K}[x,y]/(xy,x^3,y^2-x^2)$ is isomorphic to $\mathbb{K}[x,y]/(x^2,y^2)$. To see this, one should take $\tilde{x} = y - ix$ and $\tilde{y} = y + ix$, then we get

$$\mathbb{K}[x,y]/(xy,x^3,y^2-x^2) = \mathbb{K}[\tilde{x},\tilde{y}]/(\tilde{x}^2,\tilde{y}^2)$$

We are ready to state our first main result.

Theorem 3. Let (A, U) be an *H*-pair and *X* be the corresponding hypersurface in $\mathbb{P}(A)$. Then there are finitely many *U*-orbits on *X* if and only if the pair (A, U) is isomorphic to one of the following pairs:

$$(\mathbb{K}[x]/(x^{n+1}), U_i), \text{ where } U_i := \langle x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^n \rangle \text{ with } i \ge n-1, \text{ or } (\mathbb{K}[x, y]/(x^2, y^2), W), \text{ where } W = \langle x, y \rangle.$$

To prove Theorem 3, we need the following lemma.

Lemma 1. Let (A, U) be one of the following H-pairs:

$$(\mathbb{K}[x]/(x^{n+1}), U_i), where \ U_i := \langle x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^n \rangle$$
 with $i > 1$, or $(\mathbb{K}[x, y]/(x^2, y^2), W), where \ W = \langle x, y \rangle.$

And consider the corresponding hypersurface X. Then, in both cases, there are finitely many U-orbits on X if and only if $i \ge n-1$.

Proof. First consider the case when an *H*-pair (A, U) equals to $(\mathbb{K}[x]/(x^{n+1}), U_i)$. By Theorem 1, the degree of X is equal to *i*. By Proposition 3, the complement to the open U-orbit in X is the set

$$\{\pi(z) \mid z \in \mathfrak{m} \text{ such that } z^i \in U\} = \pi(\mathfrak{m}^2).$$

By Corollary 1, for each point $\pi(z)$ from this set the **m**-orbit of $\pi(z)$ is contained in X. The total number of **m**-orbits on $\mathbb{P}(A)$ is finite, see [16, Proposition 3.7]. Each **m**-orbit either coincides with an U-orbit or is the union of infinite number of U-orbits. Therefore, the total number of U-orbits in X is finite if and only if for all $z \in \mathfrak{m}^2 \setminus \{0\}$ the **m**-orbit of $\pi(z)$ is equal to U-orbit of $\pi(z)$. By Proposition 4 this is equivalent to

Ann
$$(z) + U = \mathfrak{m}, \forall z \in \mathfrak{m}^2$$

For $z \in \mathfrak{m}^2 \setminus \mathfrak{m}^3$ we have $\operatorname{Ann}(z) = \mathfrak{m}^{n-1} = \langle x^{n-1} \rangle$ and $\operatorname{Ann}(z) \supseteq \langle x^{n-1} \rangle$ for all other $z \in \mathfrak{m}^2$. Therefore, the total number of U-orbits in X is finite if and only if

$$\langle x^{n-1} \rangle + U_i = \mathfrak{m}$$

It implies that i = n or n - 1.

In the case $(A, U) = (\mathbb{K}[x, y]/(x^2, y^2), \langle x, y \rangle)$, the degree of X is 2. Three **m**-orbits are contained in the complement to the open U-orbit in X. They are $\pi(x + \mathbb{K}xy), \pi(y + \mathbb{K}xy)$ and $\pi(xy)$. It is easy to see that all these **m**-orbits coincide with U-orbits.

Proof. (of Theorem 3) Let (A, U) be an *H*-pair and suppose that corresponding hypersurface $X \subseteq \mathbb{P}^n$ contains only a finite number of *U*-orbits. By Proposition 6 and Remark 1 the algebra *A* is isomorphic to $\mathbb{K}[x]/(x^{n+1})$ or $\mathbb{K}[x,y]/(x^2,y^2)$.

Consider the case $A \simeq \mathbb{K}[x]/(x^{n+1})$. Let U be an (n-1)-dimensional subspace in \mathfrak{m} , which generates A. Suppose that $\langle x^n \rangle \not\subseteq U$. Then

$$U = \langle x + \alpha_1 x^n, x^2 + \alpha_2 x^n, \dots, x^{n-1} + \alpha_{n-1} x^n \rangle$$

for some $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{K}$. For all $\beta_2, \ldots, \beta_n \in \mathbb{K}$ we consider an automorphism φ of A, $\varphi \colon x \mapsto x + \beta_2 x^2 + \ldots + \beta_n x^n$. Then

$$\varphi(x^k + \alpha_k x^n) = (k\beta_{n-k+1} + h_k(\beta_2, \dots, \beta_{n-k}) + \alpha_k)x^n + O(x^{n-1}).$$

We take $\beta_{n-k+1} = -\frac{1}{k}(\alpha_k + h_k(\beta_2, \dots, \beta_{n-k}))$ for all $k = 1, \dots, n-1$. Then

$$\varphi(x^k + \alpha_k x^n) \in \langle x, \dots, x^{n-1} \rangle \quad \forall k = 1, \dots, n-1$$

Therefore, $\varphi(U) = \langle x, \dots, x^{n-1} \rangle$.

If $\langle x^n \rangle \subseteq U$ we can consider the canonical homomorphism $\pi \colon A \to A/\langle x^n \rangle \simeq \mathbb{K}[x]/(x^n)$. Then $\pi(U)$ is an (n-2)-dimensional subspace that generates $A/\langle x^n \rangle$. Proceeding by induction we obtain that up to automorphism of $A/\langle x^n \rangle$

$$\pi(U) = \langle x + \langle x^n \rangle, x^2 + \langle x^n \rangle, \dots, x^{i-1} + \langle x^n \rangle, x^{i+1} + \langle x^n \rangle, \dots \rangle$$

But then $U = U_i$.

Now we consider the case $A \simeq \mathbb{K}[x, y]/(x^2, y^2)$. If a 2-dimensional subspace W in $\langle x, y, xy \rangle$ generates A then $W = \langle x + \alpha xy, y + \beta xy \rangle$. Applying the automorphism of A

$$x \mapsto x - \alpha xy, \quad y \mapsto y - \beta xy,$$

we obtain that $W = \langle x, y \rangle$. The statement of Lemma 1 completes the proof.

By an *H*-pair we can find the equation of the corresponding hypersurface X. For example, we consider the *H*-pair $(A, U) = (\mathbb{K}[x]/(x^3), \langle x \rangle)$. Then we apply Theorem 2. If we choose a basis $1, x, x^2$ in A then the map $\pi \colon A \to A/U$ can be given as follows:

$$z_0 + z_1 x + z_2 x^2 \mapsto z_2$$

In this case, the degree of X is 2. If we denote $z = z_1 x + z_2 x^2$ we obtain

$$\ln(1+\frac{z}{z_0}) = \frac{z}{z_0} - \frac{z^2}{2z_0^2} = \frac{z_1}{z_0}x + \frac{2z_2z_0 - z_1^2}{2z_0^2}x^2.$$

The hypersurface X is then given by the following equation:

$$z_0^2 \cdot \pi(\ln(1+\frac{z}{z_0})) = 2z_1z_0 + z_2z_0 - 2z_1^2 = 0.$$

This is a non-degenerate quadric of rank 3. Below we recall the definition of a nondegenerate hypersurface.

Definition 2. [6, Definition 2.22] Suppose a projective hypersurface $X \subseteq \mathbb{P}^n$ of degree d is given by an equation $f(z_0, z_1, \ldots, z_n) = 0$. Then X is called *non-degenerate* if one of the following equivalent conditions holds:

- (a) $\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}$ are linearly independent (d-1)-linear forms; (b) there is no linear transformation of variables z_0, \ldots, z_n that reduces the number of variables in f.

An *H*-pair (A, U) defines a non-degenerate hypersurface if and only if dim(Soc(A)) = 1and $\mathfrak{m} = U \oplus \operatorname{Soc}(A)$, see [6, Theorem 2.30]. As a corollary we have the following result.

Corollary 4. Let $X \subseteq \mathbb{P}^n$ be a projective hypersurface admitting an induced additive action with a finite number of orbits.

- (1) When n = 2, X is a non-degenerate quadric of rank 3.
- (2) When n = 3, X is one of the following projective surfaces: (a) $\mathbb{P}^1 \times \mathbb{P}^1$ embedded to \mathbb{P}^3 as a non-degenerate quadric of rank 4 via Segre embedding;
 - (b) A non-degenerate cubic $z_0^2 z_3 z_0 z_1 z_2 + \frac{z_1^3}{3} = 0.$
 - (c) A degenerate quadric of rank 3.
- (3) When n > 3, X is either a non-degenerate hypersurface of degree n or a degenerate hypersurface of degree n-1.

In Table 1 one can find the equations of the resulting non-degenerate hypersurfaces of dimensions 2–5.

$\dim(X)$	The equation of a hypersurface
2	$2z_1z_0 + z_2z_0 - 2z_1^2 = 0$
3	$z_0 z_3 - z_1 z_2 = 0$
3	$z_0^2 z_3 - z_0 z_1 z_2 + \frac{z_1^3}{3} = 0$
4	$z_0^3 z_4 - z_0^2 z_1 z_3 + \frac{z_0^2 z_2^2}{2} + z_0 z_1^2 z_2 - \frac{z_1^4}{4} = 0$
5	$z_0^4 z_5 - z_0^3 z_4 z_5 + z_0^2 z_1^2 z_3 + z_0^2 z_1 z_2^2 - z_0 z_1^3 z_2 + z_1^5 = 0$

TABLE 1.

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