# RIESZ ENERGY WITH A RADIAL EXTERNAL FIELD: WHEN IS THE EQUILIBRIUM SUPPORT A SPHERE? 

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#### Abstract

We consider Riesz energy problems with radial external fields. We study the question of whether or not the equilibrium is the uniform distribution on a sphere. We develop general necessary as well as general sufficient conditions on the external field that apply to powers of the Euclidean norm as well as certain Lennard-Jones type fields. Additionally, in the former case, we completely characterize the values of the power for which dimension reduction occurs in the sense that the support of the equilibrium measure becomes a sphere. We also briefly discuss the relation between these problems and certain constrained optimization problems. Our approach involves the Frostman characterization, the Funk-Hecke formula, and the calculus of hypergeometric functions.


## 1. Introduction

1.1. Background. We consider Riesz equilibrium problems with an external field on the whole Euclidean space $\mathbb{R}^{d}$ and, unless otherwise stated, assume $d \geq 2$. The Riesz $s$-kernel ${ }^{1}$ $K_{s}: \mathbb{R}^{d} \rightarrow(-\infty, \infty]$ is defined by

$$
K_{s}(x):=\left\{\begin{array}{ll}
\frac{1}{s\|x\|^{s}} & \text { if } s \neq 0 \\
-\log (\|x\|) & \text { if } s=0
\end{array},\right.
$$

where $\|x\|:=\sqrt{x_{1}^{2}+\cdots+x_{d}^{2}}=\sqrt{\langle x, x\rangle}$ is the Euclidean norm. This kernel is continuous for $s<0$, singular but integrable when $0 \leq s<d$, and hypersingular (non-integrable) for $s \geq d$. Throughout, we assume $-2<s<d$, which ensures that the kernel is integrable and conditionally strictly positive definite on compact sets, see [7].

We assume throughout that our external field $V: \mathbb{R}^{d} \rightarrow(-\infty,+\infty]$ is radial, of the form

$$
\begin{equation*}
V(x)=v\left(r^{2}\right), \quad r:=\|x\| \tag{1.1}
\end{equation*}
$$

where $v:[0,+\infty) \rightarrow(-\infty,+\infty]$ is lower semi-continuous, bounded from below, and finite on some interval $(a, b)$.

Let $\mathcal{P}\left(\mathbb{R}^{d}\right)$ be the set of probability measures on $\mathbb{R}^{d}$. For $s<d$, the energy of $\mu \in \mathcal{P}\left(\mathbb{R}^{d}\right)$ is defined by

$$
\begin{equation*}
I_{s, V}(\mu):=\iint\left(K_{s}(x-y)+V(x)+V(y)\right) \mathrm{d} \mu(x) \mathrm{d} \mu(y) \in(-\infty,+\infty] \tag{1.2}
\end{equation*}
$$

[^0]When $V \equiv 0$, we simply write $I_{s}(\mu)$. We shall also denote by $\mathcal{W}_{s, V}$ the minimum of the energy over all probability measures, which is known as the Wiener constant,

$$
\mathcal{W}_{s, V}:=\inf _{\mu \in \mathcal{P}\left(\mathbb{R}^{d}\right)} I_{s, V}(\mu) .
$$

When they exist, the minimizers, called equilibrium measures, are denoted by $\mu_{\text {eq }}$. When unique, these measures must have radially symmetric support, due to the radial symmetry of $K_{s}$ and $V$.

The potential of $\mu$ is the locally integrable function $U_{s}^{\mu}: \mathbb{R}^{d} \rightarrow(-\infty,+\infty]$ defined by

$$
U_{s}^{\mu}(x):=\int K_{s}(x-y) \mathrm{d} \mu(y) .
$$

We will call the quantity $U_{s}^{\mu}+V$ the modified potential of $\mu$.
We concentrate on confining potentials where the external field ensures that the support of the equilibrium measure is compact. The following is a list of some sufficient conditions that ensure the equilibrium measure exists and has compact support:
(a) $0<s<d$ and either $v(\infty):=\lim _{r \rightarrow \infty} v\left(r^{2}\right)=+\infty$,
or $v(\infty)<+\infty$ and $\lim _{r \rightarrow \infty} s r^{s}\left(v\left(r^{r}\right)-v(\infty)\right)<-1$,
(b) $s=0$ and $\lim _{r \rightarrow \infty}\left(v\left(r^{2}\right)-\log r\right)=+\infty$,
(c) $-2<s<0$ and $\limsup _{r \rightarrow \infty} s r^{s} v\left(r^{2}\right)<-2^{-s}$.

Moreover, in each of these cases $\mathcal{W}_{s, V}<\infty$ and the equilibrium measure $\mu_{\text {eq }}$ is unique. See, for example, [16, Theorems 2.1 and 2.4], [36, Theorem I.1.3], and [7, Corollary 4.4.16]. Furthermore, $\mu=\mu_{\text {eq }}$ is characterized by the variational Frostman conditions: for some finite constant $C$,

$$
\begin{array}{ll}
U_{s}^{\mu}+V \geq C, & \text { q.e. on } \mathbb{R}^{d},  \tag{1.3}\\
U_{s}^{\mu}+V \leq C, & \text { on } \operatorname{supp} \mu,
\end{array}
$$

where q.e. denotes a property holding except on sets of $s$-capacity zero (see $[30,7]$ ).
1.2. Main results. In the statement of our results we use the following notation: $\mathrm{d} x$ denotes Lebesgue measure in $\mathbb{R}^{d}$ and $\sigma_{R}$ is the uniform probability measure on the sphere $\mathbb{S}_{R}^{d-1}:=$ $\left\{x \in \mathbb{R}^{d}:\|x\|=R\right\}$ in $\mathbb{R}^{d}$. For $R=1$, we further put $\sigma:=\sigma_{1}$ and $\mathbb{S}^{d-1}:=\mathbb{S}_{1}^{d-1}$. For an interval $I \subseteq \mathbb{R}$, a function $f: I \rightarrow \mathbb{R} \cup\{ \pm \infty\}$ is $\mathcal{C}^{k}$ in the extended sense on $I$ when for each $\ell \in\{0, \ldots, k\}$ and $y \in I, f^{(\ell)}(y)$ exists as an element of $\mathbb{R} \cup\{ \pm \infty\}$ and is finite except at a finite set of values in $I$, and $\lim _{x \rightarrow y} f^{(\ell)}(x)=f^{(\ell)}(y)$, with the limit being one-sided for the endpoints of $I$. When we do not specify the interval, it is assumed to be $[0, \infty)$.

Our first result below gives, as a special case, conditions when the support of equilibrium measure $\mu_{\text {eq }}$ for the energy (1.2) cannot be a sphere.

Theorem 1.1 (Structure of $\mu_{\mathrm{eq}}$ for $s \geq d-3$ ). Suppose that $d-3 \leq s<d$ and $v$ is bounded from below, $\mathcal{C}^{2}$ in the extended sense, with $v^{\prime \prime}$ finite on $(0, \infty)$, and such that $\mathrm{I}_{s, V}$ has a unique equilibrium measure $\mu_{\mathrm{eq}}$. If $d=2$ and $-1 \leq s<0$, assume also that $\lim _{\rho \rightarrow 0^{+}} \rho^{\frac{s}{2}+1} v^{\prime}(\rho)=0$. With $x=r \theta, r \in[0, \infty)$, and $\theta \in \mathbb{S}^{d-1}$, let $\nu \in \mathcal{P}([0, \infty))$ be such that

$$
\mathrm{d} \mu_{\mathrm{eq}}(x)=\mathrm{d} \sigma(\theta) \mathrm{d} \nu(r) .
$$

Then $\operatorname{supp}(\nu)$ is a perfect set, i.e. closed with no isolated points.

In particular $\mathcal{H}_{d-1}\left(\operatorname{supp}\left(\mu_{\mathrm{eq}}\right)\right)=\infty$, where $\mathcal{H}_{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. The additional assumption for $d=2,-1 \leq s<0$ is necessary to rule out the possibility of an isolated point at 0 , as when $s<0$, it is possible for $\delta_{0}$ to be the equilibrium measure, as shown in [12, Remark 1.1]. We also note that the conditions on $v$ can be weakened if we assume $s \geq d-2$, as we show in Lemma 2.3.

When $-2<s<d-3$, a dimension reduction phenomenon may occur. In particular, the following result addresses this possibility when the external field is a power of the Euclidean norm. For this case, our results completely answer the question of when $\mu_{\mathrm{eq}}$ is supported on a sphere, as encountered in [27, Theorem 17] and [12, Theorem 1.2]. For the statement of this result, we introduce the following constants, involving the classical hypergeometric functions ${ }_{2} \mathrm{~F}_{1}$ and ${ }_{3} \mathrm{~F}_{2}$ (see Appendix B ) :

$$
\begin{equation*}
c_{s, d}:={ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}, \frac{2+s-d}{2} ; \frac{d}{2} ; 1\right)=\frac{\Gamma\left(\frac{d}{2}\right) \Gamma(d-s-1)}{\Gamma\left(\frac{d-s}{2}\right) \Gamma\left(d-\frac{s}{2}-1\right)}, \quad \text { for }-2<s<d-1 . \tag{1.4}
\end{equation*}
$$

Note that $c_{s, d}>0, c_{s, d}$ is a strictly convex function of $s$ on $-2<s<d-1$, and $c_{0, d}=1$, $c_{d-2, d}=1$ so $c_{s, d}<1$ for $0<s<d-2$. For $s=0$, we also define, with $\psi_{0}$ denoting the digamma function,

$$
\begin{equation*}
b_{d}:=-\log (2)+\frac{1}{4}{ }_{3} \mathrm{~F}_{2}\left(1,1, \frac{d+1}{2} ; 2, d ; 1\right)=-\log (2)+\frac{1}{2} \psi_{0}(d-1)-\frac{1}{2} \psi_{0}\left(\frac{d-1}{2}\right) . \tag{1.5}
\end{equation*}
$$

Theorem 1.2 (Power-law external fields for which $\operatorname{supp}\left(\mu_{\mathrm{eq}}\right)$ is a sphere). Suppose that $-2<s<d-3$ and $V(x)=\frac{\gamma}{\alpha}\|x\|^{\alpha}$, where $\gamma>0$ and $\alpha>\max \{-s, 0\}$. Define

$$
\alpha_{s, d}:= \begin{cases}\max \left\{\frac{s c_{s, d}}{2-2 c_{s, d}}, 2-\frac{(s+2)(d-s-4)}{2(d-s-3)}\right\} & s \neq 0  \tag{1.6}\\ \max \left\{-\frac{1}{2 b_{d}}, 2-\frac{(d-4)}{(d-3)}\right\} & s=0\end{cases}
$$

If $\alpha \geq \alpha_{s, d}$, then $\mu_{\mathrm{eq}}=\sigma_{R_{*}}$, where

$$
\begin{equation*}
R_{*}=\left(\frac{c_{s, d}}{2 \gamma}\right)^{\frac{1}{\alpha+s}}=\left(\frac{\Gamma\left(\frac{d}{2}\right) \Gamma(d-s-1)}{2 \gamma \Gamma\left(\frac{d-s}{2}\right) \Gamma\left(d-\frac{s}{2}-1\right)}\right)^{\frac{1}{\alpha+s}} \tag{1.7}
\end{equation*}
$$

The energy is then

$$
I_{s, V}\left(\sigma_{R_{*}}\right)= \begin{cases}\frac{(\alpha+s)(2 \gamma)^{\frac{s}{\alpha+s}}}{\alpha s}\left(\frac{\Gamma\left(\frac{d}{2}\right) \Gamma(d-s-1)}{\Gamma\left(\frac{d-s}{2}\right) \Gamma\left(d-\frac{s}{2}-1\right)}\right)^{\frac{\alpha}{\alpha+s}} & s \neq 0  \tag{1.8}\\ \frac{1+\log (2 \gamma)}{\alpha}-\log (2)+\frac{1}{2}\left(\psi_{0}(d-1)-\psi_{0}\left(\frac{d-1}{2}\right)\right) & s=0\end{cases}
$$

Furthermore, the threshold $\alpha_{s, d}$ is a sharp bound for $\alpha$, meaning that if $\max \{-s, 0\}<\alpha<$ $\alpha_{s, d}$, then for all $R>0, \sigma_{R}$ is not a minimizer of $I_{s, V}$.

The hypotheses require that $\alpha+s>0$. When $s<0, \alpha+s=0$, and $\gamma>1$, the equilibrium measure is a point mass at the origin, otherwise the equilibrium measure does not exist, see [12, Remark 1.1]. The lower bounds (1.6) on the parameter $\alpha$ are illustrated in the four graphics of Figure 1. The active bound in (1.6) changes at $s=d-4$, for which $\alpha_{s, d}=2$, and as $d$ increases the bound on $\alpha$ goes to 0 for $-2<s<d-4$. We remark that the properties of $c_{s, d}$ ensure that $\alpha_{s, d}>\max \{-s, 0\}$ and $\alpha_{s, d}$ is continuous at $s=0$, see the discussion at the beginning of section 2.5 .

We next provide necessary conditions for a general twice continuously differentiable radial external field to yield $\sigma_{R_{*}}$ as the equilibrium measure.


Figure 1. Graphics for four values of $d$ (see also next page). The color gives the value of $R_{*}$ when the equilibrium support is $\mathbb{S}_{R_{*}}^{d-1}$, for $d=2,3,4,10$. The plots are for Riesz parameter $-2<s<d-3$ and the external field power $\alpha \geq \alpha_{s, d}$, as in Theorem 1.2 with $\gamma=1$. The two curves are the terms in the maximum in (1.6). Outside the colored region the support is not a sphere.


Theorem 1.3 (Necessary conditions). Suppose that $-2<s<d-3$ and $V(x)=v\left(\|x\|^{2}\right)$, where $v(\cdot)$ is $\mathcal{C}^{2}$ in the extended sense. If $\mu_{\mathrm{eq}}=\sigma_{R_{*}}$ for some $R_{*}>0$, then the following must hold for $R=R_{*}$ :
(i) $R^{s+2} v^{\prime}\left(R^{2}\right)=\frac{c_{s, d}}{4}$
(ii) $R^{2} \frac{v^{\prime \prime}\left(R^{2}\right)}{v^{\prime}\left(R^{2}\right)} \geq-\frac{(s+2)(d-s-4)}{4(d-s-3)}$
(iii) if $s \neq 0$, then

$$
\lim _{r \rightarrow 0^{+}} R^{s}\left(v\left(r^{2}\right)-v\left(R^{2}\right)\right) \geq \frac{c_{s, d}-1}{s}
$$

if $s=0$, then

$$
\lim _{r \rightarrow 0^{+}} v\left(r^{2}\right)-v\left(R^{2}\right) \geq b_{d}
$$

(iv) if $s \neq 0$, then

$$
v\left(R^{2}\right)+\frac{R^{-s}}{s} c_{s, d} \leq \lim _{r \rightarrow \infty}\left[\frac{(R+r)^{-s}}{s}+v\left(r^{2}\right)\right]
$$

if $s=0$, then

$$
-\log (R)+b_{d}+v\left(R^{2}\right) \leq \lim _{r \rightarrow \infty}\left[-\log (R+r)+v\left(r^{2}\right)\right]
$$

In Theorem 1.3, condition (i) arises from the requirement that $R_{*}$ be a stationary point of the modified potential, while (ii) corresponds to non-negativity of a second derivative at $R_{*}$. Conditions (iii) and (iv) arise from boundary conditions at the origin and infinity, respectively. Note that (i) may have no solutions or more than one solution, as shown in Appendix A.2. Additionally, (iii) is trivially satisfied when $\lim _{r \rightarrow 0^{+}} v\left(r^{2}\right)=\infty$, while (iv) is trivially satisfied when $s>0$ and $\lim _{r \rightarrow \infty} v\left(r^{2}\right)=\infty$.

Theorem 1.4 (Sufficient conditions). Suppose that $-2<s \leq d-4$, and $v(\cdot)$ is $\mathcal{C}^{2}$ in the extended sense, and such that $v^{\prime \prime}(r) \geq 0$ for all $r \in[0, \infty)$. If there exists $R_{*} \in(0, \infty)$ that satisfies Theorem 1.3(i), then $\mu_{\mathrm{eq}}=\sigma_{R_{*}}$.

As some examples, Theorem 1.4 ensures that $\mu_{\mathrm{eq}}=\sigma_{R_{*}}$ for the following external fields:

- Lennard-Jones type: $V(x)=\frac{\gamma}{\alpha}\|x\|^{\alpha}-\frac{\gamma \eta}{\beta}\|x\|^{\beta}$, where $\gamma, \eta>0$, so that

$$
v(\rho)=\frac{\gamma}{\alpha} \rho^{\frac{\alpha}{2}}-\frac{\gamma \eta}{\beta} \rho^{\frac{\beta}{2}} .
$$

Theorem 1.4 is satisfied for $\alpha \geq 2 \geq \beta$ with $\alpha>\beta$ and $R_{*}$ the unique solution to

$$
\begin{equation*}
R^{\alpha+s}-\eta R^{\beta+s}=\frac{c_{s, d}}{2 \gamma} . \tag{1.9}
\end{equation*}
$$

- Exponential: $V(x)=\frac{\gamma}{\alpha \beta} \exp \left(\alpha\|x\|^{\beta}\right)$, where $\gamma>0$, so that

$$
v(\rho)=\frac{\gamma}{\alpha \beta} \exp \left(\alpha \rho^{\frac{\beta}{2}}\right) .
$$

Theorem 1.4 is satisfied for $\alpha>0$ and $\beta \geq 2$ and $R_{*}$ the unique solution to

$$
R^{\beta+s} \exp \left(\alpha R^{\beta}\right)=\frac{c_{s, d}}{2 \gamma}
$$

- Power law with logarithm: $V(x)=\gamma\|x\|^{\alpha} \log \left(\|x\|^{2}\right)$, where $\gamma>0$, so that

$$
v(\rho)=\gamma \rho^{\frac{\alpha}{2}} \log (\rho)
$$

Theorem 1.4 is satisfied for $\alpha \geq 2$ and $R_{*}$ the unique solution to

$$
R^{\alpha+s}(1+\alpha \log (R))=\frac{c_{s, d}}{4 \gamma}
$$

- Power-law with a sink: $V(x)=\frac{\gamma}{\alpha}\left|\|x\|^{2}-R_{0}^{2}\right|^{\alpha / 2}$, where $\gamma, R_{0}>0$, so that

$$
v(\rho)=\frac{\gamma}{\alpha}\left|\rho-R_{0}^{2}\right|^{\alpha / 2} .
$$

Theorem 1.4 is satisfied for $\alpha \geq 2$ and $R_{*}>R_{0}$ the unique solution to

$$
R^{s+2}\left(R^{2}-R_{0}^{2}\right)^{\frac{\alpha}{2}-1}=\frac{c_{s, d}}{2 \gamma} .
$$

In contrast to these examples, Lemmas 2.6, 2.7 and Appendix D give some sufficient conditions that can be used when $v$ is not convex. Appendix A discusses an example of a Lennard-Jones type external field with $0>\alpha>\beta$, for which $v(\rho)$ has a finite limit as $\rho \rightarrow \infty$ and hence $v$ is not convex, but $\sigma_{R_{*}}$ is still the equilibrium measure.
1.3. Connection to other works. One goal of this article is to provide more insight into the following question introduced in [12, 11]: when does dimension reduction of the support of the equilibrium measure occur for Riesz energies with external fields? For $V(x)=c\|x\|^{\alpha}$, Theorem 1.2 provides a characterization of when the support of the equilibrium measure is a sphere, which leaves open the question of whether dimension reduction occurs for other combinations of $-2<s<d$ and $\alpha>\max \{0,-s\}$. Thus far, the answer appears to be negative, with [12, Theorem 1.2] showing that for $s=d-4$ and $\alpha<\alpha_{d-4, d}^{*}=2, \mu_{\mathrm{eq}}$ has a full dimensional component, and with [6, Theorem II.5], [11, Theorem 1.4], [27, Theorem 17], [31, Proposition 2.13], [33, Theorem 3.2, Example 3.2], and [2] (which studies the onedimensional setting) finding that for certain values of $\alpha$ and $s \geq d-3$, the support of the equilibrium measure is $d$-dimensional and connected, providing more information than our general Theorem 1.1 in these specific cases. While this work focuses on when the uniform measure on a sphere minimizes the energy, if the support instead has interior points, the distribution of the equilibrium measure on the interior is closely related to the fractional Laplacian of the external field $V$, see e.g. [17, 29].
The equilibrium measures for $I_{s, V}$ for power-law external fields, $V(x)=\gamma \frac{\|x\|^{\alpha}}{\alpha}$, appear in other contexts. For instance, for $s=0, d \geq 1$, and $\alpha=2$, the equilibrium measure is the asymptotic spectral distribution of the vectors of eigenvalues of $d$-tuples of commuting Hermitian $n \times n$ random matrices, as shown in [32]. Thus, the equilibrium measures obtained by Chafaï, Saff, and Womersley in [11, Theorem 1.4] and [12, Theorem $1.2(\mathrm{i})$-(b)] extend to higher dimensions the Wigner semicircular law ( $d=1$, Gaussian Unitary Ensemble) and the circular law ( $d=2$, for the planar Coulomb gas of the Complex Ginibre Ensemble).

Wasserstein gradient flows and steepest descent flows for nonsingular Riesz energies with external fields arising from the maximum mean discrepancy functional have been studied in $[3,25,26,27,28]$, in relation to the halftoning problem in image processing. In this setting, the external field is the negative of the potential of some probability measure, i.e. $V(x)=-U_{s}^{\nu}(x)$, acting as an attractive sink for mass. The Wasserstein steepest descent flows of the Riesz energy functional $I_{s}$ for $-2<s<0$ and an initial measure $\mu_{0}=\delta_{0}$ was
also considered in [27], where it was emphasized that determining the direction of steepest descent requires one to solve the following constrained optimization problem:

$$
\begin{equation*}
\underset{\nu \in \mathcal{P}\left(\mathbb{R}^{d}\right)}{\operatorname{minimize}} I_{s}(\nu) \quad \text { such that } \quad \int_{\mathbb{R}^{d}}\|x\|^{\alpha} \mathrm{d} \nu(x)=1 \tag{1.10}
\end{equation*}
$$

It was further shown, in the case that $\alpha=2$, that this is equivalent to determining the initial step of a minimizing moment scheme (Jordan-Kinderlehrer-Otto scheme):

$$
\begin{equation*}
\underset{\mu \in \mathcal{P}\left(\mathbb{R}^{d}\right)}{\operatorname{minimize}} I_{s}(\mu)+\frac{\gamma}{\alpha} W_{\alpha}^{\alpha}\left(\delta_{0}, \mu\right) \tag{1.11}
\end{equation*}
$$

where $W_{\alpha}$ is the Wasserstein or Kantorovich distance of order $\alpha$, defined in (1.12).
For $\alpha>0$, let $\mathcal{P}_{\alpha}\left(\mathbb{R}^{d}\right)$ be the set of probability measures on $\mathbb{R}^{d}$ with finite $\alpha$-moment, i.e. $\int\|x\|^{\alpha} \mathrm{d} \mu(x)<\infty$, and for $c>0$, let $\mathcal{P}_{\alpha, c}\left(\mathbb{R}^{d}\right)$ be the set of probability measures with $\int\|x\|^{\alpha} \mathrm{d} \mu(x)=c^{\alpha}$. For $\alpha \geq 1$, the Wasserstein distance of order $\alpha$, denoted $W_{\alpha}: \mathcal{P}_{\alpha}\left(\mathbb{R}^{d}\right) \times$ $\mathcal{P}_{\alpha}\left(\mathbb{R}^{d}\right) \rightarrow[0, \infty)$, is defined by

$$
\begin{equation*}
W_{\alpha}(\mu, \nu):=\left(\min _{\omega \in \Pi(\mu, \nu)} \int_{\mathbb{R}^{d} \times \mathbb{R}^{d}}\|x-y\|^{\alpha} \mathrm{d} \omega(x, y)\right)^{1 / \alpha}, \quad \mu, \nu \in \mathcal{P}_{\alpha}\left(\mathbb{R}^{d}\right) \tag{1.12}
\end{equation*}
$$

where $\Pi(\mu, \nu)$ is the set of probability measures on the product space $\mathbb{R}^{d} \times \mathbb{R}^{d}$ with marginal distributions $\mu$ and $\nu$. This set is convex and nonempty, since it contains the tensor product $\mu \otimes \nu$. The definition of $W_{\alpha}$ can be extended to $0<\alpha<1$.

The constrained optimization problem (1.11) can be interpreted as an energy minimization problem for a Riesz kernel with a power-law external field, since

$$
W_{\alpha}^{\alpha}\left(\delta_{0}, \mu\right)=\int_{\mathbb{R}^{d}}\|x\|^{\alpha} \mathrm{d} \mu(x)
$$

In the following proposition, we generalize [27, Proposition 16], showing the equivalence of (1.10) and (1.11) for a wider range of $s$ and $\alpha$. Its proof can be found in Section 2.6.

Proposition 1.5. Consider the external field $V(x)=\frac{\gamma}{\alpha}\|x\|^{\alpha}$ with $\gamma>0$ and $\alpha>\max \{-s, 0\}$. Suppose that $s \in(-\infty, d), s \neq 0$, and denote by $f_{\#} \mu$ the pushforward of $\mu$ by a map $f$. Then

- The energy $I_{s}$ is minimized over $\mathcal{P}_{\alpha, 1}\left(\mathbb{R}^{d}\right)$ by $\nu$ if and only if $\nu=(c \mathrm{Id})_{\#} \mu$ minimizes $I_{s, V}$ over $\mathcal{P}\left(\mathbb{R}^{d}\right)$, with $c=\left(\frac{s I_{s}(\nu)}{2 \gamma}\right)^{\frac{1}{s+\alpha}}$.
- The energy $I_{s, V}$ is minimized over $\mathcal{P}\left(\mathbb{R}^{d}\right)$ by $\mu$
if and only if $\nu=\left(c^{-1} \mathrm{Id}\right)_{\#} \mu$ minimizes $I_{s}$ over $\mathcal{P}_{\alpha, 1}\left(\mathbb{R}^{d}\right)$, with $c=\left(\int_{\mathbb{R}^{d}}\|x\|^{\alpha}\right)^{\frac{1}{\alpha}}$.
Suppose instead that $s=0$ and $\alpha>0$. Then
- The energy $I_{0}$ is minimized over $\mathcal{P}_{\alpha, 1}\left(\mathbb{R}^{d}\right)$ by $\nu$ if and only if $\mu=(c \mathrm{Id})_{\# \nu}$ minimizes $I_{0, V}$ over $\mathcal{P}\left(\mathbb{R}^{d}\right)$, with $c=\left(\frac{1}{\gamma}\right)^{\frac{1}{\alpha}}$.
- The energy $I_{0, V}$ is minimized over $\mathcal{P}\left(\mathbb{R}^{d}\right)$ by $\mu$
if and only if $\nu=\left(c^{-1} \mathrm{Id}\right)_{\#} \mu$ minimizes $I_{0}$ over $\mathcal{P}_{\alpha, 1}\left(\mathbb{R}^{d}\right)$, with $c=\left(\int_{\mathbb{R}^{d}}\|x\|^{\alpha}\right)^{\frac{1}{\alpha}}$.
The problem of finding the equilibrium of $I_{s, V}$ for power-law external fields, such as considered in Theorem 1.2, has a natural analogue, when replacing the Riesz kernel by

$$
L_{\alpha, \beta}(x-y)=\frac{\|x-y\|^{\alpha}}{\alpha}-\frac{\|x-y\|^{\beta}}{\beta}, \quad \alpha>\beta>-d
$$

without the presence of an external field. This Lennard - Jones type kernel forces particles to repel each other at short range and attract each other when far apart. The minimization of
these power-law energies has been the subject of much study, see e.g. [5, 8, $9,21,13,23,24$, $14,10,22]$. In particular, the results of [5, Theorem 1, Remarks 1 and 2] and [8, Theorems 3.4 and 3.10] imply that for $-d<\beta \leq 3-d$ the support of any equilibrium measure cannot have finite $(d-1)$-dimensional Hausdorff measure, much like our Theorem 1.1. Furthermore, [22, Theorem 1] shows that if the repulsion is sufficiently weak and the attraction sufficiently strong, $\sigma_{R}$ is the equilibrium measure of the power-law energy, much like Theorem 1.2.

## 2. Proofs

In what follows, we will always assume that the external field $V$ is a radial function, i.e. there is some function $v$ as in (1.1). Since $K_{s}$ and $V$ are both rotationally invariant, and since $\mu_{\text {eq }}$ is unique if it exists, the equilibrium measure must be radial, and there is some $\nu \in \mathcal{P}([0, \infty))$ such that, writing $x=r \theta$ with $r \in[0, \infty)$ and $\theta \in \mathbb{S}^{d-1}$,

$$
\mathrm{d} \mu_{\mathrm{eq}}(x)=\mathrm{d} \sigma(\theta) \mathrm{d} \nu(r)
$$

From Proposition C. 1 of the Appendix, the modified potential then becomes

$$
U_{s}^{\mu_{\mathrm{eq}}}(x)+V(x)=\int_{0}^{\infty} r^{-s} h_{s, d}\left(\frac{\|x\|^{2}}{r^{2}}\right) \mathrm{d} \nu(r)+v\left(\|x\|^{2}\right)
$$

where

$$
h_{s, d}(\lambda):= \begin{cases}\frac{1}{s}(1+\sqrt{\lambda})^{-s}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}, \frac{d-1}{2} ; d-1 ; \frac{4 \sqrt{\lambda}}{(1+\sqrt{\lambda})^{2}}\right) & s \neq 0  \tag{2.1}\\ -\log (1+\sqrt{\lambda})+\frac{\sqrt{\lambda}}{(1+\sqrt{\lambda})^{2}}{ }_{3} \mathrm{~F}_{2}\left(1,1, \frac{d+1}{2} ; 2, d ; \frac{4 \sqrt{\lambda}}{(1+\sqrt{\lambda})^{2}}\right) & s=0\end{cases}
$$

To prove that $\mu_{\mathrm{eq}}$ is indeed the equilibrium measure for the energy, we need only show that the Frostman conditions (1.3) are satisfied. The radial symmetry of both $\mu_{\mathrm{eq}}$ and the modified potential then turn this into a one-dimensional problem. In particular, if we want to show that $\sigma_{R}$ is the equilibrium measure, we need only show that $R^{-s} h_{s, d}\left(\frac{\|x\|^{2}}{R^{2}}\right)+v\left(\|x\|^{2}\right)$ achieves its global minimum at $\|x\|=R$. For ease of computation and notation, we set

$$
\lambda:=\frac{\|x\|^{2}}{R^{2}}
$$

and check that $R^{-s} h_{s, d}(\lambda)+v\left(\lambda R^{2}\right)$ has a global minimum at $\lambda=1$.
In Section 2.1, we introduce the notation that will facilitate the computation of derivatives that are needed in our proofs. In Section 2.2 we prove Theorem 1.3 giving necessary conditions for the uniform measure on a sphere to minimize the energy when $-2<s<d-3$ and $v$ is sufficiently smooth. We prove Theorem 1.1 in Section 2.3 , showing that we should not expect the uniform measure on a sphere to minimize the energy when $s \geq d-3$. Section 2.4 then provides a variety of sufficient conditions so that the equilibrium measure has spherical support. Each lemma in that section considers the behavior of the modified potential $U_{s}^{\sigma_{R}}(x)+V(x)$ for specific values of $s$, either outside or inside the sphere $\mathbb{S}_{R}^{d-1}$, and provides conditions on $V$ that guarantee the modified potential achieves its global minimum on the sphere. In Sections 2.4 and 2.5 we use some of these sufficiency results to prove Theorems 1.4 and 1.2 , respectively. Throughout the paper, increasing means nondecreasing and decreasing means nonincreasing.
2.1. Notation for proofs. For $-2<s<d-1$ and $R>0$, we use Proposition C. 1 to rewrite the modified potential of $\sigma_{R}$ as

$$
U_{s}^{\sigma_{R}}(x)+V(x)= \begin{cases}R^{-s} h_{s, d}\left(\frac{\|x\|^{2}}{R^{2}}\right)+v\left(\frac{\|x\|^{2}}{R^{2}} \cdot R^{2}\right) & s \neq 0 \\ -\log (R)+h_{0, d}\left(\frac{\|x\|^{2}}{R^{2}}\right)+v\left(\frac{\|x\|^{2}}{R^{2}} \cdot R^{2}\right) & s=0\end{cases}
$$

where $h_{s, d}$ is as in (2.1). This characterization is useful due to the structure of the derivative of $h_{s, d}$ :

$$
h_{s, d}^{\prime}(\lambda)=\left\{\begin{array}{ll}
-\frac{d-s-2}{2 d}{ }_{2} \mathrm{~F}_{1}\left(1+\frac{s}{2}, \frac{4+s-d}{2} ; \frac{d+2}{2} ; \lambda\right) & \lambda \leq 1  \tag{2.2}\\
-\frac{1}{2} \lambda^{-\frac{s}{2}-1}{ }_{2} \mathrm{~F}_{1}\left(1+\frac{s}{2}, \frac{2+s-d}{2} ; \frac{d}{2} ; \lambda^{-1}\right) & \lambda \geq 1
\end{array},\right.
$$

which holds for $-2<s<d-2$. See Appendix C. 3 for more details.
The modified potential for $\sigma_{R}$ is, as a function of $\lambda \geq 0$,

$$
f(\lambda)=f_{s, d, R, v}(\lambda):=\left\{\begin{array}{ll}
R^{-s} h_{s, d}(\lambda)+v\left(R^{2} \lambda\right) & s \neq 0  \tag{2.3}\\
-\log (R)+h_{0, d}(\lambda)+v\left(R^{2} \lambda\right) & s=0
\end{array},\right.
$$

so that

$$
f\left(\frac{\|x\|^{2}}{R^{2}}\right)=U_{s}^{\sigma_{R}}(x)+V(x)
$$

Using (2.2) and (B.11) from Appendix B, we know that for $\ell \geq 1$ and $\lambda \in[0,1)$

$$
\begin{align*}
f^{(\ell)}(\lambda)= & R^{-s} h_{s, d}^{(\ell)}(\lambda)+R^{2 \ell} v^{(\ell)}\left(R^{2} \lambda\right) \\
= & R^{-s} \frac{2+s-d}{2^{\ell} d}\left(\prod_{j=1}^{\ell-1} \frac{(2+2 j+s-d)(s+2 j)}{d+2 j}\right){ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+\ell, \frac{2+s-d}{2}+\ell ; \frac{d}{2}+\ell ; \lambda\right) \\
& \quad+R^{2 \ell} v^{(\ell)}\left(R^{2} \lambda\right) . \tag{2.4}
\end{align*}
$$

Similarly, using (2.2) and (B.12), we know that for $\ell \geq 1$ and $\lambda \in(1, \infty)$

$$
\begin{align*}
f^{(\ell)}(\lambda) & =R^{-s} h_{s, d}^{(\ell)}(\lambda)+R^{2 \ell} v^{(\ell)}\left(R^{2} \lambda\right) \\
& =R^{-s} \frac{(-1)^{\ell}}{2^{\ell}} \lambda^{-\frac{s}{2}-\ell}\left(\prod_{j=1}^{\ell-1}(s+2 j)\right){ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+\ell, \frac{2+s-d}{2} ; \frac{d}{2} ; \lambda^{-1}\right)+R^{2 \ell} v^{(\ell)}\left(R^{2} \lambda\right) . \tag{2.5}
\end{align*}
$$

When $\ell<d-s-1$ the limit exists and we can set

$$
f^{(\ell)}(1):=\lim _{\lambda \rightarrow 1} f^{(\ell)}(\lambda)
$$

In many of our lemmas below, we will be restricting to $[0,1]$ or $[1, \infty)$. In those cases, we will replace the limit with the appropriate one-sided limit.

For $\kappa \in[0,1)$, let

$$
q(\kappa):= \begin{cases}2 R^{s+2} \kappa^{-\frac{s}{2}-1} v^{\prime}\left(R^{2} \kappa^{-1}\right) & \kappa>0  \tag{2.6}\\ \lim _{t \rightarrow 0^{+}} 2 R^{s+2} t^{-\frac{s}{2}-1} v^{\prime}\left(R^{2} t^{-1}\right) & \kappa=0 .\end{cases}
$$

and

$$
\begin{equation*}
y_{s, d}(\kappa):=-{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+1, \frac{2+s-d}{2} ; \frac{d}{2} ; \kappa\right) \tag{2.7}
\end{equation*}
$$

and let

$$
\begin{equation*}
g(\kappa)=g_{s, d, q}(\kappa):=y_{s, d}(\kappa)+q(\kappa), \tag{2.8}
\end{equation*}
$$

so

$$
\begin{equation*}
g(\kappa)=2 R^{s} \kappa^{-\frac{s}{2}-1} f^{\prime}\left(\kappa^{-1}\right) \tag{2.9}
\end{equation*}
$$

For all $\ell \geq 0$ and $\kappa \in[0,1$ ), (B.11) yields

$$
\begin{align*}
g^{(\ell)}(\kappa)= & y_{s, d}^{(\ell)}(\kappa)+q^{(\ell)}(\kappa) \\
= & -\frac{1}{2^{\ell}}\left(\prod_{j=0}^{\ell-1} \frac{(2+2 j+s-d)(s+2 j+2)}{d+2 j}\right) \times \\
& { }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+\ell+1, \frac{2+s-d}{2}+\ell ; \frac{d}{2}+\ell ; \kappa\right)+q^{(\ell)}(\kappa), \tag{2.10}
\end{align*}
$$

and when $\ell<d-s-2$ the limit exists and we set

$$
g^{(\ell)}(1):=\lim _{\kappa \rightarrow 1^{-}} g^{(\ell)}(\kappa) .
$$

2.2. Proof of Theorem 1.3 (necessary conditions). When $\sigma_{R}$ is the equilibrium measure, the Frostman conditions (1.3) imply that $\lambda=1$ must be a global minimizer of $f$ on $[0, \infty)$, see the discussion at the start of Section 2. For $s<d-3, f$ is twice continuously differentiable, so the following four conditions must be satisfied: $f^{\prime}(1)=0, f^{\prime \prime}(1) \geq 0$, $f(0) \geq f(1)$, and $\lim _{\lambda \rightarrow \infty} f(\lambda) \geq f(1)$.

From (2.2), (B.11), (B.3), (B.6), and (1.4), we have

$$
\begin{aligned}
h_{s, d}^{\prime}(1) & =\frac{s-d+2}{2 d}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+1, \frac{s-d+2}{2}+1 ; \frac{d}{2}+1 ; 1\right) \\
& =-\frac{1}{4}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}, \frac{s-d+2}{2} ; \frac{d}{2} ; 1\right)=-\frac{c_{s, d}}{4},
\end{aligned}
$$

and

$$
\begin{aligned}
h_{s, d}^{\prime \prime}(1) & =\frac{(s-d+2)(s+2)(s-d+4)}{4 d(d+2)}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+2, \frac{s-d+2}{2}+2 ; \frac{d}{2}+2 ; 1\right) \\
& =\frac{(s+2)(d-s-4)}{16(d-s-3)}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}, \frac{s-d+2}{2} ; \frac{d}{2} ; 1\right)=\frac{(s+2)(d-s-4) c_{s, d}}{16(d-s-3)} .
\end{aligned}
$$

For the first condition $f^{\prime}(1)=0$, we have

$$
0=R^{-s} h_{s, d}^{\prime}(1)+R^{2} v^{\prime}\left(R^{2}\right)=-\frac{c_{s, d}}{4} R^{-s}+R^{2} v^{\prime}\left(R^{2}\right)
$$

which implies (i).
For the second condition $f^{\prime \prime}(1) \geq 0$, we have

$$
0 \leq R^{-s} h_{s, d}^{\prime \prime}(1)+R^{4} v^{\prime \prime}\left(R^{2}\right)=\frac{(s+2)(d-s-4)}{4(d-s-3)} \frac{c_{s, d}}{4} R^{-s}+R^{4} v^{\prime \prime}\left(R^{2}\right)
$$

Then for $R>0, v^{\prime}\left(R^{2}\right)=\frac{c_{s, d}}{4} R^{-s-2}>0$ gives (ii).
If $s \neq 0, f(0) \geq f(1)$ is equivalent to

$$
R^{-s} h_{s, d}(0)+v(0) \geq R^{-s} h_{s, d}(1)+v\left(R^{2}\right) .
$$

Note that by (1.4), (C.5), and (2.1), $h_{s, d}(0)=1 / s$ and $h_{s, d}(1)=c_{s, d} / s$. On the other hand, if $s=0, f(0) \geq f(1)$ is equivalent to

$$
h_{0, d}(0)+v(0) \geq h_{0, d}(1)+v\left(R^{2}\right) .
$$

Again, by (1.5) and (2.1), $h_{0, d}(0)=0$ and $h_{0, d}(1)=b_{d}$. We thus obtain (iii).

If $s \neq 0$, then (1.4), (2.1), and (B.1) show us that our last condition on $f$ is equivalent to

$$
\begin{aligned}
\frac{R^{-s}}{s} c_{s, d}+v\left(R^{2}\right) & \leq \lim _{\lambda \rightarrow \infty}\left(v\left(R^{2} \lambda\right)+\frac{R^{-s}}{s}(1+\sqrt{\lambda})^{-s}\left(1+\frac{s \sqrt{\lambda}}{(1+\sqrt{\lambda})^{2}}+\mathcal{O}\left(\frac{1}{\lambda}\right)\right)\right) \\
& =\lim _{r \rightarrow \infty}\left(v\left(r^{2}\right)+\frac{(R+r)^{-s}}{s}\right) .
\end{aligned}
$$

If $s=0$, due to (1.5), (2.1), and (B.2), our last condition is instead equivalent to

$$
\begin{aligned}
-\log (R)+b_{d}+v\left(R^{2}\right) & \leq \lim _{\lambda \rightarrow \infty}\left(-\log (R)-\log (1+\sqrt{\lambda})+\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)+v\left(R^{2} \lambda\right)\right) \\
& =\lim _{r \rightarrow \infty}\left(v\left(r^{2}\right)-\log (R+r)\right)
\end{aligned}
$$

We now obtain (iv).
Remark 2.1 (Alternative viewpoint). Assuming that $\sigma_{R}$ is the equilibrium measure, the energy (1.2) of $\sigma_{R}$, using Proposition C.2, is

$$
I_{s, V}\left(\sigma_{R}\right)=2 v\left(R^{2}\right)+\left\{\begin{array}{ll}
\frac{R^{-s}}{s} c_{s, d}, & s \neq 0 \\
-\log (R)+b_{d}, & s=0
\end{array} .\right.
$$

Treating this as a function of $R$, the derivative (remembering that $c_{0, d}=1$ )

$$
\frac{\partial}{\partial R} I_{s, V}\left(\sigma_{R}\right)=4 R v^{\prime}\left(R^{2}\right)-c_{s, d} R^{-s-1}
$$

shows that the necessary condition (i) must be satisfied at a stationary point. Moreover,

$$
\frac{\partial^{2}}{\partial R^{2}} I_{s, V}\left(\sigma_{R}\right)=4 v^{\prime}\left(R^{2}\right)+8 R^{2} v^{\prime \prime}\left(R^{2}\right)+(s+1) c_{s, d} R^{-s-2} \geq 0
$$

at a minimum, gives $\frac{R^{2} v^{\prime \prime}\left(R^{2}\right)}{v^{\prime}\left(R^{2}\right)} \geq-\frac{s+2}{2}$ at a stationary point, which is weaker than (ii).
2.3. Proof of Theorem 1.1. In order to prove Theorem 1.1, we must consider three cases separately. For the first, we will need the following well-known result (see, e.g., [19, Theorem 4.3.1]).

Theorem 2.2. Let $A \subset \mathbb{R}^{d}$ be compact and $s>0$. If the $s$-dimensional Hausdorff measure of $A, \mathcal{H}_{s}(A)$, is finite then for every probability measure $\mu \in \mathcal{P}(A)$,

$$
\int_{A} \int_{A} K_{s}(x, y) \mathrm{d} \mu(x) \mathrm{d} \mu(y)=\infty .
$$

In other word, if there exists some measure $\mu \in \mathcal{P}(A)$ such that

$$
\int_{A} \int_{A} K_{s}(x, y) \mathrm{d} \mu(x) \mathrm{d} \mu(y)<\infty
$$

then either $\operatorname{dim}_{H}(A)>s$, or $\operatorname{dim}_{H}(A)=s$ and $\mathcal{H}_{s}(A)=\infty$, where $\operatorname{dim}_{H}(A)$ is the Hausdorff dimension of $A$.

Lemma 2.3. Suppose $I_{s, V}$ has a unique compactly supported equilibrium measure $\mu_{\mathrm{eq}}$ and one of the following is true:
(a) $d-1 \leq s<d$ and $v$ is lower semi-continuous and bounded from below,
(b) $d-2 \leq s<d-1$ and $v$ is bounded from below and $\mathcal{C}^{1}$ in the extended sense,
(c) $d \geq 3, d-3 \leq s<d-2$, and $v$ is bounded from below and $\mathcal{C}^{2}$ in the extended sense such that $v^{\prime \prime}$ is finite on $(0, \infty)$.
(d) $d=2,-1 \leq s<0$, and $v$ is bounded from below, $\mathcal{C}^{2}$ in the extended sense, and such that $\lim _{\rho \rightarrow 0^{+}} \rho^{\frac{s}{2}+1} v^{\prime}(\rho)=0$ and $v^{\prime \prime}$ is finite on $(0, \infty)$.
With $x=r \theta, r \in[0, \infty)$, and $\theta \in \mathbb{S}^{d-1}$, let $\nu \in \mathcal{P}([0, \infty))$ such that

$$
\mathrm{d} \mu_{\mathrm{eq}}(x)=\mathrm{d} \sigma(\theta) \mathrm{d} \nu(r) .
$$

Then $\operatorname{supp}(\nu)$ must be a perfect set (i.e. have no isolated points).
Proof of Theorem 1.1. Our claim follows immediately from Lemma 2.3.
Overview of the proof of Lemma 2.3. We need to rule out that there is any isolated point in the support of $\nu$. We first address the possibility that there is an isolated point mass at 0 , using that fact that $K_{s}$ is singular at 0 in the first three cases, (a)-(c), and the fact that the modified potential is strictly decreasing near 0 in the last case (d). We then consider the possibility of an isolated sphere $\mathbb{S}_{R}^{d-1}$ in the support of $\mu_{\mathrm{eq}}$, and find contradictions in each setting.

In case (a), Theorem 2.2 immediately tells us that this contributes an infinite amount of energy to our total energy, contradicting that we have finite energy. In the remaining three cases, we consider the potential $U_{s}^{\mu_{\mathrm{eq}}}$, which may have contributions from the support from the interior of the sphere, the support from the exterior of the sphere, and the support on the sphere itself.

To handle case (b), we write the modified potential as a one-dimensional function and take a derivative. We then find that as $\|x\|$ approaches $R$, the contributions to the derivative from any support inside or outside the sphere are finite, but the contribution from the sphere itself goes to $\infty$ as $\|x\|$ approaches $R$ from below, and $-\infty$ as $\|x\|$ approaches $R$ from above. Since, by the Frostman conditions (1.3), the modified potential must achieve its minimum at $\|x\|=R$, this means that $v^{\prime}\left(\|x\|^{2}\right)$ must approach $-\infty$ and $\infty$ as $\|x\|$ approaches $R$ from below and above, respectively. This breaks the continuous differentiability of $v$.

For cases (c) and (d), we also interpret the modified potential as a one-dimensional function and utilize the second derivative. We then see that as $\|x\|$ approaches $R$, the contributions from any support inside or outside the sphere are finite, but the contribution from the sphere itself goes to $-\infty$ as $\|x\|$ approaches $R$ from below. Since, by the Frostman conditions (1.3), the modified potential must achieve its minimum at $\|x\|=R$, this means that $v^{\prime \prime}\left(\|x\|^{2}\right)$ must approach $\infty$ as $\|x\|$ approaches $R$ from below. This breaks the finiteness and continuity of the second derivative of $v$.

Proof of Lemma 2.3. Let $m_{0}:=\nu(\{0\})$. Suppose, for the sake of contradiction, that $s \geq 0$ and $m_{0}>0$. Since $v$ is bounded from below by some constant $C$

$$
I_{s, V}\left(\mu_{\mathrm{eq}}\right) \geq 2 C+m_{0}^{2} I_{s}\left(\delta_{0}\right)=\infty
$$

which contradicts $I_{s, V}\left(\mu_{\mathrm{eq}}\right)$ being finite, and so $m_{0}=0$. This means that 0 cannot be an isolated point in the support of $\nu$.

In case (d), $K_{s}$ is not singular, so we may have $m_{0}>0$. Suppose, for the sake of contradiction, that 0 is an isolated point in the support of $v$. Then our modified potential is

$$
m_{0} \frac{\|x\|^{-s}}{s}+v\left(\|x\|^{2}\right)+\int_{r_{2}}^{\infty} r^{-s} h_{s, d}\left(\frac{\|x\|^{2}}{r^{2}}\right) \mathrm{d} \nu(r)
$$

for some radius $r_{2}>0$. We can rewrite this as

$$
q(\rho)=m_{0} \frac{\rho^{-\frac{s}{2}}}{s}+v(\rho)+\int_{r_{2}}^{\infty} r^{-s} h_{s, d}\left(\frac{\rho}{r^{2}}\right) \mathrm{d} \nu(r)
$$

so for $\rho \in\left(0, r_{2}^{2}\right)$,

$$
q^{\prime}(\rho)=-\frac{m_{0}}{2} \rho^{-\frac{s}{2}-1}+v^{\prime}(\rho)+\int_{r_{2}}^{\infty} r^{-s-2} h_{s, d}^{\prime}\left(\frac{\rho}{r^{2}}\right) \mathrm{d} \nu(r) .
$$

From (B.9) and (B.11), we have that ${ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+1, \frac{2+s-d}{2}+1 ; \frac{d}{2}+1 ; t\right)$ is positive and increasing on $[0,1)$, so for $0 \leq \rho<r_{2}^{2}$, we have that

$$
\begin{aligned}
0 & \leq \int_{r_{2}}^{\infty} r^{-s-2}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+1, \frac{2+s-d}{2}+1 ; \frac{d}{2}+1 ; \frac{\rho}{r^{2}}\right) \mathrm{d} \nu(r) \\
& \leq \nu\left(\left[r_{2}, \infty\right)\right) r_{2}^{-s-2}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+1, \frac{2+s-d}{2}+1 ; \frac{d}{2}+1 ; \frac{\rho}{r_{2}^{2}}\right) \\
& \leq r_{2}^{-s-2}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+1, \frac{2+s-d}{2}+1 ; \frac{d}{2}+1 ; 1\right)<\infty .
\end{aligned}
$$

Since $\lim _{\rho \rightarrow 0^{+}}-\frac{m_{0}}{2} \rho^{-\frac{s}{2}-1}+v^{\prime}(\rho)=-\infty$, we see that $q$ is strictly decreasing near 0 , and so cannot achieve its minimum at 0 , contradicting the Frostman conditions. Thus 0 is not an isolated point of $\operatorname{supp} \nu$.

Now suppose, for the sake of contradiction, that $R>0$ is an isolated point in the support of $\nu$, so $m_{R}:=\nu(\{R\})>0$. Let $0 \leq r_{1}<R<r_{2}$ be such that $\left(r_{1}, r_{2}\right) \cap \operatorname{supp}(\nu)=\{R\}$ and let $\tilde{\nu}=\nu-m_{0} \delta_{0}$. Then

$$
\begin{aligned}
U_{s}^{\mu_{\mathrm{eq}}}(x)= & m_{0} \frac{\|x\|^{-s}}{s}+\int_{0}^{r_{1}} r^{-s} h_{s, d}\left(\frac{\|x\|^{2}}{r^{2}}\right) \mathrm{d} \tilde{\nu}(r) \\
& +m_{R} R^{-s} h_{s, d}\left(\frac{\|x\|^{2}}{R^{2}}\right)+\int_{r_{2}}^{\infty} r^{-s} h_{s, d}\left(\frac{\|x\|^{2}}{r^{2}}\right) \mathrm{d} \tilde{\nu}(r) .
\end{aligned}
$$

Analogous to (2.3), the modified potential, as a function of $\lambda=\frac{\|x\|^{2}}{R^{2}}$, is then

$$
\begin{align*}
p(\lambda):= & m_{0} \frac{R^{-s} \lambda^{-\frac{s}{2}}}{s}+\int_{0}^{r_{1}} r^{-s} h_{s, d}\left(\frac{R^{2}}{r^{2}} \lambda\right) \mathrm{d} \tilde{\nu}(r) \\
& +m_{R} R^{-s} h_{s, d}(\lambda)+\int_{r_{2}}^{\infty} r^{-s} h_{s, d}\left(\frac{R^{2}}{r^{2}} \lambda\right) \mathrm{d} \tilde{\nu}(r)+v\left(R^{2} \lambda\right) . \tag{2.11}
\end{align*}
$$

Recall that in cases (a), (b), and (c), $m_{0}=0$ and $\tilde{\nu}=\nu$.
We now need to show a contradiction in all four cases, by showing that $R$ cannot be an isolated point, and proving our claim.

Case (a): Since adding a constant to the external field does not change the equilibrium measure, we may assume, without loss of generality, that there is a constant $C$ such that $V(x) \geq C$ for all $x \in \mathbb{R}^{d}$, and $K_{s}(x-y)+2 C \geq 0$ for $x, y \in \operatorname{supp}\left(\mu_{\text {eq }}\right)$. We then have that

$$
\infty>I_{s, V}\left(\mu_{\mathrm{eq}}\right) \geq m^{2}\left(I_{s}\left(\sigma_{R}\right)+2 C\right) .
$$

However, Theorem 2.2 tells us that $I_{s}\left(\sigma_{R}\right)=\infty$, which is a contradiction.
Case (b): With $p$ as in (2.11), and using (2.4) and (2.5), we see that for $\lambda \in\left(\frac{r_{1}^{2}}{R^{2}}, 1\right) \cup\left(1, \frac{r_{2}^{2}}{R^{2}}\right)$,

$$
p^{\prime}(\lambda)=\int_{0}^{r_{1}} R^{2} r^{-s-2} h_{s, d}^{\prime}\left(\frac{R^{2}}{r^{2}} \lambda\right) \mathrm{d} \nu(r)+m_{R} R^{-s} h_{s, d}^{\prime}(\lambda)
$$

$$
\begin{aligned}
& +\int_{r_{2}}^{\infty} R^{2} r^{-s-2} h_{s, d}^{\prime}\left(\lambda \frac{R^{2}}{r^{2}}\right) \mathrm{d} \nu(r)+R^{2} v^{\prime}\left(R^{2} \lambda\right) \\
= & -\frac{1}{2} \int_{0}^{r_{1}} R^{-s} \lambda^{-\frac{s}{2}-1}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+1, \frac{2+s-d}{2} ; \frac{d}{2} ; \frac{r^{2}}{R^{2}} \lambda^{-1}\right) \mathrm{d} \nu(r) \\
& +\frac{2+s-d}{2 d} \int_{r_{2}}^{\infty} R^{2} r^{-s-2}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+1, \frac{2+s-d}{2}+1 ; \frac{d}{2}+1 ; \frac{R^{2}}{r^{2}} \lambda\right) \mathrm{d} \nu(r) \\
& +m_{R} R^{-s} h_{s, d}^{\prime}(\lambda)+R^{2} v^{\prime}\left(R^{2} \lambda\right) .
\end{aligned}
$$

Suppose that $d-2<s<d-1$. For the first integral, (B.9) and (B.11) tell us that ${ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+1, \frac{2+s-d}{2} ; \frac{d}{2} ; t\right)$ is positive and increasing on $[0,1)$, and combining this with the facts that $\frac{r_{1}^{2}}{R^{2}}<\lambda$ and $r_{1}<R$, we have

$$
\begin{aligned}
0 & \leq \lim _{\lambda \rightarrow 1} \int_{0}^{r_{1}} \lambda^{-\frac{s}{2}-1}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+1, \frac{2+s-d}{2} ; \frac{d}{2} ; \frac{r^{2}}{R^{2}} \lambda^{-1}\right) \mathrm{d} \nu(r) \\
& \leq \lim _{\lambda \rightarrow 1} \nu\left(\left[0, r_{1}\right]\right) \lambda^{-\frac{s}{2}-1}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+1, \frac{2+s-d}{2} ; \frac{d}{2} ; \frac{r_{1}^{2}}{R^{2}} \lambda^{-1}\right) \\
& \leq{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+1, \frac{2+s-d}{2} ; \frac{d}{2} ; \frac{r_{1}^{2}}{R^{2}}\right)<\infty
\end{aligned}
$$

For the second integral, (B.9) and (B.11) tell us that ${ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+1, \frac{2+s-d}{2}+1 ; \frac{d}{2}+1 ; t\right)$ is positive and increasing on $[0,1)$, and since $\lambda<\frac{r_{2}^{2}}{R^{2}}$ and $r_{2}>R$, we have

$$
\begin{aligned}
0 & \leq \lim _{\lambda \rightarrow 1} \int_{r_{2}}^{\infty} r^{-s-2}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+1, \frac{2+s-d}{2}+1 ; \frac{d}{2}+1 ; \frac{R^{2}}{r^{2}} \lambda\right) \mathrm{d} \nu(r) \\
& \leq \lim _{\lambda \rightarrow 1} \nu\left(\left[r_{2}, \infty\right)\right) r_{2}^{-s-2}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+1, \frac{2+s-d}{2}+1 ; \frac{d}{2}+1 ; \frac{R^{2}}{r_{2}^{2}} \lambda\right) \\
& \leq r_{2}^{-s-2}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+1, \frac{2+s-d}{2}+1 ; \frac{d}{2}+1 ; \frac{R^{2}}{r_{2}^{2}}\right)<\infty .
\end{aligned}
$$

Thus, the limits of the two integrals, as $\lambda \rightarrow 1$, are finite.
From (2.4) and (B.5), we see that $\lim _{\lambda \rightarrow 1^{+}} h_{s, d}^{\prime}(\lambda)=-\infty$ and $\lim _{\lambda \rightarrow 1^{-}} h_{s, d}^{\prime}(\lambda)=\infty$. But we know that $p$ must achieve its minimum at 1 , so we must have that $\lim _{\lambda \rightarrow 1^{+}} p^{\prime}(\lambda) \geq 0$ and $\lim _{\lambda \rightarrow 1^{-}} p^{\prime}(\lambda) \leq 0$. This, in turn, then means that $\lim _{\lambda \rightarrow 1^{+}} v^{\prime}\left(R^{2} \lambda\right)=\infty$ and $\lim _{\lambda \rightarrow 1^{-}} v^{\prime}\left(R^{2} \lambda\right)=-\infty$, but then $v$ is not continuously differentiable at 1 , a contradiction.

In the case $s=d-2$, we see that

$$
\begin{aligned}
p^{\prime}(\lambda) & =-\frac{1}{2} R^{-s} \int_{0}^{r_{1}} \lambda^{-\frac{s}{2}-1} \mathrm{~d} \nu(r)+R^{-s} h_{s, d}^{\prime}(\lambda)+R^{2} v^{\prime}\left(R^{2} \lambda\right) \\
& =-\frac{1}{2} R^{-s} \nu\left(\left[0, r_{1}\right]\right) \lambda^{-\frac{s}{2}-1}+R^{-s} h_{s, d}^{\prime}(\lambda)+R^{2} v^{\prime}\left(R^{2} \lambda\right)
\end{aligned}
$$

Since $p$ must achieve its minimum at 1 , we must have $\lim _{\lambda \rightarrow 1^{+}} p^{\prime}(\lambda) \geq 0$ and $\lim _{\lambda \rightarrow 1^{-}} p^{\prime}(\lambda) \leq 0$. But then we must have (using (2.5))

$$
\lim _{\lambda \rightarrow 1^{+}} R^{2} v^{\prime}\left(R^{2} \lambda\right) \geq \frac{1}{2} R^{-s}\left(m+\nu\left(\left[0, r_{1}\right]\right)\right)
$$

and, due to (2.4),

$$
\lim _{\lambda \rightarrow 1^{-}} R^{2} v^{\prime}\left(R^{2} \lambda\right) \leq \frac{1}{2} R^{-s} \nu\left(\left[0, r_{1}\right]\right)
$$

But then $v$ is not continuously differentiable at 1 , giving us a contradiction.

Cases (c) and (d): Again with $p$ as in (2.11) and using (2.5) and (2.4), we find that for $\lambda \in\left(\frac{r_{1}^{2}}{R^{2}}, 1\right) \cup\left(1, \frac{r_{2}^{2}}{R^{2}}\right)$,

$$
\begin{aligned}
p^{\prime \prime}(\lambda)= & \frac{m_{0}}{4} R^{-s}(s+2) \lambda^{-\frac{s}{2}-2}+\int_{0}^{r_{1}} R^{4} r^{-s-4} h_{s, d}^{\prime \prime}\left(\frac{R^{2}}{r^{2}} \lambda\right) \mathrm{d} \tilde{\nu}(r) \\
& +m_{R} R^{-s} h_{s, d}^{\prime \prime}(\lambda)+\int_{r_{2}}^{\infty} R^{4} r^{-s-4} h_{s, d}^{\prime \prime}\left(\lambda \frac{R^{2}}{r^{2}}\right) \mathrm{d} \nu(r)+R^{4} v^{\prime \prime}\left(R^{2} \lambda\right) \\
= & \frac{m_{0}}{4} R^{-s}(s+2) \lambda^{-\frac{s}{2}-2}+\frac{s+2}{4} \int_{0}^{r_{1}} R^{-s} \lambda^{-\frac{s}{2}-2}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+2, \frac{2+s-d}{2} ; \frac{d}{2} ; \frac{r^{2}}{R^{2}} \lambda^{-1}\right) \mathrm{d} \tilde{\nu}(r) \\
& +\frac{(2+s-d)(4+s-d)(s+2)}{4 d(d+2)} \int_{r_{2}}^{\infty} R^{4} r^{-s-4}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+2, \frac{2+s-d}{2}+2 ; \frac{d}{2}+2 ; \frac{R^{2}}{r^{2}} \lambda\right) \mathrm{d} \nu(r) \\
& +m_{R} R^{-s} h_{s, d}^{\prime \prime}(\lambda)+R^{4} v^{\prime \prime}\left(R^{2} \lambda\right) .
\end{aligned}
$$

For the first integral, from 0 to $r_{1}$, by (B.9) and (B.11), ${ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+2, \frac{2+s-d}{2} ; \frac{d}{2} ; t\right)$ is decreasing on $[0,1)$, so it achieves its maximum at $t=0$, and since $\frac{r_{1}^{2}}{R^{2}}<\lambda$ and $r_{1}<R$, we have

$$
\begin{aligned}
& \tilde{\nu}\left(\left[0, r_{1}\right]\right){ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+2, \frac{2+s-d}{2} ; \frac{d}{2} ; \frac{r_{1}^{2}}{R^{2}}\right) \\
&=\lim _{\lambda \rightarrow 1} \tilde{\nu}\left(\left[0, r_{1}\right]\right) \lambda^{-\frac{s}{2}-2}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+2, \frac{2+s-d}{2} ; \frac{d}{2} ; \frac{r_{1}^{2}}{R^{2}} \lambda^{-1}\right) \\
& \leq \lim _{\lambda \rightarrow 1} \int_{0}^{r_{1}} \lambda^{-\frac{s}{2}-2}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+2, \frac{2+s-d}{2} ; \frac{d}{2} ; \frac{r^{2}}{R^{2}} \lambda^{-1}\right) \mathrm{d} \tilde{\nu}(r) \\
& \leq \lim _{\lambda \rightarrow 1} \tilde{\nu}\left(\left[0, r_{1}\right]\right) \lambda^{-\frac{s}{2}-2} \\
&=\tilde{\nu}\left(\left[0, r_{1}\right]\right) .
\end{aligned}
$$

For the second integral, from $r_{2}$ to $\infty$, (B.9) and (B.11) tell us that ${ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+2, \frac{2+s-d}{2}+2 ; \frac{d}{2}+\right.$ $2 ; t)$ is positive and increasing on $[0,1)$, and since $\lambda<\frac{r_{2}^{2}}{R^{2}}$ and $r_{2}>R$, we have

$$
\begin{aligned}
0 & \leq \lim _{\lambda \rightarrow 1} \int_{r_{2}}^{\infty}{ }_{r^{-s-4}}^{{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+2, \frac{2+s-d}{2}+2 ; \frac{d}{2}+2 ; \frac{R^{2}}{r^{2}} \lambda\right) \mathrm{d} \nu(r)} \\
& \leq \lim _{\lambda \rightarrow 1} \nu\left(\left[r_{2}, \infty\right)\right) r_{2}^{-s-4}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+2, \frac{2+s-d}{2}+2 ; \frac{d}{2}+2 ; \frac{R^{2}}{r_{2}^{2}} \lambda\right) \\
& \leq r_{2}^{-s-4}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+2, \frac{2+s-d}{2}+2 ; \frac{d}{2}+2 ; \frac{R^{2}}{r_{2}^{2}}\right)<\infty .
\end{aligned}
$$

Thus, the limits as $\lambda \rightarrow 1$ of $\frac{m_{0}}{4} R^{-s}(s+2) \lambda^{-\frac{s}{2}-2}$ and the two integrals, from 0 to $r_{1}$ and $r_{2}$ to $\infty$, are finite.

From (2.4) and (B.5) (or (B.4) in the case $s=d-3$ ), we see that $\lim _{\lambda \rightarrow 1^{-}} h_{s, d}^{\prime \prime}(\lambda)=-\infty$. But we know that $p$ must achieve its minimum at 1 , so we must have that $\lim _{\lambda \rightarrow 1^{-}} p^{\prime \prime}(\lambda) \geq 0$, so $\lim _{\lambda \rightarrow 1^{-}} v^{\prime \prime}\left(R^{2} \lambda\right)=\infty$. But this contradicts $v^{\prime \prime}$ being continuous and finite on $(0, \infty)$.

### 2.4. Proof of Theorem 1.4 and more about sufficient conditions.

Proof of Theorem 1.4. The result follows immediately from Corollaries 2.5 and 2.8 below, and the Frostman conditions and uniqueness of the equilibrium measure (see the discussion at the start of Section 2).

For the rest of this section, we provide a useful tool to verify that the function $f:=f_{s, d, R, v}$ from (2.3) indeed achieves its global minimum at 1, i.e. the uniform measure $\sigma_{R}$ on the sphere of radius $R$ is the equilibrium measure for $I_{s, V}$. Roughly speaking, when the necessary conditions of Theorem 1.3 are satisfied, in particular the endpoint behavior at 0 (see part (iii)), and $f$ is increasing then decreasing on $[0,1]$ we deduce that 1 is the minimizer on $[0,1]$. Similarly on the interval $[1, \infty)$, taking into account Theorem 1.3(iv), if $f$ is increasing then decreasing we deduce that 1 is the minimizer on $[1, \infty$ ). (This includes the straightforward case when $f$ is decreasing on $[0,1]$ and increasing on $[1, \infty)$ ).

On each of the intervals $[0,1]$ and $[1, \infty)$ such functions are unimodal, a term originating in probability and statistics (see, for example, [4, 34]). More precisely, for an interval $I \subset \mathbb{R}$, a function $\varphi: I \rightarrow \mathbb{R} \cup\{ \pm \infty\}$ is unimodal on $I$ if there is a point $\xi \in I$ such that $\varphi$ is increasing on $(-\infty, \xi] \cap I$ and decreasing on $[\xi, \infty) \cap I$. Note that the degenerate cases when $\xi$ is an end-point of $I$ are treated as unimodal.

Showing the unimodality of a function is not always easy. In Proposition 2.4 below, we give a useful condition to verify unimodality in the proofs of Theorem 1.2 and Theorem 1.4. For convenience, we first introduce the following term: for an interval $I \subset \mathbb{R}$ and integers $k_{0}, k$ with $k \geq k_{0} \geq 1$, a function $\varphi: I \rightarrow \mathbb{R} \cup\{ \pm \infty\}$ is called (negatively) half-monotone of order ( $k_{0}, k$ ) at a point $a \in I$ if the following conditions hold:
(1) $\varphi \in C^{k}(I)$ in the extended sense and $(-1)^{k} \varphi^{(k)} \leq 0$ on $I$;
(2) $(-1)^{\ell} \varphi^{(\ell)}(a) \geq 0$ for all $1 \leq \ell<k_{0}$;
(3) $(-1)^{\ell} \varphi^{(\ell)}(a) \leq 0$ for all $k_{0} \leq \ell \leq k$.

In addition, $\varphi$ is called strictly half-monotone of order $\left(k_{0}, k\right)$ at $a$ if $\varphi^{\left(k_{0}-1\right)}(a) \neq 0$. We assume, without loss of generality, that $k_{0}$ is the smallest integer such that these properties hold.

Proposition 2.4 (From half-monotone to unimodal). Let $k_{0}, k$ be integers with $k \geq k_{0} \geq 1$. If $\varphi:[0,1] \rightarrow \mathbb{R} \cup\{ \pm \infty\}$ is a half-monotone function of order $\left(k_{0}, k\right)$ at 1 , then it is unimodal on $[0,1]$. If, in addition, $k_{0}=1$, then $\varphi$ is increasing on $[0,1]$. Otherwise, when $\varphi$ is strictly half-monotone at 1 with $k_{0} \geq 2$, then $\varphi$ is not increasing on the whole interval $[0,1]$.
Proof. We have $(-1)^{k} \varphi^{(k)} \leq 0$ on $[0,1]$. Suppose for some $\ell \in\left[k_{0}, k\right) \cap \mathbb{N}$, we know that $(-1)^{\ell+1} \varphi^{(\ell+1)}$ is nonpositive on $[0,1]$. Then $(-1)^{\ell} \varphi^{(\ell)}$ is an increasing function on $[0,1]$. Since $(-1)^{\ell} \varphi^{(\ell)}(1) \leq 0$, it follows that $(-1)^{\ell} \varphi^{(\ell)} \leq 0$ on $[0,1]$. Inductively, we see that $(-1)^{k_{0}} \varphi^{\left(k_{0}\right)} \leq 0$ on $[0,1]$, and so $(-1)^{k_{0}-1} \varphi^{\left(k_{0}-1\right)}$ is an increasing function (which is also true if $k=k_{0}$ ). When $k_{0}=1$, we obtain the claim.

For the rest of the proof, we only consider $k_{0} \geq 2$, in which case $\varphi$ is strictly half-monotone. Since $(-1)^{k_{0}-1} \varphi^{\left(k_{0}-1\right)}(1)>0$, there must be some $\lambda_{k_{0}-1} \in[0,1)$ such that $(-1)^{k_{0}-1} \varphi^{\left(k_{0}-1\right)}$ is nonpositive on $\left[0, \lambda_{k_{0}-1}\right)$ and positive on ( $\left.\lambda_{k_{0}-1}, 1\right]$, where the former interval may be empty.

Suppose that for some $\ell \in\left[1, k_{0}-1\right) \cap \mathbb{N}$, there exists some $\lambda_{\ell+1} \in[0,1)$ such that $(-1)^{\ell+1} \varphi^{(\ell+1)}$ is nonpositive on $\left[0, \lambda_{\ell+1}\right)$ and positive on $\left(\lambda_{\ell+1}, 1\right)$, and therefore $(-1)^{\ell} \varphi^{(\ell)}$ is unimodal on $[0,1]$, and strictly decreasing on $\left(\lambda_{\ell+1}, 1\right]$. Since $(-1)^{\ell} \varphi^{(\ell)}(1) \geq 0$, there exists some $\lambda_{\ell} \in\left[0, \lambda_{\ell+1}\right]$ such that $(-1)^{\ell} \varphi^{(\ell)}$ is nonpositive on $\left[0, \lambda_{\ell}\right)$ and positive on $\left(\lambda_{\ell}, 1\right)$. Thus, by induction there exists some $\lambda_{1} \in[0,1)$ such that $\varphi^{\prime}$ is nonnegative on $\left[0, \lambda_{1}\right)$ and negative on $\left(\lambda_{1}, 1\right)$, so $\varphi$ is unimodal on $[0,1]$. This completes the proof.

In Corollary 2.5 and Lemma 2.6 we provide sufficient conditions for 1 to be the global minimizer of $f$ on $[0,1]$ (i.e. inside the sphere, $\|x\| \leq R$ ). In particular, Corollary 2.5 implies that convexity of $v$, together with Theorem 1.3(i), is sufficient.

Corollary 2.5. Suppose $-2<s \leq d-4$ and $v$ is $\mathcal{C}^{2}$ in the extended sense on $\left[0, R^{2}\right]$, where $R>0$ satisfies Theorem 1.3(i). If $v^{\prime \prime}$ is nonnegative on $\left[0, R^{2}\right]$, then the modified potential $f$ defined in (2.3) achieves its minimum on $[0,1]$ at 1.

Proof. From (2.4), (B.9), and (B.3), we have $f^{\prime \prime} \geq 0$ on $[0,1]$. Since $f^{\prime}(1)=0$, the global minimum of the convex function $f$ on the convex set $[0,1]$ is attained at 1. (See also Proposition 2.4 which implies $f$ is decreasing on $[0,1]$.)

Lemma 2.6. Suppose $d-4<s<d-3$, and for some $k>2$, $v$ is $\mathcal{C}^{k}$ in the extended sense on $\left[0, R^{2}\right]$, where $R>0$ satisfies Theorem 1.3(i). If in addition, Theorem 1.3(ii) holds, $v^{(k)}\left(R^{2} \lambda\right) \leq 0$ for $\lambda \in[0,1]$, and, for $3 \leq \ell \leq k-1$, $f^{(\ell)}(0) \leq 0$, that is

$$
\begin{equation*}
R^{s+2 \ell} v^{(\ell)}(0) \leq-h_{s, d}^{(\ell)}(0) \tag{2.12}
\end{equation*}
$$

then the modified potential $f$ defined in (2.3) achieves its minimum on $[0,1]$ at 1 .
Proof. Since $d-4<s<d-3$, we see that $\lim _{\lambda \rightarrow 1^{-}} f^{(k)}(\lambda)=-\infty$ due to (2.4), (B.5), and the fact that $v^{(k)}$ is nonpositive at 1 . Combining this with (2.4), (B.9), and our assumption that $v^{(k)}$ is nonpositive on $[0,1)$, we have that $f^{(k)} \leq 0$ on $[0,1]$. Let $\varphi(\lambda):=f^{\prime \prime}(1-\lambda)$. Then, $\varphi$ is half-monotone of order $(1, k-2)$ at 1 . Proposition 2.4 implies that $\varphi$ is increasing, or $f^{\prime \prime}$ is decreasing on $[0,1]$. By Theorem $1.3(\mathrm{ii}), f^{\prime \prime}(1) \geq 0$, so $f^{\prime \prime}$ is nonnegative on $[0,1]$. Consequently, since $f^{\prime}(1)=0$ from Theorem 1.3(i), it follows that $f$ is decreasing on $[0,1]$, so the minimum of $f$ on $[0,1]$ occurs at 1 .

In Lemma 2.7 and Corollary 2.8, we provide sufficient conditions for 1 to be the global minimizer of $f$ on $[1, \infty$ ) (i.e. outside the sphere, $\|x\| \geq R$ ).

Lemma 2.7. Suppose $R>0$ satisfies Theorem 1.3(i). If one of the following sets of conditions holds, then $f$ achieves its infimum on $[1, \infty)$ at 1 .
(a) $-2<s<d-4, k \in 2 \mathbb{N} \cup\left[2, \frac{d-s}{2}\right)$, $v$ is $\mathcal{C}^{k}$ in the extended sense on $\left[R^{2}, \infty\right)$ such that $v^{(k)}\left(R^{2} \lambda\right) \geq 0$ for $\lambda \in[1, \infty)$, and for $2 \leq \ell<k$, $f^{(\ell)}(1) \geq 0$, that is

$$
\begin{equation*}
h_{s, d}^{(\ell)}(1) \geq-R^{2 \ell+s} v^{(\ell)}\left(R^{2}\right) \tag{2.13}
\end{equation*}
$$

(b) $d-4<s<d-3$, v is $\mathcal{C}^{2}$ in the extended sense on $\left[R^{2}, \infty\right)$, Theorem 1.3(ii) is satisfied, and $\lambda \mapsto \lambda^{\frac{s}{2}+2} v^{\prime \prime}\left(R^{2} \lambda\right)$ is increasing for $\lambda>1$.
(The case $s=d-4$ is covered in Corollary 2.8).
Proof. We will handle each case individually.
Case (a): In this case the conditions on $v$ plus (2.5), (B.9), and (B.3), imply that $f^{(k)}(\lambda) \geq$ 0 on $[1, \infty)$. Suppose that for some $\ell \in[1, k) \cap \mathbb{N}$, we have that $f^{(\ell+1)}(\lambda) \geq 0$ on $[1, \infty)$. Then $f^{(\ell)}(\lambda)$ is an increasing function, and since $f^{(\ell)}(1) \geq 0$ (due to assumption (2.13) or Theorem 1.3(i) if $\ell=1$ ), we find that $f^{(\ell)}$ is nonnegative on $[1, \infty)$. By induction, $f$ is an increasing function, establishing our claim.

Case (b): We want to show that $f^{\prime \prime}$ is nonnegative for $\lambda \geq 1$, or equivalently that the following quantity is nonnegative:

$$
\lambda^{\frac{s}{2}+2} f^{\prime \prime}(\lambda)=R^{-s} \frac{s+2}{4}{ }_{2} \mathrm{~F}_{1}\left(2+\frac{s}{2}, \frac{2+s-d}{2} ; \frac{d}{2} ; \lambda^{-1}\right)+R^{4} \lambda^{\frac{s}{2}+2} v^{\prime \prime}\left(R^{2} \lambda\right) .
$$

From Theorem 1.3(ii), this quantity is nonnegative at $\lambda=1$. Since $R^{4} \lambda^{\frac{s}{2}+2} v^{\prime \prime}\left(R^{2} \lambda\right)$ is an increasing function for $\lambda \geq 1$, it is sufficient to show that ${ }_{2} \mathrm{~F}_{1}\left(2+\frac{s}{2}, \frac{2+s-d}{2} ; \frac{d}{2} ; \lambda^{-1}\right)$ is also
an increasing function, or equivalently, that ${ }_{2} \mathrm{~F}_{1}\left(2+\frac{s}{2}, \frac{2+s-d}{2} ; \frac{d}{2} ; \lambda\right)$ is a decreasing function on $[0,1)$. Taking a derivative, we have

$$
\frac{d}{d \lambda}{ }_{2} \mathrm{~F}_{1}\left(2+\frac{s}{2}, \frac{2+s-d}{2} ; \frac{d}{2} ; \lambda\right)=-\frac{(s+4)(d-s-2)}{2 d}{ }_{2} \mathrm{~F}_{1}\left(3+\frac{s}{2}, \frac{4+s-d}{2} ; \frac{d+2}{2} ; \lambda\right) .
$$

Since $d-4<s<d-3, c=\frac{d+2}{2}>b=\frac{4+s-d}{2}>0$, and $\frac{(s+4)(d-s-2)}{2 d}>0$, inequality (B.9) implies the derivative is negative. Hence $R^{-s} \frac{s+2}{4}{ }_{2} \mathrm{~F}_{1}\left(2+\frac{s}{2}, \frac{2+s-d}{2} ; \frac{d}{2} ; \lambda^{-1}\right)$ is an increasing function on $(1, \infty)$. Thus $f$ is convex, and therefore increasing function on $(1, \infty)$, so its global minimum occurs at 1 .

As a corollary of Lemma 2.7(a), we have that convexity of $v$, together with Theorem 1.3(i), is sufficient on $[1, \infty)$.

Corollary 2.8. Suppose $-2<s \leq d-4, R>0$ and $v$ is $\mathcal{C}^{2}$ in the extended sense on $\left[R^{2}, \infty\right)$. If $v^{\prime \prime}$ is nonnegative on $\left[R^{2}, \infty\right)$ where $R>0$ satisfies Theorem 1.3(i), then $f$ achieves its minimum on $[1, \infty)$ at 1 .

Proof. The case where $-2<s<d-4$ is handled in Lemma 2.7 case (a), so assume that $s=d-4$. Then, from (2.5), we see that

$$
f^{\prime \prime}(\lambda)=\frac{R^{4-d}(d-2)}{4} \lambda^{-\frac{d}{2}}\left(1-\frac{1}{\lambda}\right)+R^{4} v^{\prime \prime}\left(R^{2} \lambda\right)
$$

which is nonnegative on $[1, \infty)$. Since $f_{d-4, d, R, v}^{\prime}(1)=0$, we see that $f_{d-4, d, R, v}^{\prime}$ is nonnegative on $[1, \infty)$, so $f_{d-4, d, R, v}$ is increasing, giving us our claim.

We note that in case (b) of Lemma 2.7, since $\lambda^{\frac{s}{2}+2} v^{\prime \prime}\left(R^{2} \lambda\right)$ is an increasing function, and it must be at least 0 at 1 , this does tell us implicitly that $v$ is convex outside the sphere. However, convexity alone is not sufficient here, which is why the range $d-4<s<d-3$ is excluded in Corollary 2.8.
2.5. Proof of Theorem 1.2. For this result, we combine Theorem 1.4, case (b) of Lemma 2.7, and Lemma 2.6. The proof is mainly breaking the ranges of $\alpha$ and $s$ into three cases, and showing that each of these cases satisfies some of the results listed above.

Before the proof of Theorem 1.2, we first show that $\alpha_{s, d}$ in (1.6) is continuous in $s$, by showing that $\lim _{s \rightarrow 0} \frac{s c_{s, d}}{2-2 c_{s, d}}=-\frac{1}{2 b_{d}}$, where $c_{s, d}$ and $b_{d}$ are defined in (1.4) and (1.5), respectively. On the one hand, using

$$
\frac{\partial c_{s, d}}{\partial s}=c_{s, d}\left[\frac{1}{2} \psi_{0}\left(\frac{d-s}{2}\right)+\frac{1}{2} \psi_{0}\left(d-1-\frac{s}{2}\right)-\psi_{0}(d-s-1)\right]
$$

we get

$$
\lim _{s \rightarrow 0} \frac{2\left(1-c_{s, d}\right)}{s c_{s, d}}=-2 \lim _{s \rightarrow 0} \frac{\frac{\partial c_{s, d}}{\partial s}}{c_{s, d}+s \frac{\partial c_{s, d}}{\partial s}}=\psi_{0}(d-1)-\psi_{0}\left(\frac{d}{2}\right)
$$

On the other hand, $\psi_{0}(2 z)=\frac{1}{2}\left(\psi_{0}(z)+\psi_{0}\left(z+\frac{1}{2}\right)\right)+\log (2)$, see [15, Eq. 5.5.8], giving, with $z=\frac{d-1}{2}$,

$$
b_{d}=\frac{1}{2}\left(\psi_{0}\left(\frac{d}{2}\right)-\psi_{0}(d-1)\right)
$$

Proof of Theorem 1.2. We have $V(x)=v\left(\|x\|^{2}\right)$, where $v(r)=\frac{\gamma}{\alpha} r^{\alpha / 2}$. If for some $R>0$, $\sigma_{R}$ is the equilibrium measure of $I_{s, V}$, then $R$ must satisfy (1.7), due to Theorem 1.3(i). Then (1.8) follows from (1.7) and Proposition C.2. It remains to show that $\sigma_{R}$ is indeed the equilibrium measure of $I_{s, V}$ with $R$ is given by (1.7).

Since $\alpha \geq \alpha_{s, d}$

$$
R^{s}\left(v(0)-v\left(R^{2}\right)\right)=-R^{s+\alpha} \frac{\gamma}{\alpha}=-\frac{c_{s, d}}{2 \alpha} \geq \begin{cases}\frac{c_{s, d}-1}{s} & s \neq 0 \\ b_{d} & s=0\end{cases}
$$

meaning that Theorem 1.3(iii) is satisfied. Likewise, we see that

$$
R^{2} \frac{v^{\prime \prime}\left(R^{2}\right)}{v^{\prime}\left(R^{2}\right)}=\frac{1}{2}(\alpha-2) \geq-\frac{1}{2} \frac{(s+2)(d-s-4)}{2(d-s-3)}
$$

so Theorem 1.3(ii) is also satisfied.
First we set up some useful identities. For all $\ell \in \mathbb{N}$

$$
\begin{equation*}
R^{2 \ell} v^{(\ell)}\left(R^{2} \lambda\right)=R^{\alpha} \frac{\gamma}{2^{\ell}} \lambda^{\frac{\alpha}{2}-\ell} \prod_{j=1}^{\ell-1}(\alpha-2 j) \tag{2.14}
\end{equation*}
$$

and

$$
q(\kappa)=R^{s+\alpha} \gamma \kappa^{-\frac{s+\alpha}{2}}=\frac{c_{s, d}}{2} \kappa^{-\frac{s+\alpha}{2}}
$$

where $q$ is as in (2.6), so for all $\ell \in \mathbb{N}$

$$
\begin{equation*}
q^{(\ell)}(\kappa)=\frac{c_{s, d}}{2} \kappa^{-\frac{s+\alpha}{2}-\ell}\left(-\frac{1}{2}\right)^{\ell} \prod_{j=0}^{\ell-1}(\alpha+s+2 j) \tag{2.15}
\end{equation*}
$$

As discussed at the beginning of Section 2, the Frostman conditions (1.3) and the uniqueness of the equilibrium measure tell us it is sufficient to show that $f$ achieves its global minimum on $[0, \infty)$ at 1 to prove that $\sigma_{R}$ is the equilibrium measure. In order to do this, we consider three cases, and for each one, we show that $v$ satisfies certain results from Section 2.4, which indeed gives us that 1 is the global minimum of $f$.

Case 1: Suppose first that $-2<s \leq d-4$ and $\alpha \geq 2$. Then $v$ is $\mathcal{C}^{2}$ in the extended sense and convex on $[0, \infty)$, so the desired result follows from Theorem 1.4.

Case 2: For our second case, suppose that $-2<s<d-4$ and $\alpha_{s, d} \leq \alpha<2$. Let $k=\left\lceil\frac{d-s}{2}\right\rceil$. Note that $k \geq 3$ since $s<d-4$.

From (2.14), we have that $(-1)^{k} v^{(k)}\left(R^{2} \lambda\right)$ is nonpositive on $[0,1]$. Since $k=\left\lceil\frac{d-s}{2}\right\rceil$, $(-1)^{k} f^{(k)}$ is negative on $[0,1]$, by (2.4), (B.9), and (B.3). Combining (2.4), (1.7), (2.14), and (B.6), we have that for $2 \leq \ell<k$

$$
\begin{aligned}
(-1)^{\ell} R^{s} f^{(\ell)}(1) & =\frac{c_{s, d}}{2^{\ell+1}}\left(\prod_{j=1}^{\ell-1} \frac{(d-s-2-2 j)(s+2 j)}{2(d-s-2-j)}-\prod_{j=1}^{\ell-1}(2 j-\alpha)\right) \\
& =\frac{c_{s, d}}{2^{\ell+1}}\left(\prod_{j=1}^{\ell-1} \frac{(d-s-2-2 j)(s+2 j)}{2(d-s-2-j)(2 j-\alpha)}-1\right) \prod_{j=1}^{\ell-1}(2 j-\alpha) .
\end{aligned}
$$

We then see that for $1<j<\frac{d-s-2}{2}$,

$$
\begin{aligned}
\frac{d}{d j} \log \left(\frac{(d-s-2-2 j)(s+2 j)}{2(d-s-2-j)(2 j-\alpha)}\right) & =\frac{-2}{d-s-2-2 j}+\frac{2}{d-s-2-j}+\frac{2}{2 j+s}-\frac{2}{2 j-\alpha} \\
& =\frac{-2 j}{(d-s-2-2 j)(d-s-2-j)}-\frac{2(s+\alpha)}{(2 j+s)(2 j-\alpha)}
\end{aligned}
$$

which is negative, since $2>\alpha>-s$. Thus, $\frac{(d-s-2-2 j)(s+2 j)}{2(d-s-2-j)(2 j-\alpha)}$ is strictly decreasing for $j \in[1, k-1]$.

Now we will show that $\alpha>2-\frac{(s+2)(d-s-4)}{2(d-s-3)}$. Suppose to the contrary that $\alpha=2-$ $\frac{(s+2)(d-s-4)}{2(d-s-3)}$, so that $(-1)^{2} f^{(2)}(1)=0$. Then, since $\frac{(d-s-2-2 j)(s+2 j)}{2(d-s-2-j)(2 j-\alpha)}$ is strictly decreasing, we see that $(-1)^{\ell} f^{(\ell)}(1)<0$ for $2<\ell<k$. Recall that $(-1)^{k} f^{(k)}(\lambda)<0$ on [0, 1$]$. Now, suppose that for some $\ell \in\{2, \ldots, k-1\}$, we know that $(-1)^{\ell+1} f^{(\ell+1)}(\lambda)$ is negative $[0,1]$. Then $(-1)^{\ell} f^{(\ell)}(\lambda)$ is strictly increasing on $[0,1]$, and since $(-1)^{\ell} f^{(\ell)}(1) \leq 0$ (the inequality being strict for $\ell>2),(-1)^{\ell} f^{(\ell)}(\lambda)$ is negative on $[0,1)$. Thus, we can conclude inductively that $f^{(2)}(\lambda)$ is negative on $[0,1)$, and so, $f^{\prime}(\lambda)$ is strictly decreasing on $[0,1]$. Since $f^{\prime}(1)=0$, we see that $f$ is strictly decreasing on $[0,1]$, so $f(0)<f(1)$, which is a contradiction to Theorem 1.3(iii). Thus, we have that $\alpha>2-\frac{(s+2)(d-s-4)}{2(d-s-3)}$, and so $(-1)^{2} f^{(2)}(1)>0$.

Since $\frac{(d-s-2-2 j)(s+2 j)}{2(d-s-2-j)(2 j-\alpha)}$ is strictly decreasing for $j \in[1, k-1]$, and $(-1)^{2} f^{(2)}(1)>0$, we can now conclude that there must be some $\ell_{0} \in\{2, \ldots, k-1\}$ such that for $3 \leq \ell \leq \ell_{0}$

$$
\prod_{j=1}^{\ell-2} \frac{(d-s-2-2 j)(s+2 j)}{2(d-s-2-j)(2 j-\alpha)} \leq \prod_{j=1}^{\ell-1} \frac{(d-s-2-2 j)(s+2 j)}{2(d-s-2-j)(2 j-\alpha)}
$$

and for $\ell_{0}<\ell \leq k-1$

$$
\prod_{j=1}^{\ell-2} \frac{(d-s-2-2 j)(s+2 j)}{2(d-s-2-j)(2 j-\alpha)} \geq \prod_{j=1}^{\ell-1} \frac{(d-s-2-2 j)(s+2 j)}{2(d-s-2-j)(2 j-\alpha)}
$$

This then implies that there is some $k_{0} \in\{3, \ldots, k\}$ such that $(-1)^{\ell} f^{(\ell)}(1) \leq 0$ for $\ell \in$ $\left[k_{0}, k\right) \cap \mathbb{N}$ and $(-1)^{\ell} f^{(\ell)}(1)>0$ for all $\ell \in\left[2, k_{0}\right) \cap \mathbb{N}$. In other words, $f$ is strictly halfmonotone of order $\left(k_{0}, k\right)$ at 1 (note that $f^{\prime}(1)=0$ due to Theorem 1.3(i)). By Proposition 2.4 , the global minimum of $f$ is at 0 or 1 ; however, by Theorem 1.3 (iii), we know $f(0) \geq f(1)$, giving our desired result for $[0,1]$.

Next, consider the function $g$ as in (2.8). From (2.15), we have that $(-1)^{k} q^{(k)}(\kappa)$ is nonnegative on $[0,1]$. Since $k=\left\lceil\frac{d-s}{2}\right\rceil$, (2.10), (B.9), and (B.3)-(B.5) imply that $(-1)^{k} g^{(k)}$ is nonnegative on $[0,1]$. To be precise, we use (B.3) for when $s<d-5$ and $s \neq d-6$; (B.4) for when $s=d-5$ or $s=d-6$; (B.5) for when $d-5<s<d-4$. Combining (2.10), (2.15), (B.7), and the fact that

$$
{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+1, \frac{2+s-d}{2} ; \frac{d}{2} ; 1\right)=\frac{c_{s, d}}{2}
$$

we have that for $1 \leq \ell<k$

$$
(-1)^{\ell} g^{(\ell)}(1)=\frac{c_{s, d}}{2^{\ell+1}}\left(-\prod_{j=0}^{\ell-1} \frac{(d-s-2-2 j)(s+2 j+2)}{2(d-s-3-j)}+\prod_{j=0}^{\ell-1}(\alpha+s+2 j)\right)
$$

Since $\alpha>2-\frac{(s+2)(d-s-4)}{2(d-s-3)}$, we see that

$$
\alpha+s>\frac{(d-s-2)(s+2)}{2(d-s-3)}
$$

and for $j \geq 1$

$$
\frac{(d-s-2-2 j)(s+2 j+2)}{2(d-s-3-j)} \leq \frac{s}{2}+j+1<\alpha+s+2 j
$$

Thus $(-1)^{\ell} g^{(\ell)}(1) \geq 0$ for all $\ell \in\{1, \ldots, k-1\}$. This shows that $-g$ is half-monotone of order $(1, k)$ at 1 . Thus, $g$ is decreasing on $[0,1]$ by Proposition 2.4. From (2.9) and Theorem 1.3(i), we have $g(1)=2 R^{s} f^{\prime}(1)=0$, which implies $g \geq 0$ on $[0,1]$. Again by (2.9), $f$ must be increasing on $[1, \infty)$, giving our desired result.

Case 3: For our final case, suppose that $d-4<s<d-3$ and $\alpha_{s, d} \leq \alpha$. Combining (2.4), (1.7), (B.6), and the fact that $\alpha \geq \alpha_{s, d} \geq 2-\frac{(d-s-4)(s+2)}{2(d-s-3)}$, we have

$$
R^{s} f^{\prime \prime}(1)=\frac{c_{s, d}}{2^{3}}\left(\frac{(d-s-4)(s+2)}{2(d-s-3)}+(\alpha-2)\right) \geq 0 .
$$

We then see that

$$
\lambda^{\frac{s}{2}+2} \frac{d^{2}}{d \lambda^{2}} v\left(R^{2} \lambda\right)=R^{\alpha} \frac{\gamma(\alpha-2)}{4} \lambda^{\frac{s+\alpha}{2}}
$$

which is an increasing function on $[1, \infty)$, since $\alpha>2>-s$. Thus, the conditions of Lemma 2.7 case (b) have now been satisfied.
Let $k=\left\lceil\frac{\alpha}{2}\right\rceil+1$. For $0 \leq \lambda \leq 1$, and $\ell \in\{1, \ldots, k\}$

$$
\frac{d^{\ell}}{d \lambda^{\ell}} v\left(R^{2} \lambda\right)=R^{\alpha} \frac{\gamma}{2^{\ell}} \lambda^{\frac{\alpha}{2}-\ell} \prod_{j=1}^{\ell-1}(\alpha-2 j),
$$

so $v^{(k)}\left(R^{2} \lambda\right) \leq 0$ on $[0,1)$, and $v^{(\ell)}\left(R^{2}\right)<\infty$. For $3 \leq \ell \leq k-2, v^{(\ell)}(0)=0<-h_{s, d}^{(\ell)}(0)$.
We have that

$$
\begin{aligned}
R^{s} f^{\prime \prime}(1) & =\frac{c_{s, d}}{4}\left(\frac{(d-s-4)(s+2)}{2(d-s-3)}+\alpha-2\right) \\
& \geq \frac{c_{s, d}}{4}\left(\frac{(d-s-4)(s+2)}{2(d-s-3)}+\left(2-\frac{(d-s-4)(s+2)}{2(d-s-3)}\right)-2\right) \\
& =0 .
\end{aligned}
$$

Thus all the conditions of Lemma 2.6 are satisfied, giving our desired result.

### 2.6. Proof of Proposition 1.5.

Proof. Note that for $V(x)=\frac{\gamma}{\alpha}\|x\|^{\alpha}$, minimizing $I_{s, V}$ over $\mathcal{P}\left(\mathbb{R}^{d}\right)$ is the same as minimizing the energy over $\mathcal{P}_{\alpha, c}\left(\mathbb{R}^{d}\right)$ for each $c$, then minimizing over positive $c$. So, for $s<d, s \neq 0$,

$$
\begin{aligned}
\inf _{\mu \in \mathcal{P}\left(\mathbb{R}^{d}\right)} I_{s, V}(\mu) & =\inf _{c>0} \inf _{\mu \in \mathcal{P}_{\alpha, c}\left(\mathbb{R}^{d}\right)} I_{s, 0}(\mu)+2 \frac{\gamma}{\alpha} c^{\alpha} \\
& =\inf _{c>0} \inf _{\mu \in \mathcal{P}_{\alpha, c}\left(\mathbb{R}^{d}\right)} c^{-s} I_{s}\left(\left(c^{-1} I d\right) \# \mu\right)+2 \frac{\gamma}{\alpha} c^{\alpha} \\
& =\inf _{c>0}\left(c^{-s} \inf _{\nu \in \mathcal{P}_{\alpha, 1}\left(\mathbb{R}^{d}\right)} I_{s}(\nu)+2 \frac{\gamma}{\alpha} c^{\alpha}\right) .
\end{aligned}
$$

Let $A=\inf _{\nu \in \mathcal{P}_{\alpha, 1}\left(\mathbb{R}^{d}\right)} I_{s}(\nu)$, and assume $A$ is finite. Taking a derivative with respect to $c$, we see that

$$
-s c^{-s-1} A+2 \gamma c^{\alpha-1}=0
$$

is equivalent to

$$
c=\left(\frac{s A}{2 \gamma}\right)^{\frac{1}{s+\alpha}}
$$

so there is exactly one critical point ( $s$ and $A$ are either both nonnegative or both nonpositive, so this value is positive). Since $\alpha>-s$, this also means that $c^{-s} A+2 \frac{\gamma}{\alpha} c^{\alpha}$ is strictly decreasing on $\left(0,\left(\frac{s A}{2 \gamma}\right)^{\frac{1}{s+\alpha}}\right)$ and strictly increasing on $\left(\left(\frac{s A}{2 \gamma}\right)^{\frac{1}{s+\alpha}}, \infty\right)$, so the minimum occurs at this point. This proves our first claim for $s \neq 0$, and the second claim is a similar proof in reverse.

For the logarithmic case, we have

$$
\begin{aligned}
\inf _{\mu \in \mathcal{P}\left(\mathbb{R}^{d}\right)} I_{s, V}(\mu) & =\inf _{c>0} \inf _{\mu \in \mathcal{P}_{\alpha, c}\left(\mathbb{R}^{d}\right)} I_{0,0}(\mu)+2 \frac{\gamma}{\alpha} c^{\alpha} \\
& =\inf _{c>0} \inf _{\mu \in \mathcal{P}_{\alpha, c}\left(\mathbb{R}^{d}\right)} I_{0}\left(\left(c^{-1} I d\right) \# \mu\right)-\log (c)+2 \frac{\gamma}{\alpha} c^{\alpha} \\
& =\inf _{c>0} \inf _{\nu \in \mathcal{P}_{\alpha, 1}\left(\mathbb{R}^{d}\right)} I_{0}(\nu)-\log (c)+2 \frac{\gamma}{\alpha} c^{\alpha} .
\end{aligned}
$$

Let $A=\inf _{\nu \in \mathcal{P}_{\alpha, 1}\left(\mathbb{R}^{d}\right)} I_{0}(\nu)$, and assume $A$ is finite. Taking a derivative with respect to $c$, we see that

$$
-c^{-1}+2 \gamma c^{\alpha-1}=0
$$

is equivalent to

$$
c=\left(\frac{1}{2 \gamma}\right)^{\frac{1}{\alpha}}
$$

so there is exactly one critical point. Since $\alpha>0$, this also means that $A-\log (c)+2 \frac{\gamma}{\alpha} c^{\alpha}$ is strictly decreasing on $\left(0,\left(\frac{1}{2 \gamma}\right)^{\frac{1}{\alpha}}\right)$ and strictly increasing on $\left(\left(\frac{1}{2 \gamma}\right)^{\frac{1}{\alpha}}, \infty\right)$, so the minimum occurs at this point. This proves our first claim about logarithmic energy, and the second claim is a similar proof in reverse.

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## Appendix A. Lennard - Jones type external fields

This section considers some additional examples of Lennard - Jones type external fields, namely with $\alpha>\beta$, and $\gamma, \eta>0$,

$$
\begin{equation*}
v(\rho)=\frac{\gamma}{\alpha} \rho^{\frac{\alpha}{2}}-\frac{\gamma \eta}{\beta} \rho^{\frac{\beta}{2}}, \quad \rho=\|x\|^{2} . \tag{A.1}
\end{equation*}
$$

We present a case where the equilibrium measure has compact support despite the fact that $v(\infty):=\lim _{\rho \rightarrow \infty} v(\rho)=0$. This can only occur if $0>\alpha$, in which case Theorem 1.4, which requires $v$ to be convex, fails. We also provide examples when there are no solutions or multiple solutions for equation (1.9), which must be satisfied by an optimal sphere radius $R_{*}$.

Note that

$$
\begin{equation*}
v^{\prime}(\rho)=\frac{\gamma}{2} \rho^{\frac{\beta}{2}-1}\left(\rho^{\frac{\alpha-\beta}{2}}-\eta\right), \quad v^{\prime \prime}(\rho)=\frac{\gamma}{4} \rho^{\frac{\beta}{2}-2}\left((\alpha-2) \rho^{\frac{\alpha-\beta}{2}}-\eta(\beta-2)\right), \tag{A.2}
\end{equation*}
$$

so $v$ has a minimum $v^{*}=-\gamma\left(\frac{\alpha-\beta}{\alpha \beta}\right) \eta^{\frac{\alpha}{\alpha-\beta}}<0$ at $\rho^{*}=\eta^{\frac{2}{\alpha-\beta}}$. Moreover, for $\alpha \geq 2 \geq \beta$, $v^{\prime \prime}(\rho)>0$ for all $\rho>0$, so $v$ is convex on $(0, \infty)$, while for $\alpha<2$, there is a single point of inflection

$$
\tilde{\rho}:=\left(\eta \frac{2-\beta}{2-\alpha}\right)^{\frac{2}{\alpha-\beta}}
$$

so $v$ is convex on $[0, \tilde{\rho}]$ and concave on $[\tilde{\rho}, \infty)$.
A.1. A particular case with sphere radius 1. Let $-2<s \leq d-4, \beta=-b-s$ for some $b>2$ and $\alpha=-2-s$, and $\gamma, \eta>0$. The necessary condition Theorem 1.3(i) is that $R_{*}$ is a solution of

$$
\begin{equation*}
R^{-2}-\eta R^{-b}=R^{\alpha+s}-\eta R^{\beta+s}=\frac{c_{s, d}}{2 \gamma} \tag{A.3}
\end{equation*}
$$

If we choose $\gamma, \eta>0$ such that

$$
\begin{equation*}
1-\eta=\frac{c_{s, d}}{2 \gamma} \tag{A.4}
\end{equation*}
$$

then one solution is $R_{*}=1$, for which $\lambda=\frac{\|x\|^{2}}{R_{*}^{2}}=\rho$, see (A.1).
Corollary A. 1 (A Special Lennard-Jones Field). Let $-2<s \leq d-4$,

$$
\begin{equation*}
\gamma>\frac{c_{s, d}}{2} \max \left\{1, \frac{(2 b+s)(2+s)}{s(b-2)}\right\} \tag{A.5}
\end{equation*}
$$

$\eta=1-\frac{c_{s, d}}{2 \gamma}, \alpha=-2-s$ and $\beta=-b-s$ with
$b>\max \left\{2,\left(\frac{s+4}{\eta}-s-2\right), \frac{1}{\eta}\left(\frac{(d-s-2)(s+2)}{d \gamma}{ }_{2} \mathrm{~F}_{1}\left(\frac{s+4}{2}, \frac{4+s-d}{2} ; \frac{d+2}{2} ; 1\right)+2\right)\right\}$.
Then $\sigma_{1}$ is the equilibrium measure for the Lennard-Jones type external field (A.1).
Proof. If $b \geq \frac{s+4}{\eta}-s-2$, then the points of inflection $\tilde{\lambda}=\tilde{\rho} \geq 1$, so $v$ is convex on $[0,1]$ and, by Corollary 2.5, the modified potential $f$ attains its global minimum on $[0,1]$ at 1 .

With $q$ as in (2.6), (A.2) and $\beta-\alpha=2-b$,

$$
q(\kappa)=2 \kappa^{-\frac{s}{2}-1} v^{\prime}\left(\kappa^{-1}\right)=\gamma \kappa\left(1-\eta \kappa^{\frac{b}{2}-1}\right)
$$

so

$$
q^{\prime}(\kappa)=\frac{\gamma}{2}\left(2-\eta b \kappa^{\frac{b-2}{2}}\right), \quad q^{\prime \prime}(\kappa)=-\frac{\gamma \eta}{4} b(b-2) \kappa^{\frac{b-4}{2}}
$$

and $b>2$ implies that $q^{\prime \prime}$ is negative on $(0,1]$. With $g$ as in (2.8), we see that $g^{\prime \prime}<0$ on $(0,1]$ as well, due to (2.10) and (B.9).

If

$$
b>\frac{1}{\eta}\left(\frac{(d-s-2)(s+2)}{d \gamma}{ }_{2} \mathrm{~F}_{1}\left(\frac{s+4}{2}, \frac{4+s-d}{2} ; \frac{d+2}{2} ; 1\right)+2\right),
$$

then, using (2.10),

$$
-g^{\prime}(1)=-\frac{1}{2}\left(\frac{(d-s-2)(s+2)}{d}\right){ }_{2} \mathrm{~F}_{1}\left(\frac{s+4}{2}, \frac{4+s-d}{2} ; \frac{d+2}{2} ; 1\right)+\frac{\gamma}{2}(\eta b-2)>0 .
$$

Thus, by Proposition 2.4, $g$ is unimodal, and since $g(1)=0$, due to (2.9) and the fact that $f^{\prime}(1)=0$, we know that there is some $\kappa \in[0,1]$ so that $g$ is nonpositive on $[0, \kappa)$ and nonnegative on $[\kappa, 1]$. Again using (2.9), we see that $f^{\prime}$ is nonnegative on $\left[1, \frac{1}{\kappa}\right.$ ) and nonpositive on $\left(\frac{1}{\kappa}, \infty\right)$, with the second interval being empty if $\kappa=0$. Thus, $f$ is unimodal on $[1, \infty)$, and, with $R_{*}=1$,

$$
f(1)=\frac{c_{s, d}}{s}+v(1)=\frac{c_{s, d}}{2}\left(\frac{2 b+s}{s(b+s)}\right)-\gamma \frac{(b-2)}{(b+s)(2+s)}<0=\lim _{\lambda \rightarrow \infty} f(\lambda)
$$

so the infimum of $f$ on $[1, \infty)$ is attained at 1 . This then implies that $\mu_{\mathrm{eq}}=\sigma_{1}$.
A.2. A numerical example. Let $d=8, s=4$, so $c_{s, d}=\frac{1}{2}$ and consider the Lennard-Jones type external field (A.1) with parameters

$$
\alpha=-6, \quad \beta=-12, \quad \gamma=5 \quad \text { and } \quad \eta=\frac{19}{20}
$$

These values of $\alpha, \beta$ correspond to the classical Lennard-Jones field, see [20] for example. Then (A.4) is satisfied, so $R_{*}=1$ satisfies (A.3), and $\lambda=1$ is the global minimum of the corresponding modified potential, as illustrated in Figure 2 (B). These parameter values give $b=8$, the bound (A.5) is $\gamma>\frac{5}{4}$ and all the conditions of Corollary A. 1 are satisfied, so $\sigma_{1}$ is the equilibrium measure. Note, however, that there is another solution to (A.3), around $R=4.47$, as illustrated in Figure $2(\mathrm{~A})$.

(A) Graph of $R^{s+2} v^{\prime \prime}(R)-\frac{c_{s, d}}{4}$ whose zeros satisfy Theorem 1.3(i) for the optimal radius, including $R_{*}=1$

(в) The modified potential $f(\lambda)$ for $R_{*}=1$ defined in (2.3), with global minimum at $\lambda=1$

Figure 2. Lennard-Jones type external field with $d=8, s=4, \alpha=-6$, $\beta=-12, \gamma=5$ and $\eta=\frac{19}{20}$

For these parameter values, but with $\gamma=1$ and $\eta=\frac{3}{4}$ then (A.4) is satisfied so $R_{*}=1$ is a solution of (A.3), but $\lambda=1$ is a local, not global, minimizer of the modified potential, as (A.5) and hence Theorem 1.3(iv) are not satisfied. Also, for $\gamma=\frac{1}{3}$ and $\eta=\frac{1}{2}$, equation (A.3) has no solutions. In these instances the support of the equilibrium measure is not a sphere.

## Appendix B. Properties of Hypergeometric Functions

B.1. Gauss hypergeometric function. Throughout this paper, we make use of two types of hypergeometric series,

$$
\begin{equation*}
{ }_{2} \mathrm{~F}_{1}(a, b ; c ; z):=\sum_{k=0}^{\infty} \frac{(a)_{k}(b)_{k}}{(c)_{k}} \frac{z^{k}}{k!}, \tag{B.1}
\end{equation*}
$$

and

$$
\begin{equation*}
{ }_{3} \mathrm{~F}_{2}(a, b, p ; c, q ; z):=\sum_{k=0}^{\infty} \frac{(a)_{k}(b)_{k}(p)_{k}}{(c)_{k}(q)_{k}} \frac{z^{k}}{k!} \tag{B.2}
\end{equation*}
$$

where $(a)_{k}=a(a+1) \cdots(a+k-1)$ is the Pochhammer symbol, which are both absolutely convergent for $|z|<1$.
B.2. Behavior at 1. The behaviour at $z=1$ depends on the value of $c-a-b$ :

- If $\Re(c-a-b)>0$, then $[15,15.4 .20]$ (also known as Gauss summation theorem)

$$
\begin{equation*}
{ }_{2} \mathrm{~F}_{1}(a, b ; c ; 1)=\frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} . \tag{B.3}
\end{equation*}
$$

- If $c=a+b$, then $[15,15.4 .21]$

$$
\begin{equation*}
\lim _{z \rightarrow 1^{-}} \frac{{ }_{2} \mathrm{~F}_{1}(a, b ; a+b ; z)}{-\log (1-z)}=\frac{\Gamma(a+b)}{\Gamma(a) \Gamma(b)} \tag{B.4}
\end{equation*}
$$

- If $\Re(c-a-b)<0$, then $[15,15.4 .23]$

$$
\begin{equation*}
\lim _{z \rightarrow 1^{-}} \frac{{ }_{2} \mathrm{~F}_{1}(a, b ; c ; z)}{(1-z)^{c-a-b}}=\frac{\Gamma(c) \Gamma(a+b-c)}{\Gamma(a) \Gamma(b)} \tag{B.5}
\end{equation*}
$$

As an immediate consequence, we have that for all $k \in \mathbb{N}_{0}$, if $s<d-k-2$,

$$
\begin{align*}
& \frac{1}{2}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+k, \frac{2+s-d}{2}+k ; \frac{d}{2}+k ; 1\right) \\
& \quad=\frac{d-s-k-2}{d+2 k}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+k+1, \frac{2+s-d}{2}+k+1 ; \frac{d}{2}+k+1 ; 1\right) . \tag{B.6}
\end{align*}
$$

Likewise, for all $k \in \mathbb{N}_{0}$, if $s<d-k-3$,

$$
\begin{align*}
& \frac{1}{2}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+k+1, \frac{2+s-d}{2}+k ; \frac{d}{2}+k ; 1\right) \\
& \quad=\frac{d-s-k-3}{d+2 k}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}+k+2, \frac{2+s-d}{2}+k+1 ; \frac{d}{2}+k+1 ; 1\right) . \tag{B.7}
\end{align*}
$$

B.3. Euler integral representation. Euler's integral formula [15, 15.6.1], for $\Re(c)>$ $\Re(b)>0$ and $|\arg (1-z)|<\pi$, is

$$
\begin{equation*}
{ }_{2} \mathrm{~F}_{1}(a, b ; c ; z)=\frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_{0}^{1} \frac{u^{b-1}(1-u)^{c-b-1}}{(1-z u)^{a}} \mathrm{~d} u \tag{B.8}
\end{equation*}
$$

This formulation leads us to the following result, which is used repeatedly throughout the text:

$$
\begin{equation*}
{ }_{2} \mathrm{~F}_{1}(a, b ; c ; z)>0 \text { for } c>b>0 \text { and } z \in[0,1) \tag{B.9}
\end{equation*}
$$

A ${ }_{3} \mathrm{~F}_{2}$ can be written with a similar integral formula, involving a ${ }_{2} \mathrm{~F}_{1}$ (see $[15,16.5 .2]$ with $p=2, q=1)$ : For $\Re\left(b_{0}\right)>\Re\left(a_{0}\right)>0$ and $|\arg (1-z)|<\pi$,

$$
{ }_{3} \mathrm{~F}_{2}\left(a_{0}, a_{1}, a_{2} ; b_{0}, b_{1} ; z\right)=\frac{\Gamma\left(b_{0}\right)}{\Gamma\left(a_{0}\right) \Gamma\left(b_{0}-a_{0}\right)} \int_{0}^{1} t^{a_{0}-1}(1-t)^{b_{0}-a_{0}-1}{ }_{2} \mathrm{~F}_{1}\left(a_{1}, a_{2} ; b_{1} ; z t\right) \mathrm{d} t
$$

In particular, for $a_{1}=a_{2}=1$ and $b_{1}=2$ and using ${ }_{2} \mathrm{~F}_{1}(1,1 ; 2 ; z)=-\frac{\log (1-z)}{z}$, we find

$$
\begin{equation*}
{ }_{3} \mathrm{~F}_{2}\left(a_{0}, 1,1 ; b_{0}, 2 ; z\right)=-\frac{\Gamma\left(b_{0}\right)}{z \Gamma\left(a_{0}\right) \Gamma\left(b_{0}-a_{0}\right)} \int_{0}^{1} t^{a_{0}-2}(1-t)^{b_{0}-a_{0}-1} \log (1-z t) \mathrm{d} t \tag{B.10}
\end{equation*}
$$

B.4. Derivatives. The derivative of a hypergeometric function is, [15, 15.5.1],

$$
\begin{gather*}
\frac{\mathrm{d}}{\mathrm{~d} z}{ }_{2} \mathrm{~F}_{1}(a, b ; c ; z)=\frac{a b}{c}{ }_{2} \mathrm{~F}_{1}(a+1, b+1 ; c+1 ; z) .  \tag{B.11}\\
\frac{\mathrm{d}}{\mathrm{~d} z}{ }_{3} \mathrm{~F}_{2}(a, b, c ; p, q ; z)=\frac{a b c}{p q}{ }_{3} \mathrm{~F}_{2}(a+1, b+1, c+1 ; p+1, q+1 ; z) .
\end{gather*}
$$

We also have

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} z}\left(z^{-a}{ }_{2} \mathrm{~F}_{1}\left(a, b ; c ; z^{-1}\right)=-a z^{-a-1}{ }_{2} \mathrm{~F}_{1}\left(a+1, b ; c ; z^{-1}\right)\right. \tag{B.12}
\end{equation*}
$$

which follows from [1, Eq 15.2.3].
B.5. Gauss quadratic transformation. See [18, 2.11 (5)]

$$
\begin{equation*}
{ }_{2} \mathrm{~F}_{1}\left(a, b ; 2 b ; \frac{4 z}{(1+z)^{2}}\right)=(1+z)^{2 a}{ }_{2} \mathrm{~F}_{1}\left(a, a-b+\frac{1}{2} ; b+\frac{1}{2} ; z^{2}\right), \quad z \in[0,1], \tag{B.13}
\end{equation*}
$$

with the series absolutely convergent at $z=1$ when $b-a>0$.

## Appendix C. Riesz energy and potential

C.1. Legendre duplication formula for the Euler Gamma function. The Legendre duplication formula, [15, Eq. 5.5.5], is

$$
\begin{equation*}
\Gamma\left(\frac{1}{2}\right) \Gamma(2 z)=2^{2 z-1} \Gamma(z) \Gamma\left(z+\frac{1}{2}\right), \quad \text { for } 2 z \neq 0,-1,-2, \ldots \tag{C.1}
\end{equation*}
$$

C.2. Funk - Hecke formula. See, for instance, [35, p. 18], [7, Eq. (5.1.9), p. 197]. Recall that $\sigma_{1}$ denotes the uniform probability measure on $\mathbb{S}^{d-1}, d \geq 2$. Then, for all $x \in \mathbb{S}^{d-1}$,

$$
\begin{equation*}
\int_{\mathbb{S}^{d-1}} p(x \cdot y) \sigma_{1}(\mathrm{~d} y)=\tau_{d-1} \int_{-1}^{1} p(t)\left(1-t^{2}\right)^{\frac{d-3}{2}} \mathrm{~d} t \tag{C.2}
\end{equation*}
$$

where

$$
\begin{equation*}
\frac{1}{\tau_{d-1}}=\int_{-1}^{1}\left(1-t^{2}\right)^{\frac{d-3}{2}} \mathrm{~d} t=\operatorname{Beta}\left(\frac{1}{2}, \frac{d-1}{2}\right):=\frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} . \tag{C.3}
\end{equation*}
$$

In probabilistic terms, this means that if $Y$ is a random vector of $\mathbb{R}^{d}$ uniformly distributed on $\mathbb{S}^{d-1}$ then for all $x \in \mathbb{S}^{d-1}$, the law of $x \cdot Y$ has density $\tau_{d-1}\left(1-t^{2}\right)^{\frac{d-3}{2}} \mathbb{1}_{t \in[-1,1]}$. This is an arcsine law when $d=2$, a uniform law when $d=3$, a semicircle law when $d=4$, and more generally, for arbitrary values of $d \geq 2$, a Beta law on $[-1,1]$.

## C.3. Riesz Potential for uniform measure on a Sphere.

Proposition C.1. For $-2<s<d-1$ and $R>0$

$$
U_{s}^{\sigma_{R}}(x)= \begin{cases}R^{-s} h_{s, d}\left(\frac{\|x\|^{2}}{R^{2}}\right) & s \neq 0 \\ -\log (R)+h_{0, d}\left(\frac{\|x\|^{2}}{R^{2}}\right) & s=0 .\end{cases}
$$

This follows immediately from Lemmas C. 3 and C.4, and as a consequence, we can easily compute then Riesz energy of $\sigma_{R}$ using (2.1).

Proposition C.2. Let $-2<s<d-1$ and $R>0$. Then

$$
I_{s}\left(\sigma_{R}\right)=\left\{\begin{array}{ll}
\frac{R^{-s}}{s} c_{s, d}, & s \neq 0 \\
-\log (R)+b_{d}, & s=0
\end{array} .\right.
$$

Lemma C.3. For $s<d-1, s \neq 0, R>0$, and $\rho=\frac{\|x\|}{R}$

$$
\begin{align*}
U_{s}^{\sigma_{R}}(x) & =\frac{R^{-s}}{s}(1+\rho)^{-s}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}, \frac{d-1}{2} ; d-1 ; \frac{4 \rho}{(1+\rho)^{2}}\right)  \tag{C.4}\\
& = \begin{cases}\frac{R^{-s}}{s}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}, \frac{s-d}{2}+1 ; \frac{d}{2} ; \rho^{2}\right) & 0 \leq \rho \leq 1 \\
\frac{R^{-s}}{s} \rho^{-s}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}, \frac{s-d}{2}+1 ; \frac{d}{2} ; \rho^{-2}\right) & \rho \geq 1\end{cases} \tag{C.5}
\end{align*}
$$

Proof. For $x=0$, the potential $U_{s}^{\sigma_{R}}$ is $\frac{R^{-s}}{s}$.
For $x \neq 0$, using the Funk-Hecke formula (C.2) and the substitution $t=2 u-1$, gives

$$
\begin{aligned}
\int_{\mathbb{R}^{d}} \frac{1}{s}\|x-y\|^{-s} \mathrm{~d} \sigma_{R}(y) & =\int_{\mathbb{S}^{d-1}} \frac{1}{s}\|x-R y\|^{-s} \mathrm{~d} \sigma_{1}(y) \\
& =\int_{\mathbb{S}^{d-1}} \frac{R^{-s}}{s}\left(\frac{\|x\|^{2}}{R^{2}}+\|y\|^{2}-2 \frac{\|x\|}{R}\left\langle\frac{x}{\|x\|}, y\right\rangle\right)^{-\frac{s}{2}} \mathrm{~d} \sigma(y) \\
& =\tau_{d-1} \frac{R^{-s}}{s} \int_{-1}^{1}\left(1+\rho^{2}-2 \rho t\right)^{-\frac{s}{2}}\left(1-t^{2}\right)^{\frac{d-3}{2}} \mathrm{~d} t \\
& =2 \tau_{d-1} \frac{R^{-s}}{s} \int_{0}^{1}\left((1+\rho)^{2}-4 \rho u\right)^{-\frac{s}{2}}(4 u(1-u))^{\frac{d-3}{2}} \mathrm{~d} u \\
& =2^{d-2} \tau_{d-1} \frac{R^{-s}}{s}(1+\rho)^{-s} \int_{0}^{1}\left(1-\frac{4 \rho u}{(1+\rho)^{2}}\right)^{-\frac{s}{2}} u^{\frac{d-3}{2}}(1-u)^{\frac{d-3}{2}} \mathrm{~d} u
\end{aligned}
$$

Euler's integral formula (B.8) with $a=\frac{s}{2}, b=\frac{d-1}{2}$, and $c=d-1$, and (C.3) plus the Legendre duplication formula (C.1) with $z=\frac{d-1}{2}$, gives (C.4). The quadratic transformation (B.13) gives the first case in (C.5) with $a=\frac{s}{2}, b=\frac{d-1}{2}$, so $b-a=\frac{d-1-s}{2}>0$ if and only if $s<d-1$. The second case follows using the transformation $u=1 / \lambda$.

Lemma C.4. For $s=0, R>0$, and $\rho=\frac{\|x\|}{R}$

$$
\begin{align*}
\int_{\mathbb{R}^{d}}-\log (\|x-y\|) \mathrm{d} \sigma_{R}(y)= & -\log (R)-\log (1+\rho)+ \\
& \frac{\rho}{(1+\rho)^{2}}{ }_{3} \mathrm{~F}_{2}\left(1,1, \frac{d+1}{2} ; 2, d ; \frac{4 \rho}{(1+\rho)^{2}}\right) \tag{C.6}
\end{align*}
$$

Proof. For $x=0$, the potential is clearly $-\log (R)$.
For $x \neq 0$, using the Funk-Hecke formula (C.2), Euler's integral representation (B.8), and (C.3), with the substitution $t=2 u-1$, gives

$$
\begin{aligned}
-\int_{\mathbb{R}^{d}} & \log (\|x-y\|) \mathrm{d} \sigma_{R}(y)=-\int_{\mathbb{S}^{d-1}} \log (\|x-R y\|) \mathrm{d} \sigma_{1}(y) \\
& =-\int_{\mathbb{S}^{d-1}}\left(\log (R)+\frac{1}{2} \log \left(\frac{\|x\|^{2}}{R^{2}}+\|y\|^{2}-2 \frac{\|x\|}{R}\left\langle\frac{x}{\|x\|}, y\right\rangle\right) \mathrm{d} \sigma(y)\right. \\
& =-\tau_{d-1} \int_{-1}^{1}\left(\log (R)+\frac{1}{2} \log \left(1+\rho^{2}-2 \rho t\right)\right)\left(1-t^{2}\right)^{\frac{d-3}{2}} \mathrm{~d} t \\
& =-\log (R)-\tau_{d-1} \int_{0}^{1} \log \left((1+\rho)^{2}-4 \rho u\right)(4 u(1-u))^{\frac{d-3}{2}} \mathrm{~d} u \\
& =-\log (R)-\log (1+\rho)-2^{d-3} \tau_{d-1} \int_{0}^{1} \log \left(1-\frac{4 \rho u}{(1+\rho)^{2}}\right) u^{\frac{d-3}{2}}(1-u)^{\frac{d-3}{2}} \mathrm{~d} u
\end{aligned}
$$

Euler's integral formula (B.10) with $a_{0}=\frac{d+1}{2}$ and $b_{0}=d$ plus the Legendre duplication formula (C.1) with $z=\frac{d-1}{2}$ gives (C.6).

Thus, for $s<d-1$, setting

$$
g_{s, d}(\rho):=\left\{\begin{array}{ll}
\frac{1}{s}(1+\rho)^{-s}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}, \frac{d-1}{2} ; d-1 ; \frac{4 \rho}{\left(1+\rho^{2}\right.}\right) & s \neq 0 \\
-\log (1+\rho)+\frac{\rho}{(1+\rho)^{2}}{ }^{2} \mathrm{~F}_{2}\left(1,1, \frac{d+1}{2} ; 2, d ; \frac{4 \rho}{(1+\rho)^{2}}\right) & s=0
\end{array},\right.
$$

and taking the derivative (and after series expansion and some algebra) we get the single formula for $-2<s<d-2$

$$
\begin{aligned}
g_{s, d}^{\prime}(\rho)= & -\frac{1}{(1+\rho)^{s+3}}\left((1+\rho)^{2}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}, \frac{d-1}{2} ; d-1 ; \frac{4 \rho}{(1+\rho)^{2}}\right)\right. \\
& \left.\quad+(\rho-1){ }_{2} \mathrm{~F}_{1}\left(1+\frac{s}{2}, \frac{d+1}{2} ; d ; \frac{4 \rho}{(1+\rho)^{2}}\right)\right) \\
= & \begin{cases}-\frac{d-s-2}{d} \rho_{2} \mathrm{~F}_{1}\left(1+\frac{s}{2}, \frac{4+s-d}{2} ; \frac{d+2}{2} ; \rho^{2}\right) & \rho \leq 1 \\
-\rho^{-s-1}{ }_{2} \mathrm{~F}_{1}\left(1+\frac{s}{2}, \frac{2+s-d}{2} ; \frac{d}{2} ; \rho^{-2}\right) & \rho \geq 1\end{cases}
\end{aligned}
$$

Let us define $h_{s, d}:[0, \infty) \rightarrow \mathbb{R}$ by,

$$
h_{s, d}(\lambda)=\left\{\begin{array}{ll}
\frac{1}{s}(1+\sqrt{\lambda})^{-s}{ }_{2} \mathrm{~F}_{1}\left(\frac{s}{2}, \frac{d-1}{2} ; d-1 ; \frac{4 \sqrt{\lambda}}{(1+\sqrt{\lambda})^{2}}\right) & s \neq 0 \\
-\log (1+\sqrt{\lambda})+\frac{\sqrt{\lambda}}{(1+\sqrt{\lambda})^{2}}{ }_{3} \mathrm{~F}_{2}\left(1,1, \frac{d+1}{2} ; 2, d ; \frac{4 \sqrt{\lambda}}{(1+\sqrt{\lambda})^{2}}\right) & s=0
\end{array} .\right.
$$

In particular we obtain, using $h_{s, d}(\lambda)=g_{s, d}(\sqrt{\lambda})$, that for $-2<s<d-2$

$$
h_{s, d}^{\prime}(\lambda)=\left\{\begin{array}{ll}
-\frac{d-s-2}{2 d}{ }_{2} \mathrm{~F}_{1}\left(1+\frac{s}{2}, \frac{4+s-d}{2} ; \frac{d+2}{2} ; \lambda\right) & \lambda \leq 1 \\
-\frac{1}{2} \lambda^{-\frac{s}{2}-1}{ }_{2} \mathrm{~F}_{1}\left(1+\frac{s}{2}, \frac{2+s-d}{2} ; \frac{d}{2} ; \lambda^{-1}\right) & \lambda \geq 1
\end{array} .\right.
$$

This expression is useful in determining if $\sigma_{R}$ is the equilibrium measure, since $h_{s, d}$ describes the potential of the uniform measure on the unit sphere.

## Appendix D. Additional sufficient conditions

In this appendix, we gather some additional results giving conditions for which $f$ achieves its minimum on $[0,1]$ or $[1, \infty)$ at 1 , which are not used elsewhere in the paper, but may be useful for the study of other external fields.

We first provide some sufficient conditions for the behavior inside the sphere.
Lemma D.1. If one of the following conditions hold, then $f$ achieves its minimum on $[0,1]$ at 1.
(a) For $-2<s<d-4, R>0, k=\left\lceil\frac{d-s}{2}\right\rceil$, and $v$ is $\mathcal{C}^{k}$ in the extended sense on $\left[0, R^{2}\right]$, with $(-1)^{k} v^{(k)}$ nonpositive on $\left[0, R^{2}\right]$, such that (i) and (iii) in Theorem 1.3 are satisfied. In addition, there is some $k_{0} \in\{3, \ldots, k\}$ such that for $k_{0} \leq \ell<k$, $(-1)^{\ell} f^{(\ell)}(1) \leq 0$, that is

$$
\begin{equation*}
(-1)^{\ell+1} v^{(\ell)}\left(R^{2}\right) \geq(-1)^{\ell} R^{-s-2 \ell} h_{s, d}^{(\ell)}(1) \tag{D.1}
\end{equation*}
$$

and for $2 \leq \ell<k_{0},(-1)^{\ell} f^{(\ell)}(1) \geq 0$, that is

$$
\begin{equation*}
(-1)^{\ell+1} v^{(\ell)}\left(R^{2}\right) \leq(-1)^{\ell} R^{-s-2 \ell} h_{s, d}^{(\ell)}(1) . \tag{D.2}
\end{equation*}
$$

(b) For $-2<s \leq d-4, R>0$, and for some $2 \leq k \leq\left\lfloor\frac{d-s}{2}\right\rfloor$, $v$ is $\mathcal{C}^{k}$ in the extended sense on $\left[0, R^{2}\right]$, with $(-1)^{k} v^{(k)}$ nonnegative on $\left[0, R^{2}\right]$. In addition, Theorem 1.3(i) is satisfied, and for $2 \leq \ell \leq k-1,(-1)^{\ell} f^{(\ell)}(1) \geq 0$, that is

$$
\begin{equation*}
(-1)^{\ell+1} v^{(\ell)}\left(R^{2}\right) \leq(-1)^{\ell} R^{-s-2 \ell} h_{s, d}^{(\ell)}(1) \tag{D.3}
\end{equation*}
$$

(c) For $d-4<s<d-3, R>0, k>2$, and $v$ is $\mathcal{C}^{k}$ in the extended sense on $\left[0, R^{2}\right]$, with $v^{(k)}$ nonnegative on $\left[0, R^{2}\right]$, such that conditions (i), (ii), and (iii) of Theorem 1.3 are satisfied. In addition, for $3 \leq \ell \leq k-1$, $v^{(\ell)}\left(R^{2}\right)<\infty$ and $f^{(\ell)}(0) \geq 0$, that is

$$
\begin{equation*}
R^{s+2 \ell} v^{(\ell)}(0) \geq-h_{s, d}^{(\ell)}(0) \tag{D.4}
\end{equation*}
$$

(d) For $d \geq 3, s=d-4, R>0$, and $v$ is $\mathcal{C}^{2}$ in the extended sense on $\left[0, R^{2}\right]$ such that conditions (i) and (iii) of Theorem 1.3 are satisfied. In addition, there is some $\lambda_{2} \in[0,1)$ such that $v^{\prime \prime}\left(R^{2} \lambda\right)$ is nonpositive on $\left[0, \lambda_{2}\right)$ and nonnegative on $\left(\lambda_{2}, 1\right]$.
Proof. We handle each of the cases separately.
Case (a): Since $k=\left\lceil\frac{d-s}{2}\right\rceil$ (note that $s<d-4$ means that $\left.k \geq 3\right),(-1)^{k} f^{(k)}$ is nonpositive on $[0,1]$, by (2.4), (B.9), and (B.3). This along with (D.1), (D.2), and Theorem 1.3(i) imply that $f$ is half-monotone of order $\left(k_{0}, k\right)$ at 1 on $[0,1]$. By Proposition 2.4, $f$ is unimodal on $[0,1]$. Thus, its global minimum on $[0,1]$ must occur at an endpoint. From Theorem 1.3(iii), we know $f(0) \geq f(1)$, giving us our claim.

Case (b): Since $k \leq\left\lfloor\frac{d-s}{2}\right\rfloor,(-1)^{k} f^{(k)}$ is nonnegative on $[0,1]$, which follows from (2.4), (B.9), and (B.3). Thus, $-f$ is half-monotone of order $(1, k)$ at 1 on $[0,1]$, which follows from (D.3) and Theorem 1.3(i). By Proposition 2.4, $f$ is decreasing on [ 0,1 , giving us our claim.

Case (c): When $k=3$, the claim follows from Lemma 2.6. Now, consider the case when $k \geq 4$. Since $d-4<s<d-3$, we see that $f^{(\ell)}(1):=\lim _{\lambda \rightarrow 1^{-}} f^{(\ell)}(\lambda)$ is $-\infty$ for $3 \leq \ell \leq k$, due to $(2.4)$, (B.5), and the fact that $v^{(\ell)}\left(R^{2}\right)<\infty$. Combining this with (2.4), (B.9), and the assumption $v^{(k)}\left(R^{2} \lambda\right) \leq 0$ on $[0,1]$, we see that $f^{(k)} \leq 0$ on $[0,1]$. Let $\varphi(\lambda):=f^{\prime \prime}(1-\lambda)$. Then, $\varphi$ is half-monotone of order $(k-2, k-2)$ at 1 . By Proposition 2.4, $\varphi$ and also $f^{\prime \prime}$ are unimodal on $[0,1]$. Theorem 1.3 (ii) gives us that $f^{(2)}(1) \geq 0$, so $-f^{\prime}$ is unimodal on $[0,1]$. This in turn implies that $f$ is unimodal on $[0,1]$ due to the fact that $f^{\prime}(1)=0$ from Theorem 1.3(i). Theorem 1.3(iii) now gives us our claim.

Case (d): Note, from (2.4), $f^{(2)}(\lambda)=R^{4} v^{(2)}\left(R^{2} \lambda\right)$ on $[0,1]$. Thus, $f^{\prime}$ is decreasing on $\left[0, \lambda_{2}\right)$ and increasing on $\left(\lambda_{2}, 1\right]$, and since $f^{\prime}(1)=0$ (due to Theorem $1.3(\mathrm{i})$ ), there is some $\lambda_{1} \in[0,1)$ such that $f^{\prime}$ is nonnegative on $\left[0, \lambda_{1}\right)$ and nonpositive on $\left(\lambda_{1}, 1\right]$, so $f$ is increasing and decreasing on those intervals, respectively. Thus, the minimum can only occur at 0 or 1, and our claim then follows from Theorem 1.3(iii).

We now determine some sufficient conditions for behavior outside of the sphere. In what follows, $q, y_{s, d}$, and $g$ are as in (2.6), (2.7), and (2.8), respectively.

Lemma D.2. If one of the following conditions hold, then $f$ achieves its infimum on $[1, \infty)$ at 1 .
(a) For $-2<s<d-4, R>0, k=\left\lceil\frac{d-s}{2}\right\rceil$, and $q$ is $\mathcal{C}^{k}$ in the extended sense on $[0,1]$ such that $q^{(k)}(\kappa) \geq 0$ on $[0,1] .0$ In addition, Theorem 1.3(i) is satisfied, and for $\ell \in\{1, \ldots, k-1\},(-1)^{\ell} g^{(\ell)}(1) \geq 0$, that is

$$
\begin{equation*}
(-1)^{\ell} q^{(\ell)}(1) \geq(-1)^{\ell+1} y_{s, d}^{(\ell)}(1) \tag{D.5}
\end{equation*}
$$

(b) For $-2<s \leq d-4, R>0$, and for some $2 \leq k \leq\left\lfloor\frac{d-s}{2}\right\rfloor, q$ is $\mathcal{C}^{k}$ in the extended sense on $[0,1]$, with $(-1)^{k} q^{(k)} \leq 0$ on $[0,1]$. In addition, Theorem 1.3(i) and (iv) are satisfied and there is some $k_{0} \in\{2, \ldots, k\}$ such that for $k_{0} \leq \ell<k,(-1)^{\ell} g^{(\ell)}(1) \leq 0$, that is

$$
\begin{equation*}
(-1)^{\ell+1} q^{(\ell)}(1) \geq(-1)^{\ell} y_{s, d}^{(\ell)}(1), \tag{D.6}
\end{equation*}
$$

and for $1 \leq \ell<k_{0},(-1)^{\ell} g^{(\ell)}(1) \geq 0$, that is

$$
\begin{equation*}
(-1)^{\ell+1} q^{(\ell)}(1) \leq(-1)^{\ell} y_{s, d}^{(\ell)}(1) . \tag{D.7}
\end{equation*}
$$

(c) For $-2<s<d-4, R>0, k \in 2 \mathbb{N}+1 \cup\left[3, \frac{d-s}{2}\right)$, and $v$ is $\mathcal{C}^{k}$ in the extended sense on $\left[R^{2}, \infty\right)$ such that $v^{(k)}$ is nonnegative on $\left[R^{2}, \infty\right)$. In addition, Theorem 1.3(i) and (iv) are satisfied, and there is some $k_{0} \in\{3, \ldots, k\}$ such that for $k_{0} \leq \ell<k$, $f^{(\ell)}(1) \leq 0$, that is

$$
\begin{equation*}
h_{s, d}^{(\ell)}(1) \leq-R^{2 \ell+s} v^{(\ell)}\left(R^{2}\right) \tag{D.8}
\end{equation*}
$$

and for $2 \leq \ell<k_{0}, f^{(\ell)}(1) \geq 0$, that is

$$
\begin{equation*}
h_{s, d}^{(\ell)}(1) \geq-R^{2 \ell+s} v^{(\ell)}\left(R^{2}\right) \tag{D.9}
\end{equation*}
$$

(d) For $d \geq 3, s=d-4, R>0$, and $v$ is $\mathcal{C}^{3}$ in the extended sense on $\left[R^{2}, \infty\right)$ such that Theorem 1.3(i) and (iv) are satisfied. In addition, $g^{\prime}(1) \leq 0$ and there is some $\kappa_{2} \in[0,1)$ such that $q^{\prime \prime}(\kappa)$ is nonpositive on $\left[0, \kappa_{2}\right)$ and nonnegative on $\left(\kappa_{2}, 1\right]$.
(e) For $d \geq 3, s=d-4, R>0$, and $v$ is $\mathcal{C}^{3}$ in the extended sense on $\left[R^{2}, \infty\right)$ such that Theorem 1.3(i) and (iv) are satisfied. In addition, $g^{\prime}(1) \leq 0, g^{\prime}(0) \geq 0$, and there is some $\kappa_{2} \in[0,1)$ such that $q^{\prime \prime}(\kappa)$ is nonnegative on $\left[0, \kappa_{2}\right)$ and nonpositive on $\left(\kappa_{2}, 1\right]$.
(f) For $d \geq 3, s=d-4, R>0$, and $v$ is $\mathcal{C}^{3}$ in the extended sense on $\left[R^{2}, \infty\right)$ such that Theorem 1.3(i) is satisfied. In addition, there is some $\kappa_{2} \in(0,1)$ such that $q^{\prime \prime}(\kappa)$ is nonnegative on $\left[0, \kappa_{2}\right)$ and nonpositive on $\left(\kappa_{2}, 1\right]$ and $g^{\prime}\left(\kappa_{2}\right) \leq 0$.

Proof. We handle each case separately.
Case (a): Since $k=\left\lceil\frac{d-s}{2}\right\rceil$ (note that since $s<d-4, k \geq 3$ ), $(-1)^{k} g^{(k)}$ is nonnegative on $[0,1]$, by (2.10), (B.9), and (B.3)-(B.5) (to be precise, we use (B.3) for when $s<d-5$ and $s \neq d-6$; (B.4) for when $s=d-5$ or $s=d-6$; (B.5) for when $d-5<s<d-4$ ). Combining this with (D.5), we have $-g$ is half-monotone of order $(1, k)$ at 1 . By Proposition $2.4, g$ is decreasing on $[0,1]$. From (2.9) and Theorem 1.3(i), we have $g(1)=2 R^{s} f^{\prime}(1)=0$, which implies $g \geq 0$ on $[0,1]$. Thus, $f$ must be increasing on $[1, \infty)$ by (2.9), giving us our claim.

Case (b): Since $1 \leq k \leq\left\lfloor\frac{d-s}{2}\right\rfloor,(-1)^{k} g^{(k)} \leq 0$ on $[0,1$ ), by (2.10), (B.9), and (B.3). Combining this with (D.6) and (D.7), we have $g$ is half-monotone of order $\left(k_{0}, k\right)$ at 1 . By Proposition 2.4, $g$ is unimodal on $[0,1]$. On the other hand, we have $g(1)=2 R^{s} f^{\prime}(1)=0$, which follows from (2.9) and Theorem 1.3(i). Thus, there exists some $\kappa_{0} \in[0,1]$ such that $g \leq 0$ on $\left[0, \kappa_{0}\right)$ and $g \geq 0$ on ( $\left.\kappa_{0}, 1\right]$. Due to (2.9), $f^{\prime} \geq 0$ on $\left[1, \kappa_{0}^{-1}\right)$ and $f^{\prime} \leq 0$ on $\left(\kappa_{0}^{-1}, \infty\right)$, so $f$ is unimodal on $[1, \infty)$. Note that we interpret $\kappa_{0}^{-1}$ as $\infty$ if $\kappa_{0}=0$. Our claim now follows from Theorem 1.3(iv).

Case (c): We see, due to (2.5), (B.9), and (B.3), that $f^{(k)} \leq 0$ on $[1, \infty$ ). This could be: For arbitrary $a>1$, let $\varphi_{a}(\xi):=f(a-(a-1) \xi)$ be a function defined on $[0,1]$. Then, $\varphi_{a}$ is half-monotone of order $\left(k_{0}, k\right)$ at 1 , which follows from Theorem 1.3(i), (D.8), and (D.9). Thus, $\varphi_{a}$ is unimodal on [ 0,1 ] by Proposition 2.4. This shows that $f$ is unimodal on
$[1, a]$ for all $a>1$. In other words, $f$ is unimodal on $[1, \infty)$. Our claim now follows from Theorem 1.3(iv).

Case (d): From (2.10), $g^{(2)}(\kappa)=q^{(2)}(\kappa)$. Thus, $g^{\prime}$ is decreasing on $\left[0, \kappa_{2}\right)$ and increasing on $\left(\kappa_{2}, 1\right]$. Since $g^{\prime}(1) \leq 0$, there is some $\kappa_{1} \in[0,1)$ such that $g^{\prime}$ is nonnegative on $\left[0, \kappa_{1}\right)$ and nonpositive on ( $\kappa_{1}, 1$ ], so $g$ is increasing and decreasing on those intervals, respectively. Using (2.9) and Theorem 1.3(i), we see that there must be some $\lambda_{1} \in(1, \infty]$ such that $f^{\prime}$ is nonnegative on $\left[1, \lambda_{1}\right.$ ) and nonpositive on ( $\lambda_{1}, \infty$ ), meaning that $f$ is increasing and decreasing on those same intervals, respectively. Our claim now follows from Theorem 1.3(iv).

Case (e): From (2.10), $g^{(2)}(\kappa)=q^{(2)}(\kappa)$. Thus, $g^{\prime}$ is increasing on $\left[0, \kappa_{2}\right)$ and decreasing on ( $\left.\kappa_{2}, 1\right]$. Since $g^{\prime}(1) \leq 0$ and $g^{\prime}(0) \geq 0$, there is some $\kappa_{1} \in[0,1)$ such that $g^{\prime}$ is nonnegative on $\left[0, \kappa_{1}\right)$ and nonpositive on ( $\left.\kappa_{1}, 1\right]$, so $g$ is increasing and decreasing on those intervals, respectively. By (2.9) and Theorem $1.3(\mathrm{i})$, we see that $g(1)=0$, so there exists some $\kappa_{0} \in[0,1)$ such that $g_{d-4, d, q}$ is nonpositive on $\left[0, \kappa_{0}\right)$ and nonnegative on ( $\left.\kappa_{0}, 1\right]$. Using (2.9), we have that $f^{\prime}$ is nonnegative on $\left[1, \kappa_{0}^{-1}\right.$ ) and nonpositive on ( $\left.\kappa_{0}^{-1}, \infty\right)$, meaning that $f$ is increasing and decreasing on those same intervals, respectively. Our claim now follows from Theorem 1.3(iv).

Case (f): From (2.10), $g^{(2)}(\kappa)=q^{(2)}(\kappa)$. Thus, $g^{\prime}$ is increasing on $\left[0, \kappa_{2}\right)$ and decreasing on ( $\kappa_{2}, 1$ ], achieving its maximum at $\kappa_{2}$. Thus, $g^{\prime}$ is nonpositive on $[0,1]$, so $g$ is decreasing. Using (2.9) and Theorem 1.3(i), we see that $g(1)=0$, so $g$ is nonnegative on $[0,1]$. Employing (2.9), this means that $f^{\prime}$ is nonnegative on $[1, \infty)$, so $f$ is increasing, which finishes the proof.

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    ${ }^{1}$ Our definition of $K_{s}$, for $s \neq 0$, differs from the more traditional version in that it includes $s$ in its denominator. This has the advantage to produce formulas which are continuous as $s=0$.

