SLICE RANK AND ANALYTIC RANK FOR TRILINEAR FORMS

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ABSTRACT. In this note, we present an elementary proof of the fact that the slice rank of a trilinear form over a finite field is bounded above by a linear expression in the analytic rank. The existing proofs by Adiprasito-Kazhdan-Ziegler and Cohen-Moshkovitz both rely on results of Derksen via geometric invariant theory. A novel feature of our proof is that the linear forms appearing in the slice rank decomposition are obtained from the trilinear form by fixing coordinates.

1. The theorem

Let $\mathbf{k} = \mathbb{F}_q$ be a finite field and let U, V, W be finite dimensional vector spaces over \mathbf{k} .

Definition 1.1. For a trilinear form $f: U \times V \times W \to \mathbf{k}$ we are interested in two kinds of rank.

• The *slice rank* of f is

$$\operatorname{srk}(f) = \min\left\{r: f = \sum_{i=1}^{r} \alpha_i \cdot h_i\right\},\$$

where α_i, h_i are linear and bilinear, respectively, in disjoint sets of variables. • The *analytic rank* of f is

$$\operatorname{ark}(f) = -\log_q \frac{|Z|}{|U \times V|},$$

where $Z = \{(u, v) \in U \times V : f(u, v, \cdot) \equiv 0\}.$

The definition of slice rank goes back to the work of Schmidt [9] on systems of polynomial equations. It was reidiscovered in [10] and used to give a reformulation of Ellenberg and Gijswijt's work on the capset problem. Analytic rank was introduced in [4]. The inequality $\operatorname{ark}(f) \leq \operatorname{srk}(f)$ is straightforward, see [6] or [7]. In the other direction, Adiprasito-Kazhdan-Ziegler [1] and Cohen-Moshkovitz [2] independently proved that $\operatorname{srk}(f) \ll \operatorname{ark}(f)$. Their proofs use results of Derksen [3] which rely on the powerful tools of geometric invariant theory. The goal of this note is to give an elementary proof.

Theorem 1.2. For any finite field $\mathbf{k} = \mathbb{F}_q$ and trilinear form f, we have

 $\operatorname{srk}(f) \le 5 \cdot \operatorname{ark}(f) + 4 \cdot \log_q(\operatorname{ark}(f) + 1) + 29.$

In addition, the linear forms in the corresponding slice rank decomposition are obtained from f by fixing coordinates.

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Remark 1.3. This is slightly weaker than the best current bound $\operatorname{srk}(f) \leq 3 \cdot \operatorname{ark}(f)$, obtained in [1].

Theorem 1.2 is the first nontrivial case of a more general conjecture relating analytic rank to another kind of rank, *partition rank*. To state the general conjecture, we need some definitions. Let V_1, \ldots, V_d be finite-dimensional vector spaces over \mathbf{k} and let $f: V_1 \times \ldots \times V_d \to \mathbf{k}$ be a multilinear form.

Definition 1.4. • The *partition rank* of *f* is

$$\operatorname{prk}(f) = \min\left\{r: f = \sum_{i=1}^{r} g_i \cdot h_i\right\},\$$

where g_i, h_i are multilinear in disjoint sets of variables. Note that if d = 3 this agrees with the slice rank.

• The analytic rank of f is

$$\operatorname{ark}(f) = -\log_q \frac{|Z|}{|V_1 \times \ldots \times V_{d-1}|},$$

where $Z = \{(x_1, \ldots, x_{d-1}) \in V_1 \times \ldots \times V_{d-1} : f(x_1, \ldots, x_{d-1}, \cdot) \equiv 0\}.$

Again, the inequality $\operatorname{ark}(f) \leq \operatorname{prk}(f)$ is not difficult, see [6] and [7]. Lovett [7] and Adiprasito-Kazhdan-Ziegler [1] conjectured that the reverse inequality also holds, up to a constant factor.

Conjecture 1.5. There exists a constant C_d such that for any finite field and multilinear form f we have

$$\operatorname{prk}(f) \le C_d \cdot \operatorname{ark}(f).$$

The current best bound in this direction is

$$\operatorname{prk}(f) \le C_d \cdot \operatorname{ark}(f) \cdot \log^{d-1}(1 + \operatorname{ark}(f)),$$

due to Moshkovitz-Zhu [8].

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2. Proof of theorem 1.2

Let $r = \operatorname{ark}(f)$. For $u \in U$ we write f[u] for the bilinear form $f[u] : V \times W \to \mathbf{k}$ obtained by fixing the U coordinate. For $v \in V$ we write $f\langle v \rangle$ for the bilinear form $f\langle v \rangle : U \times W \to \mathbf{k}$ obtained by fixing the V coordinate. For a finite set X, denote $\mathbb{E}_{x \in X} = \frac{1}{|X|} \sum_{x \in X}$. Our first lemma is inspired by a recent observation of Moshkovitz-Zhu [8].

Lemma 2.1.

$$\mathop{\mathbb{E}}_{(u,v)\in Z} q^{\operatorname{rk} f[u]} = \mathop{\mathbb{E}}_{(u,v)\in Z} q^{\operatorname{rk} f\langle v\rangle} = q^r.$$

Proof. A straightforward computation. Let

$$p(u) = \frac{|\{v \in V : (u, v) \in Z\}|}{|Z|} = q^{-\mathrm{rk}f[u]}\frac{|V|}{|Z|}$$

be the marginal probability mass function for $u \in U$. Then

$$\mathbb{E}_{\substack{(u,v)\in Z\\u}} q^{\operatorname{rk} f[u]} = \sum_{u} p(u) \cdot q^{\operatorname{rk} f[u]}$$
$$= \sum_{u} \frac{|V|}{|Z|} = \frac{|U| \cdot |V|}{|Z|} = q^{r}.$$

The proof for $\operatorname{rk} f \langle v \rangle$ is identical.

This allows us to find a large subspace $U' \subset U$ where $\operatorname{rk} f[u]$ is almost surely small.

Lemma 2.2. There exists a subspace $U' \subset U$ with $\operatorname{codim} U' \leq r+1$ and

$$\mathbb{P}_{u \in U'}(\mathrm{rk}f[u] > r+s) < q^{1-1}$$

for every s > 0. The linear forms defining U' are obtained from f by fixing coordinates.

Proof. By linearity of expectation,

$$\mathop{\mathbb{E}}_{(u,v)\in Z}(q^{\operatorname{rk} f[u]} + q^{\operatorname{rk} f\langle v\rangle}) = 2q^r \le q^{r+1}.$$

Therefore, there must exist some $v_0 \in V$ with

$$q^{\operatorname{rk} f\langle v_0 \rangle} + \mathop{\mathbb{E}}_{u \in Z(v_0)} q^{\operatorname{rk} f[u]} \le q^{r+1},$$

where $Z(v_0) = \{u \in U : (u, v_0) \in Z\}$. Choosing $U' = Z(v_0)$, we have $\operatorname{codim} U' = \operatorname{rk} f\langle v_0 \rangle \leq r+1$ and

$$\mathbb{P}_{u \in U'}(\mathrm{rk}f[u] > r + s) = \mathbb{P}_{u \in U'}(q^{\mathrm{rk}f[u]} > q^{r+s}) < q^{1-s}$$

by Markov's inequality.

The final ingredient is a lemma of Shpilka-Haramaty [5] regarding subspaces of bilinear forms of bounded rank. We include the proof for the reader's convenience.

Lemma 2.3. If $g: U \times V \times W \rightarrow k$ is a trilinear form with

$$\mathbb{P}_{u \in U}(\mathrm{rk}g[u] > t) < \frac{q-1}{2qt}$$

for some positive integer t then $\operatorname{srk}(g) \leq 4t$. Moreover, the linear forms in the slice rank decomposition are obtained by fixing coordinates of g.

Proof. Let $A = \{u \in U : \operatorname{rk} g[u] \leq t\}$. Our assumption is that $\frac{|A|}{|U|} > 1 - \frac{q-1}{2qt}$. The proof proceeds by induction on t.

Base case t = 1: Assume there is some $u_0 \in A$ with $\operatorname{rk} g[u_0] = 1$ (otherwise $g \equiv 0$ and there is nothing to prove). Writing $g[u_0] = \alpha(v)\beta(w)$, we claim that for all $u \in A \cap (A - u_0)$ the bilinear form g[u] is contained in the ideal (α, β) . Indeed, if $g[u] = \gamma(v)\delta(w)$ with $\gamma \notin \operatorname{span}(\alpha)$ and $\delta \notin \operatorname{span}(\beta)$ then

$$\operatorname{rk} g[u_0 + u] = \operatorname{rk}(\alpha(v)\beta(w) + \gamma(v)\delta(w)) = 2,$$

contradicting the fact that $u_0 + u \in A$. Therefore, if $V_0 = \{v : \alpha(v) = 0\}$ and $W_0 = \{w : \beta(w) = 0\}$ and $\tilde{g} = g \upharpoonright_{U \times V_0 \times W_0}$, we get that $\tilde{g}[u] = 0$ whenever $u \in A \cap (A - u_0)$, a set of density greater than $1 - 2 \cdot \frac{q-1}{2q} = \frac{1}{q}$. This implies $\tilde{g} = 0$ so $g \in (\alpha, \beta) \implies \operatorname{srk}(g) \leq 2$.

AMICHAI LAMPERT

Inductive step: We may assume that there is some $u_0 \in A$ with $\operatorname{rkg}[u_0] = t$, otherwise we're done by the inductive hypothesis. Write $g[u_0] = \sum_{i=1}^t \alpha_i(v)\beta_i(w)$ and set $V_0 = \{v : \alpha_i(v) = 0 \ \forall i \in [t]\}, W_0 = \{w : \beta_i(w) = 0 \ \forall i \in [t]\}$. We claim that the form $\tilde{g} = g \quad [U \times V_0 \times W_0$ satisfies $\operatorname{rk}\tilde{g}[u] \leq t/2$ for all $u \in A \cap (A - u_0)$. This will complete the proof because this set of u's has density greater than $1 - 2 \cdot \frac{q-1}{2qt} \geq 1 - \frac{q-1}{2q[t/2]}$, so the inductive hypothesis implies

$$\operatorname{srk}(\tilde{g}) \le 4\lfloor t/2 \rfloor \le 2t \implies \operatorname{srk}(g) \le 4t.$$

To prove the claim about $rk\tilde{g}[u]$, suppose it has rank s and write $\tilde{g}[u] = \sum_{i=1}^{s} \gamma_i(v)\delta_i(w)$. Note that $\alpha_i, \beta_i, \gamma_i, \delta_i$ are linearly independent. Choosing bases, we may identify $V = \mathbf{k}^n, W = \mathbf{k}^m$ and

$$\gamma_i = v_i, \ \delta_i = w_i, \ \alpha_i = v_{s+i}, \ \beta_i = w_{s+i}.$$

This yields an equation

$$g[u] = \sum_{i=1}^{s} v_i w_i + \sum_{j=1}^{t} v_{s+j} \phi_j(w) + \sum_{j=1}^{t} w_{s+j} \tau_j(v).$$

Decomposing

$$\phi_j(w) = \sum_{i=1}^s a_{i,j} w_i + \phi'(w_{s+1}, \dots, w_m), \ \tau_j(v) = \sum_{i=1}^s b_{i,j} v_i + \tau'(v_{s+1}, \dots, v_n),$$

we get

$$g[u] = \sum_{i=1}^{s} \gamma'_{i}(v)\delta'_{i}(w) + q(v_{s+1}, \dots, v_n, w_{s+1}, \dots, w_m),$$

where

$$\gamma'_i(v) = v_i + \sum_{j=1}^t a_{i,j} v_{s+j}, \ \delta'_i(w) = w_i + \sum_{j=1}^t b_{i,j} w_{s+j}$$

The collection of linear forms $\gamma'_1, \ldots, \gamma'_s, v_{s+1}, \ldots, v_n$ spans V^* and so is linearly independent, likewise for $\delta'_1, \ldots, \delta'_s, w_{s+1}, \ldots, w_m$. This means that

(1)
$$t \ge \mathrm{rk}g[u] = s + \mathrm{rk}(q).$$

Since $u + u_0 \in A$, we get

$$t \ge \operatorname{rk}(g[u] + g[u_0]) = \operatorname{rk}\left(\sum_{i=1}^s \gamma'_i(v)\delta'_i(w) + \sum_{i=1}^t v_{s+i}w_{s+i} + q\right)$$
$$= s + \operatorname{rk}\left(\sum_{i=1}^t v_{s+i}w_{s+i} + q\right) \ge s + t - \operatorname{rk}(q) \ge 2s$$

where the last inequality used (1). This proves that $\mathrm{rk}\tilde{g}[u] = s \leq t/2$, completing the proof of the lemma.

Now we are ready to put everything together.

Proof of theorem 1.2. Let U' be the subspace of lemma 2.2 and let $g = f \mid_{U' \times V \times W}$. For $s = \lceil \log_q(r+1) \rceil + 6$, we get

$$\mathbb{P}_{u \in U'}(\operatorname{rk} f[u] > r+s) < q^{1-s} \le \frac{1}{q^5(r+1)} < \frac{1}{4(r+s)} \le \frac{q-1}{2q} \cdot \frac{1}{r+s},$$

where the inequality at the start of the second line follows from

$$q^{5}(r+1) > 4r + 8r + 28 > 4(r + \lceil \log_{q}(r+1) \rceil + 6).$$

By lemma 2.3, we deduce that $srk(g) \leq 4(r+s)$ and so

$$\operatorname{srk}(f) \le r + 1 + 4(r+s) \le 5r + 4\log_a(r+1) + 29.$$

The proof of theorem 1.2 is asymmetric in its treatment of U, V, W. While only r + 1 U-linear forms are needed, we require $\approx 2r$ linear forms on each of V, W. One might hope that a more symmetric treatment would require only r + 1 linear forms in each of U, V, W. This would essentially match the best bound currently known.

Question 2.4. Is there an elementary proof that $\operatorname{srk}(f) \leq 3 \cdot \operatorname{ark}(f) + o(\operatorname{ark}(f))$?

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