ON THE DERIVATIVES OF THE LIOUVILLE CURRENTS

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ABSTRACT. The Liouville map, introduced by Bonahon, assigns to each point in the Teichmüller space a natural Radon measure on the space of geodesics of the base surface. The Liouville map is real analytic and it even extends to a holomorphic map of a neighborhood of the Teichmüller space in the Quasi-Fuchsian space of an arbitrary conformally hyperbolic Riemann surface. The earthquake paths and by their extension quake-bends, introduced by Thurston, are particularly nice real-analytic and holomorphic paths in the Teichmüller and the Quasi-Fuchsian space, respectively. We find a geometric expression for the derivative of the Liouville map along earthquake paths.

Denote by X an arbitrary conformally hyperbolic Riemann surface. In particular, X can be the upper half-plane \mathbb{H} or a surface with an infinitely generated fundamental group or an infinite area surface with finitely generated fundamental group, or a finite area surface. The universal covering \widetilde{X} of X is isometric to the upper half-plane $\mathbb{H} = \{z = x + iy : y > 0\}$ and X is isometrically equivalent to \mathbb{H}/Γ for a Fuchsian group Γ . The Teichmüller space $\mathcal{T}(X)$ is the space of quasiconformal maps from the base surface X onto variable Riemann surfaces up to isometries and homotopies.

The space of geodesics $G(\widetilde{X})$ of the universal covering \widetilde{X} supports a natural *Liouville* measure which is the unique (up to scalar multiple) measure of full support that is invariant under the isometries of \widetilde{X} which completely determines the Riemann surface X. In general, a $\pi_1(X)$ -invariant Radon measure on $G(\widetilde{X})$ is called a *geodesic current* for X. Bonahon [4] introduced the *Liouville map*

$$\mathcal{L}: \mathfrak{T}(X) \to \mathfrak{G}(X)$$

from the Teichmüller space $\mathfrak{T}(X)$ to the space of geodesic currents $\mathfrak{G}(X)$ that assigns to each quasiconformal deformation $[f: X \to Y] \in \mathfrak{T}(X)$ of the Riemann surface X the pullback of the Liouville measure of Y under the deformation f.

When the Riemann surface X is compact, Bonahon [4] used the Liouville map to introduce an alternative description of the Thurston boundary to $\mathcal{T}(X)$. The second author [18], and more recently, Bonahon and the second author [5] introduced the Thurston boundary to Teichmüller spaces of arbitrary conformally hyperbolic Riemann surfaces. Using the description of the topology on the space of bounded geodesic currents $\mathcal{G}_b(X)$ for a Riemann surface X given in [5], the first two authors introduced a space of bounded Hölder distributions $\mathcal{H}_b(X)$ (see [9] and §2) which contains bounded geodesic currents and they proved that the Liouville map

$$\mathcal{L}: \mathfrak{T}(X) \to \mathcal{H}_b(X)$$

is real analytic. Prior to [9], Bonahon and Sözen [6] proved that the Liouville map is differentiable when X is a compact surface, the second author [18] extended this result to

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all conformally hyperbolic Riemann surfaces. Using Bers' simultaneous uniformization, Otal [17] proved that the Liouville map is real-analytic in the topology introduced in [19] by extending the Liouville map to an open neighborhood of $\mathcal{T}(X)$ inside the QuasiFuchsian space $\mathcal{QF}(X)$, where $\mathcal{T}(X)$ is realized as a totally real analytic submanifold of $\mathcal{QF}(X)$. The space of bounded Hölder distributions $\mathcal{H}_b(X)$ simplifies the description of the topology in [19].

In addition to the Liouville currents, a conformally hyperbolic Riemann surface supports measured laminations. Thurston [23] introduced a natural deformation of Riemann surfaces by left shearing along geodesics of the support of a measured lamination μ with the amount given by the transverse measure called an *earthquake map* $E^{\mu} : X \to X^{\mu}$ (see §3). Thurston [23] proved that any homeomorphic deformation of X onto another Riemann surface can be obtained by a unique earthquake. A quasiconformal deformation of a Riemann surface is obtained by a unique earthquake E^{μ} with *bounded* measured lamination (see [20]). In addition, Miyachi and the second author [16] proved that the natural correspondence between the quasiconformal deformations of X (namely, the Teichmüller space $\mathcal{T}(X)$) and the space of bounded measured lamination is a homeomorphism.

Let $\tilde{\mu}$ be the lift of μ . If $\tilde{\mu}$ is a bounded measured lamination and t > 0, then $t\tilde{\mu}$ is also a bounded measured lamination. Therefore $t \mapsto E^{t\tilde{\mu}}$ is a path in the Teichmüller space called an *earthquake path*. The second author [20] proved that an earthquake path is a real analytic path in the Teichmüller space $\mathcal{T}(X)$. Therefore the composite path

$$t \mapsto \mathcal{L}(E^{t\mu})$$

is a real analytic path in $\mathcal{H}_b(X)$ and we compute its tangent vector.

Theorem 1. Let μ be a bounded measured lamination on a conformally hyperbolic Riemann surface X and let $\mathcal{L} : \mathfrak{I}(X) \to \mathcal{H}_b(X)$ be the Liouville map. The image of the tangent vector $\dot{E}^{\tilde{\mu}} := \frac{d}{dt} E^{t\tilde{\mu}}|_{t=0}$ to the earthquake path $t \mapsto E^{t\tilde{\mu}}$ is given by the formula

$$d\mathcal{L}(\dot{E}^{\widetilde{\mu}})(\xi) = \int_{G(\widetilde{X})} \int_{G(\widetilde{X})} \xi(h) \cos(g, h) dL_{\widetilde{X}}(h) d\widetilde{\mu}(g)$$

where $\tilde{\mu}$ is the lift of μ to \tilde{X} , $h \in G(\tilde{X})$, $g \in G(\tilde{X})$, $\xi : G(\tilde{X}) \to \mathbb{C}$ is a Hölder continuous function with compact support and $L_{\tilde{X}}$ is the Liouville measure on $G(\tilde{X})$.

Each tangent vector at a point of the Teichmüller space $\mathcal{T}(X)$ is obtained by taking the derivative along an earthquake path passing through that point. Moreover, the tangent space at a point of $\mathcal{T}(X)$ is homeomorphic to the space of bounded measured lamination on the surface corresponding to that point (see [16]). Theorem 1 gives an explicit formula in the geometric terms for the tangent map $d\mathcal{L}$ to the Liouville map $\mathcal{L}: \mathcal{T}(X) \to \mathcal{H}_b(X)$.

We call the extension of \mathcal{L} defined by Otal [17] the extended Liouville map \mathcal{L} . The image under $\hat{\mathcal{L}}$ of the points not in $\mathcal{T}(X)$ consists of complex-valued distributions or, rather, finitely additive complex-valued measures. These objects usually require more restrictive analytic settings to be able to integrate against functions and to be able to take their derivatives.

For $0 < \lambda \leq 1$, let $H^{\lambda}(\tilde{X})$ be the space of all Hölder continuous functions $\xi : G(\tilde{X}) \to \mathbb{C}$ with compact support and Hölder exponent λ . Thurston introduced a natural complexification of earthquakes called quake-bends. The naturality of the geometric setting and the holomorphic motions allows us to extend the formula (given by a limit) for the first derivative of the Liouville map to the neighborhood of $\Upsilon(X)$ in $Q\mathcal{F}(X)$ along the quake-bends. **Theorem 2.** Let $\xi \in H^{\lambda}(\widetilde{X})$ and let $\delta > 0$ be the radius for which $\hat{\mathcal{L}}([E^{\tau \widetilde{\mu}}])(\xi)$ is defined. Then, for $\tau \in \mathbb{C}$ with $|\tau| < \delta$,

$$\frac{d}{d\tau}\hat{\mathcal{L}}([E^{\tau\tilde{\mu}}])(\xi) = \lim_{j \to \infty} \int_{G(\tilde{X})} \int_{G(\tilde{X})} \xi(h) \cosh \boldsymbol{d}(E^{\tau\tilde{\mu}|_{K_j}}(g), E^{\tau\tilde{\mu}|_{K_j}}(h)) dL_{[E^{\tau\tilde{\mu}|_{K_j}}]}(h) d\tilde{\mu}|_{K_j}(g)$$

where $\widetilde{\mu}|_{K_i} \to \widetilde{\mu}$ in the weak* topology as $j \to \infty$ (see §4) and **d** is the complex distance.

There is also a nice formula for the second derivative of $\mathcal{L}(E^{t\tilde{\mu}})$ at t = 0. However, there is no corresponding second derivative formula (given by a limit) along quake-bends, see §6.

Theorem 3. Let μ be a bounded measured lamination on a conformally hyperbolic Riemann surface X and let $\mathcal{L} : \mathfrak{T}(X) \to \mathcal{H}_b(X)$ be the Liouville map. Then,

$$\begin{aligned} \frac{d^2}{dt^2} \mathcal{L}([E^{t\widetilde{\mu}}])(\xi)\Big|_{t=0} &= \int_{G(\widetilde{X})} \int_{G(\widetilde{X})} \\ &\left\{ \int_{G(\widetilde{X})} \xi(h) \big[\cos(g,h) \cos(g',h) - \frac{1}{2} \sin(g,h) \sin(g',h) e^{-d_h} \big] dL_{\widetilde{X}}(h) \right\} \\ &d\widetilde{\mu}(g) d\widetilde{\mu}(g') \end{aligned}$$

where $\widetilde{\mu}$ is the lift of μ to \widetilde{X} and d_h is the hyperbolic distance along h from $g \cap h$ to $g' \cap h$.

1. The Teichmüller space

Fix a conformally hyperbolic Riemann surface X of possibly infinite hyperbolic area. The Riemann surface X is identified with \mathbb{H}/Γ , where $\Gamma < PSL_2(\mathbb{R})$ is a Fuchsian group acting on the upper half-plane \mathbb{H} . A quasiconformal map $f : \mathbb{H} \to \mathbb{H}$ is normalized if it fixes 0, 1 and ∞ . Two normalized quasiconformal map $f : \mathbb{H} \to \mathbb{H}$ that conjugate the Fuchsian group Γ onto another Fuchsian group are *Teichmüller equivalent* if they agree on the ideal boundary $\mathbb{R} = \mathbb{R} \cup \{\infty\}$ of the upper half-plane \mathbb{H} . The Teichmüller space $\mathcal{T}(X)$ consists of all Teichmüller equivalence classes [f] of normalized quasiconformal maps conjugating Γ onto another Fuchsian group.

The Beltrami coefficient of a quasiconformal map $f : \mathbb{H} \to \mathbb{H}$ is given by $\mu = \frac{f_{\overline{z}}}{f_z}$ and it satisfies $\|\mu\|_{\infty} < 1$. Conversely, given $\mu \in L^{\infty}(\mathbb{H})$ with $\|\mu\|_{\infty} < 1$ there exists a unique (normalized) quasiconformal map $f : \mathbb{H} \to \mathbb{H}$ that fixes 0, 1 and ∞ whose Beltrami coefficient is μ . A quasiconformal map f conjugates Γ onto another Fuchsian group if and only if $\mu \circ \gamma \frac{\gamma'}{\gamma'} = \mu$ for all $\gamma \in \Gamma$. Two Beltrami coefficients are *Teichmüller equivalent* if the corresponding normalized quasiconformal maps are equal on $\hat{\mathbb{R}}$. Therefore we can define $\mathcal{T}(X)$ to be a set of all Teichmüller classes $[\mu]$ of Beltrami coefficients that satisfy $\mu \circ \gamma(z) \frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z)$ for all $\gamma \in \Gamma$ and $z \in \mathbb{H}$ (for example, see [13]).

A normalized quasiconformal map $f : \mathbb{C} \to \mathbb{C}$ that conjugates $\Gamma < PSL_2(\mathbb{R})$ onto a subgroup of $PSL_2(\mathbb{C})$ represents an element of the quasi-Fuchsian space $\mathfrak{QF}(\Gamma)$. Two normalized quasiconformal maps f and g that conjugate Γ onto a subgroup of $PSL_2(\mathbb{C})$ are *equivalent* if they agree on $\mathbb{R} = \mathbb{R} \cup \{\infty\}$. Denote by $[f] \in \mathfrak{QF}(\Gamma)$ the corresponding equivalence class. Equivalently, we can define $\mathfrak{QF}(\Gamma)$ to consist of all equivalence classes $[\mu]$ of Beltrami coefficients on \mathbb{C} where two Beltrami coefficients are equivalent if their corresponding normalized quasiconformal maps agree on \mathbb{R} . If Γ is trivial then f is a quasiconformal map fixing 0, 1 and ∞ which does not necessarily preserve the upper half-plane \mathbb{H} . When $X = \mathbb{H}/\Gamma$, then we set $\mathfrak{QF}(X) = \mathfrak{QF}(\Gamma)$.

Bers introduced a complex Banach manifold structure to the Teichmüller space $\mathcal{T}(X)$. The complex chart around the basepoint $[0] \in \mathcal{T}(X)$ is obtained as follows. Let $\tilde{\mu}$ be the Beltrami coefficient which equals μ in the upper half-plane \mathbb{H} and equals zero in the lower half-plane \mathbb{H}^- . The solution $f = f^{\tilde{\mu}}$ to the Beltrami equation $f_{\bar{z}} = \tilde{\mu} f_z$ is conformal in the lower half-plane \mathbb{H}^- . The Schwarzian derivative

$$S(f^{\tilde{\mu}})(z) = \frac{(f^{\tilde{\mu}})'''(z)}{(f^{\tilde{\mu}})'(z)} - \frac{3}{2} \left(\frac{(f^{\tilde{\mu}})''(z)}{(f^{\tilde{\mu}})'(z)}\right)^2$$

for $z \in \mathbb{H}^-$ defines a holomorphic function $\varphi(z) = S(f^{\tilde{\mu}})(z)$ which satisfies $(\varphi \circ \gamma)(z)\gamma'(z)^2 = \varphi(z)$ and $\|\varphi\|_b := \sup_{z \in \mathbb{H}^-} |y^2 \varphi(z)| < \infty$, called a *cusped form* for X. The space of all cusped forms $\varphi : \mathbb{H}^- \to \mathbb{C}$ for X is a complex Banach space $\Omega_b(X)$ with the norm $\|\cdot\|_b$ (see [13]).

The Schwarzian derivative maps the unit ball in $L^{\infty}(\mathbb{H})$ onto an open subset of $\mathcal{Q}_b(X)$ and it projects to a homeomorphism Φ from $\mathfrak{T}(X)$ to an open subset of $\mathcal{Q}_b(X)$ containing the origin. The open ball $B_{[0]}(\frac{1}{2}\log 2)$ in $\mathfrak{T}(X)$ of radius $\frac{1}{2}\log 2$ and center [0] maps under Φ onto an open set in $\mathcal{Q}_b(X)$ which contains the ball of radius $\frac{2}{3}$ and is contained in the ball of radius 2 with center $0 \in \mathcal{Q}_b(X)$. The map $\Phi : B_{[0]}(\frac{1}{2}\log 2) \to \mathcal{Q}_b(X)$ is a chart map for the base point $[0] \in \mathfrak{T}(X)$ (see [13, §6]). The Ahlfors-Weill section provides an explicit formula for Φ^{-1} on the ball of radius $\frac{1}{2}$ and center 0 in $\mathcal{Q}_b(X)$. Namely if $\varphi \in \mathcal{Q}_b(X)$ with $\|\varphi\|_b < \frac{1}{2}$ then Ahlfors and Weill prove that $\Phi^{-1}(\varphi) = [-2y^2\varphi(\bar{z})]$ (see [13, §6]). The Beltrami coefficient $\eta_{\varphi}(z) := -2y^2\varphi(\bar{z})$ is said to be *harmonic*. The Ahlfors-Weill formula gives an explicit expression of Beltrami coefficients that are representing points in $\mathfrak{T}(X)$ corresponding to the holomorphic disks $\{t\varphi : |t| < 1, \|\varphi\|_b < \frac{1}{2}\}$ in the chart in $\mathcal{Q}_b(X)$, namely

$$\Phi^{-1}(\{t\varphi : |t| < 1\}) = \{[t\eta_{\varphi}] \in \mathfrak{T}(X) : |t| < 1\}.$$

By the Bers simultaneous uniformization theorem, the Quasi-Fuchsian space $\mathfrak{QF}(X)$ is identified with $\mathfrak{T}(X) \times \mathfrak{T}(\bar{X})$, where \bar{X} is the Riemann surface which is anti-conformal to X. A complex chart for $\mathfrak{QF}(X)$ at the basepoint [0] is the product of two open balls in $\mathfrak{Q}_b(X)$ and in $\mathfrak{Q}_b(\bar{X})$ with centers at the origins. The Teichmüller space $\mathfrak{T}(X)$ embeds as a totally real submanifold of $\mathfrak{QF}(X)$.

2. The Liouville map and uniform Hölder topology

In this section we define the Liouville map, the space of bounded geodesic currents and the space of bounded Hölder distributions for the Riemann surface X (see [4], [5] and [9]).

Recall that the conformally hyperbolic Riemann surface X is identified with \mathbb{H}/Γ , where Γ is a Fuchsian group acting on the upper half-plane \mathbb{H} . The space of oriented geodesics $G(\widetilde{X}) = (\partial_{\infty}\widetilde{X} \times \partial_{\infty}\widetilde{X}) \setminus \text{diagonal}$ is identified with $(\mathbb{R} \times \mathbb{R}) \setminus \text{diagonal}$.

The angle distance d(x, y) for $x, y \in \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ with respect to a reference point $z_0 \in \mathbb{H}$ is the smaller of the two angles between the geodesic rays starting at z_0 and ending at xand y, respectively. The angle distance d depends on the choice of the reference point z_0 . The identity map of $\hat{\mathbb{R}}$ is bi-Lipschitz for any two angle distances given by different choices of the reference points. The angle distance induces the product metric on $G(\tilde{X})$ via the identification with $(\hat{\mathbb{R}} \times \hat{\mathbb{R}}) \setminus \text{diagonal called the angle metric.}$ The identity map on $G(\tilde{X})$ is bi-Lipschitz for any two angle metrics given by two different reference points. It is also possible to isometrically identify \tilde{X} with the unit disk model \mathbb{D} of the hyperbolic plane. In that case $G(\tilde{X})$ is identified with $(S^1 \times S^1) \setminus \text{diagonal}$.

A geodesic current for X is a positive Radon measure on $G(\widetilde{X}) = (\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}) \setminus \text{diagonal}$ that is invariant under the action of the covering group Γ and the change of the orientation of the geodesics (see [4]). The space of geodesic currents for X is denoted by $\mathcal{G}(X)$. The Liouville measure $L_{\widetilde{X}}$ for X is the unique (up to scalar multiple) full support geodesic current that is invariant under the isometries of the universal covering \widetilde{X} and the change of the orientation of the geodesics. More precisely, the Liouville measure of a Borel set $A \subset (\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}) \setminus \text{diagonal} = G(\widetilde{X})$ is given by

$$L_{\widetilde{X}}(A) = \iint_{A} \frac{dxdy}{(x-y)^2}.$$

Given two disjoint intervals [a, b] and [c, d] of $\hat{\mathbb{R}}$, the set of geodesics $A = [a, b] \times [c, d] \subset G(\tilde{X})$ with one endpoint in $[a, b] \subset \hat{\mathbb{R}}$ and another endpoint in $[c, d] \subset \hat{\mathbb{R}}$ is called a *box of geodesics*. Then

$$L_{\widetilde{X}}([a,b] \times [c,d]) = \log cr(a,b,c,d).$$

It follows from the definition that $L_{\widetilde{X}}([a,b] \times [c,d]) = L_{\widetilde{X}}([c,d] \times [a,b]).$

A quasiconformal map $f: X \to X_1$ induces a quasisymmetric map of the ideal boundaries of the universal covers \widetilde{X} and \widetilde{X}_1 of X and X_1 , respectively. The induced map in turn induces a homeomorphism of the space of geodesics $G(\widetilde{X})$ and $G(\widetilde{X}_1)$ which is equivariant for the actions of the covering groups. The Teichmüller equivalence class $[f: X \to X_1]$ induces the pull-back of the Liouville measure $L_{\widetilde{X}_1}$ to the space of geodesics $G(\widetilde{X})$ denoted by $L_{[f]}$ under the induced homeomorphism of $G(\widetilde{X})$ and $G(\widetilde{X}_1)$. In general, $L_{[f]}$ is invariant under $\pi_1(X)$ but not invariant under all isometries of \widetilde{X} and thus different from $L_{\widetilde{X}}$. In fact, $L_{[f]}$ completely recovers the Riemann surface X_1 and the Teichmüller class of $f: X \to X_1$.

Bonahon [4] introduced the *Liouville map*

$$\mathcal{L}: \mathfrak{T}(X) \to \mathfrak{G}(X)$$

by

$$\mathcal{L}([f]) = L_{[f]}.$$

and he used the Liouville map in order to give an alternative description of the Thurston boundary to the Teichmüller space of a compact surface. When X is a compact surface the space of geodesic currents $\mathcal{G}(X)$ is equipped with the standard weak* topology for which the Liouville map is an embedding onto its image (see Bonahon [4]). Bonahon and the second author [5] introduced the *uniform weak* topology* on the space of geodesic currents in order to introduce a Thurston boundary to Teichmüller spaces of arbitrary Riemann surfaces. This is a simplification of the topology that was introduced by the second author [18].

Let H(X) be the space of all Hölder continuous functions $\xi : G(X) \to \mathbb{C}$ with respect to the product metric on $G(\widetilde{X})$ that are of compact support. A linear functional $\mathbf{W} : H(\widetilde{X}) \to \mathbb{C}$ is said to be *bounded* if, for every $\xi \in H(\widetilde{X})$,

$$\|\mathbf{W}\|_{\xi} := \sup_{\gamma \in PSL_2(\mathbb{R})} |\mathbf{W}(\xi \circ \gamma)| < \infty.$$

The space of bounded Hölder distributions $\mathcal{H}_b(X)$ is the space of all bounded complex linear functionals on the space of Hölder continuous functions $H(\tilde{X})$ with compact support in $G(\tilde{X})$.

Since the space of bounded geodesic currents is a subset of the space of bounded Hölder distributions $\mathcal{H}_b(X)$, we can consider the Liouville map

$$\mathcal{L}: \mathfrak{T}(X) \to \mathcal{H}_b(X).$$

The Liouville map is a homeomorphisms onto its image in $\mathcal{H}_b(X)$ because continuous functions are well approximated by Hölder continuous functions (see [8, Theorem 4.2.1]).

The first two authors [9] complexified the Liouville map and proved that the complexification is holomorphic. For a fixed $0 < \lambda \leq 1$, let $\mathcal{H}_b^{\lambda}(X)$ be the space of complex linear functional on the space $H^{\lambda}(\widetilde{X})$ of λ -Hölder continuous functions with compact support that are bounded (for the semi-norms given by the λ -Hölder continuous functions with compact support). Given $\delta > 0$, define \mathcal{V}_{δ} to be the set of all $[\mu] \in \mathcal{QF}(X)$ with $\|\mu\|_{\infty} < \delta$. Then (see [9, Theorem 7 and 8])

Theorem 4. For a fixed $0 < \lambda \leq 1$, there exists $\delta = \delta(\lambda) > 0$ such that the Liouville map $\mathcal{L} : \mathfrak{T}(X) \to \mathcal{H}_b(X)$ extends to a holomorphic map

$$\hat{\mathcal{L}}: \mathcal{V}_{\delta} \to \mathcal{H}_{b}^{\lambda}(X).$$

3. Earthquakes, quake-bends and tangent vectors

Earthquakes are geometrically natural deformations of conformally hyperbolic Riemann surfaces. Thurston first introduced earthquakes on compact hyperbolic surfaces as completions of sequences of real positive twists along longer and longer geodesics (see Kerckhoff [15] and Thurston [23]). Later on, Thurston [24] defined earthquakes on the hyperbolic plane and using lifts to the universal covering he extended the definition of earthquakes to any conformally hyperbolic Riemann surface.

A geodesic lamination on X is a closed subset of X with an assigned foliation by complete geodesics. When X is of finite hyperbolic area any geodesic lamination of X has zero area and its foliation is unique (see [23]). This is also true for infinite area hyperbolic surfaces whose covering group Γ is of the first kind (see [21, Proposition 3.1]). However, in general we need to include the foliation in the definition of a geodesic lamination. For example, the hyperbolic plane \mathbb{H} can be foliated by complete simple geodesics in infinitely many ways.

A measured lamination μ on X is a geodesic lamination $|\mu|$ together with an assignment of a positive Radon measure to each geodesic arc I transverse to $|\mu|$ such that the measure is supported on $I \cap |\mu|$ and the assignment is invariant under homotopies relative the leaves of $|\mu|$. Equivalently, we can define a measured lamination μ on X to be a geodesic current $\tilde{\mu} \in \mathcal{G}(X)$ for X whose support $|\tilde{\mu}|$ is a geodesic lamination on \mathbb{H} (which is necessarily invariant under $\pi_1(X)$). To be precise, we take geodesics of the support $|\mu|$ with both orientations and make the measure invariant under the change of orientation.

An earthquake map of X is defined using a measured lamination μ on X. We define the earthquake map $E^{\tilde{\mu}} : \mathbb{H} \to \mathbb{H}$ using the lift $\tilde{\mu}$ of the measured lamination μ to the universal covering $\tilde{X} = \mathbb{H}$. It is necessarily true that the measured lamination $\tilde{\mu}$ is Γ invariant and is supported on complete geodesics of \mathbb{H} .

We first define a simple earthquake E_g^{δ} supported on a single geodesic $g \in G(\widetilde{X}) = G(\mathbb{H})$ corresponding to a measured lamination $\delta \mathbf{1}_g + \delta \mathbf{1}_{\widetilde{g}}$, where $\delta > 0$, g and \widetilde{g} have opposite orientations, and $\mathbf{1}_g$ is a Dirac measure on $G(\widetilde{X})$ with support g-i.e., $\mathbf{1}_g(h) = 0$ for $h \neq g$ and $\mathbf{1}_g(g) = 1$. Assign an orientation to g. The oriented geodesic g divides \mathbb{H} into the left and right geodesic half-planes. Define $E_g^{\delta} : \mathbb{H} \to \mathbb{H}$ to be the identity on the left geodesic half-plane and to be the hyperbolic translation with the oriented axis g and the translation length δ on the right geodesic half-plane.

To define $E_g^{\delta}: \mathfrak{T}(X) \to \mathfrak{T}(X)$, we set $E_g^{\delta}([id]) := E_g^{\delta}$. For a quasisymmetric boundary map $\tilde{f}: \hat{\mathbb{R}} \to \hat{\mathbb{R}}$ corresponding to $[f] \in \mathfrak{T}(X)$, we set $E_g^{\delta}([\tilde{f}]) := E_{\tilde{f}(g)}^{\delta}$ where $\tilde{f}(g)$ is the geodesic of \mathbb{H} whose endpoints are the image of endpoints of g under \tilde{f} . This defines a simple earthquake on the whole Teichmüller space $\mathfrak{T}(X)$ (for example, see [5]).

Let $\{g_1, \tilde{g}_1, \ldots, g_n, \tilde{g}_n\}$ be a finite set of pairwise disjoint geodesics in \mathbb{H} where g_i and \tilde{g}_i have opposite orientations. Let $\{\delta_1, \ldots, \delta_n\}$ be positive real numbers. Consider a measured lamination $\sigma = \sum_{i=1}^n \delta_i (\mathbf{1}_{g_i} + \mathbf{1}_{\tilde{g}_i})$ and define an *elementary earthquake* with measured lamination σ

$$E^{\sigma}: \mathfrak{T}(X) \to \mathfrak{T}(X)$$

by (see [5])

$$[\widetilde{f}] \mapsto E_{g_1}^{\delta_1} \circ E_{g_2}^{\delta_2} \circ \cdots \circ E_{g_n}^{\delta_n}([\widetilde{f}]).$$

We note that the order of the terms $E_{g_i}^{\delta_i}$ in the above definition is irrelevant due to the fact that we defined $E_g^{\delta}([\tilde{f}]) := E_{\tilde{f}(g)}^{\delta}$ (see [5]). Since the support of σ is finite, the elementary earthquake extends to a homeomorphism of $\hat{\mathbb{R}}$.

Let $\tilde{\mu}$ be a measured lamination on \mathbb{H} that is the lift of a measured lamination μ on X. A stratum of the geodesic lamination $|\tilde{\mu}|$ is either a geodesic of $|\tilde{\mu}|$ or a connected component of its complement. The upper half-plane \mathbb{H} is partitioned by strata of $|\tilde{\mu}|$ and we define the earthquake map $E^{\tilde{\mu}} : \mathbb{H} \to \mathbb{H}$ by assigning a hyperbolic isometry on each stratum. The measured lamination $\tilde{\mu}$ can be obtained as a limit in the weak* topology of a sequence of finitely supported measured laminations δ_n . For example, one can choose finitely many geodesics of the support of $\tilde{\mu}$ and assign to each of them a positive weight of all nearby geodesics (see [24], or [16]). Then, on each stratum of $|\tilde{\mu}|$, the limit of elementary earthquakes E^{δ_n} as $n \to \infty$ exists and the earthquake $E^{\tilde{\mu}}$ restricted to the stratum is defined to be the limit (see [24] and [10]).

It turns out that $E^{\tilde{\mu}}$ does not always extend to a homeomorphism of the ideal boundary $\hat{\mathbb{R}}$ (see [24], [14]). Thurston [24] showed that $E^{\tilde{\mu}}$ is, up to post-composition by an isometry, uniquely determined by $\tilde{\mu}$. More importantly, Thurston [24] proved that every orientation preserving homeomorphism of the ideal boundary $\hat{\mathbb{R}}$ can be obtained by the continuous extension of an earthquake map $E^{\tilde{\mu}} : \mathbb{H} \to \mathbb{H}$. It is an open problem to classify measured laminations whose earthquakes give rise to homeomorphisms of $\hat{\mathbb{R}}$.

For the considerations involving $\mathcal{T}(X)$, it is important to classify which measured laminations give rise to earthquakes whose continuous extensions to $\hat{\mathbb{R}}$ are quasisymmetric maps. The *Thurston norm* of a measured lamination $\tilde{\mu}$ on \mathbb{H} is given by

$$\|\widetilde{\mu}\|_{Th} := \sup \widetilde{\mu}(I)$$

where the supremum is over all geodesic arcs I of length 1 that are transverse to $\tilde{\mu}$. The quantity $\tilde{\mu}(I)$ is the $\tilde{\mu}$ -measure of the geodesics in \mathbb{H} that intersect I. The second author [18], Gardiner-Hu-Lakic [14] and Epstein-Marden-Markovic [11] gave different proofs of the

fact that $E^{\tilde{\mu}}$ extends to a quasisymmetric map of $\hat{\mathbb{R}}$ if and only if $\|\tilde{\mu}\|_{Th} < \infty$. We note that $\|\tilde{\mu}\|_{Th} < \infty$ is equivalent to $\tilde{\mu} \in \mathcal{H}_b(\mathbb{H})$.

When t > 0 and $\|\tilde{\mu}\|_{Th} < \infty$, the measured lamination $t\tilde{\mu}$ has finite Thurston norm and $t \mapsto E^{t\tilde{\mu}}([id])$ is a path in $\mathfrak{T}(X)$ through the basepoint [id], called an *earthquake path*. The second author [18] proved that there exists a neighborhood $V(\|\tilde{\mu}\|_{Th})$ of the real axis \mathbb{R} in the complex plane \mathbb{C} such that the real earthquake path $t \mapsto E^{t\tilde{\mu}}$ extends to a *complex earthquake*

$$\tau \mapsto E^{\tau \tilde{\mu}}([id])$$

that is a holomorphic map from $V(\|\tilde{\mu}\|_{Th})$ into $\mathfrak{QF}(X)$. This implies that the real earthquake path $t \mapsto E^{t\tilde{\mu}}$ is a real analytic path in the Teichmüller space $\mathfrak{T}(X)$.

4. The derivatives of all orders of the Liouville distributions along QUAKE-BEND PATHS

Let $\tilde{\mu}$ be the lift to \tilde{X} of a bounded measured lamination μ on X. Then $\tilde{\mu}$ is a bounded geodesic current for X. Let $\tilde{\mu}_n^d$ be a sequence of measured laminations on \tilde{X} with discrete support that converges to $\tilde{\mu}$ in the uniform weak* topology on $\mathcal{G}(\tilde{X})$ as constructed in [16]. The support of each $\tilde{\mu}_n^d$ is a subset of the support of $\tilde{\mu}$ and the Thurston norm $\|\tilde{\mu}_n^d\|_{Th}$ of the sequence $\tilde{\mu}_n^d$ is bounded above by a constant that depends only on $\|\tilde{\mu}\|_{Th}$.

Let $\{K_j\}_{j=1}^{\infty}$ be an exhaustion of $G(\widetilde{X})$ by compact sets such that $\widetilde{\mu}(\partial K_j) = 0$ for all j. For example, we can take K_j to be the set of all geodesics of $\widetilde{X} \equiv \mathbb{D}$ that intersect a Euclidean disk of radius r_j centered at the origin with $r_j \to 1$ as $j \to \infty$. The boundary ∂K_j consists of all geodesics that intersect the circle of radius r_j centered at the origin. Since we have uncountably many circles centered at the origin, there is a choice of r_j such that $\widetilde{\mu}(\partial K_j) = 0$.

We define $\tilde{\mu}_{n,j}$ to be the restriction of $\tilde{\mu}_n^d$ to K_j . Then $\tilde{\mu}_{n,j}$ are measured laminations with finite support such that $\tilde{\mu}_{n,j} \to \tilde{\mu}|_{K_j}$ in the weak* topology as $n \to \infty$ for each j. To see this, fix $\varepsilon > 0$. There exists $\varepsilon_j > 0$ small enough, such that the $\tilde{\mu}$ -measure of the set of geodesics E_j intersecting $\{r_j - \varepsilon_j \leq |z| \leq r_j + \varepsilon_j\}$ is less than ε and that $\tilde{\mu}_n^d(E_j) \to \tilde{\mu}(E_j) < \varepsilon$ (see [5, Lemma 6]). Let $\xi : G(\tilde{X}) \to \mathbb{C}$ be an arbitrary continuous function with a compact support. Let $\xi_0 : G(\tilde{X}) \to [0, 1]$ be a continuous function which is constantly equal to 1 on the set of geodesics E_j that intersect $\{|z| \leq r_j\}$ and that is equal to zero on the set of geodesics that do not intersect $\{|z| < r_j + \varepsilon_j\}$. Then we have

$$\left|\int_{G(\tilde{X})} \xi d(\widetilde{\mu}_{n,j} - \widetilde{\mu}|_{K_j})\right| \leq \left|\int_{G(\tilde{X})} \xi_0 \xi d(\widetilde{\mu}_n^d - \widetilde{\mu})\right| + 3\varepsilon$$

for n large enough. By letting $n \to \infty$ and by the convergence of $\widetilde{\mu}_n^d$ to $\widetilde{\mu}$, we conclude that

$$\lim_{n \to \infty} \left| \int_{G(\tilde{X})} \xi d(\tilde{\mu}_{n,j} - \tilde{\mu}|_{K_j}) \right| \leq 3\varepsilon.$$

Since ε was arbitrary, we conclude that $\widetilde{\mu}_{n,j}$ converge to $\widetilde{\mu}|_{K_j}$ in the weak* topology as $n \to \infty$.

By the main result in [20], there exists $\delta_1 = \delta_1(\|\mu\|_{\infty}) > 0$ such that the complex earthquakes (or quake-bends) $E^{\tau\tilde{\mu}}$, $E^{\tau\tilde{\mu}|_{K_j}}$ and $E^{\tau\tilde{\mu}_{n,j}}$ are well-defined for all $\tau \in \mathbb{C}$ with $|\tau| < \delta_1$, and they induces holomorphic maps from $\{|\tau| < \delta_1\}$ into the Quasi-Fuchsian space $\mathfrak{QF}(\mathbb{H}) \supset \mathfrak{QF}(X)$.

Given $0 < \lambda \leq 1$, by [9, Theorem 7] there is $\delta_2(\lambda) > 0$ such that the Liouville map

$$\mathcal{L}: \mathfrak{T}(X) \to \mathfrak{G}_b(X)$$

extends to a holomorphic map

 $\hat{\mathcal{L}}: \mathcal{V}_{\delta_2} \to \mathcal{H}_b^\lambda(X)$

where $\mathcal{V}_{\delta_2} = \{[\sigma] : \sigma \in L^{\infty}(\mathbb{C}) \text{ and } \|\widetilde{\sigma}\|_{\infty} < \delta_2\}$ is an open neighborhood in $\mathfrak{QF}(X)$ of the base point of the Teichmüller space $\mathfrak{T}(X)$. For some $\delta > 0$ small enough, we have that $[E^{\tau \widetilde{\mu}}]$, $[E^{\tau \widetilde{\mu}|_{K_j}}]$ and $[E^{\tau \widetilde{\mu}_{n,j}}]$ are in \mathcal{V}_{δ_2} . In particular, for any $\xi \in H^{\lambda}(\widetilde{X})$ we have that

$$\tau \mapsto \hat{\mathcal{L}}([E^{\tau \widetilde{\mu}}])(\xi),$$
$$\tau \mapsto \hat{\mathcal{L}}([E^{\tau \widetilde{\mu}|_{K_j}}])(\xi)$$

and

 $\tau \mapsto \hat{\mathcal{L}}([E^{\tau \widetilde{\mu}_{n,j}}])(\xi)$

are holomorphic maps from $\{|\tau| < \delta\}$ into the complex plane \mathbb{C} . By changing the basepoint, similar statements hold for all points of $\mathcal{T}(X)$.

By the construction of the quake-bends (and earthquakes) in [10], we have that $E^{\tau \tilde{\mu}_{n,j}}(x) \to E^{\tau \tilde{\mu}|_{K_j}}(x)$ as $n \to \infty$ for each $x \in \mathbb{R}$ and the convergence is uniform for x in a compact subset of \mathbb{R} and all $|\tau| < \delta$. It follows that, for each τ with $|\tau| < \delta$,

$$\hat{\mathcal{L}}([E^{\tau\widetilde{\mu}_{n,j}}])(\xi) \to \hat{\mathcal{L}}([E^{\tau\widetilde{\mu}|_{K_j}}])(\xi)$$

as $n \to \infty$. Since the functions $\tau \mapsto \hat{\mathcal{L}}([E^{\tau \tilde{\mu}_{n,j}}])(\xi)$ and $\tau \mapsto \hat{\mathcal{L}}([E^{\tau \tilde{\mu}|_{K_j}}])(\xi)$ are holomorphic we get

$$\frac{d^k}{d\tau^k} \hat{\mathcal{L}}([E^{\tau \widetilde{\mu}_{n,j}}])(\xi) \to \frac{d^k}{d\tau^k} \hat{\mathcal{L}}([E^{\tau \widetilde{\mu}|_{K_j}}])(\xi)$$

as $n \to \infty$ for all $k \ge 1$ and all $j \ge 1$.

We establish a formula for the computation of the derivatives $\frac{d^k}{d\tau^k} \hat{\mathcal{L}}([E^{\tau \tilde{\mu}}])(\xi)$ of all orders using the approximation of $\tilde{\mu}$ with $\tilde{\mu}_{n,j}$.

Theorem 5. Let μ be a bounded measured lamination on X and $\tilde{\mu}$ its lift to the universal covering \widetilde{X} . Let $0 < \lambda \leq 1$ be fixed and $\delta > 0$ be chosen as above depending on λ . Then, for the sequence $\{\widetilde{\mu}_{n,j}\}_{n,j=1}^{\infty}$ defined above, for any $\xi \in H^{\lambda}(\widetilde{X})$ and for any $k \geq 1$ we have

$$\frac{d^k}{d\tau^k}\hat{\mathcal{L}}([E^{\tau\widetilde{\mu}}])(\xi) = \lim_{j \to \infty} \Big[\lim_{n \to \infty} \frac{d^k}{d\tau^k}\hat{\mathcal{L}}([E^{\tau\widetilde{\mu}_{n,j}}])(\xi)\Big],$$

for all $\tau \in \mathbb{C}$ with $|\tau| < \delta$.

Proof. Since $\widetilde{\mu}_{n,j} \to \widetilde{\mu}|_{K_j}$ in the weak* topology, it follows that

$$\hat{\mathcal{L}}([E^{\tau\widetilde{\mu}_{n,j}}])(\xi) \to \hat{\mathcal{L}}([E^{\tau\widetilde{\mu}|_{K_j}}])(\xi)$$

as $n \to \infty$ for all j because $E^{\tau \tilde{\mu}_{n,j}}|_{\mathbb{R}} \to E^{\tau \tilde{\mu}|_{K_j}}|_{\mathbb{R}}$ and the definition of $\hat{\mathcal{L}}$. The above functions are belowerphic in τ which implies that

The above functions are holomorphic in τ which implies that

$$\frac{d^k}{d\tau^k} \hat{\mathcal{L}}([E^{\tau \widetilde{\mu}_{n,j}}])(\xi) \to \frac{d^k}{d\tau^k} \hat{\mathcal{L}}([E^{\tau \widetilde{\mu}|_{K_j}}])(\xi)$$

as $n \to \infty$ for all j and k.

To finish the proof it remains to prove that $\frac{d^k}{d\tau^k} \hat{\mathcal{L}}([E^{\tau \widetilde{\mu}|_{K_j}}])(\xi) \to \frac{d^k}{d\tau^k} \hat{\mathcal{L}}([E^{\tau \widetilde{\mu}}])(\xi)$ as $j \to \infty$. Since $\hat{\mathcal{L}}$ is holomorphic, it is enough to prove that

(1)
$$\lim_{j \to \infty} \hat{\mathcal{L}}([E^{\tau \tilde{\mu}|_{K_j}}])(\xi) = \hat{\mathcal{L}}([E^{\tau \tilde{\mu}}])(\xi)$$

for all $|\tau| < \delta$.

Using the partition of unity, it is enough to prove (1) under the assumption that the support of ξ is in a box of geodesics $[a, b] \times [c, d]$, where $[a, b], [c, d] \subset \mathbb{R}$ with $[a, b] \cap [c, d] = \emptyset$. The angle metric on $[a, b] \times [c, d]$ is biLipschitz to the Euclidean metric. By the construction of the holomorphic motion $E^{\tau \tilde{\mu}}$ restricted to the real line \mathbb{R} in [20], it follows that $E^{\tau \tilde{\mu}|_{K_j}}$ converges to $E^{\tau \tilde{\mu}}$ as $j \to \infty$ uniformly on the compact subsets of \mathbb{R} for the Euclidean metric.

Divide the box of geodesics $[a, b] \times [c, d]$ into 4^n sub-boxes $\{[a_{s-1}, a_s] \times [c_{t-1}, c_t]\}_{s,t=1}^{2^n}$ with disjoint interiors whose Liouville measures $L_{\widetilde{X}}([a_{s-1}, a_s] \times [c_{t-1}, c_t])$ are of the order 4^{-n} (see [9]). Let

$$I_n := \sum_{s,t=1}^{2^n} \xi(a_s, c_t) \log cr(E^{\tau \tilde{\mu}}(a_{s-1}, a_s, c_{t-1}, c_t)),$$
$$I_n^j := \sum_{s,t=1}^{2^n} \xi(a_s, c_t) \log cr(E^{\tau \tilde{\mu}|_{K_j}}(a_{s-1}, a_s, c_{t-1}, c_t))$$

and recall that (see [9, Lemma 6])

$$\hat{\mathcal{L}}([E^{\tau \widetilde{\mu}}])(\xi) = I_{n_0} + \sum_{n=n_0}^{\infty} (I_{n+1} - I_n),$$
$$\hat{\mathcal{L}}([E^{\tau \widetilde{\mu}|_{K_j}}])(\xi) = I_{n_0}^j + \sum_{n=n_0}^{\infty} (I_{n+1}^j - I_n^j).$$

By [9], the sums $\sum_{n=n_0}^{\infty} |I_{n+1} - I_n|$ and $\sum_{n=n_0}^{\infty} |I_{n+1}^j - I_n^j|$ are arbitrary small for n_0 sufficiently large and for all j. The pointwise convergence of $E^{\tau \tilde{\mu}|_{K_j}}$ to $E^{\tau \tilde{\mu}}$ implies that $I_{n_0}^j \to I_{n_0}$. We conclude that (1) holds and the proof is finished.

5. A Geometric formula for the first derivative of the Liouville measure

In this section, we use Theorem 5 to derive a geometric formula for the first derivative of the Liouville distributions along the earthquake and quake-bend paths.

Let $\widetilde{\mu}_n = \sum_{i=1}^{s_n} \delta_i(\mathbf{1}_{g_i} + \mathbf{1}_{\widetilde{g}_i})$, where $\mathbf{1}_{g_i}$ is the Dirac measure on $G(\widetilde{X})$ with support $\{g_i\}$. Define

$$h_{i,k}(\tau) = \frac{d^k}{d\tau^k} \hat{\mathcal{L}}([E_{g_i}^{\tau\delta_i}])(\xi)$$

and note that

$$\frac{d^k}{d\tau^k} \hat{\mathcal{L}}([E^{\tau\tilde{\mu}_n}])(\xi) = \sum_{i=1}^{s_n} h_{i,k}(\tau)$$

Denote by $V_i = \frac{d}{dt} E_{g_i}^t|_{t=0}$ the tangent vector to the simple earthquake path $t \mapsto E_{g_i}^t$ at the point t = 0. Then we have (for example, see [16])

$$\frac{d}{dt}E^{t\widetilde{\mu}_n}\Big|_{t=0} = \sum_{i=1}^{s_n} \delta_i V_i.$$

We give a formula for the derivatives along a simple earthquake $t \mapsto E_g^t$ of the Liouville map evaluated at $\xi \in H(\widetilde{X})$.

In order to find a geometric formula for the first derivative $\frac{d}{dt}\mathcal{L}([E^{t\tilde{\mu}}])(\xi)$ along an earthquake $t \mapsto E^{t\tilde{\mu}}$ for $t \in \mathbb{R}$, we first need the derivative along a simple earthquake path $t \mapsto E_g^{ta}$, where a > 0 and $g \in G(\widetilde{X})$. For $h \in G(\widetilde{X})$, we define $\cos(g, h)$ to be the cosine of the angle from g to h if they intersect and to be zero if h and g do not intersect. If g is the positive y-axis and h is the geodesic from x to y, then $\cos(g, h) = \frac{-x-y}{x-y}$. The following lemma was established by Bonahon and Sözen [6].

Lemma 6 (see [6]). Let $\xi \in H(\widetilde{X})$ and $g \in G(\widetilde{X})$ be a fixed geodesic. Then, for $t \in \mathbb{R}$ and $\omega > 0$,

$$\frac{d}{dt}\mathcal{L}([E_g^{t\omega}])(\xi)\Big|_{t=0} = \omega \int_{G(\widetilde{X})} \xi(h) \cos(g,h) dL_{\widetilde{X}}(h).$$

For a proof, see Lemma 14 where we prove a generalization of the above lemma.

Let $\widetilde{\mu}_{n,j} = \sum_{i=1}^{p(n,j)} \omega_i(\mathbf{1}_{g_i} + \mathbf{1}_{\widetilde{g}_i})$. Denote by $V_i = \frac{d}{dt} E_{g_i}^t|_{t=0}$ the tangent vector to the simple earthquake path $t \mapsto E_{g_i}^t$ at the point t = 0. Then we have (for example, see [16])

$$\frac{d}{dt} E^{t\widetilde{\mu}_{n,j}} \Big|_{t=0} = \sum_{i=1}^{p(n,j)} \omega_i V_i.$$

Since $d\mathcal{L}: T_{[id]}\mathcal{T}(\widetilde{X}) \to \mathcal{H}_b(\widetilde{X})$ is linear, by Lemma 6, we get that

(2)
$$d\mathcal{L}\left(\frac{d}{dt}E^{t\tilde{\mu}_{n,j}}|_{t=0}\right)(\xi) = \sum_{i=1}^{p(n,j)} \omega_i \int_{G(\tilde{X})} \xi(h) \cos(g_i, h) dL_{\tilde{X}}(h)$$
$$= \int_{G(\tilde{X})} \int_{G(\tilde{X})} \xi(h) \cos(g, h) dL_{\tilde{X}}(h) d\tilde{\mu}_{n,j}(g).$$

Remark 7. We claim that $g \mapsto \int_{G(\widetilde{X})} \xi(h) \cos(g, h) dL_{\widetilde{X}}(h)$ is a continuous function in g. It is enough to assume that $Supp(\xi) = [a, b] \times [c, d]$ is a box of geodesics. To see this, let $g, g' \in G(\widetilde{X})$ where $g \neq g'$ and suppose that $g \in G(\widetilde{X})$ has endpoints s and t. We define the closure of δ -neighborhood of g to be the box $\overline{B_{\delta}(g)} = [s - \delta, s + \delta] \times [t - \delta, t + \delta]$. We say that $g' \to g$ if $g' \in \overline{B_{\delta}(g)}$ as $\delta \to 0$. For all $h \in Supp(\xi)$ with no endpoints in $\overline{B_{\delta}(g)}$, we have $\int_{G(\widetilde{X})} |\xi(h)|| \cos(g', h) - \cos(g, h)| dL_{\widetilde{X}}(h) \to 0$ as $g' \to g$ since $|\cos(g', h) - \cos(g, h)| \to 0$ uniformly as $g' \to g$. For all $h \in Supp(\xi)$ with at least one endpoint in $\overline{B_{\delta}(g)}$, we also have $\int_{G(\widetilde{X})} |\xi(h)|| \cos(g', h) - \cos(g, h)| dL_{\widetilde{X}}(h) \to 0$ as $g' \to g$ since the Liouville measure $L([s - \delta, s + \delta] \times [c, d]) = L([a, b] \times [t - \delta, t + \delta]) \to 0$ as $\delta \to 0$.

Since $\widetilde{\mu}_{n,j}$ converges in the weak^{*} topology to $\widetilde{\mu}|_{K_j}$ as $n \to \infty$, it follows that

$$\lim_{n \to \infty} \int_{G(\widetilde{X})} \int_{G(\widetilde{X})} \xi(h) \cos(g, h) dL_{\widetilde{X}}(h) d\widetilde{\mu}_{n,j}(g) = \int_{G(\widetilde{X})} \int_{G(\widetilde{X})} \xi(h) \cos(g, h) dL_{\widetilde{X}}(h) d\widetilde{\mu}|_{K_j}(g).$$

By Theorem 5, we have that

$$\frac{d}{dt}\mathcal{L}([E^{t\widetilde{\mu}}])(\xi)\Big|_{t=0} = \lim_{j \to \infty} \int_{G(\widetilde{X})} \int_{G(\widetilde{X})} \xi(h) \cos(g,h) dL_{\widetilde{X}}(h) d\widetilde{\mu}|_{K_j}(g).$$

We point out that even though the limit on the right-hand side of the above equation exists and that $\tilde{\mu}|_{K_j} \to \tilde{\mu}$ in the weak* topology as $j \to \infty$, we are not guaranteed that the limit is the Lebesgue integral with respect to $\tilde{\mu}$ because $\int_{G(\tilde{X})} \xi(h) \cos(g, h) dL_{\tilde{X}}(h)$ is neither a positive function nor of compact support. In order to prove the convergence toward the integral with respect to the measure $\tilde{\mu}$, we need the following lemma proved by Bonahon and Sözen [6]. We include a proof for the reader's convenience.

Lemma 8. Fix an isometric identification of \widetilde{X} with the unit disk \mathbb{D} . Let $\xi : G(\mathbb{D}) \to \mathbb{R}$ be a λ -Hölder continuous function whose support is in a box of geodesics $[a,b] \times [c,d] \subset$ $S^1 \times S^1 \setminus \text{diag.}$ Let $g \in G(\mathbb{D})$ be a geodesic with at least one endpoint in $[a,b] \cup [c,d]$. Then

$$\left|\int_{G(\mathbb{D})} \xi(h) \cos(g, h) dL_{\mathbb{D}}(h)\right| \leq \int_{G(\mathbb{D})} \left|\xi(h) \cos(g, h)\right| dL_{\mathbb{D}}(h) \leq C e^{-(1+\lambda)d_g}$$

where $d_g \ge 0$ is the hyperbolic distance between $0 \in \mathbb{D}$ and the geodesic g.

Proof. To show that $\left|\int_{G(\mathbb{D})} \xi(h) \cos(g, h) dL_{\mathbb{D}}(h)\right|$ is bounded, it is enough to show that for fixed ξ the integral

$$I(y) = 2\int_{s}^{t} \xi(h(x,y)) \cos(g,h(x,y)) \frac{dx}{|x-y|^{2}}$$

is bounded for every $y \in (a, b)$. The factor 2 is here to avoid dragging cumbersome constants in the computation below.



FIGURE 1. Earthquake along g and Möbius map ψ .

For ease of computation, we switch to the upper half-plane \mathbb{H} such that origin is sent to i and y is sent to ∞ by the Möbius map $\psi : \mathbb{D} \to \mathbb{H}$ given by $\psi(z) = \frac{i(y+z)}{y-z}$. A simple calculation shows that the infinitesimal form $\frac{dx}{|x-y|^2}$ is sent to $\frac{1}{2}dx'$.

Then we have

$$I(\infty) = \int_{s'}^{t'} \xi \circ \psi^{-1}(h(x',\infty)) \cos(\psi(g), h(x',\infty)) dx'.$$

To simplify the notation, let $\xi'(x') = \xi \circ \psi^{-1}(h(x',\infty))$. Note that there exists some $r' \in (s',t')$ such that $\xi'(r') = 0$. That is, $(r',\infty) \notin Supp(\xi \circ \psi^{-1})$, see Figure 1.

Estimated from above, we have

$$|I(\infty)| = \left| \int_{s'}^{t'} \xi'(x') \cos(\psi(g), h(x', \infty)) dx' \right| \leq \left\| \xi'(x') \right\|_{\infty} |t' - s'|$$

Note that ξ' is Hölder continuous with Hölder exponent λ since $\xi \circ \psi^{-1}$ is. By definition of λ -Hölder continuity, we have $\frac{\left|\xi'(x') - \xi'(r')\right|}{|x' - r'|^{\lambda}} \leq \|\xi'\|_{\lambda}$, then

$$|\xi'(x')| = |\xi'(x') - \xi'(r')| \le ||\xi'||_{\lambda} |x' - r'|^{\lambda}$$

for all x' which implies that $\|\xi'(x')\|_{\infty} \leq \|\xi'\|_{\lambda} |x' - r'|^{\lambda}$. Hence,

$$|I(\infty)| \leq ||\xi'||_{\lambda} |x' - r'|^{\lambda} |t' - s'|$$
$$\leq ||\xi'||_{\lambda} |t' - s'|^{1+\lambda}.$$



FIGURE 2. Orthogonal projections of i and ∞ to $\psi(g)$.

And the rest of the proof follows from [6] where it gives the estimate $|t'-s'| \leq C \cdot e^{-d_g}$. To see this, we suppose that $I(\infty) \neq 0$ and $d_g = d_{\mathbb{D}}(0,g) = d_{\mathbb{H}}(i,\psi(g)) = d_{\mathbb{H}}(i,p) \geq R$ for some constant R. By our assumption, s' and t' stay in a closed interval [-R', R'] where R' depends only on R, see Figure 2. Consider a point q which is the orthogonal projection of the point ∞ to $\psi(g)$, namely $q = \frac{t'+s'}{2} + \frac{t'-s'}{2}i$. Also consider the horocycle centered at ∞ passing through i, namely the Euclidean horizontal line passing through i. Let q' be the point of this horocycle which lies on the same vertical line as q, namely $q' = \frac{t'+s'}{2} + i$. Note that the piece of horocycle joining i and q' has length $|\frac{t'+s'}{2}| \leq R'$. It follows that $d_{\mathbb{H}}(i,q') \leq R'$ since (i,q') is a geodesic. Thus,

$$d_{\mathbb{H}}(i,p) = d_{\mathbb{H}}(i,\psi(g)) \leqslant d_{\mathbb{H}}(i,q)$$
$$\leqslant d_{\mathbb{H}}(i,q') + d_{\mathbb{H}}(q',q)$$
$$\leqslant R' + d_{\mathbb{H}}(q',q)$$
$$= R' + \log \frac{t'-s'}{2}.$$

Since $\frac{t'-s'}{2} \leqslant R'$, we have

$$e^{-d_{\mathbb{H}}(i,p)} \ge e^{-R' - \log \frac{t'-s'}{2}}$$
$$= e^{-R'} \cdot \frac{2}{t'-s'}$$
$$\ge \frac{e^{-R'}}{2(R')^2} \cdot (t'-s')$$

This proves $|t' - s'| \leq C \cdot e^{-d_g}$.

Remark 9. We similarly denote by sin(g, h) to be the sine of the angle from g to h if they intersect and to be zero if h and g do not intersect. The following estimate (similar to Lemma 8) can be made

$$\Big|\int_{G(\mathbb{D})}\xi(h)\sin(g,h)dL_{\mathbb{D}}(h)\Big| \leqslant \int_{G(\mathbb{D})}\Big|\xi(h)\sin(g,h)\Big|dL_{\mathbb{D}}(h)\leqslant Ce^{-(1+\lambda)dg}$$

where $d_g \ge 0$ is the hyperbolic distance between $0 \in \mathbb{D}$ and the geodesic g.

And now we prove

Theorem 10. Let μ be a bounded measured lamination on a conformally hyperbolic Riemann surface X and let $\tilde{\mu}$ be its lift to the universal covering \tilde{X} . Then

(3)
$$\frac{d}{dt}\mathcal{L}([E^{t\widetilde{\mu}}])(\xi)\Big|_{t=0} = \int_{G(\widetilde{X})} \int_{G(\widetilde{X})} \xi(h) \cos(g,h) dL_{\widetilde{X}}(h) d\widetilde{\mu}(g)$$

Proof. In order to finish the proof it remains to prove that

$$\lim_{j \to \infty} \int_{G(\widetilde{X})} \int_{G(\widetilde{X})} \xi(h) \cos(g, h) dL_{\widetilde{X}}(h) d\widetilde{\mu}|_{K_j}(g)$$

is equal to the double integral in the statement of the theorem. By the dominated convergence theorem, it is enough to show that

$$\int_{G(\widetilde{X})} \Big| \int_{G(\widetilde{X})} \xi(h) \cos(g,h) dL_{\widetilde{X}}(h) \Big| d\widetilde{\mu}(g) < \infty.$$

We identify \widetilde{X} with \mathbb{D} by an isometry. Let $C_n = \{z \in \mathbb{D} : n < \rho_{\mathbb{D}}(0, z) \leq n + 1\}$ be the half-closed annulus around 0 with inner radius n and outer radius n + 1. For each $n \geq 2$, let \mathcal{F}_n be the family of geodesics of the support $|\widetilde{\mu}|$ of $\widetilde{\mu}$ that intersect C_n but do not intersect C_{n-1} . We partition \mathcal{F}_n into finitely many subfamilies, see Figure 3. Namely, if $g, g' \in \mathcal{F}_n$ and either g separates g' and 0, or g' separates g and 0 then g and g' belongs to the same subfamily. Each subfamily has the unique geodesic that is farthest away from 0.

Let g be one geodesic in a subfamily of \mathcal{F}_n that is farthest away from 0 and let $z \in g$ be the closest point of g to 0. Then the shortest distance $\rho(0, z)$ from g to 0 is less than or equal to n + 1. This is equivalent to $\log \frac{1+|z|}{1-|z|} \leq n+1$ which implies $1-|z| \geq C_1 e^{-n}$ for some universal constant $C_1 > 0$. Therefore the Euclidean circle which contains the geodesic g has the Euclidean radius $r \geq C_1 e^{-n}$. Therefore the arc length of the unit circle S^1 cut out by the geodesic g is at least $C_1 e^{-n}$ by the inequality $\tan^{-1}(x) > \frac{1}{2}x$ for $0 < x \leq 1$ and some elementary Euclidean considerations. We conclude that the number of subfamilies of \mathcal{F}_n is at most $\frac{2\pi}{C_1}e^n$.



FIGURE 3. Construction of \mathcal{F}_n and corresponding subfamilies.

To simplify the notation, set $I(g) = \left| \int_{G(\tilde{X})} \xi(h) \cos(g, h) dL_{\tilde{X}}(h) \right|$. By Lemma 8, we have that $I(g) \leq Ce^{-(1+\lambda)n}$. The total measure of the geodesics in each subfamily \mathcal{F}_n^i of \mathcal{F}_n is at most $\|\tilde{\mu}\|_{Th}$ for $\tilde{\mu}$ because each subfamily intersects an arc of the radius of hyperbolic length 1. It follows that $\int_{\mathcal{F}_n^i} I(g) d\tilde{\mu}(g) \leq Ce^{-(1+\lambda)n}$ for each subfamily \mathcal{F}_n^i and the total integral $\int_{\mathcal{F}_n} I(g) d\tilde{\mu}(g)$ is bounded by $C_2 e^{-\lambda n}$ for some universal constant $C_2 > 0$. Since $\sum_{n=1}^{\infty} e^{-\lambda n} < \infty$, the theorem is proved.

By changing the basepoint of $\mathcal{T}(X)$, Theorem 10 gives

Corollary 11. Let μ be a bounded measured lamination on a conformally hyperbolic Riemann surface X and let $\tilde{\mu}$ be its lift to the universal covering \tilde{X} . Then

(4)
$$\frac{d}{dt}\mathcal{L}([E^{t\tilde{\mu}}])(\xi) = \int_{G(\tilde{X})} \int_{G(\tilde{X})} \xi(h) \cos(E^{t\tilde{\mu}}(g), E^{t\tilde{\mu}}(h)) dL_{[E^{t\tilde{\mu}}]}(h) d\tilde{\mu}(g)$$

In order to extend the above formula to quake-bends, we first need an extension of Lemma 6. We begin with the definition of signed complex distance in \mathbb{H}^3 , see [22]. Let α be any oriented line in \mathbb{H}^3 , and let $P_1, P_2 \in \alpha$. Let $d(P_1, P_2) \ge 0$ denote the hyperbolic distance between P_1 and P_2 . We define the signed real hyperbolic distance $\delta_{\alpha}(P_1, P_2)$ as $d(P_1, P_2)$ if the orientation of the arc from P_1 to P_2 coincides with that of α and $-d(P_1, P_2)$ otherwise. Now let $L_1, L_2 \subset \mathbb{H}^3$ be oriented lines with distinct endpoints on the Riemann sphere $\hat{\mathbb{C}}$, with oriented common perpendicular α meeting L_1, L_2 in points Q_1, Q_2 respectively, where if L_1, L_2 intersect we take $Q_1 = Q_2$. Let $\mathbf{v_i}$ be tangent vectors to the positive directions of L_i at $Q_i, i = 1, 2$. Let Π be the hyperbolic plane through Q_2 orthogonal to α and let $\mathbf{w_1}$ denote the parallel translate of $\mathbf{v_1}$ along α to Q_2 . Let \mathbf{n} be a unit vector at Q_2 pointing in the positive direction of α . The signed complex distance between L_1 and L_2 is

$$\boldsymbol{d}(L_1,L_2) = \boldsymbol{\delta}_{\boldsymbol{n}}(L_1,L_2) = \boldsymbol{\delta}_{\alpha}(L_1,L_2) = \delta_{\alpha}(Q_1,Q_2) + i\boldsymbol{\delta}_{\alpha}(Q_1,Q_2) + i\boldsymbol{\delta}_{\alpha}(Q_1,Q$$

where θ , measured modulo $2\pi\mathbb{Z}$, is the angle between $\mathbf{w_1}$ and $\mathbf{v_2}$ measured anticlockwise in the plane spanned by $\mathbf{w_1}, \mathbf{v_2}$ and oriented by \mathbf{n} .

Lemma 12. Let $x, y \in \mathbb{R}$ and $\omega > 0, \tau \in \mathbb{C}$ with $|\tau| < \delta$. Let g be the geodesic with endpoints oriented from 0 to ∞ and h be the geodesic with endpoints oriented from $xe^{\tau\omega}$ to y. Then

$$\cosh(\boldsymbol{d}(g,h)) = \frac{-xe^{\tau\omega} - y}{xe^{\tau\omega} - y} = 1 - \frac{2}{cr(y,0,xe^{\tau\omega},\infty)}$$

where d is the complex distance.

Proof. For $|\tau| < \delta$, $xe^{\tau\omega}$ is in a small neighborhood of x. A simple computation shows that there is a unique common perpendicular geodesic l with endpoints $-\sqrt{xye^{\tau\omega}}, \sqrt{xye^{\tau\omega}}$ intersecting g, h at p, q, respectively.

Define a Möbius map $f(z) = \frac{z - \sqrt{xye^{\tau\omega}}}{z + \sqrt{xye^{\tau\omega}}}$, which sends l to the geodesic $l' = (0, \infty)$, g to the geodesic g' = (-1, 1) and h to the geodesic $h' = (f(xe^{\tau\omega}), f(y))$, see Figure 4. Note that l' intersects g', h' at p' = f(p), q' = f(q), respectively. A direction computation shows that $f(y) = \frac{y - \sqrt{xye^{\tau\omega}}}{y + \sqrt{xye^{\tau\omega}}} = -f(xe^{\tau\omega})$, this implies that they are diametrically opposite of the origin.



FIGURE 4. Hyperbolic distance in upper-half space.

Now we have the hyperbolic distance,

$$d(g,h) = d(g',h') = d(p',q') = \begin{cases} \log |f(y)| & \text{if } |f(y)| \ge 1\\ \\ \log \frac{1}{|f(y)|} & \text{if } |f(y)| < 1. \end{cases}$$

By our definition of signed complex distance in \mathbb{H}^3 ,

$$\begin{aligned} \boldsymbol{d}(g,h) &= \boldsymbol{d}(g',h') = \delta_{l'}(p',q') + i\theta \\ &= \log |f(y)| + i(\arg(f(y))) \\ &= \log(f(y)). \end{aligned}$$

Finally,

$$\cosh(\boldsymbol{d}(g,h)) = \frac{e^{\log(f(y))} + e^{-\log(f(y))}}{2} = \frac{f(y) + \frac{1}{f(y)}}{2}$$
$$= \frac{1}{2} \left(\frac{y - \sqrt{xye^{\tau\omega}}}{y + \sqrt{xye^{\tau\omega}}} + \frac{y + \sqrt{xye^{\tau\omega}}}{y - \sqrt{xye^{\tau\omega}}} \right)$$
$$= \frac{-xe^{\tau\omega} - y}{xe^{\tau\omega} - y}$$

and the second equality is straightforward.

The following result will be useful in our paper.

Lemma 13 (see [8]). Let X be an open subset of \mathbb{C} and Ω be a measure space. Suppose that $f: X \times \Omega \to \mathbb{C}$ satisfies the following conditions:

- (i) $f(\tau, \omega)$ is a Lebesgue integrable function of ω for each $\tau \in X$.
- (ii) For almost all $\omega \in \Omega$, the derivative $\frac{\partial f(\tau,\omega)}{\partial \tau}$ exists for all $\tau \in X$.
- (iii) There is an integrable function $\Theta: \Omega \to \mathbb{C}$ such that $\left|\frac{\partial f(\tau,\omega)}{\partial \tau}\right| \leqslant \Theta$ for all $\tau \in X$.

Then for all $\tau \in X$, $\frac{d}{d\tau} \int_{\Omega} f(\tau, \omega) d\omega = \int_{\Omega} \frac{\partial}{\partial \tau} f(\tau, \omega) d\omega$.

We first find the derivative along a simple quake-bend (which generalizes Lemma 6):

Lemma 14. Let $\xi \in H(\widetilde{X})$ and $g \in G(\widetilde{X})$ be a fixed geodesic. Let $\delta > 0$ be the radius for which $\hat{\mathcal{L}}([E_g^{\tau\omega}])(\xi)$ is defined. Then, for $\tau \in \mathbb{C}$ with $|\tau| < \delta$ and $\omega > 0$,

$$\frac{d}{d\tau}\hat{\mathcal{L}}([E_g^{\tau\omega}])(\xi) = \omega \int_{G(\widetilde{X})} \xi(h) \cosh(\boldsymbol{d}(E_g^{\tau\omega}(g), E_g^{\tau\omega}(h))) dL_{[E_g^{\tau\omega}]}(h),$$

where $E_g^{\tau\omega}(g) = g$ and $\mathbf{d}(E_g^{\tau\omega}(g), E_g^{\tau\omega}(h))$ is the complex distance between $E_g^{\tau\omega}(g)$ and $E_g^{\tau\omega}(h)$.

Proof. We identify \widetilde{X} with the upper half-plane \mathbb{H} such that g is identified with the positive y-axis. Without loss of generality, we assume that ξ has support in a box of geodesics and let $[a, b] \times [c, d] \subset G(\mathbb{H})$ be the sub-box of the support of ξ such that each geodesic of its interior intersects g. A computation for t real gives that

(5)
$$\frac{d}{dt}\mathcal{L}([E_g^{t\omega}])(\xi) = \frac{d}{dt} \iint_{[a,b]\times[c,d]} \xi \circ (E_g^{t\omega})^{-1}(h(x,y)) \frac{dxdy}{(x-y)^2}$$
$$= \frac{d}{dt} \iint_{[a,b]\times[c,d]} \xi(h(e^{-t\omega}x,y)) \frac{dxdy}{(x-y)^2}$$
$$= \omega \iint_{[a,b]\times[c,d]} \xi(h(x,y)) \frac{-xe^{t\omega}-y}{xe^{t\omega}-y} \frac{e^{t\omega}dxdy}{(xe^{t\omega}-y)^2}.$$

Since $\tau \mapsto \hat{\mathcal{L}}([E_g^{\tau\omega}])(\xi)$ is holomorphic in τ and by Lemma 13 the integral

$$\omega \iint_{[a,b]\times[c,d]} \xi(h(x,y)) \frac{-xe^{\tau\omega} - y}{xe^{\tau\omega} - y} \frac{e^{\tau\omega} dxdy}{(xe^{\tau\omega} - y)^2}$$

is also holomorphic in τ , hence by the uniqueness of holomorphic maps we have

(6)
$$\frac{d}{d\tau}\hat{\mathcal{L}}([E_g^{\tau\omega}])(\xi) = \omega \iint_{[a,b]\times[c,d]} \xi(h(x,y)) \frac{-xe^{\tau\omega} - y}{xe^{\tau\omega} - y} \frac{e^{\tau\omega}dxdy}{(xe^{\tau\omega} - y)^2}$$

Note that the density

$$\frac{e^{\tau\omega}dxdy}{(xe^{\tau\omega}-y)^2}$$

defines a countably additive complex measure on $[a, b] \times [c, d]$ of finite variation. The result follows immediately from Lemma 12.

By Lemma 14, we have

$$\frac{d}{d\tau}\hat{\mathcal{L}}([E^{\tau\tilde{\mu}_{n,j}}])(\xi) = \sum_{i=1}^{p(n,j)} \omega_i \int_{G(\tilde{X})} \xi(h) \cosh d(E^{\tau\tilde{\mu}_{n,j}}(g_i), E^{\tau\tilde{\mu}_{n,j}}(h)) dL_{[E^{\tau\tilde{\mu}_{n,j}}]}(h),$$

where $\widetilde{\mu}_{n,j} = \sum_{i=1}^{p(n,j)} \omega_i (\mathbf{1}_{g_i} + \mathbf{1}_{\widetilde{g}_i})$. The above formula can be written as

$$\frac{d}{d\tau}\hat{\mathcal{L}}([E^{\tau\tilde{\mu}_{n,j}}])(\xi) = \int_{G(\tilde{X})} \int_{G(\tilde{X})} \xi(h) \cosh d(E^{\tau\tilde{\mu}_{n,j}}(g), E^{\tau\tilde{\mu}_{n,j}}(h)) dL_{[E^{\tau\tilde{\mu}_{n,j}}]}(h) d\tilde{\mu}_{n,j}(g).$$

By Remark 7, $g \mapsto \int_{G(\widetilde{X})} \xi(h) \cosh d(E^{\tau \widetilde{\mu}_{n,j}}(g), E^{\tau \widetilde{\mu}_{n,j}}(h)) dL_{[E^{\tau \widetilde{\mu}_{n,j}}]}(h)$ is a continuous function in g. Since $\widetilde{\mu}_{n,j}$ converges in the weak* topology to $\widetilde{\mu}|_{K_j}$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty}\frac{d}{d\tau}\hat{\mathcal{L}}([E^{\tau\widetilde{\mu}_{n,j}}])(\xi) = \int_{G(\widetilde{X})}\int_{G(\widetilde{X})}\xi(h)\cosh \boldsymbol{d}(E^{\tau\widetilde{\mu}|_{K_j}}(g), E^{\tau\widetilde{\mu}|_{K_j}}(h))dL_{[E^{\tau\widetilde{\mu}|_{K_j}}]}(h)d\widetilde{\mu}|_{K_j}(g).$$

By Theorem 5, we obtain a formula for the derivative of the Liouville Hölder distributions at a point of the Quasi-Fuachsian space corresponding to a quake-bend with small imaginary parts. Note that the convergence of the formula is conditional (given by a limit) unlike the convergence at Teichmüller space where the convergence is absolute (given by a Lebesgue integral in the measure $\tilde{\mu}$). One would expect that the convergence is conditional due to the fact that the quasicircles do not give (bounded variation complex) measures but rather only Hölder distributions as in [9].

Theorem 15. Let $\xi \in H(\tilde{X})$ and let $\delta > 0$ be the radius for which $\hat{\mathcal{L}}([E^{\tau \tilde{\mu}}])(\xi)$ is defined. Then, for $\tau \in \mathbb{C}$ with $|\tau| < \delta$,

$$\frac{d}{d\tau}\hat{\mathcal{L}}([E^{\tau\tilde{\mu}}])(\xi) = \lim_{j \to \infty} \int_{G(\tilde{X})} \int_{G(\tilde{X})} \xi(h) \cosh \boldsymbol{d}(E^{\tau\tilde{\mu}|_{K_j}}(g), E^{\tau\tilde{\mu}|_{K_j}}(h)) dL_{[E^{\tau\tilde{\mu}|_{K_j}}]}(h) d\tilde{\mu}|_{K_j}(g)$$

where $\widetilde{\mu}|_{K_j} \to \widetilde{\mu}$ in the weak* topology as $j \to \infty$ and d is the complex distance.

6. A Geometric formula for the second derivative of the Liouville measure

By taking derivatives of the equations (5) and (6), we obtain

Corollary 16. Let
$$\xi \in H(X)$$
 and $g \in G(X)$ be a fixed geodesic. Then, for $t \in \mathbb{R}$ and $\omega > 0$,

$$\frac{d^2}{dt^2} \mathcal{L}([E_g^{t\omega}])(\xi)\Big|_{t=0} = \omega^2 \int_{G(\tilde{X})} \xi(h) [\cos^2(g,h) - \frac{1}{2}\sin^2(g,h)] dL_{\tilde{X}}(h).$$

Corollary 17. Let $\xi \in H(\widetilde{X})$ and $g \in G(\widetilde{X})$ be a fixed geodesic. Let $\delta > 0$ be the radius for which the quake-bend is defined. Then, for $\tau \in \mathbb{C}$ with $|\tau| < \delta$ and $\omega > 0$,

$$\begin{split} \frac{d^2}{d\tau^2} \hat{\mathcal{L}}([E_g^{\tau\omega}])(\xi) &= \omega^2 \int_{G(\tilde{X})} \xi(h) [\cosh^2 \mathbf{d}(E_g^{\tau\omega}(g), E_g^{\tau\omega}(h)) - \frac{1}{2} \sinh^2 \mathbf{d}(E_g^{\tau\omega}(g), E_g^{\tau\omega}(h))] dL_{[E_g^{\tau\omega}]}(h) \\ \text{where } E_g^{\tau\omega}(g) &= g \text{ and } \mathbf{d} \text{ is the complex distance.} \end{split}$$

However, the above two corollaries do not provide the second derivatives in the case when the measures are finite combinations of Dirac measures due to the interactions of different terms. We would like to find the formula for the second derivative of the Liouville measure along an elementary earthquake, i.e. an earthquake whose measure is a finite sum of Dirac measures. Since the full computations are much more involved, we point out that the second derivative captures all pairwise interactions of the supporting geodesics in $|\tilde{\mu}_{n,j}|$.

To see this, let $E^{t\sigma} = E_{g_i}^{t\omega_i} \circ E_{g_k}^{t\omega_k}$ be an elementary earthquake with measured lamination σ supported on two disjoint geodesics g_i and g_k as defined in section 3. Suppose that

 $g_i = (0, \infty), g_k = (\alpha, \beta) \text{ and } h(x, y) \text{ is a geodesic intersecting both } g_i \text{ and } g_k.$ Define a Möbius map $\varphi(z) = \frac{\alpha - z}{z - \beta}$ which sends α to 0 and β to ∞ . By definition, $E_{g_i}^{t\omega_i} = \begin{cases} x & \text{if } x \leq 0 \\ e^{t\omega_i} x & \text{if } x > 0. \end{cases}$ For $t \in \mathbb{R}$, we have

$$\begin{aligned} \left. \frac{d}{dt} \mathcal{L}([E_g^{t\sigma}])(\xi) \right|_{t=0} &= \left. \frac{d}{dt} \int_{G(\tilde{X})} \xi \circ (E^{t\sigma})^{-1} (h(x,y)) \frac{dxdy}{(x-y)^2} \right|_{t=0} \\ &= \left. \frac{d}{dt} \int_{G(\tilde{X})} \xi \circ \varphi^{-1} \circ \varphi \circ (E_{g_k}^{t\omega_k})^{-1} \circ \varphi^{-1} \circ \varphi (h(e^{-t\omega_i}x,y)) \frac{dxdy}{(x-y)^2} \right|_{t=0} \\ &= \left. \frac{d}{dt} \int_{G(\tilde{X})} \xi \circ \varphi^{-1} \circ (E_{\varphi(g_k)}^{t\omega_k})^{-1} \circ \varphi (h(e^{-t\omega_i}x,y)) \frac{dxdy}{(x-y)^2} \right|_{t=0} \\ &= \left. \int_{G(\tilde{X})} \xi (h) \left[\omega_i \cos(g_i,h) + \omega_k \cos(g_k,h) \right] dL_{\tilde{X}}(h) \end{aligned}$$

and

$$\begin{aligned} \frac{d^2}{dt^2} \mathcal{L}([E_g^{t\sigma}])(\xi) \Big|_{t=0} &= \int_{G(\tilde{X})} \xi(h) \\ & \left[\omega_i^2 \left[\cos^2(g_i, h) - \frac{1}{2} \sin^2(g_i, h) \right] + \omega_k^2 \left[\cos^2(g_k, h) - \frac{1}{2} \sin^2(g_k, h) \right] \\ & + \omega_i \omega_k \left[\cos(g_i, h) \cos(g_k, h) - \frac{1}{2} \sin(g_i, h) \sin(g_k, h) e^{-d_h} \right] \right] dL_{\tilde{X}}(h) \end{aligned}$$

where d_h is the hyperbolic distance along h from $g_i \cap h$ to $g_k \cap h$ and e^{-d_h} is obtained by some hyperbolic geometry considerations.

Recall that $\widetilde{\mu}_{n,j} = \sum_{i=1}^{p(n,j)} \omega_i(\mathbf{1}_{g_i} + \mathbf{1}_{\widetilde{g}_i})$ are measured laminations with finite support as previously defined. Let $\xi \in H(\widetilde{X})$ and for $t \in \mathbb{R}$, we have

$$\begin{aligned} \frac{d^2}{dt^2} \mathcal{L}([E^{t\widetilde{\mu}_{n,j}}])(\xi)\Big|_{t=0} &= \int_{G(\widetilde{X})} \xi(h) \\ & \left[\sum_{i,k=1}^{p(n,j)} \omega_i \omega_k \left[\cos(g_i,h)\cos(g_k,h) - \frac{1}{2}\sin(g_i,h)\sin(g_k,h)e^{-d_h}\right]\right] dL_{\widetilde{X}}(h) \\ &= \int_{G(\widetilde{X})} \int_{G(\widetilde{X})} \\ & \left\{\int_{G(\widetilde{X})} \xi(h) \left[\cos(g_i,h)\cos(g_k,h) - \frac{1}{2}\sin(g_i,h)\sin(g_k,h)e^{-d_h}\right] dL_{\widetilde{X}}(h)\right\} \\ & d\widetilde{\mu}_{n,j}(g_i) d\widetilde{\mu}_{n,j}(g_k) \end{aligned}$$

where d_h is the hyperbolic distance along h from $g_i \cap h$ to $g_k \cap h$. Using similar arguments as in Remark 7, $(g_i, g_k) \mapsto \int_{G(\widetilde{X})} \xi(h) \left[\cos(g_i, h) \cos(g_k, h) - \frac{1}{2} \sin(g_i, h) \sin(g_k, h) e^{-d_h} \right] dL_{\widetilde{X}}(h)$ is a continuous function in (g_i, g_k) . Since $\widetilde{\mu}_{n,j}$ converges in the weak* topology to $\widetilde{\mu}|_{K_j}$ as $n \to \infty$, similar to the first derivative it follows that

$$\begin{split} &\lim_{n\to\infty} \int_{G(\widetilde{X})} \int_{G(\widetilde{X})} \\ & \left\{ \int_{G(\widetilde{X})} \xi(h) \big[\cos(g_i, h) \cos(g_k, h) - \frac{1}{2} \sin(g_i, h) \sin(g_k, h) e^{-d_h} \big] dL_{\widetilde{X}}(h) \right\} d\widetilde{\mu}_{n,j}(g_i) d\widetilde{\mu}_{n,j}(g_k) \\ &= \int_{G(\widetilde{X})} \int_{G(\widetilde{X})} \\ & \left\{ \int_{G(\widetilde{X})} \xi(h) \big[\cos(g_i, h) \cos(g_k, h) - \frac{1}{2} \sin(g_i, h) \sin(g_k, h) e^{-d_h} \big] dL_{\widetilde{X}}(h) \right\} d\widetilde{\mu}|_{K_j}(g_i) d\widetilde{\mu}|_{K_j}(g_k). \end{split}$$

By Theorem 5, we have that

$$\frac{d^2}{dt^2} \mathcal{L}([E^{t\widetilde{\mu}}])(\xi)\Big|_{t=0} = \lim_{j \to \infty} \int_{G(\widetilde{X})} \int_{G(\widetilde{X})} \\ \left\{ \int_{G(\widetilde{X})} \xi(h) \Big[\cos(g_i, h) \cos(g_k, h) - \frac{1}{2} \sin(g_i, h) \sin(g_k, h) e^{-d_h} \Big] dL_{\widetilde{X}}(h) \right\} \\ d\widetilde{\mu}|_{K_j}(g_i) d\widetilde{\mu}|_{K_j}(g_k).$$

To simplify the notation, for the rest of the paper, we denote $A \leq B$ if $A \leq CB$ for some universal constant C. And now we prove

Theorem 18. Let μ be a bounded measured lamination on a conformally hyperbolic Riemann surface X and let $\tilde{\mu}$ be its lift to the universal covering \tilde{X} . Then

$$\begin{split} \frac{d^2}{dt^2} \mathcal{L}([E^{t\widetilde{\mu}}])(\xi)\Big|_{t=0} &= \int_{G(\widetilde{X})} \int_{G(\widetilde{X})} \\ & \left\{ \int_{G(\widetilde{X})} \xi(h) \big[\cos(g,h)\cos(g',h) - \frac{1}{2}\sin(g,h)\sin(g',h)e^{-d_h} \big] dL_{\widetilde{X}}(h) \right\} \\ & d\widetilde{\mu}(g)d\widetilde{\mu}(g') \end{split}$$

where d_h is the hyperbolic distance along h from $g \cap h$ to $g' \cap h$.

To simplify the notation, set

$$I(g,g') = \left| \int_{G(\tilde{X})} \xi(h) \Big[\cos(g,h) \cos(g',h) - \frac{1}{2} \sin(g,h) \sin(g',h) e^{-d_h} \Big] dL_{\tilde{X}}(h) \right|$$

In particular, I(g,g') = 0 when h intersects only g or g' but not both since then we have $\cos(g,h) = \sin(g,h) = 0$ or $\cos(g',h) = \sin(g',h) = 0$. When h intersects both g and g', we obtain an upper bound for I(g,g') as follows:

$$\begin{split} I(g,g') &= \left| \int_{G(\widetilde{X})} \xi(h) \big[\cos(g,h) \cos(g',h) - \frac{1}{2} \sin(g,h) \sin(g',h) e^{-d_h} \big] dL_{\widetilde{X}}(h) \right| \\ &\leqslant \left| \int_{G(\widetilde{X})} \xi(h) \cos(g,h) \cos(g',h) dL_{\widetilde{X}}(h) \right| + \left| \int_{G(\widetilde{X})} \xi(h) \frac{1}{2} \sin(g,h) \sin(g',h) e^{-d_h} dL_{\widetilde{X}}(h) \right| \\ &\leqslant \min\{ C e^{-(1+\lambda)d_g}, C' e^{-(1+\lambda)d_{g'}} \} \quad \text{(by Lemma 8 and Remark 9)} \end{split}$$

It is enough to show that

$$\int_{G(\widetilde{X})} \int_{G(\widetilde{X})} I(g,g') d\widetilde{\mu}(g) d\widetilde{\mu}(g') < \infty.$$

We need the following estimate.

Lemma 19. Let \mathfrak{F}_n be defined as in the proof of Theorem 10 and I(g,g') be defined as above. Then

$$\int_{G(\widetilde{X})} \int_{\mathcal{F}_n} I(g,g') d\widetilde{\mu}(g) d\widetilde{\mu}(g') \lesssim n e^{-\lambda n} + e^{-\lambda n},$$

where the universal constant in \leq is the maximum of the universal constants for $ne^{-\lambda n}$ and $e^{-\lambda n}$ respectively.

Proof. We identify \widetilde{X} with \mathbb{D} by an isometry. As in the proof of Theorem 10, let $C_n = \{z \in \mathbb{D} : n < \rho_{\mathbb{D}}(0, z) \leq n + 1\}$ be the half-closed annulus around 0. For each $n \geq 2$, let \mathcal{F}_n be the family of geodesics of the support $|\widetilde{\mu}|$ of $\widetilde{\mu}$ that intersect C_n but do not intersect C_{n-1} . We partition \mathcal{F}_n into finitely many subfamilies.

We begin our estimate with a single subfamily \mathcal{F}_n^i of \mathcal{F}_n for the inner integral. Suppose that $g \in \mathcal{F}_n^i$. Since *h* intersects both *g* and *g'*, then *g'* belongs to at most 2 subfamilies of \mathcal{F}_m , see Figure 5. For indexing purpose, we denote them by \mathcal{F}_m^j and \mathcal{F}_m^k . Thus,

$$\begin{split} &\int_{G(\widetilde{X})} \int_{\mathcal{F}_{n}^{i}} I(g,g') d\widetilde{\mu}(g) d\widetilde{\mu}(g') \\ &= \sum_{m=1}^{\infty} \int_{\mathcal{F}_{m}^{j} + \mathcal{F}_{m}^{k}} \int_{\mathcal{F}_{n}^{i}} I(g,g') d\widetilde{\mu}(g) d\widetilde{\mu}(g') \\ &= \sum_{m=1}^{n} \int_{\mathcal{F}_{m}^{j} + \mathcal{F}_{m}^{k}} \int_{\mathcal{F}_{n}^{i}} I(g,g') d\widetilde{\mu}(g) d\widetilde{\mu}(g') + \sum_{m=n+1}^{\infty} \int_{\mathcal{F}_{m}^{j} + \mathcal{F}_{m}^{k}} \int_{\mathcal{F}_{n}^{i}} I(g,g') d\widetilde{\mu}(g) d\widetilde{\mu}(g') \\ &= A + B \end{split}$$



FIGURE 5. h intersecting geodesics g and g'.

Note that the total measure of the geodesics in $\mathcal{F}_m^j + \mathcal{F}_m^k$ is at most $2\|\widetilde{\mu}\|_{Th}$. Thus, $A \leq 2\|\widetilde{\mu}\|_{Th}Cne^{-(1+\lambda)n} \leq C_1ne^{-(1+\lambda)n}$. To estimate B, we consider

$$\begin{split} \sum_{m=n+1}^{\infty} \int_{\mathcal{F}_{m}^{j} + \mathcal{F}_{m}^{k}} \int_{\mathcal{F}_{n}^{i}} I(g, g') d\widetilde{\mu}(g) d\widetilde{\mu}(g') &= \int_{\mathcal{F}_{n}^{i}} \left(\sum_{m=n+1}^{\infty} \int_{\mathcal{F}_{m}^{j} + \mathcal{F}_{m}^{k}} I(g, g') d\widetilde{\mu}(g') \right) d\widetilde{\mu}(g) \\ &\leq \int_{\mathcal{F}_{n}^{i}} \left(2 \sum_{m=n+1}^{\infty} C e^{-(1+\lambda)m} \right) d\widetilde{\mu}(g) \\ &\leq 2 \|\widetilde{\mu}\|_{Th} C \frac{e^{(1+\lambda)}}{e^{(1+\lambda)} - 1} e^{-(1+\lambda)(n+1)} \\ &\leq C_{2} e^{-(1+\lambda)(n+1)}. \end{split}$$

Then,

$$\int_{G(\widetilde{X})} \int_{\mathcal{F}_n^i} I(g,g') d\widetilde{\mu}(g) d\widetilde{\mu}(g') \lesssim n e^{-(1+\lambda)n} + e^{-(1+\lambda)(n+1)},$$

where the universal constant in \leq is the $max\{C_1, C_2\}$. Since the number of subfamilies \mathcal{F}_n^i is at most $\frac{2\pi}{C}e^n$,

$$\int_{G(\widetilde{X})} \int_{\mathcal{F}_n} I(g,g') d\widetilde{\mu}(g) d\widetilde{\mu}(g') \lesssim \frac{2\pi}{C} e^n \left(n e^{-(1+\lambda)n} + e^{-(1+\lambda)(n+1)} \right)$$
$$\lesssim n e^{-\lambda n} + e^{-\lambda n}.$$

Proof of Theorem 18. It follows from Lemma 19,

$$\int_{G(\widetilde{X})} \int_{G(\widetilde{X})} I(g,g') d\widetilde{\mu}(g) d\widetilde{\mu}(g') = \sum_{n=1}^{\infty} \int_{G(\widetilde{X})} \int_{\mathcal{F}_n} I(g,g') d\widetilde{\mu}(g) d\widetilde{\mu}(g')$$
$$\lesssim \sum_{n=1}^{\infty} (ne^{-\lambda n} + e^{-\lambda n}) < \infty$$

and the theorem is proved.

By changing the basepoint of $\mathcal{T}(X)$, we obtain an immediate consequence of Theorem 18. **Corollary 20.** Let μ be a bounded measured lamination on a conformally hyperbolic Riemann surface X and let $\tilde{\mu}$ be its lift to the universal covering \tilde{X} . Then

$$\begin{split} \frac{d^2}{dt^2} \mathcal{L}([E^{t\widetilde{\mu}}])(\xi) &= \int_{G(\widetilde{X})} \int_{G(\widetilde{X})} \\ & \left\{ \int_{G(\widetilde{X})} \xi(h) \big[\cos(E^{t\widetilde{\mu}}(g), E^{t\widetilde{\mu}}(h)) \cos(E^{t\widetilde{\mu}}(g'), E^{t\widetilde{\mu}}(h)) \\ & - \frac{1}{2} \sin(E^{t\widetilde{\mu}}(g), E^{t\widetilde{\mu}}(h)) \sin(E^{t\widetilde{\mu}}(g'), E^{t\widetilde{\mu}}(h)) e^{-d_E t\widetilde{\mu}(h)} \big] dL_{[E^{t\widetilde{\mu}}]}(h) \right\} \\ & d\widetilde{\mu}(g) d\widetilde{\mu}(g') \end{split}$$

where $d_{E^{t\tilde{\mu}}(h)}$ is the hyperbolic distance from $E^{t\tilde{\mu}}(g) \cap E^{t\tilde{\mu}}(h)$ to $E^{t\tilde{\mu}}(g') \cap E^{t\tilde{\mu}}(h)$ along the geodesic $E^{t\tilde{\mu}}(h)$.

Remark 21. We would like to find the second derivative along a quake-bend of the Liouville measure in a neighborhood of the Teichmüller space $\mathcal{T}(X)$ similar to Theorem 15 (given by a limit). However, the quantity $d_{E^{\tau \widetilde{\mu}|_{K_j}}(h)}$ is not well-defined in the upper half-space \mathbb{H}^3 since there are no such intersection points $E^{\tau \widetilde{\mu}|_{K_j}}(g) \cap E^{\tau \widetilde{\mu}|_{K_j}}(h)$ and $E^{\tau \widetilde{\mu}|_{K_j}}(g') \cap E^{\tau \widetilde{\mu}|_{K_j}}(h)$.

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