# SLICE RANK AND ANALYTIC RANK FOR TRILINEAR FORMS 

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#### Abstract

In this note, we present an elementary proof of the fact that the slice rank of a trilinear form over a finite field is bounded above by a linear expression in the analytic rank. The existing proofs by Adiprasito-KazhdanZiegler and Cohen-Moshkovitz both rely on results of Derksen via geometric invariant theory. A novel feature of our proof is that the linear forms appearing in the slice rank decomposition are obtained from the trilinear form by fixing coordinates.


## 1. The Theorem

Let $\mathbf{k}=\mathbb{F}_{q}$ be a finite field and let $U, V, W$ be finite dimensional vector spaces over k.

Definition 1.1. For a trilinear form $f: U \times V \times W \rightarrow \mathbf{k}$ we are interested in two kinds of rank.

- The slice rank of $f$ is

$$
\operatorname{srk}(f)=\min \left\{r: f=\sum_{i=1}^{r} \alpha_{i} \cdot h_{i}\right\}
$$

where $\alpha_{i}, h_{i}$ are linear and bilinear, respectively, in disjoint sets of variables.

- The analytic rank of $f$ is

$$
\operatorname{ark}(f)=-\log _{q} \frac{|Z|}{|U \times V|}
$$

where $Z=\{(u, v) \in U \times V: f(u, v, \cdot) \equiv 0\}$.
The definition of slice rank goes back to the work of Schmidt 9 on systems of polynomial equations. It was reidiscovered in [10] and used to give a reformulation of Ellenberg and Gijswijt's work on the capset problem. Analytic rank was introduced in 4]. The inequality $\operatorname{ark}(f) \leq \operatorname{srk}(f)$ is straightforward, see 6] or [7]. In the other direction, Adiprasito-Kazhdan-Ziegler [1] and Cohen-Moshkovitz [2] independently proved that $\operatorname{srk}(f) \ll \operatorname{ark}(f)$. Their proofs use results of Derksen [3] which rely on the powerful tools of geometric invariant theory. The goal of this note is to give an elementary proof.

Theorem 1.2. For any finite field $\mathbf{k}=\mathbb{F}_{q}$ and trilinear form $f$, we have

$$
\operatorname{srk}(f) \leq 5 \cdot \operatorname{ark}(f)+4 \cdot \log _{q}(\operatorname{ark}(f)+1)+29
$$

In addition, the linear forms in the corresponding slice rank decomposition are obtained from $f$ by fixing coordinates.

[^0]Remark 1.3. This is slightly weaker than the best current bound $\operatorname{srk}(f) \leq 3$. $\operatorname{ark}(f)$, obtained in [1].

Theorem 1.2 is the first nontrivial case of a more general conjecture relating analytic rank to another kind of rank, partition rank. To state the general conjecture, we need some definitions. Let $V_{1}, \ldots, V_{d}$ be finite-dimensional vector spaces over $\mathbf{k}$ and let $f: V_{1} \times \ldots \times V_{d} \rightarrow \mathbf{k}$ be a multilinear form.

Definition 1.4. - The partition rank of $f$ is

$$
\operatorname{prk}(f)=\min \left\{r: f=\sum_{i=1}^{r} g_{i} \cdot h_{i}\right\}
$$

where $g_{i}, h_{i}$ are multilinear in disjoint sets of variables. Note that if $d=3$ this agrees with the slice rank.

- The analytic rank of $f$ is

$$
\operatorname{ark}(f)=-\log _{q} \frac{|Z|}{\left|V_{1} \times \ldots \times V_{d-1}\right|}
$$

where $Z=\left\{\left(x_{1}, \ldots, x_{d-1}\right) \in V_{1} \times \ldots \times V_{d-1}: f\left(x_{1}, \ldots, x_{d-1}, \cdot\right) \equiv 0\right\}$.
Again, the inequality $\operatorname{ark}(f) \leq \operatorname{prk}(f)$ is not difficult, see [6] and [7]. Lovett [7] and Adiprasito-Kazhdan-Ziegler [1] conjectured that the reverse inequality also holds, up to a constant factor.

Conjecture 1.5. There exists a constant $C_{d}$ such that for any finite field and multilinear form $f$ we have

$$
\operatorname{prk}(f) \leq C_{d} \cdot \operatorname{ark}(f)
$$

The current best bound in this direction is

$$
\operatorname{prk}(f) \leq C_{d} \cdot \operatorname{ark}(f) \cdot \log ^{d-1}(1+\operatorname{ark}(f))
$$

due to Moshkovitz-Zhu [8].
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## 2. Proof of theorem 1.2

Let $r=\operatorname{ark}(f)$. For $u \in U$ we write $f[u]$ for the bilinear form $f[u]: V \times W \rightarrow \mathbf{k}$ obtained by fixing the $U$ coordinate. For $v \in V$ we write $f\langle v\rangle$ for the bilinear form $f\langle v\rangle: U \times W \rightarrow \mathbf{k}$ obtained by fixing the $V$ coordinate. For a finite set $X$, denote $\mathbb{E}_{x \in X}=\frac{1}{|X|} \sum_{x \in X}$. Our first lemma is inspired by a recent observation of Moshkovitz-Zhu [8].

Lemma 2.1.

$$
\underset{(u, v) \in Z}{\mathbb{E}} q^{\mathrm{rkf} f u]}=\underset{(u, v) \in Z}{\mathbb{E}} q^{\mathrm{rkf} f\langle v\rangle}=q^{r}
$$

Proof. A straightforward computation. Let

$$
p(u)=\frac{|\{v \in V:(u, v) \in Z\}|}{|Z|}=q^{-\operatorname{rk} f[u]} \frac{|V|}{|Z|}
$$

be the marginal probability mass function for $u \in U$. Then

$$
\begin{aligned}
\underset{(u, v) \in Z}{\mathbb{E}} q^{\mathrm{rk} f[u]} & =\sum_{u} p(u) \cdot q^{\mathrm{rk} f[u]} \\
& =\sum_{u} \frac{|V|}{|Z|}=\frac{|U| \cdot|V|}{|Z|}=q^{r}
\end{aligned}
$$

The proof for $\operatorname{rk} f\langle v\rangle$ is identical.
This allows us to find a large subspace $U^{\prime} \subset U$ where $\operatorname{rk} f[u]$ is almost surely small.

Lemma 2.2. There exists a subspace $U^{\prime} \subset U$ with $\operatorname{codim} U^{\prime} \leq r+1$ and

$$
\underset{u \in U^{\prime}}{\mathbb{P}}(\operatorname{rk} f[u]>r+s)<q^{1-s}
$$

for every $s>0$. The linear forms defining $U^{\prime}$ are obtained from $f$ by fixing coordinates.

Proof. By linearity of expectation,

$$
\underset{(u, v) \in Z}{\mathbb{E}}\left(q^{\mathrm{rk} f[u]}+q^{\mathrm{rkf} f v\rangle}\right)=2 q^{r} \leq q^{r+1}
$$

Therefore, there must exist some $v_{0} \in V$ with

$$
q^{\mathrm{rk} f\left\langle v_{0}\right\rangle}+\underset{u \in Z\left(v_{0}\right)}{\mathbb{E}} q^{\mathrm{rk} f[u]} \leq q^{r+1}
$$

where $Z\left(v_{0}\right)=\left\{u \in U:\left(u, v_{0}\right) \in Z\right\}$. Choosing $U^{\prime}=Z\left(v_{0}\right)$, we have $\operatorname{codim} U^{\prime}=\operatorname{rk} f\left\langle v_{0}\right\rangle \leq r+1$ and

$$
\underset{u \in U^{\prime}}{\mathbb{P}}(\operatorname{rk} f[u]>r+s)=\underset{u \in U^{\prime}}{\mathbb{P}}\left(q^{\mathrm{rkf} f u]}>q^{r+s}\right)<q^{1-s}
$$

by Markov's inequality.
The final ingredient is a lemma of Shpilka-Haramaty [5] regarding subspaces of bilinear forms of bounded rank. We include the proof for the reader's convenience.

Lemma 2.3. If $g: U \times V \times W \rightarrow k$ is a trilinear form with

$$
\underset{u \in U}{\mathbb{P}}(\operatorname{rkg}[u]>t)<\frac{q-1}{2 q t}
$$

for some positive integer $t$ then $\operatorname{srk}(g) \leq 4 t$. Moreover, the linear forms in the slice rank decomposition are obtained by fixing coordinates of $g$.

Proof. Let $A=\{u \in U: \operatorname{rkg}[u] \leq t\}$. Our assumption is that $\frac{|A|}{|U|}>1-\frac{q-1}{2 q t}$. The proof proceeds by induction on $t$.

Base case $t=1$ : Assume there is some $u_{0} \in A$ with $\operatorname{rkg} g\left[u_{0}\right]=1$ (otherwise $g \equiv 0$ and there is nothing to prove). Writing $g\left[u_{0}\right]=\alpha(v) \beta(w)$, we claim that for all $u \in A \cap\left(A-u_{0}\right)$ the bilinear form $g[u]$ is contained in the ideal $(\alpha, \beta)$. Indeed, if $g[u]=\gamma(v) \delta(w)$ with $\gamma \notin \operatorname{span}(\alpha)$ and $\delta \notin \operatorname{span}(\beta)$ then

$$
\operatorname{rkg}\left[u_{0}+u\right]=\operatorname{rk}(\alpha(v) \beta(w)+\gamma(v) \delta(w))=2
$$

contradicting the fact that $u_{0}+u \in A$. Therefore, if $V_{0}=\{v: \alpha(v)=0\}$ and $W_{0}=\{w: \beta(w)=0\}$ and $\tilde{g}=g \upharpoonright_{U \times V_{0} \times W_{0}}$, we get that $\tilde{g}[u]=0$ whenever $u \in A \cap\left(A-u_{0}\right)$, a set of density greater than $1-2 \cdot \frac{q-1}{2 q}=\frac{1}{q}$. This implies $\tilde{g}=0$ so $g \in(\alpha, \beta) \Longrightarrow \operatorname{srk}(g) \leq 2$.

Inductive step: We may assume that there is some $u_{0} \in A$ with $\operatorname{rkg} g\left[u_{0}\right]=t$, otherwise we're done by the inductive hypothesis. Write $g\left[u_{0}\right]=\sum_{i=1}^{t} \alpha_{i}(v) \beta_{i}(w)$ and set $V_{0}=\left\{v: \alpha_{i}(v)=0 \forall i \in[t]\right\}, W_{0}=\left\{w: \beta_{i}(w)=0 \forall i \in[t]\right\}$. We claim that the form $\tilde{g}=g \quad{ }_{U U \times V_{0} \times W_{0}}$ satisfies $\operatorname{rk} \tilde{g}[u] \leq t / 2$ for all $u \in A \cap\left(A-u_{0}\right)$. This will complete the proof because this set of $u$ 's has density greater than $1-2 \cdot \frac{q-1}{2 q t} \geq 1-\frac{q-1}{2 q\lfloor t / 2\rfloor}$, so the inductive hypothesis implies

$$
\operatorname{srk}(\tilde{g}) \leq 4\lfloor t / 2\rfloor \leq 2 t \Longrightarrow \operatorname{srk}(g) \leq 4 t
$$

To prove the claim about $\operatorname{rk} \tilde{g}[u]$, suppose it has rank $s$ and write $\tilde{g}[u]=\sum_{i=1}^{s} \gamma_{i}(v) \delta_{i}(w)$. Note that $\alpha_{i}, \beta_{i}, \gamma_{i}, \delta_{i}$ are linearly independent. Choosing bases, we may identify $V=\mathbf{k}^{n}, W=\mathbf{k}^{m}$ and

$$
\gamma_{i}=v_{i}, \delta_{i}=w_{i}, \alpha_{i}=v_{s+i}, \beta_{i}=w_{s+i}
$$

This yields an equation

$$
g[u]=\sum_{i=1}^{s} v_{i} w_{i}+\sum_{j=1}^{t} v_{s+j} \phi_{j}(w)+\sum_{j=1}^{t} w_{s+j} \tau_{j}(v) .
$$

Decomposing

$$
\phi_{j}(w)=\sum_{i=1}^{s} a_{i, j} w_{i}+\phi^{\prime}\left(w_{s+1}, \ldots, w_{m}\right), \tau_{j}(v)=\sum_{i=1}^{s} b_{i, j} v_{i}+\tau^{\prime}\left(v_{s+1}, \ldots, v_{n}\right)
$$

we get

$$
g[u]=\sum_{i=1}^{s} \gamma_{i}^{\prime}(v) \delta_{i}^{\prime}(w)+q\left(v_{s+1}, \ldots, v_{n}, w_{s+1}, \ldots, w_{m}\right)
$$

where

$$
\gamma_{i}^{\prime}(v)=v_{i}+\sum_{j=1}^{t} a_{i, j} v_{s+j}, \quad \delta_{i}^{\prime}(w)=w_{i}+\sum_{j=1}^{t} b_{i, j} w_{s+j} .
$$

The collection of linear forms $\gamma_{1}^{\prime}, \ldots, \gamma_{s}^{\prime}, v_{s+1}, \ldots, v_{n}$ spans $V^{*}$ and so is linearly independent, likewise for $\delta_{1}^{\prime}, \ldots, \delta_{s}^{\prime}, w_{s+1}, \ldots, w_{m}$. This means that

$$
\begin{equation*}
t \geq \operatorname{rkg}[u]=s+\operatorname{rk}(q) \tag{1}
\end{equation*}
$$

Since $u+u_{0} \in A$, we get

$$
\begin{aligned}
t & \geq \operatorname{rk}\left(g[u]+g\left[u_{0}\right]\right)=\operatorname{rk}\left(\sum_{i=1}^{s} \gamma_{i}^{\prime}(v) \delta_{i}^{\prime}(w)+\sum_{i=1}^{t} v_{s+i} w_{s+i}+q\right) \\
& =s+\operatorname{rk}\left(\sum_{i=1}^{t} v_{s+i} w_{s+i}+q\right) \geq s+t-\operatorname{rk}(q) \geq 2 s
\end{aligned}
$$

where the last inequality used (11). This proves that $\operatorname{rk} \tilde{g}[u]=s \leq t / 2$, completing the proof of the lemma.

Now we are ready to put everything together.

Proof of theorem 1.2. Let $U^{\prime}$ be the subspace of lemma 2.2 and let $g=f\left\lceil_{U^{\prime} \times V \times W}\right.$. For $s=\left\lceil\log _{q}(r+1)\right\rceil+6$, we get

$$
\begin{aligned}
\underset{u \in U^{\prime}}{\mathbb{P}}(\operatorname{rk} f[u]>r+s) & <q^{1-s} \leq \frac{1}{q^{5}(r+1)} \\
& <\frac{1}{4(r+s)} \leq \frac{q-1}{2 q} \cdot \frac{1}{r+s}
\end{aligned}
$$

where the inequality at the start of the second line follows from

$$
q^{5}(r+1)>4 r+8 r+28>4\left(r+\left\lceil\log _{q}(r+1)\right\rceil+6\right)
$$

By lemma 2.3, we deduce that $\operatorname{srk}(g) \leq 4(r+s)$ and so

$$
\operatorname{srk}(f) \leq r+1+4(r+s) \leq 5 r+4 \log _{q}(r+1)+29 .
$$

The proof of theorem 1.2 is asymmetric in its treatment of $U, V, W$. While only $r+1 U$-linear forms are needed, we require $\approx 2 r$ linear forms on each of $V, W$. One might hope that a more symmetric treatment would require only $r+1$ linear forms in each of $U, V, W$. This would essentially match the best bound currently known.

Question 2.4. Is there an elementary proof that $\operatorname{srk}(f) \leq 3 \cdot \operatorname{ark}(f)+o(\operatorname{ark}(f))$ ?

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