

FIBERING POLARIZATIONS AND MABUCHI RAYS ON SYMMETRIC SPACES OF COMPACT TYPE

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ABSTRACT. In this paper, we describe holomorphic quantizations of the cotangent bundle of a symmetric space of compact type $T^*(U/K) \cong U_{\mathbb{C}}/K_{\mathbb{C}}$, along Mabuchi rays of U -invariant Kähler structures. At infinite geodesic time, the Kähler polarizations converge to a mixed polarization \mathcal{P}_{∞} . We show how a generalized coherent state transform relates the quantizations along the Mabuchi geodesics such that holomorphic sections converge, as geodesic time goes to infinity, to distributional \mathcal{P}_{∞} -polarized sections. Unlike in the case of T^*U , the gCST mapping from the Hilbert space of vertically polarized sections are not asymptotically unitary due to the appearance of representation dependent factors associated to the isotypical decomposition for the U -action. In agreement with the general program outlined in [Bai+23], we also describe how the quantization in the limit polarization \mathcal{P}_{∞} is given by the direct sum of the quantizations for all the symplectic reductions relative to the invariant torus action associated to the Hamiltonian action of U .

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1. INTRODUCTION

In this paper, we pursue the program outlined in [Bai+23] for the geometric quantization of Kähler manifolds with an Hamiltonian action of a compact Lie group U , by considering the case of the cotangent bundle of a compact symmetric space,

$$T^*(U/K) \cong U_{\mathbb{C}}/K_{\mathbb{C}}.$$

The main idea is to use Mabuchi rays of Kähler structures, generated by the Hamiltonian flows in imaginary time of Hamiltonian functions which are convex in the moment map, to relate families of holomorphic quantizations to the quantization in a mixed polarization which is attained at infinite Mabuchi geodesic time.

The case of the cotangent bundle T^*U of a Lie group U of compact type has been studied in [Hal02; KMN13; KMN14; Bai+23]. The related case of symplectic toric manifolds has been studied in [Bai+11; KMN16] and for recent developments in Kähler manifolds with T -symmetry see [LW22; LW23a; LW23b]. Applications to the case of flag manifolds are explored in [HK14] and for more general algebraic varieties in [HHK21]

In Section 2, we describe Mabuchi rays of U -invariant Kähler structures on $T^*(U/K)$, obtained by symplectic reduction from invariant Kähler structures on T^*U . These geodesic rays are generated by the Hamiltonian flow in imaginary time of convex functions on \mathfrak{u}^* which are compatible with the symmetric space involution. At infinite geodesic time along the Mabuchi rays, in Section 3, we obtain the mixed polarization \mathcal{P}_{∞} on the open dense subset of regular values of the moment map for the right K -action on T^*U , $T^*(U/K)_{\text{reg}}$. In Section 4, we show how a generalized coherent state transform relates the Kähler quantizations of $T^*(U/K)$ along the Mabuchi rays such that, in the limit of infinite geodesic time, the elements of natural basis of holomorphic sections, given by the isotypical decomposition with respect to the U -action, converge to distributional polarized sections for \mathcal{P}_{∞} . In Section 4.4, we show that independently of the Mabuchi ray which one follows, connecting the vertical, or Schrödinger, polarization to \mathcal{P}_{∞} , there is a well-defined limit for the inner product structures along the family of Kähler polarizations. This leads to a natural definition of Hilbert space structure on the space of \mathcal{P}_{∞} -polarized section \mathcal{H}_{∞} . As described in Sections 4.4 and 4.6, in contrast to the case of T^*U studied in [Bai+23], the resulting U -equivariant isomorphism $\mathcal{H}_{\text{Sch}} \rightarrow \mathcal{H}_{\infty}$ defined by the gCST is not

asymptotically unitary (even for Mabuchi rays generated by Hamiltonians quadratic in the moment map) and unitarity is achieved only by including representation-dependent correcting factors for each isotypical component. Finally, in Section 4.7, we give a quantum-geometrical interpretation of \mathcal{P}_∞ , in the line of the general program described in [Bai+23], where we relate the Kähler quantization of $T^*(U/K)$ with a direct sum over the quantizations of the coisotropic reductions for \mathcal{P}_∞ or, equivalently, over the quantizations of the symplectic reductions for the invariant moment map.

2. PRELIMINARIES

2.1. Basic definitions. Let U be a compact simply connected Lie group and

$$\sigma : U \rightarrow U$$

an involutive automorphism such that U/K is a symmetric space of compact type, where K is a closed subgroup of U and a relatively open subgroup of the set U^σ of fixed points¹. One has for the Lie algebra \mathfrak{u} of U , and for its dual \mathfrak{u}^* , orthogonal decompositions with respect to a fixed Ad_U -invariant inner product, $\langle \cdot, \cdot \rangle_{\mathfrak{u}}$, on \mathfrak{u} ,

$$\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{s}, \quad \mathfrak{u}^* = \mathfrak{k}^* \oplus \mathfrak{s}^*,$$

where, for the derived automorphism of \mathfrak{u} , also denoted by σ ,

$$\sigma|_{\mathfrak{k}} = \text{Id}_{\mathfrak{k}}, \quad \sigma|_{\mathfrak{s}} = -\text{Id}_{\mathfrak{s}},$$

with $[\mathfrak{k}, \mathfrak{s}] \subset \mathfrak{s}$ and $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{k}$.

The cotangent bundle $T^*(U/K)$ has the structure of a homogeneous vector bundle associated to the principal K -bundle $U \rightarrow U/K$,

$$T^*(U/K) = U \times_K \mathfrak{s}^*.$$

2.2. Invariant Kähler structures on $T^*(U/K)$. Here, we will recall standard facts about the symplectic geometry of the cotangent bundles $T^*(U)$ and $T^*(U/K)$, realized as the symplectic quotient

$$T^*(U/K) = (T^*U) // K.$$

Recall that there is a Hamiltonian right action of U on T^*U with moment map $\mu^{(R)}$. The moment map of the corresponding right action of the subgroup $K \subset U$ is

$$(1) \quad \mu^K = \pi_{\mathfrak{k}^*} \circ \mu^{(R)}, \quad \pi_{\mathfrak{k}^*} : \mathfrak{u}^* \rightarrow \mathfrak{k}^*.$$

Using the trivialization $T^*(U) \cong U \times \mathfrak{u}^*$ one finds that

$$(2) \quad (\mu^K)^{-1}(0) = U \times \mathfrak{s}^* \quad \text{and} \quad (T^*U) // K = (\mu^K)^{-1}(0)/K = U \times_K \mathfrak{s}^* = T^*(U/K).$$

¹Since we are taking U to be simply connected, from [Tak94, Lemma II.6.2] we also have that U^σ and hence K is connected.

The moment map of the left U -action on $T^*(U/K)$, which we will denote μ , descends from T^*U through this quotient,

$$(3) \quad U \curvearrowright T^*(U/K) \xrightarrow{\mu} \mathfrak{u}^*, \quad \mu([x, \xi]) = \text{Ad}_x^* \xi,$$

where $[x, \xi]$ denotes the K -orbit through the point (x, ξ) in $U \times \mathfrak{s}^*$.

If we equip T^*U with a $U \times U$ -invariant Kähler structure, then this symplectic quotient becomes a Kähler quotient so that $T^*(U/K)$ inherits an U -invariant Kähler structure with respect to the reduced symplectic structure. We will now describe an infinite-dimensional space of U -invariant Kähler structures on $T^*(U/K)$ obtained by Kähler reduction from the $(U \times U)$ -invariant Kähler structures on $T^*(U)$, which are described in Section 2.3 of [Bai+23] and in [Nee00b; Nee00a; KMN13]. Given a uniformly convex invariant function on \mathfrak{u}^* ,

$$g \in \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^{\text{Ad}U}),$$

one has the Legendre transform $\mathcal{L}_g : T^*U \cong U \times \mathfrak{u}^* \rightarrow U_{\mathbb{C}}$ given by

$$\mathcal{L}_g(x, \xi) = xe^{id_\xi g}.$$

The pull-back of the canonical complex structure on $U_{\mathbb{C}}$ by \mathcal{L}_g and the canonical symplectic structure on T^*U define a $U \times U$ -invariant Kähler structure on T^*U , I_g , with global Kähler potential

$$\kappa_g(x, \xi) = \langle \xi, d_\xi g \rangle - g(\xi).$$

Lemma 1. *For any uniformly convex $g \in \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^{\text{Ad}U})$, the corresponding Kähler potential $\kappa_g : U_{\mathbb{C}} \rightarrow \mathbb{R}$ is exhausting.*

Proof. We need to show that the component in \mathfrak{u}^* of the preimage of an interval $] -\infty, \lambda[$ under $\kappa_g \circ \mathcal{L}_g$ is bounded, which follows if we show that $\kappa_g \circ \mathcal{L}_g$ is not bounded above on unbounded subsets of \mathfrak{u}^* . As g is uniformly convex on \mathfrak{u}^* , its Legendre transform maps unbounded subsets to unbounded subsets in \mathfrak{u} , but on these κ_g is unbounded as it is supercoercive by a proposition of Moreau and Rockafellar [BV10, Prop. 3.5.4]. \square

Remark 1. As we will see in Theorem 1, the exhaustion property of the global Kähler potential κ_g is needed to show that all points in $U_{\mathbb{C}}$ are stable with respect to the $K_{\mathbb{C}}$ -action in the sense that all $K_{\mathbb{C}}$ -orbits intersect the zero level set of the K -moment map.

Remark 2. Let σ also denote the involution on \mathfrak{u}^* induced from $\sigma : \mathfrak{u} \rightarrow \mathfrak{u}$, so that

$$\sigma|_{\mathfrak{k}^*} = \text{Id}_{\mathfrak{k}^*}, \quad \sigma|_{\mathfrak{s}^*} = -\text{Id}_{\mathfrak{s}^*}.$$

In this paper, we will always assume that the symplectic potential g (and also h and $g_t = g + th, t > 0$, to appear below), are compatible with the involution σ , that is we assume that

$$g \circ \sigma = g.$$

This is not a major restriction since symplectic potentials built from the even degree Casimirs of \mathfrak{u} will satisfy this property, to begin with the fundamental example given by $g = \frac{1}{2} \|\xi\|^2$. In addition to the properties listed in Proposition 2.7 of [Bai+23], we will therefore also have that:

$$d_\xi g = d_{\sigma(\xi)} g \circ \sigma = \sigma(d_{\sigma(\xi)} g),$$

so that, in particular, for $\xi \in \mathfrak{s}^*$ we get

$$(d_\xi g)|_{\mathfrak{s}^*} = -(d_{-\xi} g)|_{\mathfrak{s}^*},$$

and

$$(d_\xi g)|_{\mathfrak{k}^*} = (d_{-\xi} g)|_{\mathfrak{k}^*}.$$

For the Hessian of g we also obtain (see (c) in Proposition 2.7 of [Bai+23]), for $\xi \in \mathfrak{s}^*$,

$$\text{Hess}_g(\xi)|_{\mathfrak{s}^*} = \text{Hess}_g(-\xi)|_{\mathfrak{s}^*}$$

These identities will be used below.

Remark 3. Note that, from Lemma 3.1 in [KMN13], one has that the map $\mathfrak{u}^* \ni \xi \mapsto d_\xi g \in \mathfrak{u}$ is a diffeomorphism. The compatibility condition between g and σ , imposed in Remark 2, then gives that for $\xi \in \mathfrak{k}^*$,

$$d_\xi g = \sigma d_\xi g,$$

which implies that $d_\xi g \in \mathfrak{k}$ for $\xi \in \mathfrak{k}^*$. The fact that g is compatible with σ also implies that the Kähler potential κ_g is compatible with σ so that the inverse Legendre transform, which is the Legendre transform with respect to κ_g , maps \mathfrak{k} bijectively onto \mathfrak{k}^* . It follows that, since $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{s}$,

$$\xi \in \mathfrak{s}^* \Leftrightarrow d_\xi g \in \mathfrak{s}.$$

Theorem 1. *Symplectic reduction provides us with a map*

$$(4) \quad \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^*)^{\text{Ad}_U^*} \xrightarrow{\hat{I}} \mathcal{J}(T^*(U/K), \omega_{\text{std}})^U, \quad g \mapsto \hat{I}_g,$$

where $\mathcal{J}(T^*(U/K), \omega_{\text{std}})^U$ is the space of U -invariant complex structures on $T^*(U/K)$ compatible with the standard symplectic form ω_{std} . Dually, these Kähler structures are described by a map

$$(5) \quad \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^*)^{\text{Ad}_U^*} \xrightarrow{\hat{\omega}} \mathcal{K}(U_{\mathbb{C}}/K_{\mathbb{C}})^U, \quad g \mapsto \hat{\omega}_g,$$

to the space of U -invariant Kähler forms on $U_{\mathbb{C}}/K_{\mathbb{C}}$.

Explicitly, these maps are defined by pulling back the relevant structures along the map descending from the Legendre transform \mathcal{L}_g , which we still denote by \mathcal{L}_g , through the symplectic quotient,

$$(6) \quad \begin{array}{ccc} U \times \mathfrak{s}^* & \xrightarrow{\mathcal{L}_g|_{U \times \mathfrak{s}^*}} & U \times \mathfrak{u} & \longrightarrow & U_{\mathbb{C}} \\ \downarrow & & & & \downarrow \\ U \times_K \mathfrak{s}^* & \xrightarrow{\mathcal{L}_g} & & \longrightarrow & U_{\mathbb{C}}/K_{\mathbb{C}} \\ [x, \xi] & \longmapsto & & \longrightarrow & x e^{i d_\xi g} K_{\mathbb{C}} \end{array} .$$

In particular, a Kähler potential $\widehat{\kappa}_g$ for $\widehat{\omega}_g$ is given by the restriction of κ_g to the Kempf–Ness set $(\mu^K)^{-1}(0) = U \times \mathfrak{s}^*$ (see (1)),

$$(7) \quad \widehat{\kappa}_g \circ \mathcal{L}_g([x, \xi]) = \langle \xi, d_\xi g \rangle - g(\xi).$$

Proof. Since, by Lemma 1, κ_g is a global $(U \times U)$ -invariant strictly plurisubharmonic exhaustion function on $U_{\mathbb{C}}$, it follows from [HH99, Lemma 2.4.2] that all points in $U_{\mathbb{C}}$ are $\mu^K \circ (\mathcal{L}_g)^{-1}$ -stable. Considering the restriction of \mathcal{L}_g to $(\mu^K)^{-1}(0) = U \times \mathfrak{s}^*$ as in (6), we know that both these maps are diffeomorphisms onto their images. From the equivariance of the Legendre transform (see Proposition 2.7 in [Bai+23]), we know that the map $\mathcal{L}_g|_{U \times \mathfrak{s}^*}$ descends to the K -quotients. The map \mathcal{L}_g in the lower arrow of (6) then corresponds to the map i_x in [HH99, Lemma 2.4.3] and hence defines a diffeomorphism between the symplectic and the Hilbert quotients. Thus, it defines the Kähler quotient structure on these quotients.

The form of the Kähler potential in (7) follows from the fact that the Kähler potential of a Kähler reduction is obtained by restricting the Kähler potential to the Kempf–Ness set, cf. [HH99, Proposition 2.4.6] or [BG97, Theorem 7]. \square

Remark 4. We note that the fact that the map (6) is a diffeomorphism is not at all easy to see from Lie theoretic arguments, not even in the case of $g(\xi) = |\xi|^2/2$, for which $d_\xi g = \xi$.

We denote by $\widetilde{\mathcal{K}}(U/K)$ the space of Kähler structures on $T^*(U/K)$ corresponding to the image of the map \widehat{I} in (4)

$$\widetilde{\mathcal{K}}(U/K) := \left\{ (\omega_{\text{std}}, \widehat{I}_g) \mid g \in \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^*)^{\text{Ad}_U^*} \right\}.$$

2.3. Restricted roots, Satake diagrams. Let U/K be an irreducible symmetric space of compact type and recall that $T^*(U/K) \cong U \times_K \mathfrak{s}^*$, with U -moment map given by (3). We will assume that the Cartan subalgebra \mathfrak{t} of \mathfrak{u} is chosen σ -invariant such that

$$i\mathfrak{a} := \mathfrak{t} \cap \mathfrak{s} \subset \mathfrak{s}$$

is a maximal Abelian subspace of \mathfrak{s} .

We will now follow [Ara62; War72; Hel84; Hel01] to describe aspects of the geometry of the symmetric space U/K through the properties of the symmetric pair $(\mathfrak{u}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$. Let, again, σ denote the antilinear involution extended to $\mathfrak{u}_{\mathbb{C}}$ by antilinearity from the following linear involution on \mathfrak{u} ,

$$\sigma(X) = \begin{cases} X & \text{if } X \in \mathfrak{k} \\ -X & \text{if } X \in \mathfrak{s}. \end{cases}$$

Note that $\sigma|_{\mathfrak{a}} = \text{Id}_{\mathfrak{a}}$. The antilinear involution σ defines an antilinear involution on $\mathfrak{u}_{\mathbb{C}}^*$, which leaves the root system Φ of $(\mathfrak{u}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$ invariant, and

defines an involutive isometry of Φ [Ara62, **1.3**].

$$(8) \quad \begin{aligned} \mathfrak{u}_{\mathbb{C}}^* \ni \xi &\mapsto \xi^\sigma \in \mathfrak{u}_{\mathbb{C}}^* \\ \xi^\sigma(X) &:= \overline{\xi(\sigma(X))}, \quad \forall X \in \mathfrak{u}_{\mathbb{C}}. \end{aligned}$$

Let $E_\alpha, \alpha \in \Phi$, be a choice of root vectors such that $[E_\alpha, E_{-\alpha}] = H_\alpha$, where H_α denotes the coroot vector corresponding to α , that is $\beta(H_\alpha) = \langle \alpha, \beta \rangle, \beta \in \Phi$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on \mathfrak{t} obtained from the fixed invariant inner product on \mathfrak{u} and extended to $\mathfrak{u}_{\mathbb{C}}$. If $\mathfrak{u}_\alpha := \mathbb{C}E_\alpha$ denotes the root space generated by the vector E_α , then

$$\sigma(E_\alpha) \in \mathfrak{u}_{\alpha^\sigma}.$$

The pair (Φ, σ) is called a σ -root system. The subset

$$\Phi_0 := \{\alpha \in \Phi \mid \alpha^\sigma = -\alpha\} = \{\alpha \in \Phi \mid \alpha|_{\mathfrak{a}} = 0\}$$

is a root subsystem of Φ .

The order of \mathfrak{it}^* with positive cone $\text{Cone}(\Lambda_+)$, where Λ_+ is the set of dominant integral weights induced by the choice of positive roots, $\Phi_+ \subset \Phi$, is called a σ -order if [Ara62, **2.8**], [War72, p. 23]

$$(9) \quad \alpha \in (\Phi \setminus \Phi_0) \cap \Phi_+ \Rightarrow \alpha^\sigma \in \Phi_+.$$

If Δ is a system of simple roots for a σ -order, then $\Delta_0 := \Delta \cap \Phi_0$ is a system of simple roots for Φ_0 . Let $r := \text{rank}(\Phi)$, $r_0 := \text{rank}(\Phi_0)$ and

$$\Delta = \{\alpha_1, \dots, \alpha_{r-r_0}, \alpha_{r-r_0+1}, \dots, \alpha_r\}$$

with

$$\Delta_0 = \{\alpha_{r-r_0+1}, \dots, \alpha_r\}.$$

Then σ induces a permutation of $\Delta \setminus \Delta_0$ as there is an involutive permutation $\tilde{\sigma}$ of $\{1, \dots, r - r_0\}$ such that [War72, Lemma 1.1.3.2]

$$(10) \quad \alpha_j^\sigma = \alpha_{\tilde{\sigma}(j)} + \sum_{k=r-r_0+1}^r c_k^{(j)} \alpha_k, \quad j = 1, \dots, r - r_0,$$

with coefficients $c_k^{(j)} \geq 0$. Then the restriction of roots to \mathfrak{a} ,

$$\Sigma := (\Phi \setminus \Phi_0)|_{\mathfrak{a}} \subset \mathfrak{a}^*,$$

is a root system, called the *restricted root system* of the σ -system (Φ, σ) , with system of simple roots, Δ^- given by

$$(11) \quad \Delta^- := \{\beta_1, \dots, \beta_l\} := (\Delta \setminus \Delta_0)|_{\mathfrak{a}},$$

where $l := \dim(\mathfrak{a}) = \text{rank}(U/K)$. Note that, if $\alpha \in \Phi \setminus \Phi_0$ then, since $\alpha \in \mathfrak{it}^* = i \text{Hom}_{\mathbb{R}}(\mathfrak{t}, \mathbb{R})$, one obtains $\alpha|_{\mathfrak{a}}^\sigma = \alpha|_{\mathfrak{a}}$.

We see from (10) that the simple roots α_j and $\alpha_{\tilde{\sigma}(j)}$ map to the same restricted root β_j . These are the only relations for the restriction of simple roots from $\Delta \setminus \Delta_0$ to \mathfrak{a} and the restricted root systems can be obtained from the Dynkin diagram of Δ by a *Satake diagram*: Simple roots from Δ_0 are represented by black circles and roots from $\Delta \setminus \Delta_0$ are represented by white

circles. White circles related by the permutation $\tilde{\sigma}$ are linked by a curved arrow. One has, therefore,

$$(12) \quad l = r - n_b - n_a,$$

where n_b is the number of black circles and n_a the number of arrows in the Satake diagram for U/K . The classification of irreducible symmetric spaces is then given by the possible Satake diagrams. Those corresponding to simple \mathfrak{u} are listed by the tables in [Ara62, p. 32, 33] and [War72, p. 30–32].

It follows from [Hel01, Lemma VI.3.6] that $\mathfrak{s}_{\mathbb{C}}$ has the following decomposition

$$(13) \quad \mathfrak{s}_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}} + \sum_{\alpha \in (\Phi \setminus \Phi_0) \cap \Phi_+} \mathbb{C}(E_{\alpha} - \theta E_{\alpha}),$$

where θ denotes the \mathbb{C} -linear involution on $\mathfrak{u}_{\mathbb{C}}$ that on \mathfrak{u} coincides with σ so that $\theta|_{\mathfrak{k}_{\mathbb{C}}} = \text{Id}_{\mathfrak{k}_{\mathbb{C}}}$ and $\theta|_{\mathfrak{s}_{\mathbb{C}}} = -\text{Id}_{\mathfrak{s}_{\mathbb{C}}}$.

2.4. Spherical representations. Recall that we are assuming that U is simply-connected.² (Note that since U is compact one has a splitting $\mathfrak{u} = [\mathfrak{u}, \mathfrak{u}] \oplus \mathfrak{b}$ where \mathfrak{b} is abelian; since U is simply-connected one has $\mathfrak{b} = \{0\}$ so that U is semisimple.) We have the isomorphism

$$L^2(U/K) \cong L^2(U)^K,$$

where $L^2(U)^K$ denotes the subspace of right K -invariant functions. Let \hat{U} denote the set of equivalence classes of irreducible representations V_{λ} of U , labelled by highest weight λ . Recall the Peter-Weyl theorem giving an orthogonal decomposition

$$L^2(U) \cong \widehat{\bigoplus}_{\lambda \in \hat{U}} \text{End}(V_{\lambda}),$$

where the hat over the sum denotes the norm completion. The summand

$$\text{End}(V_{\lambda}) \cong V_{\lambda} \otimes V_{\lambda}^*$$

is realized by

$$f_{\lambda, v \otimes w^*}(x) = \text{tr}(\pi_{\lambda}(x)v \otimes w^*) \in L^2(U), \quad \lambda \in \hat{U}, v \in V_{\lambda}, w^* \in V_{\lambda}^*,$$

where $\pi_{\lambda}(x) \in \text{End}(V_{\lambda})$ denotes the representative of $x \in U$. Then $L^2(U/K)$ is given by the terms which are invariant under the right K -action. From, Theorem 4.1 in Chapter 5 in [Hel84], each V_{λ} in \hat{U} contains at most a one-dimensional subspace of so-called K -spherical vectors which are invariant under K . The corresponding representations are called K -spherical representations and we denote them by \hat{U}_K .

²Just as in [Bai+23], this condition makes the presentation simpler but we expect the generalization to other cases to be straightforward. See Remark 5.

If $\lambda \in \hat{U}_K$ let $v_\lambda^K \in V_\lambda$ be a non-trivial K -spherical vector and let $V_\lambda^K = \mathbb{C} \cdot v_\lambda^K$ be the one-dimensional subspace of K -spherical vectors. Then, we obtain an orthogonal decomposition

$$L^2(U/K) \cong \widehat{\bigoplus}_{\lambda \in \hat{U}_K} \text{Hom}(V_\lambda, V_\lambda^K),$$

where the summand $\text{Hom}(V_\lambda, V_\lambda^K) \cong V_\lambda^K \otimes V_\lambda^*$ is realized by the right K -invariant functions

$$f_{\lambda, v_\lambda^K \otimes v^*} \in L^2(U), \quad v^* \in V_\lambda^*.$$

(Note that the choice of spherical vector v_λ^K is unique up to a multiplicative constant c and that $f_{\lambda, cv_\lambda^K \otimes v^*} = cf_{\lambda, v_\lambda^K \otimes v^*}$.)

From Theorem 4.1 in Chapter 5 in [Hel84] we also obtain the explicit characterization of the highest weights for the K -spherical representations. If $\mathfrak{t}_\mathbb{Z}^*$ denotes the weight lattice for U , we have that a highest weight $\lambda \in \mathfrak{t}_\mathbb{Z}^*$ is the highest weight of a K -spherical representation if and only if

$$\lambda(i(\mathfrak{t} \cap \mathfrak{k})) = 0.$$

We denote by Λ_+^K the set of highest weights of K -spherical representations. In particular, the cone generated by Λ_+^K is

$$\text{Cone}(\Lambda_+^K) = \mathfrak{a}^*,$$

where $\mathfrak{a}_+^* := i(\mathfrak{t}_+^* \cap \mathfrak{s}^*)$ and \mathfrak{t}_+^* is the closed positive Weyl chamber associated to the choice of positive roots $\Phi_+ \subset \Phi$, defined by

$$\mathfrak{t}_+^* = \{\xi \in \mathfrak{t}^* : \langle \alpha, i\xi \rangle \geq 0, \alpha \in \Phi_+\}$$

(Note that $\mathfrak{a}^* = \text{Hom}_\mathbb{R}(\mathfrak{a}, \mathbb{R}) = \text{Hom}_\mathbb{R}(\mathfrak{t} \cap \mathfrak{s}, i\mathbb{R}) = i(\mathfrak{t}^* \cap \mathfrak{s}^*)$.)

Remark 5. If U is not simply-connected, the above needs to be adapted since not all representations of \mathfrak{u} will integrate to representations of U .

Remark 6. In particular, from Remark 3, since the Legendre transform maps the positive Weyl chamber in \mathfrak{u}^* to its dual (see Lemma 4.7 in [Bai+23]), $d_{\xi_+}g \in -i\mathfrak{a}_+ = \mathfrak{t}_+ \cap \mathfrak{s}$ if and only if $\xi_+ \in -i\mathfrak{a}_+^* = \mathfrak{t}_+^* \cap \mathfrak{s}^*$.

2.5. The invariant moment map and invariant torus action. Recall the sweeping map $s : \mathfrak{u}^* \rightarrow \mathfrak{t}_+^*$, given by conjugation to the positive Weyl chamber. The invariant moment map is

$$(14) \quad \mu_{\text{inv}} := s \circ \mu,$$

whose image defines the Kirwan polytope. The μ -regular stratum $T^*(U/K)_{\text{reg}}$ is the pre-image under μ_{inv} of the relative interior of the top-dimensional face of the Kirwan polytope which is called the principal stratum. Along $T^*(U/K)_{\text{reg}}$, as we will recall later, μ_{inv} is the moment map for a smooth effective Hamiltonian torus action by a torus T_{inv} which will play a crucial role in our analysis. The torus T_{inv} is a quotient by a discrete subgroup of the torus \tilde{T}_{inv} whose Lie algebra is determined by the principal stratum. Let $\tau_{U/K}$ denote the principal stratum of $T^*(U/K) \cong U \times_K \mathfrak{s}^*$.

Proposition 1. *A point $[x, \xi] \in U \times_K \mathfrak{s}^*$ is μ -regular if and only if*

$$(15) \quad \langle s(\xi), \alpha \rangle \neq 0, \quad \forall \alpha \in \Phi \setminus \Phi_0.$$

Proof. The point $[x, \xi]$ being *not* μ -regular means that $s(\mu([x, \xi])) = s(\xi)$ is in the (relative) boundary of $\tau_{U/K}$. From the discussion in Section 2.3, we see that this happens if and only if $\langle s(\xi), \beta \rangle = 0$, for some $\beta \in \Sigma$, which, since $s(\xi) \in i\mathfrak{a}_+^* = (\mathfrak{t}_+^* \cap \mathfrak{s}^*)$, is equivalent to $\langle s(\xi), \alpha \rangle = 0$ for some root $\alpha \in \Phi \setminus \Phi_0$. \square

Concerning the momentum set of $T^*(U/K)$ one has the following.

Proposition 2. *The Kirwan polytope of $T^*(U/K)$ is $-i$ times the cone generated by dominant K -spherical weights*

$$(16) \quad \mu(T^*(U/K)) \cap \mathfrak{t}_+^* = -i\mathfrak{a}_+^*.$$

Remark 7. The principal stratum $\tau_{U/K}$ is the unique stratum with nonempty intersection with the relative interior of this cone. Therefore, the dimension of $\tau_{U/K}$ is equal to the rank of U/K .

Remark 8. Equation (16) implies that two different roots, $\alpha, \gamma \in \Phi \setminus \Phi_0$, that are equal when restricted to \mathfrak{a} correspond to the same component of μ_{inv} with respect to a basis of $i\mathfrak{a}^*$ given by restricted roots. (The components of μ_{inv} are also called Guillemin-Sternberg action coordinates.)

Proof. From the form of the moment map (3) it follows that the image of the moment map contains the cone

$$-i\mathfrak{a}_+^* = \mathfrak{t}_+^* \cap \mathfrak{s}^*.$$

On the other hand [Hel84, Theorem III.4.14] implies that any element $\xi \in \mathfrak{s}^*$ can be conjugated to $-i\mathfrak{a}_+$ by the Ad_K^* -action, and therefore also by the Ad_U^* -action, to a unique element so that we have

$$\mu(T^*(U/K)) \cap \mathfrak{t}_+^* = \mathfrak{t}_+^* \cap \mathfrak{s}^* = -i\mathfrak{a}_+^*.$$

\square

Example 1. Let us consider the example of the group K considered as a symmetric space

$$K \cong (K \times K)/K_{\text{diag}},$$

where the quotient is taken with respect to the diagonal action and $K_{\text{diag}} := \text{diag}(K) = \{(x, x) : x \in K\}$. Then, if $\mathfrak{t}_{\mathfrak{k}}$ is a Cartan subalgebra of \mathfrak{k} , we have that $\mathfrak{t} := \mathfrak{t}_{\mathfrak{k}} \oplus \mathfrak{t}_{\mathfrak{k}}$ is a Cartan subalgebra of $U = K \times K$ and we have

$$\begin{aligned} \mathfrak{k}_{\text{diag}} &= \{(X, X) \mid X \in \mathfrak{k}\} \\ \mathfrak{s} &= \{(X, -X) \mid X \in \mathfrak{k}\} \\ \mathfrak{a} &= i\{(H, -H) \mid H \in \mathfrak{t}_{\mathfrak{k}}\}. \end{aligned}$$

The antilinear involution σ is defined on $i\mathfrak{t}$ by

$$\begin{aligned} (iX, iX)^\sigma &:= -i(X, X) \\ (iY, -iY)^\sigma &:= i(Y, -Y), \quad X, Y \in \mathfrak{t}_{\mathfrak{k}}, \end{aligned}$$

and therefore, on it^* takes the form

$$(\eta_1, \eta_2)^\sigma = -(\eta_2, \eta_1), \quad \eta_1, \eta_2 \in it_{\mathfrak{k}}^* .$$

Let r_0 be the rank of K (so that $r = 2r_0$) and consider an order on $it_{\mathfrak{k}}^*$ corresponding to the simple roots $\Delta_{\mathfrak{k}} = \{\alpha_1, \dots, \alpha_{r_0}\}$ for the root system $\Phi_{\mathfrak{k}} \subset it_{\mathfrak{k}}^*$ and an order on it^* corresponding to the following simple roots for $\mathfrak{t} = \mathfrak{t}_{\mathfrak{k}} \oplus \mathfrak{t}_{\mathfrak{k}}$,

$$\Delta := \{(\alpha_1, 0), \dots, (\alpha_{r_0}, 0), (0, -\alpha_1), \dots, (0, -\alpha_{r_0})\} .$$

Then,

$$\Phi = \{(\alpha, 0), (0, \beta) \mid \alpha, \beta \in \Phi_{\mathfrak{k}}\} ,$$

and

$$\begin{aligned} \Phi_+ &= \{(\alpha, 0), (0, -\beta) \mid \alpha, \beta \in \Phi_{\mathfrak{k},+}\} , \\ \Phi_0 &= \{(\alpha, \beta) \in \Phi \mid (\alpha, \beta)|_{\mathfrak{a}} = 0\} = \emptyset . \end{aligned}$$

Then,

$$\Delta_0 = \Delta \cap \Phi_0 = \emptyset ,$$

so that there are no black circles in the Satake diagram and therefore the principal stratum is the interior of the Weyl chamber, with

$$\dim(\tau_{U/K}) = l = r_0 .$$

Let us now verify that this order is a σ -order. We have that

$$(17) \quad (\alpha_j, 0)^\sigma = (0, -\alpha_j), \quad (0, -\alpha_j)^\sigma = (\alpha_j, 0) ,$$

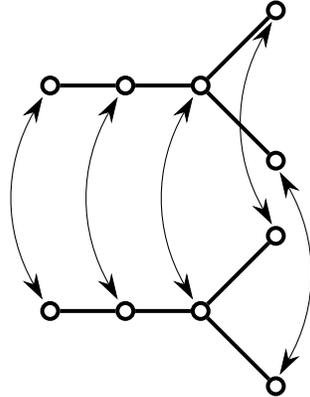
and

$$(\Phi \setminus \Phi_0) \cap \Phi_+ = \Phi_+ .$$

Then, since $(\alpha, -\beta)^\sigma = (\beta, -\alpha)$ we find that

$$\sigma : (\Phi \setminus \Phi_0) \cap \Phi_+ = \Phi_+ \longrightarrow \Phi_+ ,$$

so that indeed the condition (9) is verified. From (10) and (17) we see that $\tilde{\sigma}$ simply permutes the simple roots of one factor with the simple roots of the second factor so that the Satake diagram reads:



Satake diagram of $(\mathfrak{so}(10) \oplus \mathfrak{so}(10), \mathfrak{so}_{\text{diag}}(10))$

For the Guillemin–Sternberg coordinates we therefore have the relations, $(\alpha_j, 0) = -(0, \alpha_j)$.

Let us now describe the action of the invariant torus in more detail. Recall from [GS84; Lan17; Kno11] that, under mild conditions, if X is a smooth manifold with an Hamiltonian U -action with equivariant moment map $\mu : X \rightarrow \mathfrak{u}^*$, then along the (open dense) regular stratum $X_{\text{reg}} = \mu_{\text{inv}}^{-1}(\tau_X)$, where $\mu_{\text{inv}} = s \circ \mu$ as in (14) and τ_X is the principal stratum of the Kirwan polytope associated with the U -action on X , is the moment map for a Hamiltonian action of a torus T_{inv} ,

$$t \star p = (u^{-1}tu) \cdot p, \quad p \in X_{\text{reg}}, \quad t \in T_{\text{inv}},$$

where $\text{Ad}_u^* \mu(p) \in \tau_X$. In our case, $T^*(U/K)_{\text{reg}}$, writing $\tau_{U/K}$ as before instead of $\tau_{T^*(U/K)}$, we have that

$$\tau_{U/K} = -i\check{\mathfrak{a}}_+^*,$$

where

$$\check{\mathfrak{a}}_+^* = \{\eta \in \mathfrak{a}^* \mid \forall \beta \in \Sigma_+ : \langle \beta, \eta \rangle > 0\},$$

and $\Sigma_+ := \{\alpha|_{\mathfrak{a}} \mid \alpha \in \Phi_+ \setminus \Phi_0\}$. Using the K -action, as in the proof of Proposition 2, an element $[x, \xi] \in T^*(U/K)_{\text{reg}}$ can be written uniquely in the form $[u, \xi_+]$ where $u \in U$ and $\xi_+ \in -i\check{\mathfrak{a}}_+^*$. The invariant torus is then

$$T_{\text{inv}} \cong T_{\mathfrak{s}} := \exp(i\mathfrak{a}),$$

with action

$$[u, \xi_+] \star t = [ut^{-1}, \xi_+], \quad t \in T_{\text{inv}}.$$

2.6. Fourier harmonics for T_{inv} . Let $f : T^*(U/K)_{\text{reg}} \rightarrow \mathbb{C}$. We obtain a Fourier decomposition in terms of characters of T_{inv} ,

$$f = \sum_{\nu \in \hat{T}_{\text{inv}}} \hat{f}_{\nu},$$

where

$$\hat{f}_{\nu}(p) = \int_{T_{\text{inv}}} \chi_{\nu}(t) f(p \star t) dt,$$

and χ_{ν} is the character associated to the weight ν of T_{inv} . We have

$$\hat{f}_{\nu}(p \star t) = \chi_{\nu}(t^{-1}) \hat{f}_{\nu}(p).$$

Let now g be an invariant uniformly convex function on \mathfrak{u}^* and denote by \hat{I}_g the corresponding U -invariant Kähler structure on $T^*(U/K)$ as described in Theorem 1. The coordinate ring of $(T^*(U/K), \hat{I}_g) \cong U_{\mathbb{C}}/K_{\mathbb{C}}$ is generated by the \hat{I}_g -holomorphic functions

$$(18) \quad f_{\lambda, v_{\lambda}^K \otimes v^*}^g([x, \xi]) := \text{tr}(\pi_{\lambda}(x e^{id_{\xi}g}) v_{\lambda}^K \otimes v^*),$$

where $\lambda \in \hat{U}_K, v^* \in V_\lambda^*$. The restriction of $f_{\lambda, v_\lambda^K \otimes v^*}^g$ to the regular set $T^*(U/K)_{\text{reg}}$ can then be conveniently written as

$$(19) \quad f_{\lambda, v_\lambda^K \otimes v^*}^g([x, \xi_+]) = \text{tr}(\pi_\lambda(xe^{id_{\xi_+}g})v_\lambda^K \otimes v^*),$$

with $\xi_+ \in -i\hat{\mathfrak{a}}_+^*$. (Note that, from [Bai+23], this implies that $d_{\xi_+}g \in i\mathfrak{a}_{+ \cdot}$.)

Let $P_\nu : V_\lambda \rightarrow V_\lambda$ denote the projection onto the weight space for the weight $\nu \in \mathfrak{s}_\mathbb{Z}^* := \{\lambda \in \mathfrak{t}_\mathbb{Z}^* : \lambda|_{i(\mathfrak{t} \cap \mathfrak{k})} = 0\}$. Then, for $A \in \text{End}(V_\lambda)$,

$$\text{tr}(\pi_\lambda(xe^{id_{\xi_+}g})P_\nu A) = e^{i\langle \nu, d_{\xi_+}g \rangle} \text{tr}(\pi_\lambda(x)P_\nu A).$$

We then obtain the analog of Proposition 4.5 of [Bai+23].

Proposition 3. *The Fourier harmonics for the \hat{I}_g -holomorphic functions $f_{\lambda, v_\lambda^K \otimes v^*}^g$ read*

$$\widehat{(f_{\lambda, v_\lambda^K \otimes v^*}^g)}_\nu([x, \xi_+]) = \text{tr}(\pi_\lambda(xe^{id_{\xi_+}g})P_\nu v_\lambda^K \otimes v^*).$$

Proof. This is a special case of the proof of Proposition 4.5 in [Bai+23]. Namely, in Proposition 4.5 of [Bai+23], just take $A \in \text{End}(V_\lambda)$ to be

$$A = v_\lambda^K \otimes v^*,$$

and, moreover, consider the case when $\nu \in \mathfrak{s}_\mathbb{Z}^*$. Note that, in general, the spherical vector v_λ^K will have non-zero components along different weight spaces. \square

3. MABUCHI RAYS AND A MIXED POLARIZATION ON $T^*(U/K)$

3.1. The polarization \mathcal{P}_∞ on $T^*(U/K)_{\text{reg}}$. In this section, we will apply the concept of fibering polarization, developed in Section 3 of [Bai+23], to the symplectic manifold $T^*(U/K)_{\text{reg}}$. This will provide a generalization of the Kirwin–Wu polarization on $(T^*K)_{\text{reg}}$, \mathcal{P}_{KW} , which was studied in detail in [Bai+23], and which corresponds to the case $U = K \times K$ and $U/K \cong K$.

Consider the diagram

$$\begin{array}{ccc} & U \times_K \mathfrak{s}_{\text{reg}}^* & \\ & \parallel & \\ & T^*(U/K)_{\text{reg}} & \\ \mu \swarrow & & \searrow \mu_{\text{inv}} \\ \mathfrak{u}_{-i\mathfrak{a}^*}^* & \xrightarrow{\phi} & -i\mathfrak{a}_+^* \end{array}$$

where $\mathfrak{u}_{-i\mathfrak{a}^*}^* \subset \mathfrak{u}^*$ is the subset of elements whose coadjoint orbits intersect \mathfrak{t}^* in $-i\mathfrak{a}^*$ and ϕ is the restriction of the sweeping map s to this set. As detailed in [Bai+23], this describes a U -invariant mixed polarization \mathcal{P}_∞ , of $T^*(U/K)_{\text{reg}}$ whose real directions are given by the orbits of T_{inv} and whose

complex directions are given by the fibers of the map ϕ . In our case, these are coadjoint orbits

$$\phi^{-1}(\xi_+) = \mathcal{O}_{\xi_+}, \quad \xi_+ \in -i\mathfrak{a}_+^*,$$

which carry a canonical complex structure given by the identifications

$$\mathcal{O}_{\xi_+} \cong U_{\mathbb{C}}/B,$$

where B is a Borel subgroup. The fibers of μ_{inv} define the coisotropic distribution

$$\mathcal{E} := (\mathcal{P}_{\infty} + \overline{\mathcal{P}_{\infty}}) \cap T(T^*(U/K)_{\text{reg}}).$$

Each of the coadjoint orbits \mathcal{O}_{ξ_+} , $\xi_+ \in -i\mathfrak{a}_+^*$ is then the base of a T_{inv} -principal fiber bundle given by the corresponding fiber of μ_{inv} and corresponds to its coisotropic reduction.

Since $T^*(U/K)_{\text{reg}}$ is multiplicity free (the orbits of the U -action are separated by the values of μ) we can apply Corollary 3.19 of [Bai+23] so that any \mathcal{P}_{∞} -polarized section will be supported on the inverse image under μ_{inv} of the intersection of the interior of the Kirwan polytope with $-i$ times the ρ -translate of the character lattice of the maximal torus, where ρ is the half-sum of positive roots.

The notation \mathcal{P}_{∞} is justified below when we show that \mathcal{P}_{∞} can be obtained at infinite geodesic time along a Mabuchi ray of U -invariant Kähler polarizations of $T^*(U/K)$.

3.2. Local description of \mathcal{P}_{∞} . Let us describe the polarization \mathcal{P}_{∞} locally in terms of Hamiltonian vector fields. Consider the functions

$$F_{\lambda, v_{\lambda}^K \otimes v^*}([x, \xi_+]) := \text{tr}(\pi_{\lambda}(x) P_{\lambda} v_{\lambda}^K \otimes v^*), \quad [x, \xi_+] \in T^*(U/K),$$

for $v^* \in V_{\lambda}^*$.

Proposition 4. *Let V_{λ} be a highest weight representation of U with highest weight $\lambda \in \hat{U}_K$. For $v_1^*, v_2^* \in V_{\lambda}^*$, the complex function*

$$\frac{F_{\lambda, v_{\lambda}^K \otimes v_1^*}}{F_{\lambda, v_{\lambda}^K \otimes v_2^*}},$$

defined on the subset $O \subset T^*(U/K)_{\text{reg}}$ where the denominator does not vanish, is T_{inv} -invariant and hence descends to a function on O/T_{inv} on an open subset of \mathfrak{s}^* . The family of such functions, for all possible choices of $v_1^*, v_2^* \in V_{\lambda}^*$ generate the complex directions of the polarization \mathcal{P}_{∞} .

Proof. This analogous to the proof of Proposition 4.2 of [Bai+23]. Namely, the functions $F_{\lambda, v_{\lambda}^K \otimes v^*}$ are T_{inv} -equivariant,

$$F_{\lambda, v_{\lambda}^K \otimes v^*}([x, \xi] \star t) = \chi_{\lambda}(t^{-1}) F_{\lambda, v_{\lambda}^K \otimes v^*}[x, \xi], \quad t \in T_{\text{inv}},$$

so that their quotients will be invariant under T_{inv} . Moreover, we see that the functions $F_{\lambda, v_{\lambda}^K \otimes v^*}$ are defined on $U_{\mathbb{C}}$, are B -equivariant and that therefore they define holomorphic sections of the ample Borel–Weil line bundle $L_{\lambda} \rightarrow$

\mathcal{O}_{ξ_+} on $U_{\mathbb{C}}/B$, where $B \subset U_{\mathbb{C}}$ is the Borel subgroup associated to the choice of positive roots. For fixed λ and varying $v^* \in V_{\lambda}^*$ we thus obtain

$$H^0(\mathcal{O}_{\lambda}, L_{\lambda}) \cong V_{\lambda}^* \cong \left\{ F_{\lambda, v_{\lambda}^K \otimes v^*} \mid v^* \in V_{\lambda}^* \right\}.$$

The quotients therefore generate the structure sheaf of \mathcal{O}_{ξ_+} and the Hamiltonian vector fields of their complex conjugates define an holomorphic polarization along the coisotropic reductions. \square

As described in Section 3.1, the remaining real directions of the mixed polarization \mathcal{P}_{∞} are generated by the Hamiltonian vector fields of the Guillemin–Sternberg action coordinates given by the components of μ_{inv} .

3.3. Convergence of polarizations. We will now describe how the mixed polarization \mathcal{P}_{∞} arises at infinite geodesic time along a Mabuchi ray of U -invariant Kähler polarizations \mathcal{P}_{g_t} , $t > 0$, associated to the U -invariant Kähler structures \hat{I}_{g_t} , where $g_t := g + th$ for $g, h \in \text{Conv}_{\text{unif}}^{\infty}(\mathbf{u}^*)^{\text{Ad}_U^*}$.

Lemma 2. *For the Mabuchi ray of U -invariant Kähler polarizations on $T^*(U/K)_{\text{reg}}$ given by the complex structures \hat{I}_{g_t} , $t > 0$, one has, for $\lambda \in \hat{U}_K$, $v^* \in V_{\lambda}^*$:*

- (i) $\lim_{t \rightarrow +\infty} e^{-i\langle \lambda, \mathcal{L}_{g_t \circ \mu_{\text{inv}}} \rangle} f_{\lambda, v_{\lambda}^K \otimes v^*}^{g_t} = F_{\lambda, v_{\lambda}^K \otimes v^*},$
- (ii) $\lim_{t \rightarrow +\infty} \frac{1}{t} \ln f_{\lambda, v_{\lambda}^K \otimes v^*}^{g_t} = i\langle \lambda, \mathcal{L}_h \circ \mu_{\text{inv}} \rangle.$

Proof. This follows from Lemma 4.8 and equation (30) in [Bai+23]. \square

Theorem 2. *The family of Kähler polarizations \mathcal{P}_{g_t} , $t > 0$, on $T^*(U/K)_{\text{reg}}$ converges pointwise to the mixed polarization \mathcal{P}_{∞} as $t \rightarrow +\infty$.*

Proof. For the real directions of \mathcal{P}_{∞} , we see from (ii) in Lemma 2 that the distribution generated by the vector fields

$$(f_{\lambda, v_{\lambda}^K \otimes v^*}^{g_t})^{-1} X_{f_{\lambda, v_{\lambda}^K \otimes v^*}^{g_t}},$$

for $\lambda \in \hat{U}_K$, $v^* \in V_{\lambda}^*$, converges, as $t \rightarrow +\infty$, pointwise to the distribution generated by the Hamiltonian vector fields of the components of μ_{inv} . On the other hand, from (i) in Lemma 2, we see that the distribution generated by the Hamiltonian vector fields

$$X_{\frac{f_{\lambda, v_{\lambda}^K \otimes v^*}^{g_t}}{f_{\lambda, v_{\lambda}^K \otimes w^*}^{g_t}}}, \quad \lambda \in \hat{U}_K, v^*, w^* \in V_{\lambda}^*,$$

converges, as $t \rightarrow +\infty$, to the distribution generated by the Hamiltonian vector fields

$$X_{\frac{F_{\lambda, v_{\lambda}^K \otimes v^*}}{F_{\lambda, v_{\lambda}^K \otimes w^*}}}, \quad \lambda \in \hat{U}_K, v^*, w^* \in V_{\lambda}^*$$

which, from Proposition 4, gives the holomorphic directions of \mathcal{P}_{∞} . \square

3.4. The half-form bundle for \mathcal{P}_∞ . In this section, we will study the half-form bundle for \mathcal{P}_∞ . To do so, let us describe the sections of the canonical bundle for the Kähler polarizations $\mathcal{P}_{g_t}, t > 0$.

Recall from [Akh86] that homogeneous holomorphic vector bundles on $U_{\mathbb{C}}/K_{\mathbb{C}}$, that is vector bundles with a lift of the $U_{\mathbb{C}}$ -action on the base, are in bijective correspondence with holomorphic representations of $K_{\mathbb{C}}$. Sections of a homogeneous line bundle $L_\chi \rightarrow U_{\mathbb{C}}/K_{\mathbb{C}}$, defined by the character χ of $K_{\mathbb{C}}$, are therefore in bijection with χ -equivariant functions on $U_{\mathbb{C}}$, that is with functions $f : U_{\mathbb{C}} \rightarrow \mathbb{C}$ such that

$$f(u \cdot k) = \chi(k)^{-1} f(u), \quad u \in U_{\mathbb{C}}, k \in K_{\mathbb{C}}.$$

The canonical bundle \mathcal{K} of $U_{\mathbb{C}}/K_{\mathbb{C}}$ is associated to the character defined by, see [Akh86],

$$\det(\text{Ad}_h : \mathfrak{s}_{\mathbb{C}} \rightarrow \mathfrak{s}_{\mathbb{C}}), \quad h \in K_{\mathbb{C}}.$$

Since this determinant gets a contribution from both positive and negative roots for the adjoint action of $\mathfrak{k}_{\mathbb{C}}$ it defines the trivial character. (Note that these roots are with respect to a choice of Cartan subalgebra for \mathfrak{u} which extends a Cartan subalgebra for \mathfrak{k} and which in general will be different from \mathfrak{t} .) Therefore, \mathcal{K} is trivializable.

Recall from Theorem 1 that for $g \in \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^*)^{\text{Ad}_v^*}$ the Legendre transform \mathcal{L}_g provides a biholomorphism between $(T^*(U/K), \hat{I}_g)$ and $U_{\mathbb{C}}/K_{\mathbb{C}}$, and therefore also the canonical bundle $K_g := \mathcal{L}_g^* \mathcal{K}$ of $(T^*(U/K), \hat{I}_g)$, is trivializable.

Holomorphic trivializing sections for \mathcal{K} and K_g can be obtained by making use of the transitive holomorphic left action of $U_{\mathbb{C}}$, as follows. (See Proposition 3.3.22 in [Kay15].) On a sufficiently small open neighborhood A of $[e] \in U_{\mathbb{C}}/K_{\mathbb{C}}$ one can choose a local section $s : A \rightarrow U_{\mathbb{C}}$ of the canonical projection $p : U_{\mathbb{C}} \rightarrow U_{\mathbb{C}}/K_{\mathbb{C}}$, identifying

$$T_{[e]}(U_{\mathbb{C}}/K_{\mathbb{C}}) \cong \mathfrak{s} \oplus i\mathfrak{s} \cong \mathfrak{s}_{\mathbb{C}}.$$

Given a basis of \mathfrak{s}^* , one can then define a holomorphic form of top degree on $T_{[e]}^*(U_{\mathbb{C}}/K_{\mathbb{C}})$. When K is connected, as in the case that we are considering, this form can then be extended to a global holomorphic form of top degree, Ω , on the whole of $U_{\mathbb{C}}/K_{\mathbb{C}}$ by imposing invariance under the holomorphic transitive left action of $U_{\mathbb{C}}$. (Connectedness of K ensures that the right action of $K_{\mathbb{C}}$ preserves Ω , see Proposition 3.2.33 in [Kay15].) Ω then defines a global holomorphic frame for $\mathcal{K} \rightarrow U_{\mathbb{C}}/K_{\mathbb{C}}$, so that $\mathcal{L}_g^* \Omega$ is a global $U_{\mathbb{C}}$ -left invariant holomorphic trivializing frame for $K_g \rightarrow T^*(U/K)$.

Letting $\dim K = n, \dim U = d + n$, let us consider a basis of U -left-invariant one-forms on U , $\omega^j, j = 1, \dots, d + n$, dual to an orthonormal basis of \mathfrak{u} adapted to the decomposition $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{s}$ that is, such that $(\omega^j)|_{\mathfrak{k}} = 0$ for $j = 1, \dots, d$ and $(\omega^j)|_{\mathfrak{s}} = 0$ for $j = d + 1, \dots, d + n$.

Lemma 3. (Lemma 3.3 in [KMN13]) A $U_{\mathbb{C}}$ -left-invariant holomorphic frame of one-forms on (T^*U, I_g) is given by

$$\Omega_g^j(x, \xi) := \sum_{k=1}^{d+n} \left[e^{-i \operatorname{ad}_{d_\xi g}} \right]_k^j \omega^k + \left[\frac{1 - e^{-i \operatorname{ad}_{d_\xi g}}}{\operatorname{ad}_{d_\xi g}} \cdot \operatorname{Hess}_g(\xi) \right]_k^j d\xi^k,$$

$j = 1, \dots, d+n$. This is obtained by pulling-back a frame of left $U_{\mathbb{C}}$ -invariant holomorphic one-forms on $U_{\mathbb{C}}$ by the Legendre transform \mathcal{L}_g .

The left actions of $U_{\mathbb{C}}$ on itself and $U_{\mathbb{C}}/K_{\mathbb{C}}$, and the corresponding actions on T^*U and $T^*(U/K)$ induced by the respective Legendre transforms \mathcal{L}_g are, of course, all compatible. We have then,

Proposition 5. On T^*U ,

$$(20) \quad \Omega_g := (\mathcal{L}_g^* \circ p^*)\Omega = \bigwedge_{j=1}^d \Omega_g^j.$$

Proof. The forms on both sides of (20) are $U_{\mathbb{C}}$ -left-invariant and are therefore defined by their values at $(e, 0) \in T^*U$, where $e \in U$ is the identity. From Remark 2 we have that $\xi \in \mathfrak{s}^* \iff d_\xi g \in \mathfrak{s}$ and $\xi \in \mathfrak{k}^* \iff d_\xi g \in \mathfrak{k}^*$. This implies, in particular, that $d_{\xi=0}g = 0$.³ From (c) in Proposition 2.7 in [Bai+23], we have that, as endomorphisms of \mathfrak{u}^* ,

$$\operatorname{ad}_{\xi^*}^* = \operatorname{Hess}_g^{-1}(\xi) \operatorname{ad}_{d_\xi g}^*,$$

where $\xi^* \in \mathfrak{u}$ corresponds to $\xi \in \mathfrak{u}^*$ via the invariant form on \mathfrak{u} . It follows that the endomorphism $\operatorname{Hess}_g(\xi)$ preserves the decomposition $\mathfrak{u}^* = \mathfrak{k}^* \oplus \mathfrak{s}^*$ for any $\xi \in \mathfrak{u}^*$. Therefore, since $[\mathfrak{s}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{k}$, we have that at $(e, 0)$, for $j = 1, \dots, d$,

$$\Omega_g^j(e, 0) = \omega^j + i \sum_{k=1}^d [\operatorname{Hess}_g(0)]_k^j d\xi^k = (\mathcal{L}_g^* \circ p^*)\Omega^j(e),$$

where e also denotes the identity in $U_{\mathbb{C}}$ and where Ω^j is the $U_{\mathbb{C}}$ -left-invariant 1-form on $U_{\mathbb{C}}/K_{\mathbb{C}}$ such that $(p^*\Omega^j)(e) = d\xi^j + id\xi^j$. Then, from Proposition 3.3.22 in [Kay15], we get

$$\Omega = \bigwedge_{j=1}^d \Omega^j.$$

This ends the proof. \square

Remark 9. Note that each individual Ω_g^j is not the pull-back of an holomorphic one-form on $U_{\mathbb{C}}/K_{\mathbb{C}}$ since $K_{\mathbb{C}}$ will not act trivially on it.

³Recall that we are assuming that g is compatible with the symmetric space involution σ as in Remark 2.

Note that, while the form Ω is fixed, the left $U_{\mathbb{C}}$ actions on T^*U and $T^*(U/K)$ are induced by the Legendre transforms \mathcal{L}_g and therefore depend on the choice of g . For that reason, to study the behavior of Ω_{g+th} as $t \rightarrow +\infty$ for the families of Kähler structures considered in Section 3.3, it is not enough to consider the restriction to the identity coset $[e] \in U_{\mathbb{C}}/K_{\mathbb{C}}$. On the other hand, the restriction of $p \circ \mathcal{L}_g$ to $U \times \mathfrak{s}^*$ is surjective onto $U_{\mathbb{C}}/K_{\mathbb{C}}$ so that it is enough to study the form Ω_g in (20) along $U \times \mathfrak{s}^*$. Along $U \times \mathfrak{s}^*$, since j also runs from 1 to d , only terms with even powers of $\text{ad}_{d_{\xi}g}$ contribute, again due to the conditions $[\mathfrak{s}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{k}$. We can, moreover, using the Ad_K^* -action, restrict the study of the behavior of Ω_g even further to $U \times (-i)\mathfrak{a}^*$.

Let, for $j = 1, \dots, d$ and $\xi_+ \in (-i)\mathfrak{a}^*$,

$$(21) \quad \tilde{\Omega}_g^j(x, \xi_+) := \sum_{k=1}^d [\cosh(i\text{ad}_{d_{\xi}g})]_k^j \omega^k + \left[\frac{\sinh(i\text{ad}_{d_{\xi}g})}{\text{ad}_{d_{\xi}g}} \cdot \text{Hess}_g(\xi) \right]_k^j d\xi^k,$$

so that over $U \times (-i)\mathfrak{a}^*$ we have

$$(22) \quad \Omega_g = \bigwedge_{j=1}^d \tilde{\Omega}_g^j.$$

Remark 10. Note that under the involution $(\text{Id}_U \times \sigma)$ on T^*U , which descends to the anti-holomorphic involution $[x, \xi] \mapsto [x, -\xi]$ on $T^*(U/K)$, see [Ste90], and using Remarks 2, 3 and 6, we do obtain that $(\text{Id}_U \times \sigma)^* \tilde{\Omega}_g^j = \overline{\tilde{\Omega}_g^j}$. On the other hand, the involution σ on \mathfrak{u} lifts to a holomorphic involution σ of $T^*(U/K)$ which acts as $[x, \xi] \mapsto [\sigma(x), -\xi]$ where $x \mapsto \sigma(x)$ is the involution on U induced from σ by the exponential map. In particular, at the points (e, ξ) it is easy to check that the pull-back of $\tilde{\Omega}_g^j$ under the lift of this involution to T^*U is $-\tilde{\Omega}_g^j$, consistently with σ being holomorphic.

We then have, with $g_t = g + th, t \geq 0$,

Proposition 6. *On $T^*U_{\text{reg}} \cap (U \times (-i)\mathfrak{a}^*)$,*

$$(23) \quad \lim_{t \rightarrow +\infty} t^{-l} e^{-2\langle \hat{\rho}, d_{\xi_+}^{g_t} \rangle} \Omega_{g_t}(x, \xi_+) = c_0 i^l 2^{-\#(\Sigma \cap \Phi_+)} \det \left(\text{Hess}_h(\tilde{\xi}_+) \right)_{\mathfrak{ia}} \hat{\Omega}_{\infty},$$

where $\hat{\rho}$ is the half-sum of the positive restricted roots weighted by multiplicity, c_0 is a nonzero constant, and where

$$(24) \quad \hat{\Omega}_{\infty} := \bigwedge_{j=1}^l d\xi^j \cdot \bigwedge_{\alpha \in \Sigma \cap \Phi_+} (\omega^{\alpha} - \omega^{\alpha\sigma} - \langle \alpha, \xi_+ \rangle^{-1} (d\xi^{\alpha} - d\xi^{\alpha\sigma})).$$

The right-hand sides of (23) and (24) define smooth form on T^*U_{reg} by the K right-invariant extension from $U \times (-i)\mathfrak{a}^*$.

Proof. If $\eta \in \mathfrak{ia}$, we have

$$\text{ad}_{\eta}^2(E_{\alpha} - E_{\alpha\sigma}) = \langle \beta, \eta \rangle^2 (E_{\alpha} - E_{\alpha\sigma}),$$

where β is the restricted root obtained by restricting α to \mathfrak{a} . (Recall from Section 2.3 that for restricted roots $\alpha|_{\mathfrak{a}} = \alpha|_{\mathfrak{a}}$.) We then have, from (21), the proof of Lemma 4.14 in [Bai+23] and from Remarks 2 and 3, for regular ξ_+ ,

$$\lim_{t \rightarrow +\infty} t^{-l} \bigwedge_{j=1}^l \tilde{\Omega}_{g_t}^j(x, \xi_+) = i^l \det(\text{Hess}_h(\xi_+))_{i\mathfrak{a}} d\xi^1 \wedge \cdots \wedge d\xi^l$$

and

$$\begin{aligned} & \lim_{t \rightarrow +\infty} e^{-2\langle \hat{\rho}, d\xi_+ g_t \rangle} \bigwedge_{\alpha \in \Sigma \cap \Phi_+} \left(\tilde{\Omega}_{g_t}^\alpha - \tilde{\Omega}_{g_t}^{\alpha^\sigma} \right) (x, \xi_+) = \\ & = 2^{-\#(\Sigma \cap \Phi_+)} \bigwedge_{\alpha \in \Sigma \cap \Phi_+} (\omega^\alpha - \omega^{\alpha^\sigma} - \langle \alpha, \xi_+ \rangle^{-1} (d\xi^\alpha - d\xi^{\alpha^\sigma})). \end{aligned}$$

The constant c_0 comes from the fact that we are changing from an orthonormal basis in (21) to a Chevalley basis. \square

Let $\tilde{p} : U \times \mathfrak{s}^* \rightarrow U \times_K \mathfrak{s}^*$ be the canonical projection.

Proposition 7. *The form $\hat{\Omega}_\infty$ is the pull-back by \tilde{p} of a trivializing section $\tilde{\Omega}_\infty$ of the canonical bundle of \mathcal{P}_∞ over $T^*(U/K)_{\text{reg}}$,*

$$\hat{\Omega}_\infty = \tilde{p}^* \tilde{\Omega}_\infty.$$

Proof. We have $p \circ \mathcal{L}_{g_t} = \mathcal{L}_{g_t} \circ \tilde{p}$, $t > 0$, so that $\mathcal{L}_{g_t}^* \circ p^* = \tilde{p}^* \circ \mathcal{L}_{g_t}^*$. From Proposition 5 and since (23) extends by right K -invariance we have that the left-hand side of (23) can be written as the limit as $t \rightarrow \infty$ of a pull-back by $\tilde{p} \circ \mathcal{L}_{g_t}$ from $U_{\mathbb{C}}/K_{\mathbb{C}}$. But since \tilde{p} is a submersion this implies that the right-hand side of (23) is the pull-back under \tilde{p} of a well defined form on $T^*(U/K)_{\text{reg}}$ whence $\hat{\Omega}_\infty = \tilde{p}^* \tilde{\Omega}_\infty$.

Moreover, since $\lim_{t \rightarrow \infty} \mathcal{P}_{g_t} = \mathcal{P}_\infty$, pointwise in the Lagrangian Grassmannian over $T^*(U/K)_{\text{reg}}$ we obtain that $\tilde{\Omega}_\infty$, which is never vanishing, gives a trivializing section of the corresponding canonical bundle. \square

We also have,

Remark 11. Note that, from Theorem 4.1 in Chapter 5 in [Hel84], as recalled in Section 2.4, we obtain that $\hat{\rho}$ is a spherical weight. Moreover, one can check that the representation $V_{\hat{\rho}}$ is self-dual so that under the identification $V_{\hat{\rho}} \cong V_{\hat{\rho}}^*$ via an invariant inner product, since the subspace of spherical vectors is one-dimensional, we can take $(v_{\hat{\rho}}^K)^* = v_{\hat{\rho}}^K$.

Proposition 8. *On the subset of \tilde{W} where $F_{\hat{\rho}, v_{\hat{\rho}}^K \otimes v_{\hat{\rho}}^K}$ is nonzero,*

$$\Omega_\infty := F_{\hat{\rho}, v_{\hat{\rho}}^K \otimes v_{\hat{\rho}}^K}^{-2} \tilde{\Omega}_\infty$$

is a polarized section of \mathcal{P}_∞ .

Proof. This follows from Proposition 6 and from Lemma 2, since both Ω_{g_t} and $f_{\hat{\rho}, v_{\hat{\rho}}^K \otimes v_{\hat{\rho}}^K}^{g_t}$ are \mathcal{P}_{g_t} -polarized for all $t > 0$. \square

Since the canonical bundle of \mathcal{P}_∞ over $T^*(U/K)_{\text{reg}}$ is trivializable, with trivializing section $\tilde{\Omega}_\infty$, we can define a bundle of half-forms by taking its square root with trivializing section $\tilde{\Omega}_\infty^{\frac{1}{2}}$.

4. QUANTIZATIONS OF $T^*(U/K)$ ALONG MABUCHI RAYS

In this section we will describe the quantizations of $T^*(U/K)$ for the vertical (or Schrödinger) polarization and for the U -invariant Kähler polarizations determined by the complex structures $\hat{I}_g, g \in \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^*)^{\text{Ad}_U^*}$. We will then relate them to the space of polarized sections for the limit polarization \mathcal{P}_∞ which, as we will describe, inherits a natural Hilbert space structure, \mathcal{H}_∞ , so that we obtain an extended bundle of quantum Hilbert spaces

$$\mathcal{H} \rightarrow \{0\} \cup \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^*)^{\text{Ad}_U^*} \cup \{\infty\},$$

equipped with a natural U -invariant flat connection.

4.1. The Schrödinger and the Kähler quantizations for $\mathcal{P}_{g_t}, t > 0$. The quantization of $T^*(U/K)$ in the Schrödinger, or vertical, polarization \mathcal{P}_{Sch} is given by

$$(25) \quad \mathcal{H}_{\text{Sch}} := L^2(U/K) \otimes \sqrt{dx}$$

where dx denotes the pull-back to $T^*(U/K)$, by the canonical projection, of a unit-volume U -invariant volume form on U/K . We have then,

$$\mathcal{H}_{\text{Sch}} = \widehat{\bigoplus_{\lambda \in \hat{U}_K} \left\{ \sigma_{\lambda, v^*}^0 \mid \lambda \in \hat{U}_K, v^* \in V_\lambda^* \right\}},$$

where

$$\sigma_{\lambda, v^*}^0(x) := \text{tr}(\pi_\lambda(x) v_\lambda^K \otimes v^*) \otimes \sqrt{dx},$$

and where the hat denotes norm completion with respect to the (usual) U -invariant L^2 inner product,

$$\langle \sigma_{\lambda, v^*}^0, \sigma_{\tilde{\lambda}, \tilde{v}^*}^0 \rangle = d_\lambda^{-1} \delta_{\lambda \tilde{\lambda}} \langle v^*, \tilde{v}^* \rangle_{V_\lambda^*}.$$

From Theorem 1, the Kähler structure of $T^*(U/K)$ for the complex structure \hat{I}_g , for $g \in \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^*)^{\text{Ad}_U^*}$, is obtained from a Kähler quotient of (T^*U, \hat{I}_g) . From equations (18), (19) and Section 3.4 we also obtain that the half-form corrected quantization of $T^*(U/K)$ in the holomorphic quantization associated to the complex structure \hat{I}_g , is given by the Hilbert space

$$(26) \quad \mathcal{H}_g := \widehat{\bigoplus_{\lambda \in \hat{U}_K} \left\{ \sigma_{\lambda, v^*}^g \mid \lambda \in \hat{U}_K, v^* \in V_\lambda^* \right\}},$$

where

$$(27) \quad \sigma_{\lambda, v^*}^g := e^{-g(\lambda + \hat{\rho})} f_{\lambda, v_\lambda^K \otimes v^*}^g e^{-\frac{1}{2}\kappa g} \otimes \Omega_g^{\frac{1}{2}},$$

and where the inner product in \mathcal{H}_g is the usual inner product for half-form corrected Kähler quantization (see, for instance, Section 4 of [KMN13]). Of course, this is an instance of the celebrated results of Guillemin and

Sternberg [GS82]. (While $T^*(U/K) \cong U_{\mathbb{C}}/K_{\mathbb{C}}$ is not compact the quotient can be described in terms of an action of $K_{\mathbb{C}}$.)

As will be discussed below in Section 4.2, the first exponential factor in (27) is justified by the fact that

$$(28) \quad \left\{ \sigma_{\lambda, v_j}^g \mid \lambda \in \hat{U}_K, v_j^* \in V_{\lambda}^*, j = 1, \dots, \dim V_{\lambda} \right\},$$

with $\{v_j\}_{j=1, \dots, \dim V_{\lambda}}$ a basis of V_{λ} , gives a parallel frame for a naturally defined flat connection on the extended quantum bundle.

4.2. The flat connection on the quantum bundle. In the previous subsection, for the family of U -invariant Kähler quantizations of $T^*(U/K)$, we obtained a quantum bundle of Hilbert spaces of polarized sections, with fiber \mathcal{H}_g over $g \in \text{Conv}_{\text{unif}}^{\infty}(\mathfrak{u}^*)^{\text{Ad}_U^*}$. In the spirit of [ADPW91] (see also [Flo+05; KMN14]), this bundle comes equipped with a flat connection, ∇^Q , which relates the different U -invariant Kähler quantizations. This connection is obtained by covariantly differentiating sections of the quantum bundle along a tangent vector h to $\text{Conv}_{\text{unif}}^{\infty}(\mathfrak{u}^*)^{\text{Ad}_U^*}$ by combining the prequantum operator \hat{h} and a fiber-preserving “quantum” operator $\mathcal{Q}(h)$. In the case of the quantization of symplectic vector spaces along translation invariant Kähler polarizations, this combination defines a flat connection whose (unitary) parallel transport intertwines natural representations of the Heisenberg group on the different fibers.

For $h \in \text{Conv}_{\text{unif}}^{\infty}(\mathfrak{u}^*)^{\text{Ad}_U^*}$, consider the half-form corrected Kostant-Souriau prequantum operator (let ∇ denote the connection on the prequantum bundle obtained from the prequantum data on T^*U by Kähler reduction, à la Guillemin-Sternberg)

$$\hat{h} := (i\nabla_{X_h} + h) \otimes 1 + 1 \otimes iL_{X_h}.$$

Consider also the quantum operator $\mathcal{Q}(h)$ defined by

$$\mathcal{Q}(h)\sigma_{\lambda, v^*}^g := h(\lambda + \hat{\rho})\sigma_{\lambda, v^*}^g,$$

for any $g \in \text{Conv}_{\text{unif}}^{\infty}(\mathfrak{u}^*)^{\text{Ad}_U^*}$, $\lambda \in \hat{U}_K$, $v^* \in V_{\lambda}^*$. As in Section 2.4 of [Flo+05] and Section 5.3 of [Bai+23], consider sections of the quantum bundle of the form

$$s(g, [x, \xi]) = f(g, xe^{id_{\xi}g})e^{-\frac{1}{2}\kappa_g} \otimes \Omega_g^{\frac{1}{2}},$$

such that, for fixed g , f is holomorphic in $U_{\mathbb{C}}/K_{\mathbb{C}}$. The connection ∇^Q is then defined by

$$\nabla_h^Q s := \frac{\delta}{\delta h} s + \hat{h}s - \mathcal{Q}(h)s$$

where h is a tangent vector to $\text{Conv}_{\text{unif}}^{\infty}(\mathfrak{u}^*)^{\text{Ad}_U^*}$. (This is an analog of equation (1.30) in [ADPW91].) The evaluation of the connection form, in this frame, along the tangent vector h is therefore given by $(\hat{h} - \mathcal{Q}(h))$. Note that the quantum operator \mathcal{Q} acts through the isotypical decomposition of

s under U -representations, corresponding to the decomposition of s in the frame.

For T^*U , as described in Section 5.3 of [Bai+23], the equation of covariant constancy determined by ∇^Q (along quadratic h) corresponds to the heat equation described in [Flo+05]. (The first order differential operator in the connection form, given by the prequantum operator \hat{h} , does not show up explicitly in the heat equation in [Flo+05] because there the equation was written in $U_{\mathbb{C}}$, not in T^*U , so the that term corresponding to the Legendre transform \mathcal{L}_{g+th} is implicit there.) Also for quadratic h , the condition of covariant constancy is an analog of the heat equation satisfied by theta functions for varying moduli in the quantization of an abelian variety.

Note that the sections $\sigma_{\lambda, v^*}^{g+th}$, for $\lambda \in \hat{U}_K, v^* \in V_{\lambda}^*$, which are linear combinations with constant coefficients of the frame sections in (28), satisfy the equation of parallel transport:

$$\frac{d}{dt}\sigma_{\lambda, v^*}^{g+th} = \nabla_h^Q \sigma_{\lambda, v^*}^{g+th} = \hat{h}\sigma_{\lambda, v^*}^{g+th} - h(\lambda + \hat{\rho})\sigma_{\lambda, v^*}^{g+th},$$

which follows explicitly from (27) and from (see Proposition 3.19 in [KMN13])

$$e^{t\hat{h}} \left(f_{\lambda, v_{\lambda}^K \otimes v^*}^g e^{-\frac{1}{2}\kappa g} \otimes \Omega_g^{\frac{1}{2}} \right) = f_{\lambda, v_{\lambda}^K \otimes v^*}^{g+th} e^{-\frac{1}{2}\kappa_{g+th}} \otimes \Omega_{g+th}^{\frac{1}{2}}.$$

In particular, this shows that ∇^Q is flat.

Since the operators \hat{h} and $\mathcal{Q}(h)$ commute, the parallel transport of ∇^Q along the Mabuchi ray generated by h is given by exponentiating the connection form, which gives a generalized coherent state transform (gCST) defined by

$$C_{t,h} := e^{t\hat{h}} \circ e^{-t\mathcal{Q}(h)}.$$

In fact, one can take this parallel transport operator to act also on the Hilbert space of the vertical polarization \mathcal{H}_{Sch} . The following is an immediate corollary of [KMN13] or Section 5.2 of [Bai+23].

Proposition 9. *Let $g = 0$ or $g \in \text{Conv}_{\text{unif}}^{\infty}(\mathbf{u}^*)^{\text{Ad}_U^*}$. For $t \geq 0$,*

$$C_{t,h}\sigma_{\lambda, v^*}^g = \sigma_{\lambda, v^*}^{g+th}.$$

Therefore, $C_{t,h}$ is a U -equivariant isomorphism

$$C_{t,h} : \mathcal{H}_g \rightarrow \mathcal{H}_{g+th}, \quad t \geq 0.$$

(Here, $\mathcal{H}_0 := \mathcal{H}_{\text{Sch}}$.)

Corollary 1. *For $g = 0$ or $g \in \text{Conv}_{\text{unif}}^{\infty}(\mathbf{u}^*)^{\text{Ad}_U^*}$, letting*

$$(29) \quad \mathcal{H}_g^{\lambda} := \left\{ \sigma_{\lambda, v^*}^g \mid v^* \in V_{\lambda}^* \right\}, \quad \lambda \in \hat{U}_K,$$

we obtain from U -equivariance that there exist constants $a_{t,h,\lambda} \in \mathbb{R} \setminus \{0\}$, for $t > 0, \lambda \in \hat{U}_K$, such that

$$a_{t,h,\lambda} C_{t,h} : \mathcal{H}_g^{\lambda} \rightarrow \mathcal{H}_{g+th}^{\lambda}$$

are unitary isomorphisms.

As we describe next, parallel transport along Mabuchi rays of the frame $\{\sigma_{\lambda, v^*}^g \mid v^* \in V_\lambda^*, \lambda \in \hat{U}_K\}$ has a nice behavior at infinite geodesic time.

4.3. Convergence of quantizations as $t \rightarrow +\infty$. In this section, in the spirit of [Bai+23], we want to study the limit of the family of Kähler quantizations \mathcal{H}_{g+th} as $t \rightarrow +\infty$.

Definition 1. The distributions $\delta_{\mu_{\text{inv}}^{-1}(\lambda+\hat{\rho})}$, $\lambda \in \hat{U}_K$, are defined by the unit mass measure supported on $\mu_{\text{inv}}^{-1}(\lambda+\hat{\rho})$ such that, for $f \in C_c(T^*(U/K))$,

$$(30) \quad \int_{T^*(U/K)} f \delta_{\mu_{\text{inv}}^{-1}(\lambda+\hat{\rho})} = \int_{U/K} f([x, \lambda + \hat{\rho}]) dx,$$

where dx is the normalized U -invariant measure on U/K .

$$(31) \quad \sigma_{\lambda, v^*}^\infty := c_0(2\pi)^{l/2} i^l 2^{-\#(\Sigma \cap \Phi_+)} P(\lambda + \hat{\rho})^2 F_{\lambda, v_\lambda^K \otimes v^*} F_{\hat{\rho}, v_{\hat{\rho}}^K \otimes v_{\hat{\rho}}^K} \delta_{\mu_{\text{inv}}^{-1}(\lambda+\hat{\rho})} \otimes F_{\hat{\rho}, v_{\hat{\rho}}^K \otimes v_{\hat{\rho}}^K}^{-1} \tilde{\Omega}_\infty^{1/2},$$

where $P(\lambda + \hat{\rho}) := \prod_{\alpha \in \Sigma \cap \Phi_+} \langle \alpha, \lambda + \hat{\rho} \rangle$.

Theorem 3. Let $g = 0$ or $g \in \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^*)^{\text{Ad}_U^*}$ and $h \in \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^*)^{\text{Ad}_U^*}$. In the distributional sense,

$$\lim_{t \rightarrow +\infty} C_{t,h} \sigma_{\lambda, v^*}^g = \sigma_{\lambda, v^*}^\infty, \quad \lambda \in \hat{U}_K, v^* \in V_\lambda^*.$$

Proof. The proof follows the same calculations as in the proof of theorems 5.5 and 5.8 of [Bai+23]. \square

Corollary 2. The distributional sections $\sigma_{\lambda, v^*}^\infty$, $\lambda \in \hat{U}_K, v^* \in V_\lambda^*$ are \mathcal{P}_∞ -polarized.

Proof. This is an analog of the proof of Corollary 5.7 of [Bai+23]: one can generate \mathcal{P}_∞ by limits of Hamiltonian vector fields generating the Kähler polarizations and $\sigma_{\lambda, v^*}^\infty$ arises as a limit of Kähler polarized sections. \square

In fact, these distributional sections obtained at infinite Mabuchi geodesic time comprise all of the polarized sections of \mathcal{P}_∞ .

Theorem 4. The vector space of \mathcal{P}_∞ -polarized sections is given by the closure of

$$W_\infty := \bigoplus_{\lambda \in \hat{U}_K} \left\{ \sigma_{\lambda, v^*}^\infty \mid v^* \in V_\lambda^*, \lambda \in \hat{U}_K \right\}.$$

in $(C_c(T^*(U/K)))^* \otimes \tilde{\Omega}_\infty^{\frac{1}{2}}$.

Proof. Half-form corrected \mathcal{P}_∞ -polarized sections can be written in the form

$$\sigma = s \otimes \Omega_\infty^{\frac{1}{2}},$$

where, from Proposition 8, $\Omega_\infty^{\frac{1}{2}}$ is already \mathcal{P}_∞ -polarized. Note that instead of $\Omega_\infty^{\frac{1}{2}}$ one can use other polarized sections of the half-form bundle of \mathcal{P}_∞ , with different divisors, by multiplying Ω_∞ by (\mathcal{P}_∞ -polarized) factors of the form $F_{\hat{\rho}, v_{\hat{\rho}K} \otimes v_{\hat{\rho}}^K} F_{\hat{\rho}, v_{\hat{\rho}} \otimes z^*}^{-1}$, $z^* \in V_{\hat{\rho}}^*$ and by multiplying with the inverse factor on the left factor of the tensor product. The section σ will then be \mathcal{P}_∞ -polarized iff s is \mathcal{P}_∞ -polarized. The prequantum connection on T^*U is given by $\nabla = d + i\theta$ where the connection form can be written as

$$\theta = \sum_{j=1}^n \xi_j \tilde{\omega}_j,$$

where $\{\tilde{\omega}_j\}_{j=1, \dots, n}$ is a basis of right-invariant 1-forms on U , corresponding to an orthonormal basis of \mathfrak{u} , pulled-back to T^*U by the canonical projection $T^*U \rightarrow U$ (see Sections 2 and 4 of [KMN13]), and ξ_j are the coordinates of ξ in the corresponding dual basis of \mathfrak{u}^* . By symplectic reduction [GS82], this induces the prequantum connection on the prequantum line bundle on $T^*(U/K)$, so that over a point $[u, \xi_+] \in T^*(U/K)$, with $\xi_+ \in -i\mathfrak{a}_+^*$, the connection form reads

$$\sum_{j=1}^l (\xi_+)_j \tilde{\omega}_j,$$

where indices have been chosen such that $j = 1, \dots, l$ runs over a basis of $\mathfrak{a} = i(\mathfrak{t} \cap \mathfrak{s})$. This connection form exactly matches the one that one obtains for symplectic toric manifolds, along the open dense subset diffeomorphic to $T^*((S^1)^l)$. Therefore, using Fourier decomposition of sections with respect to the action of T_{inv} , the equations for covariant constancy along the real directions of \mathcal{P}_∞ , given precisely by the orbits of T_{inv} , match the equations of covariant constancy along the real toric polarization of a symplectic toric manifold. From Proposition 3.1 in [Bai+11], the solutions are proportional to a Dirac delta distribution supported on the Bohr-Sommerfeld set, which in our case is given by the level sets $\mu_{\text{inv}}^{-1}(\lambda + \hat{\rho})$ for highest weights $\lambda \in \hat{U}_K$, as mentioned in Section 3.1 and as can be explicitly checked from the previous formula for the connection form. In particular, there are no solutions proportional to derivatives of Dirac delta distributions. On the other hand, covariant constancy along the holomorphic directions of \mathcal{P}_∞ just correspond to holomorphicity along the coadjoint orbits $\mathcal{O}_{\lambda+\hat{\rho}}$. But then all such holomorphic sections arise by taking the sections $\sigma_{\lambda, v^*}^\infty$ above, with $v^* \in V_\lambda^*$. Indeed, $V_{\lambda+\hat{\rho}}^*$ is an irreducible component in the tensor product $V_\lambda^* \otimes V_{\hat{\rho}}^*$. Thus $F_{\lambda+\hat{\rho}, v_{\lambda+\hat{\rho}}^K} \otimes w^*$, for $w \in V_{\lambda+\hat{\rho}}^*$, decomposes into sums of products of $F_{\lambda, v_\lambda^K} \otimes v^* F_{\hat{\rho}, v_{\hat{\rho}}^K} \otimes z^*$, $v^* \in V_\lambda^*$, $z^* \in V_{\hat{\rho}}^*$. So, by multiplying and dividing $\Omega_\infty^{\frac{1}{2}}$ by

different $F_{\hat{\rho}, v_{\hat{\rho}}^K \otimes z^*}$, $z^* \in V_{\hat{\rho}}^*$, in (31) we can describe all holomorphic sections of the Borel-Weil line bundle $L_{\lambda+\hat{\rho}} \rightarrow \mathcal{O}_{\lambda+\hat{\rho}}$. (Note also that, as described in Proposition 4, the holomorphic directions of \mathcal{P}_{∞} along $\mu_{\text{inv}}^{-1}(\lambda + \hat{\rho})$ select as polarized functions elements of the ring generated by quotients of the form $F_{\lambda+\hat{\rho}, v_{\lambda+\hat{\rho}}^K \otimes w_1^*} F_{\lambda+\hat{\rho}, v_{\lambda+\hat{\rho}}^K \otimes w_2^*}^{-1}$, for $w_1, w_2 \in V_{\lambda+\hat{\rho}}^*$, so that no other irreducible components of $V_{\lambda}^* \otimes V_{\hat{\rho}}^*$ contribute to give \mathcal{P}_{∞} -polarized sections supported on that fiber of μ_{inv} . \square

Since U itself can be described as a symmetric space of compact type, we obtain the

Corollary 3. *The Hilbert space*

$$\mathcal{H}_{\text{KW}} = \widehat{\bigoplus}_{\lambda \in \hat{U}_K} \left\{ \sigma_{\lambda, A}^{\infty} \mid \lambda \in \hat{U}, A \in \text{End}(V_{\lambda}) \right\},$$

as described in Section 5.3 of [Bai+23], is the Hilbert space of polarized sections for the Kirwin-Wu polarization of T^*U .

4.4. The Hilbert space \mathcal{H}_{∞} . In this section, we will see that the inner products on the fibers of the quantum bundle induce naturally an inner product structure on W_{∞} so that one obtains the Hilbert space of \mathcal{P}_{∞} -polarized sections, \mathcal{H}_{∞} , by taking the norm completion of W_{∞} .

For the Hilbert space of the Schrödinger quantization $\mathcal{H}_{\text{Sch}} = L^2(U/K) \otimes \sqrt{dx}$, we have the U -invariant inner product,

$$\langle \sigma_{\lambda, v^*}^0, \sigma_{\lambda, \tilde{v}^*}^0 \rangle = d_{\lambda}^{-1} \delta_{\lambda \tilde{\lambda}} \langle v^*, \tilde{v}^* \rangle_{V_{\lambda}^*}.$$

Let us now study the evolution of the norms of $\sigma_{\lambda, v^*}^{th}$, $t > 0$, for $h \in \text{Conv}_{\text{unif}}^{\infty}(\mathfrak{u}^*)^{\text{Ad}^*_{\tilde{v}}}$ and $\lambda \in \hat{U}_K$. From Theorem 3.3.25 in [Kay15] we obtain that, for $g_{\text{std}} = \frac{1}{2} \|\xi\|^2$, $\|\mathcal{L}_{g_{\text{std}}}^* \Omega^{\frac{1}{2}}\|$ is an Ad_K^* -invariant function

$$\|\mathcal{L}_{g_{\text{std}}}^* \Omega^{\frac{1}{2}}\|^2([x, \xi]) = \eta(\xi),$$

where

$$\eta(\xi_+) := \prod_{\alpha \in \Sigma \cap \Phi_+} \left(\frac{\sinh(2\alpha(\xi_+))}{2\alpha(\xi_+)} \right)^{\frac{m_{\alpha}}{2}},$$

for $\xi_+ \in -i\mathfrak{a}_+^*$. For other Kähler structures determined by a symplectic potential $g \in \text{Conv}_{\text{unif}}^{\infty}(\mathfrak{u}^*)^{\text{Ad}^*_{\tilde{v}}}$, we need to pull-back by the composition of Legendre transforms $\mathcal{L}_g \circ \mathcal{L}_{g_{\text{std}}}^{-1}$. This composition of Legendre transforms is not a symplectomorphism of $T^*(U/K)$ but it is straightforward to obtain the correcting factor for the pull-back of the Liouville measure (see, for instance, Lemma 2.4 in [KMN14]) to obtain for the contribution of the half-form

$$\|\mathcal{L}_g^* \Omega^{\frac{1}{2}}\|^2([x, \xi_+]) = \eta(d_{\xi_+} g) \cdot \det(\text{Hess}_{g|_{\mathfrak{g}^*}}(\xi_+))^{\frac{1}{2}}.$$

We will now study the behavior of the norms of the polarized sections $\sigma_{\lambda, v^*}^{gt}$ as $t \rightarrow +\infty$ where, throughout this section we will take

$$g_t = th, \quad t > 0, \quad h \in \text{Conv}_{\text{unif}}^{\infty}(\mathfrak{u}^*)^{\text{Ad}^*_{\tilde{v}}},$$

so that we are considering a Mabuchi geodesic ray that connects the Schrödinger (vertical) polarization to the polarization \mathcal{P}_∞ . Since the half-form corrected inner product on \mathcal{H}_{g_t} is U -invariant, we obtain that two different isotypical components $\mathcal{H}_{g_t}^\lambda, \mathcal{H}_{g_t}^{\tilde{\lambda}}$ are orthogonal for $\lambda \neq \tilde{\lambda}$. Moreover, we have that on each isotypical component \mathcal{H}_g^λ the inner product is fixed up to a multiplicative constant and it is determined, in particular, by the norm of the K -left-invariant vectors

$$\sigma_{\lambda, (v_\lambda^K)^*}^{gt}, \quad \lambda \in \hat{U}_K$$

with $(v_\lambda^K)^*$ defined from v_λ^K via the anti-linear bijection $V_\lambda \rightarrow V_\lambda^*$ explained in Appendix A. For $\lambda \in \hat{U}_K$ let v_λ^K be a unit norm spherical vector (for an U -invariant inner product on V_λ) and consider the Harish-Chandra c -function [HPV02, §4],

$$c(\lambda + \hat{\rho}) = |\langle v_\lambda, v_\lambda^K \rangle_{V_\lambda}|^2 \neq 0.$$

Note here that v_λ and v_λ^K were assumed to be unit vectors so that the expression for $c(\lambda + \hat{\rho})$ is independent of the normalization of the inner product on V_λ .

Theorem 5. *There is a non-zero constant C_0 (independent of h and λ) such that*

$$\lim_{t \rightarrow +\infty} \|\sigma_{\lambda, (v_\lambda^K)^*}^{gt}\|_{\mathcal{H}_{g_t}}^2 = C_0 \hat{P}(\lambda + \hat{\rho})^{\frac{1}{2}} c(\lambda + \hat{\rho}) \|\sigma_{\lambda, (v_\lambda^K)^*}^0\|_{\mathcal{H}_{\text{Sch}}}^2,$$

where $\hat{P}(\xi_+) = \prod_{\alpha \in \Sigma \cap \Phi_+} \alpha(\xi_+)^{m_\alpha}$.

Proof. From the expression for the half-form corrected inner product on the space of Kähler polarized sections (see, for example, Section 4 in [KMN13]), we need to compute

$$\begin{aligned} & \|\sigma_{\lambda, (v_\lambda^K)^*}^{gt}\|_{\mathcal{H}_{g_t}}^2 = e^{-2th(\lambda + \hat{\rho})} \\ & \cdot \int_{T^*(U/K)} \overline{\text{tr}(\pi_\lambda(x e^{itd_\xi h} v_\lambda^K \otimes (v_\lambda^K)^*))} \text{tr}(\pi_\lambda(x e^{itd_\xi h} v_\lambda^K \otimes (v_\lambda^K)^*)) \\ & \cdot e^{-t\kappa_h} \eta(td_\xi h) \det(\text{Hess}_{th})|_{\mathfrak{s}^*}^{\frac{1}{2}} \varepsilon, \end{aligned}$$

where ε is the Liouville measure. From, Theorem 3.3.41 in [Kay15] (or Theorem A.4 in [Kay20]), we obtain that, for some non-zero constant a , and f any integrable function on $T^*(U/K) \cong U \times_K \mathfrak{s}^*$

$$\int_{T^*(U/K)} f \varepsilon = a \int_{U \times \mathfrak{s}^*} (p^* f) dx d\xi,$$

where $p : U \times \mathfrak{s}^* \rightarrow U \times_K \mathfrak{s}^*$ is the canonical projection, dx is the normalized Haar measure on U and $d\xi$ is the Lebesgue measure on \mathfrak{s}^* . Applying this above and using the Weyl orthogonality relations

$$\int_U \overline{\pi_\lambda(x)_{ij}} \pi_\lambda(x)_{kl} dx = d_\lambda^{-1} \delta_{ik} \delta_{jl},$$

we obtain

$$\begin{aligned} & \|\sigma_{\lambda, (v_\lambda^K)^*}^{gt}\|_{\mathcal{H}_{gt}}^2 = ad_\lambda^{-1} e^{-2th(\lambda+\hat{\rho})}. \\ & \cdot \int_{\mathfrak{s}^*} \mathrm{tr}(\pi_\lambda(e^{2itd_\xi h}) v_\lambda^K \otimes (v_\lambda^K)^*) e^{-t\kappa_h} \eta(td_\xi h) \det(\mathrm{Hess}_{th})|_{\mathfrak{s}^*}^{\frac{1}{2}} d\xi, \end{aligned}$$

where we take a spherical vector v_λ^K of norm one. The integrand is Ad_K^* -invariant. From Theorem I.5.17 in [Hel84], we obtain that for f any integrable Ad_K^* -invariant function on \mathfrak{s}^*

$$\int_{\mathfrak{s}^*} f(\xi) d\xi = a' \int_{\mathfrak{a}_+^*} f(\xi_+) \hat{P}(\xi_+) d\xi_+$$

for some non-zero constant a' . Therefore,

$$\begin{aligned} & \|\sigma_{\lambda, (v_\lambda^K)^*}^{gt}\|_{\mathcal{H}_{gt}}^2 = \\ & = a'' d_\lambda^{-1} e^{-2th(\lambda+\hat{\rho})} \int_{\mathfrak{a}_+^*} \mathrm{tr}(\pi_\lambda(e^{2itd_{\xi_+} h}) v_\lambda^K \otimes (v_\lambda^K)^*) e^{-t\kappa_h} \eta(td_{\xi_+} h) \det(\mathrm{Hess}_{th})|_{\mathfrak{s}^*}^{\frac{1}{2}} \hat{P}(\xi_+) d\xi_+, \end{aligned}$$

where a'' is a non-zero constant. Therefore, the leading term in the expression for the norm as $t \rightarrow +\infty$ is

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \|\sigma_{\lambda, (v_\lambda^K)^*}^{gt}\|_{\mathcal{H}_{gt}}^2 = a'' d_\lambda^{-1} c(\lambda + \hat{\rho}). \\ & \lim_{t \rightarrow +\infty} t^{d/2} e^{-2th(\lambda+\hat{\rho})} \int_{\mathfrak{a}_+^*} e^{-2t\langle \lambda, d_{\xi_+} h \rangle} e^{-t\kappa_h} \eta(td_{\xi_+} h) \det(\mathrm{Hess}_h)|_{\mathfrak{s}^*}^{\frac{1}{2}} \hat{P}(\xi_+) d\xi_+, \end{aligned}$$

where $c(\lambda + \hat{\rho})$ is the Harish-Chandra c -function and $d = \dim \mathfrak{s}^*$. From (13) and the proof of Proposition 5, we obtain that

$$\mathrm{Hess}_h(\xi_+)(E_\alpha - E_{\alpha\sigma}) = (2\alpha(\xi_+))^{-1} \mathrm{ad}_{(d_{\xi_+} h)^*}^*(E_\alpha + E_{\alpha\sigma}),$$

so that along the subspace

$$(\mathfrak{a}^*)^\perp = \bigoplus_{\alpha \in (\Phi \setminus \Phi_0) \cap \Phi_+} (E_\alpha - E_{\alpha\sigma}) \subset \mathfrak{s}^*$$

$\mathrm{Hess}_h(\xi_+)$ is diagonal with entries

$$\frac{\alpha(d_{\xi_+} h)}{\alpha(\xi_+)},$$

which is analogous to the case of T^*U (see the proof of Lemma 4.14 in [Bai+23]). Therefore

$$\det \mathrm{Hess}_h(\xi_+)|_{\mathfrak{s}^*} = \frac{\hat{P}((d_{\xi_+} h)^*)}{\hat{P}(\xi_+)} \det \mathrm{Hess}_h(\xi_+)|_{\mathfrak{a}^*}.$$

Therefore, the leading term for the norm as $t \rightarrow +\infty$ is

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \|\sigma_{\lambda, (v_\lambda^K)^*}^{gt}\|_{\mathcal{H}_{gt}}^2 = a'' d_\lambda^{-1} c(\lambda + \hat{\rho}). \\ & \cdot \lim_{t \rightarrow +\infty} t^{d/2} e^{-2th(\lambda+\hat{\rho})} \int_{\mathfrak{a}_+^*} e^{-2t\langle \lambda, d_{\xi_+} h \rangle} e^{-t\kappa_h} \eta(td_{\xi_+} h) \det(\mathrm{Hess}_h)|_{\mathfrak{a}^*}^{\frac{1}{2}} (\hat{P}(\xi_+))^{\frac{1}{2}} (\hat{P}((d_{\xi_+} h)^*))^{\frac{1}{2}} d\xi_+. \end{aligned}$$

where in leading order

$$\eta(td_{\xi_+}h) \sim \text{const. } t^{-\#(\Sigma \cap \Phi_+)} e^{2t\hat{\rho}(d_{\xi_+}h)} P((d_{\xi_+}h)^*)^{-\frac{1}{2}}.$$

We obtain,

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \|\sigma_{\lambda, (v_\lambda^K)^*}^{gt}\|_{\mathcal{H}_{gt}}^2 = a''' d_\lambda^{-1} c(\lambda + \hat{\rho}). \\ \cdot \lim_{t \rightarrow +\infty} t^{l/2} e^{-2th(\lambda + \hat{\rho})} \int_{\mathfrak{a}_+^*} e^{-2t\langle(\lambda + \hat{\rho}), d_{\xi_+}h\rangle} e^{-t\kappa h} \det(\text{Hess}_h)_{|\mathfrak{a}_+^*}^{\frac{1}{2}} (\hat{P}(\xi_+))^{\frac{1}{2}} d\xi_+, \end{aligned}$$

where a''' is a non-zero constant. Just as in the proof of Theorem 4.1 in [KMN14] or in the proof of Lemma 5.4 in [Bai+23], the exponentials, the power $t^{l/2}$ and the determinant of the Hessian of h along \mathfrak{a}^* produce in the limit $t \rightarrow +\infty$, up to a constant, a delta function $\delta(\lambda + \hat{\rho})$ so that

$$\lim_{t \rightarrow +\infty} \|\sigma_{\lambda, (v_\lambda^K)^*}^{gt}\|_{\mathcal{H}_{gt}}^2 = a'''' d_\lambda^{-1} \hat{P}(\lambda + \hat{\rho})^{\frac{1}{2}} c(\lambda + \hat{\rho}),$$

for a non-zero constant a'''' . \square

Remark 12. The factor of $c(\lambda + \hat{\rho})$ in the statement of Theorem 5 is consistent with the results of Stenzel in [Ste99]. By identifying the fibers of $T^*(U/K)$ with the non-compact dual symmetric space to U/K , he considers a natural measure on $T^*(U/K)$ for which the time- t CST, for the case $h = \frac{1}{2}\|\xi\|^2$, becomes as unitary transform from $L^2(U/K)$ to a space of holomorphic functions on $U_{\mathbb{C}}/K_{\mathbb{C}}$. This measure involves the time- $2t$ heat kernel on the non-compact dual symmetric space (see, for instance, [AO04]) which carries a factor of $|c(\mu)|^{-2}$ (from the Plancherel formula for non-compact symmetric spaces) and an integration along $\mu \in \mathfrak{a}^*$. From Theorem 3 in [Ste99], since this CST is unitary with respect to this choice of measure, one can evaluate the norm of $\text{tr}(\pi_\lambda(xe^{i\xi_+})v_\lambda^K \otimes v^*)$ asymptotically as $t \rightarrow +\infty$. The spherical function φ_μ which features in the heat kernel will give a factor of $c(\mu)$ in the asymptotic limit (see Section 2 of [Hel64] or Section 4 of [HPV02]). The integration along ξ_+ then produces Dirac delta functions supported on the points $(\nu + \hat{\rho})$, where ν runs over the weights of V_λ . In the asymptotic limit only the highest weight survives and the remaining factor of $|c|^{-1}$ localizes on $(\lambda + \hat{\rho})$, consistently with Theorem 5.

Using Theorems 3 and 5, we define an inner product on W_∞ induced from the asymptotic limit of the inner products on the Hilbert spaces for half-form corrected Kähler quantizations \mathcal{H}_{th} along *any* geodesic path of U -invariant Kähler structures $th, t > 0$, that is by declaring that the generators

$$\left(\frac{\hat{P}(\lambda + \hat{\rho})^{\frac{1}{2}} c(\lambda + \hat{\rho})}{d_\lambda} \right)^{-\frac{1}{2}} \sigma_{\lambda, e_j^*}^\infty, \lambda \in \hat{U}_K,$$

give an orthonormal set, where $e_j, j = 1, \dots, d_\lambda$ is an orthonormal basis for V_λ . Taking the norm completion we obtain the Hilbert space

$$\mathcal{H}_\infty := \widehat{\bigoplus_{\lambda \in \hat{U}_K} \left\{ \sigma_{\lambda, v^*}^\infty \mid \lambda \in \hat{U}_K, v^* \in V_\lambda^* \right\}}$$

of \mathcal{P}_∞ -polarized sections.

Since the inner products on U -isotypical components in \mathcal{H}_{Sch} and in \mathcal{H}_∞ are related by λ -dependent constants we obtain, as a corollary,

Theorem 6. *The maps $C_{t,h} : \mathcal{H}_{\text{Sch}} \rightarrow \mathcal{H}_{th}$ are not asymptotically unitary up to a (h -independent) constant as $t \rightarrow +\infty$ for any $h \in \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^*)^{\text{Ad}_U^*}$.*

Note, however, that for Mabuchi rays generated by quadratic Hamiltonians, the Laplace approximation used in the proof of Theorem 5 is exact so that, for strictly positive time, the corresponding gCST are unitary, that is we have

Proposition 10. *Let $h \in \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^*)^{\text{Ad}_U^*}$ be quadratic and $s > 0$. Then, the U -equivariant maps*

$$C_{t,h} : \mathcal{H}_{sh} \rightarrow \mathcal{H}_{(s+t)h}$$

are unitary for $t > -s$.

Thus, the inner product structure on \mathcal{H}_∞ can be obtained through the continuous family of unitary gCST maps for the Mabuchi geodesic going through quadratic Hamiltonians. For geodesics generated by non-quadratic Hamiltonians one obtains only asymptotic unitarity of the gCST maps $C_{t,h}$. Moreover, the U -invariant inner product structure for \mathcal{H}_{Sch} is related to the inner products for the Hilbert spaces of the Kähler polarizations and, hence also to the inner product on \mathcal{H}_∞ , through representation-dependent factors in the U -isotypical decompositions.

Remark 13. In the case of T^*U [Bai+23], the inner product for the limit Kirwin-Wu polarization is also induced asymptotically by taking the inner product along Mabuchi rays. By contrast, however, one can take the rays to begin at the vertical polarization, that is, in the case of T^*U , the inner products for \mathcal{H}_{Sch} , for the Kähler polarizations \mathcal{H}_g , $g \in \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^*)^{\text{Ad}_U^*}$, and for \mathcal{H}_{KW} are related t, h -dependent overall factors which do not vary along the U -isotypical components. (In the case of Mabuchi rays of quadratic Hamiltonians, the gCST provides a continuous family of unitary maps from \mathcal{H}_{Sch} to \mathcal{H}_{KW} .)

4.5. The extended quantum bundle. By including the fibers corresponding to the vertical polarization and to \mathcal{P}_∞ , one then obtains an extended bundle of quantization Hilbert spaces

$$\mathcal{H} \rightarrow \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^*)^{\text{Ad}_U^*} \cup \{0, \infty\}$$

with fibers \mathcal{H}_g over $g \in \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^*)^{\text{Ad}_U^*}$, $\mathcal{H}_0 = \mathcal{H}_{\text{Sch}}$ over 0 and \mathcal{H}_∞ over ∞ . As described in the previous sections, \mathcal{H} comes equipped with a global trivializing frame

$$\left\{ \sigma_{\lambda, v^*}^g \mid \lambda \in \hat{U}_K, v^* \in V_\lambda^*, g \in \{0\} \cup \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^*)^{\text{Ad}_U^*} \cup \{\infty\} \right\},$$

which is parallel with respect to the flat connection ∇^Q extended to \mathcal{H} .

The gCST maps $C_{t,h}, t \geq 0, h \in \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^*)^{\text{Ad}^*_{\hat{v}}}$, give U -equivariant isomorphisms between the fibers of \mathcal{H} and define the parallel transport operators of ∇^Q . If one excludes the fiber corresponding to the vertical polarization, the connection ∇^Q is unitary along the Mabuchi rays going through quadratic Hamiltonians while for other rays one obtains asymptotic unitarity at infinite geodesic time.

4.6. Comparing $C_{\infty,h}$ with the unitary Fourier transform. As in the case of T^*U , parallel transport in the extended quantum bundle along the geodesic paths $th, h \in \text{Conv}_{\text{unif}}^\infty(\mathfrak{u}^*)^{\text{Ad}^*_{\hat{v}}}, t \geq 0$, gives an U -equivariant isomorphism defined by the linear extension of

$$L^2(U/K, dx) \cong \mathcal{H}_{\text{Sch}} \ni \text{tr}(\pi_\lambda(x)v_\lambda^K \otimes v^*) \otimes \sqrt{dx} \mapsto \sigma_{\lambda, v^*}^\infty \in \mathcal{H}_\infty, \lambda \in \hat{U}_K, v^* \in V_\lambda^*.$$

Clearly we have an U -equivariant isomorphism

$$\Phi_{\text{GQ}} : \mathcal{H}_\infty \cong \widehat{\bigoplus_{\lambda \in \hat{U}_K} V_\lambda^*}$$

with

$$\Phi_{\text{GQ}}(\sigma_{\lambda, v^*}^\infty) = v^*.$$

From above, this defines a unitary (up to a constant) vector valued Fourier transform with respect to the inner product on $\widehat{\bigoplus_{\lambda \in \hat{U}_K} V_\lambda^*}$ such that, for a basis of $V_\lambda, \{e_j, j = 1, \dots, d_\lambda\}$, orthonormal for a choice of U -invariant inner product,

$$\|e_j^*\|_{\text{GQ}} = \left(\frac{\hat{P}(\lambda + \hat{\rho})^{\frac{1}{2}} c(\lambda + \rho)}{d_\lambda} \right)^{-\frac{1}{2}}.$$

The standard unitary vector-valued Fourier transform for compact symmetric spaces (see [Hel84] and Appendix A)

$$\mathcal{F} : L^2(U/K) \rightarrow \widehat{\bigoplus_{\lambda \in \hat{U}_K} V_\lambda} \cong \widehat{\bigoplus_{\lambda \in \hat{U}_K} V_\lambda^*}$$

can be defined by setting

$$L^2(U/K) \ni \sqrt{d_\lambda} \text{tr}(\pi_\lambda(x)v_\lambda^K \otimes v^*) \mapsto v \in V_\lambda.$$

\mathcal{F} is a unitary isomorphism of Hilbert spaces.

Therefore, unlike what happens for the case of T^*U , the vector-valued Fourier transform induced by geometric quantization along the Mabuchi rays of U -invariant Kähler structures that we describe in this paper is not identical to the standard Fourier transform. That is, if one considers the natural inner products making all the arrows in the following diagram unitary

maps,

$$\begin{array}{ccc}
 L^2(U/K) & \longrightarrow & \mathcal{H}_{\text{Sch}} \\
 \downarrow \mathcal{F} & & \downarrow \\
 \widehat{\bigoplus}_{\lambda \in \hat{U}_K} V_\lambda & & \\
 \downarrow & & \downarrow \\
 \widehat{\bigoplus}_{\lambda \in \hat{U}_K} V_\lambda^* & \xleftarrow{\Phi_{\text{GQ}}} & \mathcal{H}_\infty
 \end{array}$$

then, the diagram commutes only up to multiplication by the representation dependent factor $\hat{P}(\lambda + \hat{\rho})^{-\frac{1}{2}} c(\lambda + \hat{\rho})^{-1}$, along the isotypical component for $\lambda \in \hat{U}_K$, which is necessary to make Φ_{GQ} unitary with respect to the inner product structure on \mathcal{H}_∞ which is, as we described, induced by taking a limit of the half-form corrected inner products for the Kähler quantizations along Mabuchi geodesics connecting the vertical polarizations and \mathcal{P}_∞ .

4.7. Quantum-geometric interpretation of \mathcal{P}_∞ . The limit polarization \mathcal{P}_∞ has an important quantum-geometric interpretation, in line with the general program outlined in Section 7 of [Bai+23]. The Bohr-Sommerfeld set of \mathcal{P}_∞ , as we have seen, is given by

$$\bigcup_{\lambda \in \hat{U}_K} \mu_{\text{inv}}^{-1}(\lambda + \hat{\rho}),$$

where these level sets were called “spectral manifolds” in [Bai+23]. In the limit polarization \mathcal{P}_∞ , the quantization of the coordinate components of μ_{inv} is indeed given just by multiplication operators with spectrum determined by evaluation at $\lambda + \hat{\rho}$, $\lambda \in \hat{U}_K$.

From Section 3.1 and Proposition 4, the corresponding symplectic reductions for the Hamiltonian action of T_{inv} on $T^*(U/K)_{\text{reg}}$ give coadjoint orbits

$$\mu_{\text{inv}}^{-1}(\lambda + \hat{\rho})/T_{\text{inv}} \cong \mathcal{O}_{\lambda + \hat{\rho}}.$$

Let

$$\mathcal{H}_\infty^\lambda := \bigoplus_{v^* \in V_\lambda^*} \langle \sigma_{\lambda, v^*}^\infty \rangle_{\mathbb{C}} \subset \mathcal{H}_\infty, \quad \lambda \in \hat{U}_K.$$

From Proposition 4 and (31), for $\lambda \in \hat{U}_K$, we obtain a natural U -equivariant linear isomorphism

$$H^0(\mathcal{O}_\lambda, L_\lambda) \cong \mathcal{H}_\infty^\lambda,$$

identifying the quantization of the (integral) coisotropic reductions of \mathcal{P}_∞ with subspaces of \mathcal{H}_∞ , so that

$$\mathcal{H}_\infty = \overline{\bigoplus_{\lambda \in \hat{U}_K} \mathcal{H}_\infty^\lambda} \cong \overline{\bigoplus_{\lambda \in \mu_{\text{inv}}(T^*(U/K)_{\text{reg}}) \cap \Lambda_+^K} H^0(\mathcal{O}_\lambda, L_\lambda)}.$$

Thus, as described in Section 7 of [Bai+23], the Hilbert space for the quantization in the limit polarization \mathcal{P}_∞ “decomposes” as a sum of the holomorphic quantizations of the symplectic reductions, with respect to the

Hamiltonian action of T_{inv} , of the spectral manifolds given by the components of the Bohr-Sommerfeld set for \mathcal{P}_∞ .

APPENDIX A. THE FOURIER TRANSFORM

[Hel84, Thm. V.4.3] shows that one has a unitary vector valued Fourier transform

$$\mathcal{F} : L^2(U/K) \rightarrow \widehat{\bigoplus_{\lambda \in \hat{U}_K} V_\lambda},$$

defined by

$$\mathcal{F}f = \sum_{\lambda \in \hat{U}_K} d_\lambda \int_U \overline{\chi_\lambda}(u) L_u f \, du,$$

where L_u is the left regular representation of U on $L^2(U/K)$. Here we use the isometric embeddings

$$V_\lambda \rightarrow L^2(U/K), \quad v \mapsto \sqrt{d_\lambda} f_v$$

where

$$f_{\lambda,v}(u) := \langle v \mid \pi_\lambda(u) v_\lambda^K \rangle_{V_\lambda}$$

and $\langle \cdot \mid \cdot \rangle_{V_\lambda}$ denotes the inner product on V_λ . Note that

$$\mathcal{F}f_{\lambda,v} = f_{\lambda,v}$$

Let V_λ^* be the complex dual of V_λ . The characterization of the spherical dual \hat{U}_K given in [Hel84, Thm. V.4.1] shows that the contragredient representation $(\tilde{\pi}_\lambda, V_\lambda^*)$ of (π_λ, V_λ) is spherical as well. Recall that

$$\langle \tilde{\pi}_\lambda(u) \nu, v \rangle = \langle \nu, \tilde{\pi}_\lambda(u^{-1}) v \rangle$$

for $U \in U, v \in V_\lambda, \nu \in V_\lambda^*$ and $\langle \cdot, \cdot \rangle$ denoting the natural pairing of V_λ^* and V_λ . Writing $\langle w \mid v \rangle_{V_\lambda} = \langle v^*, w \rangle$ then defines an anti-linear bijection

$$V_\lambda \rightarrow V_\lambda^*, \quad v \mapsto v^*$$

which is U -equivariant with respect to π_λ and $\tilde{\pi}_\lambda$. We equip V_λ^* with the inner product making this map an isometry, i.e.

$$\langle v^* \mid w^* \rangle_{V_\lambda^*} = \langle w \mid v \rangle_{V_\lambda} = \langle v^*, w \rangle.$$

Collecting these maps for all $\lambda \in \hat{U}_K$ we obtain an anti-linear bijective isometry

$$\widehat{\bigoplus_{\lambda \in \hat{U}_K} V_\lambda} \longrightarrow \widehat{\bigoplus_{\lambda \in \hat{U}_K} V_\lambda^*}.$$

In view of

$$\langle v \mid \pi_\lambda(u) v_\lambda^K \rangle_{V_\lambda} = \overline{\langle \pi_\lambda(u) v_\lambda^K \mid v \rangle_{V_\lambda}} = \overline{\langle v^*, \pi_\lambda(u) v_\lambda^K \rangle}$$

we have isometric embeddings

$$V_\lambda^* \rightarrow L^2(U/K), \quad v^* \mapsto \sqrt{d_\lambda} f_{\lambda,v^*}$$

where

$$f_{\lambda, v^*}(u) := \langle v^*, \pi_{\lambda}(u)v_{\lambda}^K \rangle$$

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