# ADDITIVE ACTIONS ON PROJECTIVE HYPERSURFACES WITH A FINITE NUMBER OF ORBITS 

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#### Abstract

An induced additive action on a projective variety $X \subseteq \mathbb{P}^{n}$ is a regular action of the group $\mathbb{G}_{a}^{m}$ on $X$ with an open orbit, which can be extended to a regular action on the ambient projective space $\mathbb{P}^{n}$. In this work, we classify all projective hypersurfaces admitting an induced additive action with a finite number of orbits.


## 1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of zero characteristic. By a variety or an algebraic group we always mean an algebraic variety or an algebraic group over $\mathbb{K}$. By open and closed subsets of algebraic varieties we always mean open and closed subsets in Zariski topology. We denote by $\mathbb{G}_{a}$ the additive group of a field $(\mathbb{K},+)$ and by $\mathbb{G}_{a}^{m}$ the group

$$
\mathbb{G}_{a}^{m}=\underbrace{\mathbb{G}_{a} \times \cdots \times \mathbb{G}_{a}}_{m \text { times }} .
$$

Definition 1. An additive action on an algebraic variety $X$ is a regular effective action of $\mathbb{G}_{a}^{m}$ on $X$ with an open orbit. By an induced additive action on an embedded projective algebraic variety $X \subseteq \mathbb{P}^{n}$ we mean a regular effective action of $\mathbb{G}_{a}^{m}$ on $\mathbb{P}^{n}$ such that the variety $X$ is the closure of the open orbit of $\mathbb{G}_{a}^{m}$.

Not every additive action on a projective variety is induced. An example can be found in [3, Example 1]. However, when the projective variety $X \subseteq \mathbb{P}^{n}$ is normal and linearly normal, then this holds and every additive action of $\mathbb{G}_{a}^{m}$ on $X$ is induced and lifts to the regular effective action of $\mathbb{G}_{a}^{m}$ on the projective space $\mathbb{P}^{n}$.
In [16] a remarkable correspondence between additive actions on the projective space $\mathbb{P}^{n}$ and local algebras of dimension $n+1$ was obtained. By a local algebra we mean a commutative associative algebra over $\mathbb{K}$ with a unit and a unique maximal ideal. We will recall this correspondence in Section 2. A more general correspondence between actions of commutative algebraic groups on $\mathbb{P}^{n}$ with an open orbit and associative commutative algebras with a unit element of dimension $n+1$ was established in [17].

The systematic study of additive actions on projective and complete varieties was initiated in $[3,5,20]$. There are several results on additive actions on projective hypersurfaces. For example, it was proven in [20] that there is a unique additive action on a non-degenerate quadric. This result was generalized in [10], where actions of arbitrary algebraic commutative groups on non-degenerate quadrics with an open orbit were described. In [3] and [5] induced actions on projective hypersurfaces were studied. It was proven in [6] that a non-degenerate hypersurface (see Definition 2) admits at most one additive action. And

[^0]when a degenerate hypersurface admits an additive action, then there are at least two nonisomorphic additive actions on it, see [9]. For additive actions on degenerate hypersurfaces we refer also to [18].
Flag varieties admitting an additive action were classified in [1] and all additive actions on flag varieties were classified in [13]. Additive actions on toric varieties were studied in $[2,4,14,15,19,21,22]$. There are results on additive actions on Fano varieties in [7,8,11, 12,23]. For a detailed review of the results on additive actions we refer to [6].

In this paper we find all projective hypersurfaces admitting an induced additive action with a finite number of orbits. We use the technique developed in [3,5,6,20], generalizing the correspondence from $[16,17]$. Each hypersurface admitting an induced additive action corresponds to a pair $(A, U)$, where $A$ is a local algebra with the maximal ideal $\mathfrak{m}$ and $U$ is a subspace in $\mathfrak{m}$ of codimension 1 generating $A$ as an algebra with unit. We classify all such pairs $(A, U)$ that correspond to hypersurfaces admitting an induced additive action with a finite number of orbits, see Theorem 3. By a pair $(A, U)$, one can find an equation defining the hypersurface using [ 6 , Theorem 2.14]. Our final result is formulated in Corollary 4.

## 2. Additive actions on projective varieties

In this section we recall some of the facts on additive actions on projective varieties. We say that two induced additive actions on a projective variety $X \subseteq \mathbb{P}^{n}$ are equivalent if one is obtained from the other via an automorphism of $\mathbb{P}^{n}$ preserving $X$.

Proposition 1. [16, Proposition 2.15] There is a one-to-one correspondence between
(1) equivalence classes of additive actions on $\mathbb{P}^{n}$;
(2) isomorphism classes of local algebras of dimension $n+1$.

We now recall how to construct an additive action on $\mathbb{P}^{n}$ by an $(n+1)$-dimensional local algebra $A$. Let $\mathfrak{m}$ be the maximal ideal in $A$. Then $A=\mathbb{K} \oplus \mathfrak{m}$ (a direct sum of vector spaces) and all elements in the ideal $\mathfrak{m}$ are nilpotent. This is a well-known fact, for the proof we refer to [6, Lemma 1.2]. Consider an exponential map on $\mathfrak{m}$ :

$$
m \mapsto \exp (m)=\sum_{i \geq 0} \frac{m^{i}}{i!}, \text { for } m \in \mathfrak{m} .
$$

This map is well-defined on $\mathfrak{m}$. The additive group of $\mathfrak{m}$ is isomorphic to $\mathbb{G}_{a}^{n}$ and $\mathfrak{m}$ acts on the algebra $A$ by the following rule: $m \circ a=\exp (m) \cdot a$. This is an algebraic action. The stabilizer of a unit is trivial, so we have the following isomorphisms of algebraic varieties

$$
\mathbb{A}^{n} \simeq \mathbb{G}^{n} \simeq \exp (\mathfrak{m}) \cdot 1=1+\mathfrak{m},
$$

where the last equality is satisfied since the map

$$
1+m \mapsto \ln (1+m)=\sum_{i>0}(-1)^{i-1} \frac{m^{i}}{i}, \text { for } m \in \mathfrak{m}
$$

is well-defined on $1+\mathfrak{m}$ and $\exp (\ln (1+m))=1+m$. The action of $\mathfrak{m}$ on the algebra $A$ defines an algebraic action of $\mathbb{G}_{a}^{n}$ on the projective space $\mathbb{P}^{n}=\mathbb{P}(A)$ by the rule

$$
m \circ \pi(a)=\pi(\exp (m) \cdot a),
$$

where the map $\pi: A \backslash\{0\} \rightarrow \mathbb{P}(A)$ is the canonical projection. The orbit of $\pi(1)$ is the open orbit, so this defines an additive action on $\mathbb{P}^{n}$. See [6, Example 1.50] for further examples of this construction.

Proposition 2. [3, Proposition 3]
Let $X$ be a projective hypersurface in $\mathbb{P}^{n}$ of degree at least 2. Then there is a one-to-one correspondence between
(1) equivalence classes of induced additive actions on $X$;
(2) isomorphism classes of pairs $(A, U)$, where $A$ is a local $(n+1)$-dimensional algebra with the maximal ideal $\mathfrak{m}$ and $U$ is an $(n-1)$-dimensional subspace in $\mathfrak{m}$ that generates $A$ as an algebra with a unit.
The pairs $(A, U)$ from Proposition 2 are called $H$-pairs. We say that two $H$-pairs $\left(A_{1}, U_{1}\right)$ and $\left(A_{2}, U_{2}\right)$ are isomorphic if there is an isomorphism of local algebras $\varphi: A_{1} \rightarrow A_{2}$ such that $\varphi\left(U_{1}\right)=U_{2}$.

We can construct an additive action on a projective hypersurface $X \subseteq \mathbb{P}^{n}$ similarly to one from Proposition 1. From now on until the end of the section, we fix an $H$-pair $(A, U)$. Let $\mathfrak{m}$ be the maximal ideal of $A$. We define the action of $\mathfrak{m}$ on $\mathbb{P}(A)$ in the same way as in Proposition 1. We then restrict this action on the subgroup $U \simeq \mathbb{G}_{a}^{n-1}$ and consider the subvariety

$$
X=\pi(\overline{\exp (U) \cdot 1})
$$

Then $X$ is a hypersurface in $\mathbb{P}(A)=\mathbb{P}^{n}$ and the group $U$ acts on $X$ with an open orbit.
The following results illustrate how to find the defining equation and degree of $X$.
Theorem 1. [5, Theorem 5.1] The degree of the hypersurface $X$ is equal to the largest number $d \in \mathbb{N}$ such that $\mathfrak{m}^{d} \nsubseteq U$, where $\mathfrak{m}$ is the maximal ideal in the corresponding local algebra $A$.

Theorem 2. [6, Theorem 2.14] The hypersurface $X$ is given in $\mathbb{P}(A)$ by the following homogeneous equation:

$$
z_{0}^{d} \pi\left(\ln \left(1+\frac{z}{z_{0}}\right)\right)=0
$$

where $z_{0} \in \mathbb{K}, z \in \mathfrak{m}$ and $\pi: \mathfrak{m} \rightarrow \mathfrak{m} / U \simeq \mathbb{K}$ is the canonical projection.
It is also possible to describe elements $a \in A$ such that $\pi(a) \in X$.
Proposition 3. [6, Corollary 2.18] The complement of the open $U$-orbit in $X$ is the set

$$
\left\{\pi(z) \mid z \in \mathfrak{m} \text { such that } z^{d} \in U\right\}
$$

where $\pi: A \backslash\{0\} \rightarrow \mathbb{P}(A)$ is the canonical projection and $d$ is the degree of $X$.
Corollary 1. Suppose that the point $x \in X$ belongs to the complement of the open orbit of the group $U$. Then the $\mathfrak{m}$-orbit of $x$ is contained in $X$.
Proof. Let us take $z \in \mathfrak{m}$ such that $\pi(z)=x$ lies in $X$. Then $z^{d} \in \mathfrak{m}^{d} \cap U$. The $\mathfrak{m}$-orbit of the element $z$ is $z+z \cdot \mathfrak{m}$. But then $(z+z \cdot \mathfrak{m})^{d} \subseteq z^{d}+\mathfrak{m}^{d+1} \subseteq U$. So $\pi(z+z \cdot \mathfrak{m}) \subseteq X$.

We recall that a socle of a local algebra $A$ is the ideal $\operatorname{Soc}(A):=\{z \in A \mid z \cdot \mathfrak{m}=0\}$.
Corollary 2. The set $\{\pi(z) \mid z \in \operatorname{Soc}(A) \backslash\{0\}\}$ is contained in $X$.
Proof. For all $z \in \operatorname{Soc}(A)$ we have $z^{d}=0$ is in the group $U$.
Corollary 3. When $\operatorname{dim}(\operatorname{Soc}(A))>1$ then there are infinitely many $U$-orbits on $X$.
Proof. If $z \in \operatorname{Soc}(A)$ then $\exp (U) \cdot z=\{z\}$. So the set $\{\pi(z) \mid z \in \operatorname{Soc}(A) \backslash\{0\}\} \subseteq X$ consists of the $U$-fixed points and has dimension at least 1 .

It is also possible to describe the relationship between $\mathfrak{m}$-orbits on $X$ and $U$-orbits. For an element $z \in A$ we denote by $\operatorname{Ann}(z)$ the ideal $\{a \in A \mid a z=0\}$.

Proposition 4. Let $z \in \mathfrak{m} \backslash\{0\}$ be an element with $\pi(z) \in X$.
(1) If $\operatorname{Ann}(z)+U=\mathfrak{m}$, then the $\mathfrak{m}$-orbit of $\pi(z)$ coincides with the $U$-orbit.
(2) Otherwise, the $\mathfrak{m}$-orbit of $\pi(z)$ is the union of an infinite number of $U$-orbits.

Proof. We will show that $\operatorname{Ann}(z)$ coincides with the stabilizer $\mathrm{St}_{\mathfrak{m}}(\pi(z))$ with respect to the $\mathfrak{m}$-action. The inclusion $\operatorname{Ann}(z) \subseteq \operatorname{St}_{\mathfrak{m}}(\pi(z))$ is clear. Indeed, if $a \in \operatorname{Ann}(z)$ then we have $a z=0$ and $\exp (a) \cdot z=z$. We now show the reverse inclusion. If $a \in \operatorname{St}_{\mathfrak{m}}(\pi(z))$ then

$$
\exp (a) \cdot \pi(z)=\pi\left(z+a z+\frac{a^{2}}{2} z+\frac{a^{3}}{6} z+\ldots\right)=\pi(z),
$$

which implies

$$
a z+\frac{a^{2}}{2} z+\frac{a^{3}}{6} z+\ldots=0 .
$$

Assume $a z \neq 0$. Then there is a number $k \in \mathbb{N}$ such that $a z \in \mathfrak{m}^{k} \backslash \mathfrak{m}^{k+1}$. However, then the element $\frac{a^{2}}{2} z+\frac{a^{3}}{6} z+\ldots$ lies in the ideal $\mathfrak{m}^{k+1}$. So $a z=0$ and $a \in \operatorname{Ann}(z)$.

Thus, the $\mathfrak{m}$-orbit of $\pi(z)$ is isomorphic to $\mathfrak{m} / \operatorname{Ann}(z)$ and $U$-orbit of $\pi(z)$ is isomorphic to

$$
U /(\operatorname{Ann}(z) \cap U) \simeq(U+\operatorname{Ann}(z)) / \operatorname{Ann}(z)
$$

Hence, if $U+\operatorname{Ann}(z)=\mathfrak{m}$ then the $\mathfrak{m}$-orbit of $\pi(z)$ coincides with the $U$-orbit, and if $U+\operatorname{Ann}(z) \neq \mathfrak{m}$ then the action of $U$ on $\mathfrak{m} / \operatorname{Ann}(z)$ has infinitely many orbits.

## 3. Main result

In this section we state our main result. Recall that for a local algebra $A$ with the maximal ideal $\mathfrak{m}$ the following sequence of numbers

$$
\left(\operatorname{dim}_{\mathbb{K}} A / \mathfrak{m}, \operatorname{dim}_{\mathbb{K}} \mathfrak{m} / \mathfrak{m}^{2}, \operatorname{dim}_{\mathbb{K}} \mathfrak{m}^{2} / \mathfrak{m}^{3}, \ldots\right)
$$

is called a Hilbert-Samuel sequence.
Proposition 5. Let $(A, U)$ be an $H$-pair and $X$ be the corresponding hypersurface in $\mathbb{P}(A)$. Suppose that there are finitely many $U$-orbits in $X$. Then the Hilbert-Samuel sequence of $A$ is either $(1,1,1, \ldots, 1)$ or $(1,2,1, \ldots 1)$.

Proof. Since $\mathbb{K}$ is algebraically closed and $A / \mathfrak{m}$ is a finite dimensional field over $\mathbb{K}$ we have

$$
\operatorname{dim}_{\mathbb{K}} A / \mathfrak{m}=1
$$

Suppose there is a number $k \geq 2$ with $\operatorname{dim}_{\mathbb{K}} \mathfrak{m}^{k} / \mathfrak{m}^{k+1}>1$. For all $z \in \mathfrak{m}^{k} \backslash\{0\}$ we have

$$
z^{d} \in \mathfrak{m}^{k d} \subseteq \mathfrak{m}^{d+1} \subseteq U,
$$

where $d$ is the degree of $X$. Then $\pi(z)$ lies in $X$ for all $z \in \mathfrak{m}^{k} \backslash\{0\}$, where $\pi: A \backslash\{0\} \rightarrow \mathbb{P}(A)$ is the canonical projection. The $\mathfrak{m}$-orbit of $\pi(z)$ is $\pi(z+z \cdot \mathfrak{m}) \subseteq \pi\left(z+\mathfrak{m}^{k+1}\right)$. Thus, if the images of elements $z_{1}$ and $z_{2}$ from $\mathfrak{m}^{k}$ in $\mathfrak{m}^{k} / \mathfrak{m}^{k+1}$ are not proportional, then the $\mathfrak{m}$ orbits of $\pi\left(z_{1}\right)$ and $\pi\left(z_{2}\right)$ do not coincide, so their $U$-orbits are also different. Therefore, if $\operatorname{dim}_{\mathbb{K}} \mathfrak{m}^{k} / \mathfrak{m}^{k+1}>1$, there are infinitely many $U$-orbits on $X$, this contradicts our assumption.

It implies that the Hilbert-Samuel sequence has the following form: $(1, r, 1, \ldots, 1)$. Now suppose that $r \geq 3$ and consider a map:

$$
\begin{aligned}
\varphi: \mathfrak{m} / \mathfrak{m}^{2} & \rightarrow \mathfrak{m}^{d} / \mathfrak{m}^{d+1} \\
z+\mathfrak{m}^{2} & \mapsto z^{d}+\mathfrak{m}^{d+1} .
\end{aligned}
$$

The map $\varphi$ is a morphism between algebraic varieties $\mathfrak{m} / \mathfrak{m}^{2} \simeq \mathbb{A}^{r}$ and $\mathfrak{m}^{d} / \mathfrak{m}^{d+1} \simeq \mathbb{A}^{1}$. The set $Z:=\varphi^{-1}\left(0+\mathfrak{m}^{d+1}\right)$ is non-empty, so $\operatorname{dim}(Z) \geq r-1 \geq 2$. For all elements $z \in \mathfrak{m} \backslash\{0\}$ with $z+\mathfrak{m}^{2} \in Z$ we have $\pi(z)$ is lying in $X$. As previously, when elements $z_{1}+\mathfrak{m}^{2} \in Z$ and $z_{2}+\mathfrak{m}^{2} \in Z$ are not proportional then the $U$-orbits of $\pi\left(z_{1}\right)$ and $\pi\left(z_{2}\right)$ are different.

Since $\operatorname{dim}(Z) \geq 2$ there are infinitely many $U$-orbits on $X$. This contradicts our assumption, thus $r \leq 2$ and the Hilbert-Samuel sequence of $A$ equals $(1, \ldots, 1)$ or $(1,2,1, \ldots 1)$.

Proposition 6. Let $(A, U)$ be an $H$-pair and $X$ be the corresponding hypersurface in $\mathbb{P}(A)$. Suppose that there are finitely many $U$-orbits in $X$. Then for $n \geq 1$ we have

$$
A \simeq \mathbb{K}[x] /\left(x^{n+1}\right) \quad \text { or } \quad A \simeq \mathbb{K}[x, y] /\left(x y, x^{3}, y^{2}-x^{2}\right)
$$

Proof. First suppose the Hilbert-Samuel sequence of $A$ equals to $(1,1, \ldots, 1)$. Then $A$ is generated by one nilpotent element, so $A$ is isomorphic to $\mathbb{K}[x] /\left(x^{n+1}\right)$.

Now consider the case when the Hilbert-Samuel sequence of $A$ is $(1,2, \ldots, 1)$. Denote by $r$ the maximal number such that $\mathfrak{m}^{r} \neq 0$, where $\mathfrak{m}$ is the maximal ideal in $A$. If $r=1$ then $A \simeq \mathbb{K}[x, y] /\left(x^{2}, x y, y^{2}\right)$. In this case $\operatorname{Soc}(A)=\langle x, y\rangle$, this contradicts Corollary 3.

Now consider the case $r>1$. Then there is an element $x \in \mathfrak{m}$ such that $\left\langle x^{r}\right\rangle=\mathfrak{m}^{r}$, see [6, Lemma 2.13]. Hence, $\mathfrak{m}=\left\langle x, x^{2}, \ldots, x^{r}, y\right\rangle$ where $y \in \mathfrak{m} \backslash \mathfrak{m}^{2}$ and images of $x$ and $y$ are linearly independent in $\mathfrak{m} / \mathfrak{m}^{2}$. Thus, $x y$ lies in $\mathfrak{m}^{2}$, so $x y=f(x)$, where $f(x)$ is a polynomial divisible by $x^{2}$. We replace $y$ with $y-\frac{f(x)}{x}$ to obtain $x y=0$.

The element $y^{2}$ belongs to $\mathfrak{m}^{2}$. Thus, $y^{2}=g(x)$, where $g(x)$ is a polynomial divisible by $x^{2}$. Assume that $y^{2}=0$, then $\operatorname{Soc}(A)=\left\langle x^{r}, y\right\rangle$ which contradicts Corollary 3. On the other hand, $x y^{2}=(x y) y=0=x g(x)$. It implies that $g(x)=\lambda x^{r}$, where $\lambda \in \mathbb{K} \backslash\{0\}$. We replace $y$ with $\sqrt{\lambda} y$ to get $y^{2}=x^{r}$. Then $A$ is isomorphic to the algebra $\mathbb{K}[x, y] /\left(x y, x^{r+1}, y^{2}-x^{r}\right)$.

To complete the proof we should show that $r \leq 2$. Assume the opposite, i.e., $r>2$. If we denote by $d \geq 2$ the degree of the hypersurface $X$, then $\mathfrak{m}^{d+1} \subseteq U$. We have

$$
\left(y+\alpha x^{2}\right)^{d}=y^{d}+\alpha^{d} x^{2 d} \in \mathfrak{m}^{d+1} \quad \text { for all } \alpha \in \mathbb{K}
$$

Here we use that $y^{2}=x^{r}$ and $y^{3}=0$. Therefore, $\pi\left(y+\alpha x^{2}\right) \in X$ for all $\alpha \in \mathbb{K}$, where, as before, $\pi: A \rightarrow \mathbb{P}(A)$ is the canonical projection. The $\mathfrak{m}$-orbit of $\left(y+\alpha x^{2}\right)$ is the set

$$
y+\alpha x^{2}+\left(y+\alpha x^{2}\right) \mathfrak{m} \subseteq y+\alpha x^{2}+\mathfrak{m}^{3}
$$

That is, the $\mathfrak{m}$-orbits of the points $\pi\left(y+\alpha x^{2}\right)$ do not coincide. Hence, if $r>2$ there are infinitely many $U$-orbits on $X$, which leads to a contradiction.

Remark 1. Note that the algebra $\mathbb{K}[x, y] /\left(x y, x^{3}, y^{2}-x^{2}\right)$ is isomorphic to $\mathbb{K}[x, y] /\left(x^{2}, y^{2}\right)$. To see this, one should take $\tilde{x}=y-i x$ and $\tilde{y}=y+i x$, then we get

$$
\mathbb{K}[x, y] /\left(x y, x^{3}, y^{2}-x^{2}\right)=\mathbb{K}[\tilde{x}, \tilde{y}] /\left(\tilde{x}^{2}, \tilde{y}^{2}\right)
$$

We are ready to state our first main result.
Theorem 3. Let $(A, U)$ be an $H$-pair and $X$ be the corresponding hypersurface in $\mathbb{P}(A)$. Then there are finitely many $U$-orbits on $X$ if and only if the pair $(A, U)$ is isomorphic to
one of the following pairs:
$\left(\mathbb{K}[x] /\left(x^{n+1}\right), U_{i}\right)$, where $U_{i}:=\left\langle x^{1}, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{n}\right\rangle$ with $i \geq n-1$, or $\left(\mathbb{K}[x, y] /\left(x^{2}, y^{2}\right)\right.$, $\left.W\right)$, where $W=\langle x, y\rangle$.
To prove Theorem 3, we need the following lemma.
Lemma 1. Let $(A, U)$ be one of the following H-pairs:

$$
\begin{aligned}
\left(\mathbb{K}[x] /\left(x^{n+1}\right), U_{i}\right), \text { where } U_{i}: & =\left\langle x^{1}, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{n}\right\rangle \text { with } i>1, \text { or } \\
\left(\mathbb{K}[x, y] /\left(x^{2}, y^{2}\right), W\right), \text { where } W & =\langle x, y\rangle .
\end{aligned}
$$

And consider the corresponding hypersurface $X$. Then, in both cases, there are finitely many $U$-orbits on $X$ if and only if $i \geq n-1$.

Proof. First consider the case when an $H$-pair $(A, U)$ equals to $\left(\mathbb{K}[x] /\left(x^{n+1}\right), U_{i}\right)$. By Theorem 1, the degree of $X$ is equal to $i$. By Proposition 3, the complement to the open $U$-orbit in $X$ is the set

$$
\left\{\pi(z) \mid z \in \mathfrak{m} \text { such that } z^{i} \in U\right\}=\pi\left(\mathfrak{m}^{2}\right)
$$

By Corollary 1, for each point $\pi(z)$ from this set the $\mathfrak{m}$-orbit of $\pi(z)$ is contained in $X$. The total number of $\mathfrak{m}$-orbits on $\mathbb{P}(A)$ is finite, see [16, Proposition 3.7]. Each $\mathfrak{m}$-orbit either coincides with an $U$-orbit or is the union of infinite number of $U$-orbits. Therefore, the total number of $U$-orbits in $X$ is finite if and only if for all $z \in \mathfrak{m}^{2} \backslash\{0\}$ the $\mathfrak{m}$-orbit of $\pi(z)$ is equal to $U$-orbit of $\pi(z)$. By Proposition 4 this is equivalent to

$$
\operatorname{Ann}(z)+U=\mathfrak{m}, \forall z \in \mathfrak{m}^{2}
$$

For $z \in \mathfrak{m}^{2} \backslash \mathfrak{m}^{3}$ we have $\operatorname{Ann}(z)=\mathfrak{m}^{n-1}=\left\langle x^{n-1}\right\rangle$ and $\operatorname{Ann}(z) \supseteq\left\langle x^{n-1}\right\rangle$ for all other $z \in \mathfrak{m}^{2}$. Therefore, the total number of $U$-orbits in $X$ is finite if and only if

$$
\left\langle x^{n-1}\right\rangle+U_{i}=\mathfrak{m} .
$$

It implies that $i=n$ or $n-1$.
In the case $(A, U)=\left(\mathbb{K}[x, y] /\left(x^{2}, y^{2}\right),\langle x, y\rangle\right)$, the degree of $X$ is 2 . Three $\mathfrak{m}$-orbits are contained in the complement to the open $U$-orbit in $X$. They are $\pi(x+\mathbb{K} x y), \pi(y+\mathbb{K} x y)$ and $\pi(x y)$. It is easy to see that all these $\mathfrak{m}$-orbits coincide with $U$-orbits.

Proof. (of Theorem 3) Let $(A, U)$ be an $H$-pair and suppose that corresponding hypersurface $X \subseteq \mathbb{P}^{n}$ contains only a finite number of $U$-orbits. By Proposition 6 and Remark 1 the algebra $A$ is isomorphic to $\mathbb{K}[x] /\left(x^{n+1}\right)$ or $\mathbb{K}[x, y] /\left(x^{2}, y^{2}\right)$.

Consider the case $A \simeq \mathbb{K}[x] /\left(x^{n+1}\right)$. Let $U$ be an $(n-1)$-dimensional subspace in $\mathfrak{m}$, which generates $A$. Suppose that $\left\langle x^{n}\right\rangle \nsubseteq U$. Then

$$
U=\left\langle x+\alpha_{1} x^{n}, x^{2}+\alpha_{2} x^{n}, \ldots, x^{n-1}+\alpha_{n-1} x^{n}\right\rangle
$$

for some $\alpha_{1}, \ldots, \alpha_{n-1} \in \mathbb{K}$. For all $\beta_{2}, \ldots, \beta_{n} \in \mathbb{K}$ we consider an automorphism $\varphi$ of $A$, $\varphi: x \mapsto x+\beta_{2} x^{2}+\ldots+\beta_{n} x^{n}$. Then

$$
\varphi\left(x^{k}+\alpha_{k} x^{n}\right)=\left(k \beta_{n-k+1}+h_{k}\left(\beta_{2}, \ldots, \beta_{n-k}\right)+\alpha_{k}\right) x^{n}+O\left(x^{n-1}\right) .
$$

We take $\beta_{n-k+1}=-\frac{1}{k}\left(\alpha_{k}+h_{k}\left(\beta_{2}, \ldots, \beta_{n-k}\right)\right)$ for all $k=1, \ldots, n-1$. Then

$$
\varphi\left(x^{k}+\alpha_{k} x^{n}\right) \in\left\langle x, \ldots, x^{n-1}\right\rangle \quad \forall k=1, \ldots, n-1 .
$$

Therefore, $\varphi(U)=\left\langle x, \ldots, x^{n-1}\right\rangle$.

If $\left\langle x^{n}\right\rangle \subseteq U$ we can consider the canonical homomorphism $\pi: A \rightarrow A /\left\langle x^{n}\right\rangle \simeq \mathbb{K}[x] /\left(x^{n}\right)$. Then $\pi(U)$ is an $(n-2)$-dimensional subspace that generates $A /\left\langle x^{n}\right\rangle$. Proceeding by induction we obtain that up to automorphism of $A /\left\langle x^{n}\right\rangle$

$$
\pi(U)=\left\langle x+\left\langle x^{n}\right\rangle, x^{2}+\left\langle x^{n}\right\rangle, \ldots, x^{i-1}+\left\langle x^{n}\right\rangle, x^{i+1}+\left\langle x^{n}\right\rangle, \ldots\right\rangle .
$$

But then $U=U_{i}$.
Now we consider the case $A \simeq \mathbb{K}[x, y] /\left(x^{2}, y^{2}\right)$. If a 2-dimensional subspace $W$ in $\langle x, y, x y\rangle$ generates $A$ then $W=\langle x+\alpha x y, y+\beta x y\rangle$. Applying the automorphism of $A$

$$
x \mapsto x-\alpha x y, \quad y \mapsto y-\beta x y,
$$

we obtain that $W=\langle x, y\rangle$. The statement of Lemma 1 completes the proof.
By an $H$-pair we can find the equation of the corresponding hypersurface $X$. For example, we consider the $H$-pair $(A, U)=\left(\mathbb{K}[x] /\left(x^{3}\right),\langle x\rangle\right)$. Then we apply Theorem 2. If we choose a basis $1, x, x^{2}$ in $A$ then the map $\pi: A \rightarrow A / U$ can be given as follows:

$$
z_{0}+z_{1} x+z_{2} x^{2} \mapsto z_{2}
$$

In this case, the degree of $X$ is 2 . If we denote $z=z_{1} x+z_{2} x^{2}$ we obtain

$$
\ln \left(1+\frac{z}{z_{0}}\right)=\frac{z}{z_{0}}-\frac{z^{2}}{2 z_{0}^{2}}=\frac{z_{1}}{z_{0}} x+\frac{2 z_{2} z_{0}-z_{1}^{2}}{2 z_{0}^{2}} x^{2} .
$$

The hypersurface $X$ is then given by the following equation:

$$
z_{0}^{2} \cdot \pi\left(\ln \left(1+\frac{z}{z_{0}}\right)\right)=2 z_{1} z_{0}+z_{2} z_{0}-2 z_{1}^{2}=0 .
$$

This is a non-degenerate quadric of rank 3 . Below we recall the definition of a nondegenerate hypersurface.

Definition 2. [6, Definition 2.22] Suppose a projective hypersurface $X \subseteq \mathbb{P}^{n}$ of degree $d$ is given by an equation $f\left(z_{0}, z_{1}, \ldots, z_{n}\right)=0$. Then $X$ is called non-degenerate if one of the following equivalent conditions holds:
(a) $\frac{\partial f}{\partial z_{0}}, \ldots, \frac{\partial f}{\partial z_{n}}$ are linearly independent ( $d-1$ )-linear forms;
(b) there is no linear transformation of variables $z_{0}, \ldots, z_{n}$ that reduces the number of variables in $f$.

An $H$-pair $(A, U)$ defines a non-degenerate hypersurface if and only if $\operatorname{dim}(\operatorname{Soc}(A))=1$ and $\mathfrak{m}=U \oplus \operatorname{Soc}(A)$, see [6, Theorem 2.30]. As a corollary we have the following result.

Corollary 4. Let $X \subseteq \mathbb{P}^{n}$ be a projective hypersurface admitting an induced additive action with a finite number of orbits.
(1) When $n=2, X$ is a non-degenerate quadric of rank 3.
(2) When $n=3, X$ is one of the following projective surfaces:
(a) $\mathbb{P}^{1} \times \mathbb{P}^{1}$ embedded to $\mathbb{P}^{3}$ as a non-degenerate quadric of rank 4 via Segre embedding;
(b) A non-degenerate cubic $z_{0}^{2} z_{3}-z_{0} z_{1} z_{2}+\frac{z_{1}^{3}}{3}=0$.
(c) A degenerate quadric of rank 3.
(3) When $n>3, X$ is either a non-degenerate hypersurface of degree $n$ or a degenerate hypersurface of degree $n-1$.

In Table 1 one can find the equations of the resulting non-degenerate hypersurfaces of dimensions 2-5.

| $\operatorname{dim}(X)$ | The equation of a hypersurface |
| :---: | :---: |
| 2 | $2 z_{1} z_{0}+z_{2} z_{0}-2 z_{1}^{2}=0$ |
| 3 | $z_{0} z_{3}-z_{1} z_{2}=0$ |
| 3 | $z_{0}^{2} z_{3}-z_{0} z_{1} z_{2}+\frac{z_{1}^{3}}{3}=0$ |
| 4 | $z_{0}^{3} z_{4}-z_{0}^{2} z_{1} z_{3}+\frac{z_{0}^{2} z_{2}^{2}}{2}+z_{0} z_{1}^{2} z_{2}-\frac{z_{1}^{4}}{4}=0$ |
| 5 | $z_{0}^{4} z_{5}-z_{0}^{3} z_{4} z_{5}+z_{0}^{2} z_{1}^{2} z_{3}+z_{0}^{2} z_{1} z_{2}^{2}-z_{0} z_{1}^{3} z_{2}+z_{1}^{5}=0$ |

Table 1.

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