# On irreducibility of eccentricity matrix of graphs and construction of $\epsilon$ -equienergetic graphs

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#### Abstract

The eccentricity matrix  $\epsilon(G)$ , of a connected graph G is obtained by retaining the maximum distance from each row and column of the distance matrix of G and the other entries are assigned with 0. In this paper, we discuss the eccentricity spectrum of subdivision vertex (edge) join of regular graphs. Also, we obtain new families of graphs having irreducible or reducible eccentricity matrix. Furthermore, we use these results to construct infinitely many  $\epsilon$ -cospectral graph pairs as well as infinitely many pairs and triplets of  $\epsilon$ -cospectral  $\epsilon$ -equienergetic graphs. Moreover, we present some new family of  $\epsilon$ -integral graphs.

Keywords: eccentricity matrix, eccentricity spectrum, eccentricity equienergetic graphs, subdivision vertex (edge) join, irreducibility.

Mathematics Subject Classifications: 05C50, 05C76.

#### 1 Introduction

Let G = (V(G), E(G)) be a simple, (p,q) connected undirected graph with vertex set  $V(G) = \{v_1, v_2, \ldots, v_p\}$ , and edge set  $E(G) = \{e_1, \ldots, e_q\}$ . If two vertices  $v_i$  and  $v_j$  are adjacent in G, we write  $v_i \sim v_j$ , and the edge between them is denoted by  $e_{ij}$ . The vertex  $v_i$  is incident with  $e_j$ , if  $v_i$  is an end vertex of edge  $e_j$ . The degree of a vertex  $v_i$  is denoted by  $deg(v_i)$ . A graph G is r-regular if the degree of every vertex is r. The complement  $\overline{G}$  of a graph G has V(G) as its vertex set and two vertices are adjacent in  $\overline{G}$  if and only if they are not adjacent in G. The line graph L(G) of a graph G, is a graph with its vertex set as the edge set of G and two vertices in L(G) become adjacent in L(G) if the corresponding edges of G share a common end vertex. The line graph of the graph L(G) is represented as  $L^2(G)$ . We denote by  $K_p$  the complete graph on p vertices and by  $K_{p1,p2}$  the complete bipartite graph on  $p_1 + p_2$  vertices. Throughout, we represent J and I as the all-one and identity matrices, respectively, in an appropriate order. The equitable quotient matrix [1] of a partitioned matrix is denoted by F. The *incidence matrix* R = R(G) of a graph G, is the  $p \times q$  matrix,

 $(R(G))_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j, \\ 0 & \text{otherwise.} \end{cases}$ 

The adjacency matrix A = A(G) of a graph G, is a square symmetric matrix,

$$(A(G))_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

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The eigenvalues of A(A-eigenvalues) are real and ordered as  $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_p$ . The A-spectrum of G, spec(G), is the collection of all A - eigenvalues of G along with their multiplicities. Two nonisomorphic graphs of the same order are cospectral if they have the same A - spectrum. The energy  $E_A(G)$ , of a graph G is defined as,  $E_A(G) = \sum_{i=1}^p |\lambda_i|$ . If the energy of two graphs with the same order is equal, they are said to be equienergetic. A graph G is A-integral if all A- eigenvalues of G are integers. The adjacency matrix of the complement of a graph G is  $A(\overline{G}) = J - I - A(G)$ .

The distance matrix D = D(G) of a graph G is a symmetric matrix, whose rows and columns are indexed by the set of vertices, and whose  $(ij)^{th}$  entry represents the distance between the vertices  $v_i$ and  $v_j$ . The distance between two vertices  $v_i$  and  $v_j$  is denoted as  $d(v_i, v_j)$ .

The eccentricity  $e(v_i)$  of a vertex  $v_i$ , is defined as  $e(v_i) = max\{d(v_i, v_j); v_i \in V(G)\}$ . If  $d(v_i, v_k) = e(v_i)$ , then  $v_k$  is an eccentric vertex of  $v_i$ . The minimum and maximum eccentricity of all vertices in G is known as radius(G)(r(G)) and diameter of G(diam(G)) respectively. A graph G is self-centered if r(G) = diam(G). In [8], Randić introduced the notion of eccentricity matrix of a graph, then known as  $D_{MAX}$ - matrix and later in [11], Wang et al. renamed it as eccentricity matrix. The eccentricity matrix,  $\epsilon = \epsilon(G)$  of a graph G of order p is defined as follows,

$$(\epsilon(G))_{ij} = \begin{cases} d(v_i, v_j) & \text{if } d(v_i, v_j) = \min\{\epsilon(v_i), \epsilon(v_j)\}, \\ 0 & \text{otherwise.} \end{cases}$$

The eccentricity and adjacency matrix of a connected graph is obtained from the distance matrix by preserving only the largest and smallest non-zero distances in each row and column, respectively. Thus, the graph's eccentricity matrix is also known as its anti-adjacency matrix. A graph G is eccentricity regular ( $\epsilon$ -regular) if the row sum of  $\epsilon(G)$  is constant. A matrix N is said to be *irreducible* if it is not permutationally similar to an upper triangular block matrix, and is *reducible* if there exists a permutation matrix P such that

$$N = P^T \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} P,$$

where A and C are square block matrices.

In contrast to the adjacency and distance matrices of a connected graph, the eccentricity matrix of a graph need not be irreducible. In [11], the authors asked the following question: For which connected graphs the eccentricity matrix is either reducible or irreducible?

The eccentricity matrix of a complete bipartite graph is reducible, whereas that of a tree is irreducible [6]. One of our aim is to find families of connected graphs whose eccentricity matrix is either irreducible or reducible.

Eccentricity matrix of a graph is a real symmetric matrix and its eigenvalues ( $\epsilon$ -eigenvalues) are real. If  $\epsilon_1 > \ldots > \epsilon_k$  are the distinct  $\epsilon$ -eigenvalues of G, then the eccentricity spectrum( $\epsilon$ -spectrum) of G is defined as,

$$spec_{\epsilon}(G) = \begin{pmatrix} \epsilon_1 & \epsilon_2 & \cdots & \epsilon_k \\ s_1 & s_2 & \cdots & s_k \end{pmatrix},$$

where  $s_j$  indicates the algebraic multiplicity of the  $\epsilon$ -eigenvalue  $\epsilon_j$ . A graph G is eccentricity integral  $(\epsilon$ -integral) if all the  $\epsilon$ -eigenvalues are integers. The largest eigenvalue of  $\epsilon(G)$  is called the eccentricity spectral radius  $(\epsilon$ -spectral radius) of  $G(\rho_{\epsilon}(G))$ . The  $\epsilon$ -Wiener index [7],  $W_{\epsilon}(G)$  of a graph G, defined as

$$W_{\epsilon}(G) = \frac{1}{2} \sum_{i,j} (\epsilon(G))_{ij}.$$

Two non-isomorphic graphs of the same order are said to be eccentricity cospectral ( $\epsilon$ -cospectral) if they have the same  $\epsilon$ -spectrum, otherwise, they are non  $\epsilon$ -cospectral. In [10], Wang et al. introduced the concept of eccentricity energy ( $\epsilon$ -energy),  $E_{\epsilon}(G)$  and it is defined as the sum of absolute values of  $\epsilon$ - eigenvalues of G. Two graphs having the same order are said to be eccentricity equienergetic ( $\epsilon$ -equienergetic) if they have the same  $\epsilon$ -energy. Obviously,  $\epsilon$ -cospectral graphs are  $\epsilon$ -equienergetic. But  $\epsilon$ -equienergetic graphs need not be  $\epsilon$ -cospectral. Here, we focus on the construction of non isomorphic non  $\epsilon$ -cospectral  $\epsilon$ -equienergetic graphs.

In [4], Indulal introduced the graph operations, subdivision vertex join, and subdivision edge join of two graphs and determined its adjacency spectrum. In 2019 [3], the authors studied about the corresponding distance spectrum. Motivated from this we study the  $\epsilon$ -spectrum of subdivision vertex join and subdivision edge join of two regular graphs.

This paper is organized as follows: In Section 2, we collect the definitions and preliminary results needed. In Sections 3 and 4, we discuss the  $\epsilon$ - spectrum of subdivision vertex(edge) join and join of graphs. Also, we obtain new families of graphs having either irreducible or reducible eccentricity matrix. Moreover, we construct families of  $\epsilon$ - cospectral graphs. In Section 5, we present infinitely many pairs and triplets of non  $\epsilon$ -cospectral  $\epsilon$ -equienergetic graphs and some families of  $\epsilon$ -integral graphs.

### 2 Preliminaries

In this section, we collect some basic definitions and results which will be used in the subsequent sections.

**Definition 2.1.** [2] The subdivision graph S(G) of a graph G is obtained from G by subdividing each edge of G using a new vertex. The collection of such new vertices is denoted by I(G).

**Definition 2.2.** [4] The subdivision-vertex join  $G_1 \lor G_2$  of two vertex disjoint graphs  $G_1$  and  $G_2$  is the graph obtained from  $S(G_1)$  and  $G_2$  by joining each vertex of  $V(G_1)$  to every vertex of  $V(G_2)$ .

**Definition 2.3.** [4] The subdivision-edge join  $G_1 \underline{\lor} G_2$  of two vertex disjoint graphs  $G_1$  and  $G_2$  is the graph obtained from  $S(G_1)$  and  $G_2$  by joining each vertex of  $I(G_1)$  with every vertex of  $V(G_2)$ .

**Definition 2.4.** [9] Let G be a (p,q) graph. The eccentric graph of G is denoted by  $G^e$  whose vertex set is  $V(G^e) = V(G)$  and  $v_i \sim v_j$  in  $G^e$  if and only if  $d(v_i, v_j) = min\{e(v_i), e(v_j)\}$ .

**Theorem 2.5.** [9] Let G be any (p,q) graph. Then the matrix  $\epsilon(G)$  is an irreducible if and only if  $G^e$  is a connected graph.

**Lemma 2.6.** [2] Let G be an r-regular, (p,q) graph with an adjacency matrix A and an incidence matrix R. Let L(G) be the line graph of G. Then  $RR^T = A + rI$ ,  $R^TR = B + 2I$ , where B is the adjacency matrix of L(G). Also, if J is an all-one matrix of appropriate order, then  $JR = 2J = R^TJ$  and  $JR^T = rJ = RJ$ .

**Lemma 2.7.** [2] Let G be an r-regular, (p,q) graph with  $spec(G) = \{r, \lambda_2, \ldots, \lambda_p\}$ . Then

$$spec(L(G)) = \begin{pmatrix} 2r-2 & \lambda_2 + r - 2 & \cdots & \lambda_p + r - 2 & -2 \\ 1 & 1 & \cdots & 1 & q - p \end{pmatrix}.$$

Also, Z is an eigenvector corresponding to the eigenvalue -2 if and only if RZ = 0.

**Lemma 2.8.** [5] For every  $t \ge 3$ , there exists a pair of non-cospectral cubic graphs on 2t vertices.

**Lemma 2.9.** [5] Let G be a r - regular  $(r \ge 3)$ , (p,q) graph with  $spec(G) = \{r, \lambda_2, \ldots, \lambda_p\}$ . Then

$$spec(L^{2}(G)) = \begin{pmatrix} 4r - 6 & \lambda_{2} + 3r - 6 & \cdots & \lambda_{p} + 3r - 6 & 2r - 6 & -2 \\ 1 & 1 & \cdots & 1 & \frac{1}{2}p(r-2) & \frac{1}{2}pr(r-2) \end{pmatrix}.$$

**Lemma 2.10.** [2] Let G be a r - regular, (p,q) graph. If the spec $(G) = \{r, \lambda_2, \ldots, \lambda_p\}$ , then the spectrum of the complement of G is, spec $(\overline{G}) = \{p - r - 1, -(\lambda_2 + 1), \ldots, -(\lambda_p + 1)\}$ .

**Theorem 2.11.** [7] Let G be a connected, (p,q) graph. Then  $\rho_{\epsilon}(\epsilon(G)) \ge 2\frac{W_{\epsilon}}{p}$ . Moreover, equality holds if and only if G is  $\epsilon$ -regular.

## 3 Irreducibility and eccentricity spectrum of subdivision vertex join of two graphs

In this section, we discuss about the irreducibility,  $\epsilon$ - spectrum, and eccentricity Wiener index of subdivision vertex join of two graphs.

**Theorem 3.1.** Let  $G_i$  be a  $(p_i, q_i)$  graph, i = 1, 2. Then  $\epsilon(G_1 \lor G_2)$  is irreducible, unless  $G_1 = K_{1,1}$ .

Proof. Let  $G = G_1 \vee G_2$ ,  $V(G) = V(G_1) \cup I(G_1) \cup V(G_2)$ . The vertices in  $V(G_1)$ ,  $I(G_1)$ , and  $V(G_2)$  are denoted by  $v_j$ ,  $u_k$ , and  $w_l$  respectively. If  $G_1 = K_1$ , then clearly  $G^e$  is connected. Case 1:  $G_1$  is not a star graph

In G, it is easy to check that,

$$e(v_j) = 3, \text{ for all } v_j \in V(G_1),$$
  

$$3 \le e(u_k) \le 4, \text{ for all } u_k \in I(G_1),$$
  

$$e(w_l) = 2, \text{ for all } w_l \in V(G_2).$$

In  $G^e$ , every vertex of  $I(G_1)$  is adjacent to every vertex in  $V(G_2)$  since  $d(u_k, w_l) = 2 = min\{e(u_k), e(w_l)\}$ . Here  $G_1$  is not a star graph, so for each vertex  $v_j$  in  $V(G_1)$ , there is a vertex  $u_k$  in  $I(G_1)$  such that  $d(v_j, u_k) = 3 = min\{e(v_j), e(uk)\}$ . Therefore, each  $v_j$  in  $V(G_1)$  is adjacent to some  $u_k$  in  $I(G_1)$ . Hence  $G^e$  is connected.

Case 2:  $G_1 = K_{1,p_1-1}, p_1 \ge 3.$ 

Let  $v_1$  be the unique vertex in  $V(G_1)$  such that  $d_{eg}(v_1) = p_1 - 1$ . In G, it is easy to see that  $e(v_j) = 3$ , for all  $v_j \in V(G_1) \setminus \{v_1\}$  and  $e(v_1) = 2$ . Also  $e(u_k) = 3$  for all  $u_k \in I(G_1)$  and  $e(w_l) = 2$ , for each  $w_l$  in  $V(G_2)$ . Since  $d(u_k, w_l) = 2 = min\{e(u_k), e(w_l)\}$ , in  $G^e$ , every vertex of  $V(G_2)$  is adjacent to every vertex in  $I(G_1)$ . Also, for each vertex  $v_j, j \neq 1$ , there exist  $p_1 - 2$  vertices in  $I(G_1)$  such that  $d(v_j, u_k) = 3 = min\{e(v_j), e(u_k)\}$ . Moreover, for  $j \neq 1$ ,  $d(v_1, v_j) = 2 = min\{e(v_1, e(v_j)\})$ . Hence there is a path joining vertices of  $V(G_1)$  and  $I(G_1)$ . Therefore,  $G^e$  is a connected graph. By Theorem 2.5,  $\epsilon(G_1 \lor G_2)$  is irreducible.

**Theorem 3.2.** Let  $G_j$ , be  $r_j$ -regular  $(r_1 \ge 2)$ ,  $(p_j, q_j)$  graph, j = 1, 2. Let  $\{r_1, \lambda_2, \ldots, \lambda_{p_1}\}$  and  $\{r_2, \beta_2, \ldots, \beta_{p_2}\}$  be the A-eigenvalues of  $G_1$  and  $G_2$  respectively. Then the  $\epsilon$ -eigenvalues of  $\epsilon(G_1 \lor G_2)$  consists of

- 1. 4, with multiplicity  $q_1 p_1$ .
- 2.  $-2(1+\beta_k), k=2,3,\ldots,p_2.$
- 3.  $-3t_1 + 4 4(\lambda_i + r_1), i = 2, 3, ..., p_1, where t_1 = \frac{2(1 (\lambda_i + r_1)) + \sqrt{4(\lambda_i + r_1)^2 + (\lambda_i + r_1) + 4}}{3}$
- 4.  $-3t_2 + 4 4(\lambda_i + r_1), i = 2, 3, \dots, p_1, where t_2 = \frac{2(1 (\lambda + r_1) \sqrt{4(\lambda_i + r_1)^2 + (\lambda_i + r_1) + 4}}{3}$

5. 3 eigenvalues of the equitable quotient matrix of  $\epsilon(G_1 \lor G_2)$ ,

$$\begin{pmatrix} 0 & 3q_1 - 3r_1 & 0 \\ 3p_1 - 6 & 4q_1 - 8r_1 + 4 & 2p_2 \\ 0 & 2q_1 & 2(p_2 - r_2 - 1) \end{pmatrix}$$

*Proof.* Let  $G_1$  and  $G_2$  be regular graphs with regularity  $r_1 \geq 2$  and  $r_2$  respectively. Then, by a appropriate labelling of the vertices of  $G_1 \lor G_2$ , its eccentricity matrix can be written as

$$\epsilon(G_1 \lor G_2) = \begin{pmatrix} 0 & 3(J-R) & 0\\ 3(J-R^T) & 4(J-I-B) & 2J\\ 0 & 2J & 2(J-I-A_2) \end{pmatrix},$$

where R, B and  $A_2$  represents the incidence matrix of  $G_1$ , adjacency matrix of  $L(G_1)$  and adjacency matrix of  $G_2$  respectively.

Now, let Z be an eigenvector of B corresponding to the eigenvalue -2 (with multiplicity  $q_1 - p_1$ ). By  $\begin{pmatrix} 0 \\ \end{pmatrix}$ 

Lemma 2.7 RZ = 0. Consider  $U = \begin{pmatrix} 0 \\ Z \\ 0 \end{pmatrix}$ , then we have

$$\epsilon(G_1 \lor G_2)U = \begin{pmatrix} 0 & 3(J-R) & 0 \\ 3(J-R^T) & 4(J-I-B) & 2J \\ 0 & 2J & 2(J-I-A_2) \end{pmatrix} \begin{pmatrix} 0 \\ Z \\ 0 \end{pmatrix} = \begin{pmatrix} 3(J-R)Z \\ 4(J-I-B)Z \\ 2JZ \end{pmatrix} = 4\begin{pmatrix} 0 \\ Z \\ 0 \end{pmatrix} = 4U.$$

Thus, 4 is an eigenvalue of  $\epsilon(G_1 \lor G_2)$  with multiplicity  $q_1 - p_1$ .

Let W be an eigenvector of  $A_2$  corresponding to the eigenvalue  $\beta_k$   $(k = 2, ..., p_2)$ . Let  $V = \begin{pmatrix} 0 \\ 0 \\ W \end{pmatrix}$ , then we have

we have

$$\epsilon(G_1 \lor G_2)V = \begin{pmatrix} 0 & 3(J-R) & 0 \\ 3(J-R^T) & 4(J-I-B) & 2J \\ 0 & 2J & 2(J-I-A_2) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ W \end{pmatrix} = -2(1+\beta_i) \begin{pmatrix} 0 \\ 0 \\ W \end{pmatrix} = -2(1+\beta_i)V.$$

Thus,  $-2(1 + \beta_i)$  (for  $i = 2, 3, ..., p_2$ ) is an eigenvalue of  $\epsilon(G_1 \lor G_2)$ .

Let X be an eigenvector of  $A_1$  corresponding to the eigenvalue  $\lambda_i$  (for  $i = 2, ..., p_1$ ), where  $A_1$  is the adjacency matrix of  $G_1$ . Using Lemma 2.6, we get  $R^T X$  is an eigenvector of B corresponding to the eigenvalue  $\lambda_i + r_1 - 2$ . Thus the vectors X and  $R^T X$  are orthogonal to the vector  $J_{p_1 \times 1}$ .

Now, consider the vector  $\phi = \begin{pmatrix} tX \\ R^T X \\ 0 \end{pmatrix}$ . Next, we determine under what condition  $\phi$  is an eigenvector

of  $\epsilon(G_1 \lor G_2)$ . If  $\mu$  is an eigenvalue of  $\epsilon(G_1 \lor G_2)$  such that  $\epsilon(G_1 \lor G_2)\phi = \mu\phi$ , then

$$\begin{pmatrix} 0 & 3(J-R) & 0 \\ 3(J-R^{T}) & 4(J-I-B) & 2J \\ 0 & 2J & 2(J-I-A_{2}) \end{pmatrix} \begin{pmatrix} tX \\ R^{T}X \\ 0 \end{pmatrix} = \mu \begin{pmatrix} tX \\ R^{T}X \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -3(\lambda_{i}+r_{1})X \\ (-3t+4-4(\lambda_{i}+r_{1}))R^{T}X \\ 0 \end{pmatrix} = \mu \begin{pmatrix} tX \\ R^{T}X \\ 0 \end{pmatrix}$$

Therefore,

$$-3(\lambda_i + r_1) = \mu t$$
$$-3t + 4 - 4(\lambda_i + r_1) = \mu$$

From this we will get

$$3t^{2} + 4t((\lambda_{i} + r_{1}) - 1) - 3(\lambda_{i} + r_{1}) = 0$$

So t has two values

$$t_1 = \frac{2(1 - (\lambda_i + r_1)) + \sqrt{4(\lambda_i + r_1)^2 + (\lambda_i + r_1) + 4}}{3}.$$
$$t_2 = \frac{2(1 - (\lambda + r_1) - \sqrt{4(\lambda_i + r_1)^2 + (\lambda_i + r_1) + 4}}{3}.$$

Therefore, corresponding to each eigenvalue  $\lambda_i$  (for  $i = 2, ..., p_1$ ) of  $G_1$ , we get 2 eigenvalues for  $\epsilon(G_1 \vee G_2)$ , as  $-3t_1 + 4 - 4(\lambda_i + r_1)$  and  $-3t_2 + 4 - 4(\lambda_i + r_1)$ . Thus, we get  $2(p_1 - 1)$  eigenvalues of  $\epsilon(G_1 \vee G_2)$ .

The remaining 3 eigenvalues of  $\epsilon(G_1 \lor G_2)$  are eigenvalues of its equitable quotient matrix,

$$F = \begin{pmatrix} 0 & 3q_1 - 3r_1 & 0 \\ 3p_1 - 6 & 4q_1 - 8r_1 + 4 & 2p_2 \\ 0 & 2q_1 & 2(p_2 - r_2 - 1) \end{pmatrix}$$

This completes the proof.

As an application we construct new pairs of  $\epsilon$ -cospectral graphs in the following theorem.

**Corollary 3.3.** Let  $G_1$  and  $G_2$  be two regular cospectral graphs and H be any arbitrary regular graph. Then

- 1.  $G_1 \lor H$  and  $G_2 \lor H$  are  $\epsilon$ -cospectral.
- 2.  $H \lor G_1$  and  $H \lor G_2$  are  $\epsilon$  cospectral.

**Corollary 3.4.** Let  $G_j$  be an  $r_j$ -regular  $(r_1 \ge 2)$ ,  $(p_j, q_j)$  graph, j = 1, 2. Then the eccentricity Wiener index of  $\epsilon(G_1 \lor G_2)$  is given by,

$$W_{\epsilon}(G_1 \lor G_2) = q_1(3p_1 - 4r_1 + 2q_1 - 1) + p_2(p_2 + 2q_1 - r_2 - 1) - \frac{3}{2}r_1p_1.$$

**Corollary 3.5.** Let  $G_j$  be an  $r_j$ -regular  $(r_1 \ge 2)$ ,  $(p_j, q_j)$  graph, j = 1, 2. Then

$$\rho_{\epsilon}(G_1 \lor G_2) > \frac{2q_1(3p_1 - 4r_1 + 2q_1 - 1) + 2p_2(p_2 + 2q_1 - r_2 - 1) - 3p_1r_1}{p_1 + p_2 + q_1}$$

Proof. Proof follows from Theorem 2.11 and corollary 3.4

## 4 Irreducibility and eccentricity spectrum of subdivision edge join of two graphs

**Theorem 4.1.** Let  $G_i$  be a  $(p_i, q_i)$  graph, i = 1, 2. Then  $\epsilon(G_1 \leq G_2)$  is irreducible.

Proof. Let  $V(G_1 \underline{\lor} G_2) = V(G_1) \cup I(G_1) \cup V(G_2)$  be a partition of the vertex set of  $G = G_1 \underline{\lor} G_2$ . Let  $v_l$ ,  $u_j$  and  $w_k$  represents the vertices in  $V(G_1), I(G_1)$  and  $V(G_2)$  respectively. Case 1:  $G_1$  is not a star graph. In  $G_1 \underline{\lor} G_2$  we get,

$$3 \le e(v_l) \le 4$$
, for each  $v_l \in V(G_1)$ 

 $e(u_j) = 3$  for each  $u_j \in I(G_1)$  $e(w_k) = 2$ , for each  $w_k \in V(G_2)$ 

Since  $d(w_k, v_l) = 2 = min\{e(w_k), e(v_l)\}$ , every vertex in  $V(G_2)$  is adjacent to every vertex in  $V(G_1)$ in  $G^e$ . Also for each vertex  $u_j \in I(G_1)$ , the eccentric vertex of  $u_j$  will be some  $v_l$ . Hence in  $G^e$  every vertex in  $I(G_1)$  will be adjacent to some vertex in  $V(G_1)$ . Therefore  $G^e$  is connected. Case 2:  $G_1$  is a star graph,  $G_1 = K_{1,p_1-1}$ 

If  $p_1 = 2$ , clearly  $(G_1 \leq G_2)^e$  is connected.

Let  $v_1$  be the unique vertex in  $G_1$  such that  $deg(v_1) = p_1 - 1$ ,  $p_1 > 2$ , then

$$e(v_1) = 2$$
 and  $e(v_l) = 4$ , for  $l \neq 1$   
 $e(u_j) = 3$ , for  $u_j \in I(G_1)$   
 $e(w_k) = 2$ , for  $w_k \in V(G_2)$ .

 $d(v_l, w_k) = 2 = min\{e(v_l), e(w_k)\}$ , so every vertex in  $V(G_1)$  is adjacent to every vertex in  $V(G_2)$ . And for each  $v_l, l \neq 1$  there exist  $p_1 - 2$  vertices  $u_j$  such that  $d(v_l, u_j) = 3 = min\{e(v_l), e(u_j)\}$ . Therefore  $G^e$  is connected. Hence by Theorem 2.5  $\epsilon(G_1 \lor G_2)$  is irreducible.

**Theorem 4.2.** Let  $G_j$  be  $r_j$ -regular  $(r_1 \ge 2)$ ,  $(p_j, q_j)$  graph, j = 1, 2. Let  $\{r_1, \lambda_2, \ldots, \lambda_{p_1}\}$  and  $\{r_2, \gamma_2, \ldots, \gamma_{p_2}\}$  be the A-eigenvalues of  $G_1$  and  $G_2$  respectively. Then the  $\epsilon$ -eigenvalues of  $\epsilon(G_1 \lor G_2)$  consists of

- 1. 0, with multiplicity  $q_1 p_1$ .
- 2.  $-2(1+\gamma_k), k=2,3,\ldots,p_2.$
- 3.  $-2(1+\lambda_i) \pm \sqrt{4(1+\lambda_i)^2 + 9(\lambda_i+r_1)}, i=2,\ldots,p_1.$
- 4. 3 eigenvalues of the equitable quotient matrix of  $\epsilon(G_1 \underline{\vee} G_2)$ ,

$$\begin{pmatrix} 4(p_1 - 1 - r_1) & 3(q_1 - r_1) & 2p_2 \\ 3(p_1 - 2) & 0 & 0 \\ 2p_1 & 0 & 2(p_2 - r_2 - 1) \end{pmatrix}.$$

*Proof.* By a proper labeling of the vertices of  $G_1 \underline{\vee} G_2$ , we get

$$\epsilon(G_1 \underline{\vee} G_2) = \begin{pmatrix} 4(J - I - A_1) & 3(J - R) & 2J \\ 3(J - R^T) & 0 & 0 \\ 2J & 0 & 2(J - I - A_2) \end{pmatrix},$$

where R and  $A_j$  represents the incidence matrix of  $G_1$ , adjacency matrix of  $G_j$  for j = 1, 2. Let Z be an eigenvector of B corresponding to the eigenvalue -2 (with multiplicity  $q_1 - p_1$ ), where B is the adjacency matrix of  $L(G_1)$ . By Lemma 2.7, RZ = 0. Consider  $\psi = \begin{pmatrix} 0 \\ Z \\ 0 \end{pmatrix}$ . Then we have

$$\epsilon(G_1 \underline{\vee} G_2)\psi = \begin{pmatrix} 4(J - I - A_1) & 3(J - R) & 2J \\ 3(J - R^T) & 0 & 0 \\ 2J & 0 & 2(J - I - A_2) \end{pmatrix} \begin{pmatrix} 0 \\ Z \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 0 \\ Z \\ 0 \end{pmatrix} = 0\psi.$$

Thus, 0 is an eigenvalue of  $\epsilon(G_1 \leq G_2)$  with multiplicity  $q_1 - p_1$ .

Let W be an eigenvector of  $A_2$  corresponding to the eigenvalue  $\gamma_k$  (for  $k = 2, ..., p_2$ ). Let  $\zeta = \begin{pmatrix} 0 \\ 0 \\ W \end{pmatrix}$ , then we have  $\epsilon(G_1 \underline{\vee} G_2)\zeta = -2(1 + \gamma_k)\zeta$ . Thus,  $-2(1 + \gamma_k)$  (for  $k = 2, ..., p_2$ ) is an eigenvalue of  $\epsilon(G_1 \underline{\vee} G_2)$ .

Let X be an eigenvector of  $A_1$  corresponding to the eigenvalue  $\lambda_i$  (for  $i = 2, ..., p_1$ ). Using Lemma 2.6, we get  $R^T X$  is an eigenvector of B corresponding to the eigenvalue  $\lambda_i + r_1 - 2$ . Thus, the vectors X and  $R^T X$  are orthogonal to the vector  $J_{p_1 \times 1}$ .

Now, consider the vector  $\phi^* = \begin{pmatrix} tX \\ R^T X \\ 0 \end{pmatrix}$ . Next, we determine under what condition  $\phi^*$  is an eigenvector

of  $\epsilon(G_1 \underline{\vee} G_2)$ . If  $\mu^*$  is an eigenvalue of  $\epsilon(G_1 \underline{\vee} G_2)$  such that  $\epsilon(G_1 \underline{\vee} G_2)\phi^* = \mu^*\phi^*$ , then

$$\begin{pmatrix} 4(J-I-A_1) & 3(J-R) & 2J \\ 3(J-R^T) & 0 & 0 \\ 2J & 0 & 2(J-I-A_2) \end{pmatrix} \begin{pmatrix} tX \\ R^TX \\ 0 \end{pmatrix} = \mu^* \begin{pmatrix} tX \\ R^TX \\ 0 \end{pmatrix} \\ \begin{pmatrix} -4t(1+\lambda_i)X - 3(\lambda_i + r_1)X \\ -3tR^TX \\ 0 \end{pmatrix} = \mu^* \begin{pmatrix} tX \\ R^TX \\ 0 \end{pmatrix}.$$

Therefore

$$-(4t + 4t\lambda_i + 3\lambda_i + 3r_1) = \mu^* t$$
$$-3t = \mu^*.$$

From this we will get

$$3t^2 - 4(1 + \lambda_i)t - 3(\lambda_i + r_1) = 0$$

so that t has two values

$$t_1 = \frac{2(1+\lambda_i) + \sqrt{4(1+\lambda_i)^2 + 9(\lambda_i + r_1)}}{3}, t_2 = \frac{2(1+\lambda_i) - \sqrt{4(1+\lambda_i)^2 + 9(\lambda_i + r_1)}}{3}$$

Therefore, corresponding to each eigenvalue  $\lambda_i$  (for  $i = 2, ..., p_1$ ) of  $G_1$  we get 2 eigenvalues of  $\epsilon(G_1 \vee G_2)$  as  $-3t_1$  and  $-3t_2$ . Hence, we get  $2(p_1 - 1)$  eigenvalues of  $\epsilon(G_1 \vee G_2)$ .

The remaining 3 eigenvalues of  $\epsilon(G_1 \vee G_2)$  are eigenvalues of its equitable quotient matrix,

$$F = \begin{pmatrix} 4(p_1 - 1 - r_1) & 3(q_1 - r_1) & 2p_2 \\ 3(p_1 - 2) & 0 & 0 \\ 2p_1 & 0 & 2(p_2 - r_2 - 1) \end{pmatrix}.$$

The following corollary gives new pairs of  $\epsilon$ -cospectral graphs.

**Corollary 4.3.** Let  $G_1$  and  $G_2$  be two regular cospectral graphs and H be any arbitrary regular graph. Then

- 1.  $G_1 \underline{\lor} H$  and  $G_2 \underline{\lor} H$  are  $\epsilon$  cospectral.
- 2.  $H \underline{\lor} G_1$  and  $H \underline{\lor} G_2$  are  $\epsilon$  cospectral.

**Corollary 4.4.** Let  $G_j$ , j = 1, 2 be a  $r_j$ -regular  $(r_1 \ge 2)$ ,  $(p_i, q_i)$  graph. Then the eccentricity Wiener index of  $\epsilon(G_1 \ge G_2)$  is given by,

$$W_{\epsilon}(G_1 \underline{\vee} G_2) = 2p_1^2 + p_2^2 - \frac{p_1}{2}(4 - 4p_2 + 7r_1) + 3q_1(p_1 - 1) - p_2(r_2 + 1).$$

**Corollary 4.5.** Let  $G_j$  be an  $r_j$ -regular  $(r_1 \ge 2)$ ,  $(p_j, q_j)$  graph, j = 1, 2. Then

$$\rho_{\epsilon}(G_1 \underline{\vee} G_2) > \frac{4p_1^2 + 2p_2^2 - p_1(4 - 4p_2 + 7r_1) + 6q_1(p_1 - 1) - 2p_2(r_2 + 1)}{p_1 + p_2 + q_1}$$

The join  $G_1 \vee G_2$  of two graphs  $G_1$  and  $G_2$ , is the graph obtained from  $G_1 \cup G_2$  by joining every vertex of  $G_1$  with every vertex of  $G_2$ , where  $G_1 \cup G_2$  denotes the disjoint union of  $G_1$  and  $G_2$ . In [11] the authors established that, if G is a r-regular graph of order p and diameter 2, then

$$\epsilon(G \vee K_1) = \begin{pmatrix} 2(J - I - A) & J \\ J & 0 \end{pmatrix},$$

where A is the adjacency matrix of G. The same matrix is obtained if G is a r- regular graph with order p and  $diam(G) \ge 2$ . In the next theorem, we consider the *join* of a non-complete regular graph with  $diam(G) \ge 2$  and  $K_1$ .

**Theorem 4.6.** Let G be a r-regular, non-complete, (p,q) graph. Then

$$spec_{\epsilon}(G \lor K_{1}) = \begin{pmatrix} (p-r-1) \pm \sqrt{(p-r-1)^{2} + p} & -2(\lambda_{2}+1) & \cdots & -2(\lambda_{p}+1) \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

where  $\{r, \lambda_2, \lambda_3, \ldots, \lambda_p\}$  are A-eigenvalues of G.

*Proof.* By a proper labeling of vertices in  $G \vee K_1$ , we get  $\epsilon(G \vee K_1) = \begin{pmatrix} 2(J-I-A) & J \\ J & 0 \end{pmatrix}$ , where A is the adjacency matrix of G. Let X be an eigenvector of A corresponding to the eigenvalue  $\lambda_i$  (for i = 2, ..., p). Then we have

$$\epsilon(G \vee K_1) \begin{pmatrix} X \\ 0 \end{pmatrix} = -2(1+\lambda_i) \begin{pmatrix} X \\ 0 \end{pmatrix}.$$

Thus,  $-2(1 + \lambda_i)$  (for i = 2, ..., p) is an eigenvalue of  $\epsilon(G \vee K_1)$ . The remaining 2 eigenvalues are given by the equitable quotient matrix of  $\epsilon(G \vee K_1)$ ,  $\begin{pmatrix} 2(p-r-1) & 1 \\ p & 0 \end{pmatrix}$ .

In [11], the authors discussed the  $\epsilon$ -spectrum of  $G \lor G$ , where G is a regular graph with diam(G) = 2. Later in [6], Mahato et al. discussed the  $\epsilon$ - spectrum of  $G_1 \lor G_2$ , where  $G_1$  and  $G_2$  are non-complete connected graphs.

In this context, we will refer to the subsequent theorem for its application in Section 5.

**Theorem 4.7.** [6] Let G is a r-regular, non complete, (p,q) graph. Then the  $\epsilon$ -spectrum of  $G \lor G$  is given by,

$$Spec_{\epsilon}(G \lor G) = \begin{pmatrix} 2(p-r-1) & -2(\lambda_2+1) & \cdots & -2(\lambda_p+1) \\ 2 & 2 & \cdots & 2 \end{pmatrix},$$

where  $\lambda_i \neq r$ , i = 2, 3, ..., p are the A-eigenvalues of G.

Next, we will discuss the *join* and *subdivision* vertex(edge)join of a connected graph with a non-connected graph having two components. We will explore its  $\epsilon$ - spectrum and reveal some new graph families with irreducible or reducible eccentricity matrix.

**Theorem 4.8.** Let  $G_j$  be an  $r_j$ -regular,  $(p_j, q_j)$  graph, j = 0, 1, 2, and  $diam(G_1) \ge 2$ . Let  $\{r_0, \lambda_2, \lambda_3, \ldots, \lambda_j\}$  $\ldots, \lambda_{p_0}$ ,  $\{r_1, \beta_2, \beta_3, \ldots, \beta_{p_1}\}$  and  $\{r_2, \gamma_2, \gamma_3, \ldots, \gamma_{p_2}\}$  are the A- eigenvalues of  $G_0, G_1$  and  $G_2$  respectively. Then  $\epsilon$ - eigenvalues of  $G_0 \lor (G_1 \cup G_2)$  consists of

- 1.  $-2(1+\lambda_i), i=2,\ldots,p_0$ .
- 2.  $-2(1+\beta_k), k=2,\ldots,p_1$
- 3.  $-2(1+\gamma_l), l=2,\ldots,p_2.$
- 4. 3 eigenvalues of the equitable quotient matrix of  $\epsilon(G_0 \lor (G_1 \cup G_2))$ ,

$$\begin{pmatrix} 2(p_0 - 1 - r_0) & 0 & 0\\ 0 & 2(p_1 - 1 - r_1) & 2p_2\\ 0 & 2p_1 & 2(p_2 - 1 - r_2) \end{pmatrix}$$

*Proof.* By a proper labeling for the vertices of  $G_0 \vee (G_1 \cup G_2)$ , we get

$$\epsilon(G_0 \lor (G_1 \cup G_2)) = \begin{pmatrix} 2(J - I - A_0) & 0 & 0\\ 0 & 2(J - I - A_1) & 2J\\ 0 & 2J & 2(J - I - A_2) \end{pmatrix},$$

where  $A_j$  denotes the adjacency matrix of  $G_j$  for j = 0, 1, 2.

Let X be an eigenvector of  $A_0$  corresponding to the eigenvalue  $\lambda_i$  (for  $i = 2, ..., p_0$ ), Y be an eigenvector of  $A_1$  corresponding to the eigenvalue  $\beta_k$  (for  $k = 2, ..., p_1$ ), and Z be an eigenvector of  $A_2$  corresponding to the eigenvalue  $\gamma_l$  (for  $l = 2, ..., p_2$ ). Then we have,  $\epsilon(G_0 \vee (G_1 \cup G_2)) \begin{pmatrix} X \\ 0 \\ 0 \end{pmatrix} = -2(1 + \lambda_i) \begin{pmatrix} X \\ 0 \\ 0 \end{pmatrix}$ ,

 $\epsilon(G_0 \lor (G_1 \cup G_2)) \begin{pmatrix} 0 \\ Y \\ 0 \end{pmatrix} = -2(1+\beta_k) \begin{pmatrix} 0 \\ Y \\ 0 \end{pmatrix}, \text{ and } \epsilon(G_0 \lor (G_1 \cup G_2)) \begin{pmatrix} 0 \\ 0 \\ Z \end{pmatrix} = -2(1+\gamma_l) \begin{pmatrix} 0 \\ 0 \\ Z \end{pmatrix}. \text{ The remaining}$ 

3 eigenvalues of  $\epsilon(G_0 \lor (G_1 \cup G_2))$  are given by its equitable quotient matrix,

$$F = \begin{pmatrix} 2(p_0 - 1 - r_0) & 0 & 0\\ 0 & 2(p_1 - 1 - r_1) & 2p_2\\ 0 & 2p_1 & 2(p_2 - 1 - r_2) \end{pmatrix}.$$

**Theorem 4.9.** Let  $G_0$ ,  $G_1$  and  $G_2$  be any three graphs. The following gives some new families of graphs having either a reducible or irreducible eccentricity matrix.

- 1. If  $G_0$  is self-centered and diam $(G_0) = 2$ , then  $G_0 \lor (G_1 \cup G_2)$  is reducible.
- 2. If  $G_0$  is not self-centered and diam $(G_0) = 2$ , then  $G_0 \vee (G_1 \cup G_2)$  is irreducible.
- 3. If  $diam(G_0) \ge 3$ , then  $G_0 \lor (G_1 \cup G_2)$  is reducible.

*Proof.* Let  $G = G_0 \lor (G_1 \cup G_2)$ ,  $V(G) = V(G_0) \cup V(G_1) \cup V(G_2)$  and let  $u_i, v_j, w_k$  be the vertices in  $V(G_0), V(G_1)$ , and  $V(G_2)$  respectively. In G,

$$e(v_j) = 2$$
, for every  $v_j \in V(G_1)$   
 $e(w_k) = 2$ , for every  $w_k \in V(G_2)$ 

and  $d(v_i, w_k) = 2$ . Therefore, every vertex in  $V(G_1)$  is adjacent to every vertex of  $V(G_2)$  in  $G^e$ .

1. If  $G_0$  is self-centered and  $diam(G_0) = 2$ , clearly in G for each  $u_i$  in  $V(G_0)$ ,  $e(u_i) = 2$ . But

$$d(u_i, v_j) = 1 \neq \min\{e(u_i), e(v_j)\} d(u_i, w_k) = 1 \neq \min\{e(u_i), e(w_k)\}.$$

Therefore,  $G^e$  is a disconnected graph. Thus, the result follows from Theorem 2.5.

2. If  $G_0$  is not self-centered and diam(G) = 2, then there exist at least one  $u_l$  in  $V(G_0)$  such that  $e(u_l) = 1$ . Then we have

$$d(u_l, u_i) = 1 = min\{e(u_l), e(u_i)\}, \text{ for all } u_i \text{ in } V(G_0), d(u_l, v_j) = 1 = min\{e(u_l), e(v_j)\}, \text{ for all } v_j \text{ in } V(G_1), d(u_l, w_k) = 1 = min\{e(u_l), e(w_k)\}, \text{ for all } w_k \text{ in } V(G_2).$$

Thus, the eccentric graph  $G^e$  is connected. Therefore,  $\epsilon(G)$  is irreducible by Theorem 2.5.

3. If  $diam(G_0) \ge 3$ , then in G,  $e(u_i) = 2$ , for every  $u_i$  in  $V(G_0)$ . So in  $G^e$ , no vertex in  $V(G_0)$  is adjacent to any vertex in  $V(G_1)$  and  $V(G_2)$ . Thus,  $G^e$  is disconnected. Hence, using Theorem 2.5,  $\epsilon(G)$  is reducible.

- **Theorem 4.10.** Let  $G_j$  be an  $r_j$ -regular  $(r_0 \ge 2)$ ,  $(p_j, q_j)$  graph, j = 0, 1, 2 and let  $\{r_0, \lambda_2, \ldots, \lambda_{p_0}\}$ ,  $\{r_1, \beta_2, \beta_3, \ldots, \beta_{p_1}\}$  and  $\{r_2, \gamma_2, \gamma_3, \ldots, \gamma_{p_2}\}$  be the A- eigenvalues of  $G_0$ ,  $G_1$  and  $G_2$  respectively. Then  $\epsilon$  eigenvalues of  $G_0 \lor (G_1 \cup G_2)$  consists of
  - 1. 4 with multiplicity  $q_0 p_0$ .
  - 2.  $-2(1+\beta_k), k=2,\ldots,p_1.$
  - 3.  $-2(1+\gamma_l), l=2,\ldots,p_2.$
  - 4.  $-3t_1 4 4(\lambda_i + r_0 2), i = 2, \dots, p_0, t_1 = \frac{2(1 (\lambda_i + r_0)) + \sqrt{4(1 (\lambda_i + r_0))^2 + 9(\lambda_i + r_0)}}{3}$
  - 5.  $-3t_2 4 4(\lambda_i + r_0 2), i = 2, \dots, p_0, t_2 = \frac{2(1 (\lambda_i + r_0)) \sqrt{4(1 (\lambda_i + r_0))^2 + 9(\lambda_i + r_0)}}{3}.$
  - 6. 4 eigenvalues of the equitable quotient matrix of  $\epsilon(G_0 \lor (G_1 \cup G_2))$ ,

$$\begin{pmatrix} 0 & 3(q_0 - r_0) & 0 & 0 \\ 3(p_0 - 2) & 4(q_0 - 2r_0 + 1) & 2p_1 & 2p_2 \\ 0 & 2q_0 & 2(p_1 - r_1 - 1) & 2p_2 \\ 0 & 2q_0 & 2p_1 & 2(p_2 - r_2 - 1) \end{pmatrix}$$

*Proof.* By a proper ordering of the vertices of  $G_0 \lor (G_1 \cup G_2)$ ,

$$\epsilon(G_0 \lor (G_1 \cup G_2)) = \begin{pmatrix} 0 & 3(J - R_0) & 0 & 0 \\ 3(J - R_0^T) & 4(J - I - B_0) & 2J & 2J \\ 0 & 2J & 2(J - I - A_1) & 2J \\ 0 & 2J & 2J & 2(J - I - A_2) \end{pmatrix},$$

where  $R_0$  represents the incidence matrix of  $G_0$  and  $B_0$ ,  $A_1$ ,  $A_2$  represents the adjacency matrix of  $L(G_0)$ ,  $G_1$ ,  $G_2$  respectively. Let W be an eigenvector of  $A_1$  corresponding to the eigenvalue  $\beta_k$  (for  $k = 2, ..., p_1$ ), V be an eigenvector of  $A_2$  corresponding to the eigenvalue  $\gamma_l$  (for  $l = 2, ..., p_2$ ), and

X be the eigenvector of  $B_0$  corresponding to the eigenvalues -2 (with multiplicity  $q_0 - p_0$ ). Then we have

$$\begin{pmatrix} 0 & 3(J-R_0) & 0 & 0 \\ 3(J-R_0^T) & 4(J-I-B_0) & 2J & 2J \\ 0 & 2J & 2(J-I-A_1) & 2J \\ 0 & 2J & 2J & 2(J-I-A_2) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ W \\ 0 \end{pmatrix} = -2(1+\beta_k) \begin{pmatrix} 0 \\ 0 \\ W \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 3(J-R_0) & 0 & 0 \\ 3(J-R_0^T) & 4(J-I-B_0) & 2J & 2J \\ 0 & 2J & 2J & 2(J-I-A_1) & 2J \\ 0 & 2J & 2J & 2(J-I-A_2) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ V \end{pmatrix} = -2(1+\gamma_l) \begin{pmatrix} 0 \\ 0 \\ 0 \\ V \end{pmatrix},$$

$$\begin{pmatrix} 0 & 3(J-R_0) & 0 & 0 \\ 3(J-R_0^T) & 4(J-I-B_0) & 2J & 2J \\ 0 & 2J & 2J & 2(J-I-A_2) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ V \end{pmatrix} = -2(1+\gamma_l) \begin{pmatrix} 0 \\ 0 \\ 0 \\ V \end{pmatrix},$$

$$\begin{pmatrix} 0 & 3(J-R_0) & 0 & 0 \\ 3(J-R_0^T) & 4(J-I-B_0) & 2J & 2J \\ 0 & 2J & 2J & 2J & 2J \\ 0 & 2J & 2J & 2(J-I-A_2) \end{pmatrix} \begin{pmatrix} 0 \\ X \\ 0 \\ 0 \end{pmatrix} = 4 \begin{pmatrix} 0 \\ X \\ 0 \\ 0 \end{pmatrix}.$$

Let U be an eigenvector corresponding to the A- eigenvalue  $\lambda_i$  (for  $i = 2, ..., p_0$ ) of  $G_0$ . Using Lemma 2.6, we get  $R_0^T U$  is an eigenvector of  $B_0$  corresponding to the eigenvalue  $\lambda_i + r_0 - 2$ . Thus, the vectors U and  $R_0^T U$  are orthogonal to the vector  $J_{p_0 \times 1}$ .

Now, consider the vector  $\tilde{\phi} = \begin{pmatrix} tU \\ R_0^T U \\ 0 \\ 0 \end{pmatrix}$ . Next, we determine under what condition  $\tilde{\phi}$  is an eigenvector

of  $\epsilon(G_0 \vee (G_1 \cup G_2))$ . If  $\tilde{\mu}$  is an eigenvalue of  $\epsilon(G_0 \vee (G_1 \cup G_2))$  such that  $\epsilon(G_0 \vee (G_1 \cup G_2))\tilde{\phi} = \tilde{\mu}\tilde{\phi}$ , then

$$\begin{pmatrix} 0 & 3(J-R_0) & 0 & 0 \\ 3(J-R_0^T) & 4(J-I-B_0) & 2J & 2J \\ 0 & 2J & 2(J-I-A_1) & 2J \\ 0 & 2J & 2J & 2(J-I-A_2) \end{pmatrix} \begin{pmatrix} tU \\ R_0^T U \\ 0 \\ 0 \end{pmatrix} = \tilde{\mu} \begin{pmatrix} tU \\ R_0^T U \\ 0 \\ 0 \end{pmatrix}.$$

Therefore,

$$-3(\lambda_i + r_0) = \tilde{\mu}t$$
$$-3t - 4 - 4(\lambda_i + r_0 - 2) = \tilde{\mu}.$$

From this we will get,

$$\tilde{\mu} = -3t_1 - 4 - 4(\lambda_i + r_0 - 2), i = 2, \dots, p_0, \text{ where } t_1 = \frac{2(1 - (\lambda_i + r_0)) + \sqrt{4(1 - (\lambda_i + r_0))^2 + 9(\lambda_i + r_0)}}{3}$$

and

$$\tilde{\mu} = -3t_2 - 4 - 4(\lambda_i + r_0 - 2), i = 2, \dots p_0, \text{ where } t_2 = \frac{2(1 - (\lambda_i + r_0)) - \sqrt{4(1 - (\lambda_i + r_0))^2 + 9(\lambda_i + r_0)}}{3}$$

And the remaining 4 eigenvalues of  $\epsilon(G_0 \lor (G_1 \cup G_2))$  are given by its equitable quotient matrix,

$$F = \begin{pmatrix} 0 & 3(q_0 - r_0) & 0 & 0\\ 3(p_0 - 2) & 4(q_0 - 2r_0 + 1) & 2p_1 & 2p_2\\ 0 & 2q_0 & 2(p_1 - r_1 - 1) & 2p_2\\ 0 & 2q_0 & 2p_1 & 2(p_2 - r_2 - 1) \end{pmatrix}$$

The following remarks describes a family of graphs with an irreducible eccentricity matrix.

**Remark 4.11.** For any three connected graphs  $G_0$ ,  $G_1$  and  $G_2$  the matrix  $\epsilon(G_0 \lor (G_1 \cup G_2))$  is irreducible, unless  $G_0 = K_{1,1}$ .

**Remark 4.12.** For any connected graphs  $G_i$ , i = 0, 1, ..., p, the matrix  $\epsilon(G_0 \lor (\cup_{l=1}^p G_l))$  is irreducible, unless  $G_0 = K_{1,1}$ .

**Theorem 4.13.** Let  $G_j$  be an  $r_j$ -regular  $(r_0 \ge 2)$ ,  $(p_j, q_j)$  graph,  $j = 0, 1, 2, \{r_0, \lambda_2, \ldots, \lambda_{p_0}\}, \{r_1, \beta_2, \beta_3, \ldots, \beta_{p_1}\}$  and  $\{r_2, \gamma_2, \gamma_3, \ldots, \gamma_{p_2}\}$  are the A- eigenvalues of  $G_0$ ,  $G_1$  and  $G_2$  respectively. Then  $\epsilon$ -eigenvalues of  $G_0 \ge (G_1 \cup G_2)$  consists of

- 1. 0 with multiplicity  $q_0 p_0$ .
- 2.  $-2(1+\beta_k), k=2,\ldots,p_1.$
- 3.  $-2(1 + \gamma_l), l = 2, \dots, p_2.$
- 4.  $-2(1+\lambda_i) \pm \sqrt{4(1+\lambda_i)^2 + 9(\lambda_i+r_0)}, i=2,\ldots,p_0.$
- 5. 4 eigenvalues of the equitable quotient matrix of  $\epsilon(G_0 \lor (G_1 \cup G_2))$ ,

$$\begin{pmatrix} 4(p_0 - 1 - r_0) & 3(q_0 - r_0) & 2p_1 & 2p_2 \\ 3(p_0 - 2) & 0 & 0 & 0 \\ 2p_0 & 0 & 2(p_1 - 1 - r_1) & 2p_2 \\ 2p_0 & 0 & 2p_1 & 2(p_2 - 1 - r_2) \end{pmatrix}$$

*Proof.* Proof is similar to Theorem4.10.

**Remark 4.14.** For any three connected graphs  $G_0$ ,  $G_1$  and  $G_2$  the matrix  $\epsilon(G_0 \lor (G_1 \cup G_2))$  is irreducible. This gives a new family of graphs having irreducible eccentricity matrix.

**Remark 4.15.** For any connected graphs  $G_i$ , i = 0, 1, ..., p, the matrix  $\epsilon(G_0 \lor (\cup_{l=1}^p G_l))$  is irreducible.

### 5 Applications

This section will explore some of the applications of the results discussed in sections 3 and 4.

#### 5.1 Eccentricity equienergetic graphs

The construction of non cospectral equienergetic graphs is a significant problem in spectral graph theory. In [7], Mahato et al. constructed a pair of non  $\epsilon$ -cospectral  $\epsilon$ -equienergetic graphs for every  $p \geq 5$ . Motivated by this, in this section we will discuss the construction some non  $\epsilon$ -cospectral  $\epsilon$ -equienergetic graphs.

**Theorem 5.1.** For every  $t \ge 3$ , there exist a pair of non  $\epsilon$ - cospectral  $\epsilon$ -equienergetic graphs on 12t vertices.

Proof. Let  $G_1$  and  $G_2$  be two non-cospectral cubic graphs on 2t vertices as in Lemma 2.8 with  $spec(G_1) = \{3, \lambda_2, \ldots, \lambda_{2t}\}$  and  $spec(G_2) = \{3, \beta_2, \ldots, \beta_{2t}\}$ . Using Lemma 2.9,

$$specL^{2}(G_{1}) = \begin{pmatrix} 6 & \lambda_{2} + 3 & \cdots & \lambda_{2t} + 3 & 0 & -2 \\ 1 & 1 & \cdots & 1 & t & 3t \end{pmatrix}$$

and

$$specL^{2}(G_{2}) = \begin{pmatrix} 6 & \beta_{2} + 3 & \cdots & \beta_{2t} + 3 & 0 & -2 \\ 1 & 1 & \cdots & 1 & t & 3t \end{pmatrix}$$

Using Theorem 4.7 we have

$$spec_{\epsilon}(L^{2}(G_{1}) \lor L^{2}(G_{1}) = \begin{pmatrix} 2(6t-7) & -2(\lambda_{i}+4) & -2 & 2\\ 2 & 2 & 2t & 6t \end{pmatrix}, i = 2, \dots, 2t.$$
$$spec_{\epsilon}(L^{2}(G_{2}) \lor L^{2}(G_{2}) = \begin{pmatrix} 2(6t-7) & -2(\beta_{j}+4) & -2 & 2\\ 2 & 2 & 2t & 6t \end{pmatrix}, j = 2, \dots, 2t.$$

Then we have,

$$E_{\epsilon}(L^{2}(G_{1}) \vee L^{2}(G_{1})) = E_{\epsilon}(L^{2}(G_{2}) \vee L^{2}(G_{2})) = 72t - 56.$$

**Example 5.2.** Let  $G_1$  and  $G_2$  are two graphs as in Figure 1.



Figure 1:  $G_1$  and  $G_2$  are non isomorphic non cospectral cubic graphs on 6 vertices.

Then,  $spec(G_1) = \begin{pmatrix} 3 & 1 & 0 & -2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$  and  $spec(G_2) = \begin{pmatrix} 3 & 0 & 3 \\ 1 & 4 & 1 \end{pmatrix}$ . Now,  $L^2(G_l) \lor L^2(G_l), l = 1, 2$  is a graph having 36 vertices. Also, its eccentricity spectrum is given by,

$$spec_{\epsilon}((L^{2}(G_{1}) \lor L^{2}(G_{1})) = \begin{pmatrix} 22 & -10 & -8 & -4 & -2 & 2\\ 2 & 2 & 4 & 4 & 6 & 18 \end{pmatrix}$$
$$spec_{\epsilon}((L^{2}(G_{2}) \lor L^{2}(G_{2})) = \begin{pmatrix} 22 & -8 & -2 & 2\\ 2 & 8 & 8 & 18 \end{pmatrix}.$$
Then we have,  $E_{\epsilon}(L^{2}(G_{1}) \lor L^{2}(G_{1})) = E_{\epsilon}(L^{2}(G_{2}) \lor L^{2}(G_{2})) = 160.$ 



Figure 2: non isomorphic non  $\epsilon$ -cospectral  $\epsilon$ equienergetic graphs on 36-vertices with  $E_{\epsilon} = 160$ 

**Theorem 5.3.** For every  $t \ge 3$ , there exist a pair of non  $\epsilon$ -cospectral  $\epsilon$ -equienergetic graphs on 6t + 1 vertices.

Proof. Let  $G_1$  and  $G_2$  are two non-cospectral cubic graphs on 2t vertices, as in Lemma 2.8 with  $spec(G_1) = \{3, \lambda_2, \ldots, \lambda_{2t}\}$  and  $spec(G_2) = \{3, \beta_2, \ldots, \beta_{2t}\}$ . Then using Lemma 2.8 and Theorem 4.6,

$$spec_{\epsilon}(L^{2}(G_{1}) \vee K_{1}) = \begin{pmatrix} (6t-7) \pm \sqrt{(6t-7)^{2}+6t} & -2(\lambda_{i}+4) & -2 & 2\\ 1 & 1 & t & 3t \end{pmatrix}, i = 2, \dots, 2t.$$
$$spec_{\epsilon}(L^{2}(G_{2}) \vee K_{1}) = \begin{pmatrix} (6t-7) \pm \sqrt{(6t-7)^{2}+6t} & -2(\beta_{j}+4) & -2 & 2\\ 1 & 1 & t & 3t \end{pmatrix}, j = 2, \dots, 2t.$$

Then we have

$$E_{\epsilon}(L^{2}(G_{1}) \vee K_{1}) = E_{\epsilon}(L^{2}(G_{2}) \vee K_{1}) = 24t - 14 + 2\sqrt{(6t - 7)^{2} + 6t}$$

**Example 5.4.** Let  $G_1$  and  $G_2$  are two non cospectral cubic graphs as in figure 1. Now,  $L^2(G_l) \vee K_1$ , l = 1, 2 is a graph having 19 vertices. Also, its eccentricity spectrum is given by,

$$spec_{\epsilon}(L^{2}(G_{1}) \lor K_{1}) = \begin{pmatrix} 11 \pm \sqrt{139} & -10 & -8 & -4 & -2 & 2\\ 1 & 1 & 2 & 2 & 3 & 9 \end{pmatrix}$$
$$spec_{\epsilon}(L^{2}(G_{2}) \lor K_{1}) = \begin{pmatrix} 11 \pm \sqrt{139} & -8 & -2 & 2\\ 1 & 4 & 4 & 9 \end{pmatrix}.$$
Then we have,  $E_{\epsilon}(L^{2}(G_{1}) \lor K_{1}) = E_{\epsilon}(L^{2}(G_{2}) \lor K_{1}) = 81.579.$ 



Figure 3: non isomorphic non  $\epsilon$ -cospectral  $\epsilon$ equienergetic graphs on 19-vertices with  $E_{\epsilon} = 81.579$ 

Next, we give a method for the construction of new pairs and triplets of non  $\epsilon$ - cospectral  $\epsilon$ -equienergetic graphs.

**Theorem 5.5.** Let  $G_1$  and  $G_2$  be two non cospectral cubic graph on 2t vertices,  $t \ge 2$ , and G be any r-regular graph with  $r \ge 2$ . Let  $H_1 = L^2(G_1)$  and  $H_2 = L^2(G_2)$ . Then

- 1.  $G \lor H_1$  and  $G \lor H_2$  are non  $\epsilon$ -cospectral  $\epsilon$ -equienergetic graphs.
- 2.  $G \lor H_1$  and  $G \lor H_2$  are non  $\epsilon$ -cospectral  $\epsilon$ -equienergetic graphs.
- 3.  $G \lor (H_1 \cup H_2)$  and  $G \lor (H_1 \cup H_1)$  and  $G \lor (H_2 \cup H_2)$  are non  $\epsilon$ -cospectral  $\epsilon$ -equienergetic graphs.
- 4.  $G \leq (H_1 \cup H_2)$  and  $G \leq (H_1 \cup H_1)$  and  $G \leq (H_2 \cup H_2)$  are non  $\epsilon$ -cospectral  $\epsilon$ -equienergetic graphs.

Proof. The proof follows from Lemma 2.9, and Theorems 3.2, 4.2, 4.10, 4.13.

**Corollary 5.6.** For every  $t \ge 3$ , there exist 2 pairs of non  $\epsilon$ -cospectral  $\epsilon$ -equienergetic graph on 6+6t vertices.

*Proof.* By Theorem 5.5 we have

$$E_{\epsilon}(K_3 \underline{\vee} H_1) = E_{\epsilon}(K_3 \underline{\vee} H_2)$$
  
and  $E_{\epsilon}(K_3 \underline{\vee} H_1) = E_{\epsilon}(K_3 \underline{\vee} H_2).$ 

**Corollary 5.7.** For every  $t \ge 3$ , there exist 2 triplets of non  $\epsilon$ -cospectral  $\epsilon$ -equienergetic graphs on 6 + 12t vertices.

*Proof.* By Theorem 5.5 we have

$$E_{\epsilon}(K_3 \vee (H_1 \cup H_1)) = E_{\epsilon}(K_3 \vee (H_2 \cup H_2)) = E_{\epsilon}(K_3 \vee (H_1 \cup H_2)),$$
  
and  $E_{\epsilon}(K_3 \vee (H_1 \cup H_1)) = E_{\epsilon}(K_3 \vee (H_2 \cup H_2)) = E_{\epsilon}(K_3 \vee (H_1 \cup H_2)).$ 

#### 5.2 Some $\epsilon$ -integral graphs

The problem of finding A-integral graphs is also an interesting one in spectral graph theory. Motivated by this, in this section, using the results obtained in previous sections, we give some methods to construct some new families of  $\epsilon$ -integral graphs.

**Proposition 5.8.** Let  $G_j$  be a  $r_j$   $(r_1 \ge 2)$  regular,  $(p_j, q_j)$  graph. Then  $G_1 \lor G_2$  is  $\epsilon$ -integral graph if and only if  $G_2$  is A-integral, roots of the equation  $3t^2 + 4t((\lambda_i + r_1) - 1) - 3(\lambda_i + r_1) = 0$ ,  $i = 2, ..., p_1$ , and eigenvalues of the matrix  $\begin{pmatrix} 0 & 3q_1 - 3r_1 & 0 \\ 3p_1 - 6 & 4q_1 - 8r_1 + 4 & 2p_2 \\ 0 & 2q_1 & 2(p_2 - r_2 - 1) \end{pmatrix}$  are integers.

**Proposition 5.9.** Let  $G_j$  be a  $r_j$   $(r_1 \ge 2)$  regular graph,  $(p_i, q_i)$  graph. Then  $G_1 \le G_2$  is  $\epsilon$ -integral

graph if and only if  $G_2$  is A-integral and roots of the equation  $3t^2-4(1+\lambda_i)t-3(\lambda_i+r_1)=0, i=2,\ldots,p_1,$ and eigenvalues of the matrix  $\begin{pmatrix} 4(p_1-1-r_1) & 3(q_1-r_1) & 2p_2\\ 3(p_1-2) & 0 & 0\\ 2p_1 & 0 & 2(p_2-r_2-1) \end{pmatrix}$  are integers.

Now we present some  $\epsilon\text{-integral graphs}$ 

- 1. The graph  $K_3 \lor K_n$  is  $\epsilon$ -integral graphs if and only if 12n + 9 is a perfect square. For example  $K_3 \lor K_6$ ,  $K_3 \lor K_{18}$ ,  $K_3 \lor K_{36}$ ,  $K_3 \lor K_{60}$ ,  $K_3 \lor K_{90}$ ,  $K_3 \lor K_{126}$ .
- 2. The graph  $K_{11} \leq K_n$  is  $\epsilon$ -integral if and only if 44n + 3645 is a perfect square. For example  $K_{11} \underline{\lor} K_{45}$  is  $\epsilon$ -integral.
- 3. Using Theorem 4.8
  - If G is a A-integral graph then  $G \lor (G \cup G)$  is  $\epsilon$ -integral.
  - If  $G_i$ , i = 0, 1, 2, r- regular, A-integral graph then  $G_0 \vee (G_1 \cup G_2)$  is  $\epsilon$ -integral.

The graphs  $K_n \vee (K_m \cup K_l), K_{m,m} \vee (K_{m+1} \cup K_{m+1}), K_{m+1} \vee (K_{m,m} \cup K_{m,m}), K_{m,m} \vee (K_{m+1} \cup K_{m,m})$ are some  $\epsilon$ -integral graphs.

#### Conclusion 6

Using the graph operations subdivision vertex (edge) join and join of graphs, we have added new classes of connected graphs to the existing classes of graphs for which the irreducibility or reducibility of their eccentricity matrix is known. In addition, we analyzed their eccentricity spectra. These results enable us to provide infinitely many  $\epsilon$ -cospectral graph pairs. Furthermore, infinitely many pairs and triplets of non  $\epsilon$ -cospectral  $\epsilon$ -equienergetic graphs are constructed. Moreover, some new families of  $\epsilon$ -integral graphs are obtained.

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#### Disclosure statement

All authors declare that they have no conflicts of interest.

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