

# Optimal E-Values for Exponential Families: the Simple Case

Peter Grünwald<sup>1,2</sup>, Tyron Lardy<sup>2</sup>, Yunda Hao<sup>1</sup>, Shaul K. Bar-Lev<sup>3</sup>, and Martijn de Jong<sup>2</sup>

<sup>1</sup>Centrum Wiskunde & Informatica, Amsterdam, The Netherlands

<sup>2</sup>Leiden University, Leiden, The Netherlands

<sup>3</sup>Holon Institute of Technology, Holon, Israel

May 1, 2024

## Abstract

We provide a general condition under which e-variables in the form of a simple-vs.-simple likelihood ratio exist when the null hypothesis is a composite, multivariate exponential family. Such ‘simple’ e-variables are easy to compute and expected-log-optimal with respect to any stopping time. Simple e-variables were previously only known to exist in quite specific settings, but we offer a unifying theorem on their existence for testing exponential families. We start with a simple alternative  $Q$  and a regular exponential family null. Together these induce a second exponential family  $\mathcal{Q}$  containing  $Q$ , with the same sufficient statistic as the null. Our theorem shows that simple e-variables exist whenever the covariance matrices of  $\mathcal{Q}$  and the null are in a certain relation. Examples in which this relation holds include some  $k$ -sample tests, Gaussian location- and scale tests, and tests for more general classes of natural exponential families.

## 1 Introduction

Exponential families play a central role in statistical modelling, as they include the Bernoulli-, Gaussian-, Poisson-, and many more models. An important task is to test whether these models are well-specified, that is, whether observed data are indeed distributed by an element of an exponential family. Many classic goodness-of-fit tests are well-suited for this purpose [Anderson and Darling, 1954, Lilliefors, 1967, Stephens, 1974]. However, the vast majority of these methods are based on p-values, and thus designed for fixed sample size experiments. Here, we are instead interested in hypothesis tests that are based on e-values [Grünwald et al., 2024].

An e-value is the value taken by an e-variable, which is a test statistic that is suitable for experiments with a flexible design. That is, when sample sizes are not pre-specified, or when the decision to conduct new experiments may depend on past data; see e.g. Ramdas et al. [2023] for a comprehensive overview. The most straightforward example of e-variables are likelihood ratios between simple alternatives and simple null hypotheses. E-variables for composite hypotheses, and in particular ‘good’ e-variables, are generally more complicated. However, e-variables in the form of a likelihood ratio with a single, special element of the null representing the full, composite null sometimes still exist. We refer to such e-variables as ‘simple’ e-variables.

Simple e-variables, if they exist, can easily be computed, and are known to be optimal in an expected-log-optimality sense [Koolen and Grünwald, 2021, Grünwald et al., 2024]. That is, if we combine evidence from a repeated experiment where data is collected using a fixed stopping rule, then using the simple e-variable will asymptotically result in the most evidence against the null,

among all e-variables; details can be found in Section 1.3. As such, it is desirable to find out whether or not simple e-variables exist in specific settings.

Indeed, the main result of this paper, Theorem 1, provides a set of equivalent conditions under which simple e-variables exist for exponential family nulls. We briefly describe it here, assuming prior knowledge on e-variables and exponential families — all relevant definitions are given in Section 1.1–1.3. We fix a regular multivariate exponential family null  $\mathcal{P}$  for data  $U$  with some sufficient statistic vector  $X = t(U)$  and a distribution  $Q$  for  $U$ , outside of  $\mathcal{P}$ , and with density  $q$ . As our most important regularity condition, we assume that  $Q$  has a moment generating function and that there exists  $P_{\mu^*} \in \mathcal{P}$  with the same mean of  $X$ , say  $\mu^*$ , as  $Q$ . It is known that  $P_{\mu^*}$  is the *Reverse Information Projection (RIPr)* of  $Q$  onto  $\mathcal{P}$  [Li, 1999], that is, it achieves  $\min_{P \in \mathcal{P}} D(Q \| P)$ . Denoting the density of  $P_{\mu^*}$  by  $p_{\mu^*}$ , it follows by Theorem 1 of Grünwald et al. [2024] that  $q(U)/p_{\mu^*}(U)$  would be an e-variable in case  $\inf_{P \in \text{CONV}(\mathcal{P})} D(Q \| P) = \min_{P \in \mathcal{P}} D(Q \| P)$ . Our theorem establishes a sufficient condition for when this is actually the case. It is based on constructing a second exponential family  $\mathcal{Q}$  with densities proportional to  $\exp(\beta^T t(U))q(U)$  for varying  $\beta$ :  $\mathcal{Q}$  contains  $Q$  and has the same sufficient statistic as  $\mathcal{P}$ . In some cases, but not all,  $\mathcal{Q}$  may be thought of as the composite alternative we are interested in. Letting  $\Sigma_p(\mu)$  and  $\Sigma_q(\mu)$  denote the covariance matrices of the  $P_\mu \in \mathcal{P}$  and  $Q_\mu \in \mathcal{Q}$  with mean  $\mu$ , Theorem 1 below implies the following: under a further regularity condition on the parameter spaces of  $\mathcal{P}$  and  $\mathcal{Q}$ , simple e-variables exist whenever  $\Sigma_p(\mu) - \Sigma_q(\mu)$  is positive semidefinite for all  $\mu$  in the mean-value parameter space of  $\mathcal{Q}$  (additionally, three equivalent conditions will be given). If this happens, then we may further conclude that for *every* element  $Q_{\mu'}$  of the constructed  $\mathcal{Q}$ , the likelihood ratio  $q_{\mu'}(U)/p_{\mu'}(U)$  is an e-variable, where  $P_{\mu'}$  is the element of  $\mathcal{P}$  to which  $Q_{\mu'}$  is projected. An example pair  $(Q, \mathcal{P})$  to which the theorem applies is when, under  $Q$ ,  $U \sim N(m, s^2)$  for fixed  $m, s^2$  and  $\mathcal{P} = \{N(0, \sigma^2) : \sigma^2 > 0\}$  is the univariate (scale) family of normal distributions. This situation is illustrated in Figure 1 and is treated in detail in Section 4.3.2.

The proof of Theorem 1 is based on convex duality properties of exponential families. In the remainder of this introductory section, we fix notation and definitions of exponential families and e-variables. In Section 2 we show how, based on the constructed family  $\mathcal{Q}$ , one can often easily construct *local* e-variables, i.e. e-variables with the null restricted to a subset of  $\mathcal{P}$ . Then, in Section 3 we present our main theorem, extending the insight to global e-variables. Section 4 provides several examples. This includes cases for which simple e-variables were already established, such as certain k-sample tests [Turner et al., 2024, Hao et al., 2024], as well as cases for which it was previously unknown whether simple e-variables exist, such as for a broad class of natural exponential families. Theorem 1 can thus be seen as a unification and generalization of known results on the existence of simple e-variables, leading to deeper understanding of why they sometimes exist. Section 5 provides the proof for Theorem 1. Finally, Section 6 provides a concluding discussion and points out potential future directions.

## 1.1 Formal Setting

We study general hypothesis testing problems in which the null hypothesis  $\mathcal{P}$  is a regular (and hence full)  $d$ -dimensional exponential family. Here and in the sequel, we will freely use standard properties of exponential families without explicitly referring to their definitions and proofs, for which we refer to e.g. [Barndorff-Nielsen, 1978, Brown, 1986, Efron, 2022]. Each member of  $\mathcal{P}$  is a distribution for a random element  $U$ , that takes values in some set  $\mathcal{U}$ , with a density relative to some given underlying measure  $\nu$  on  $\mathcal{U}$ . The sufficient statistic vector is denoted by  $X = (X_1, \dots, X_d)$  with  $X_j = t_j(U)$  for given measurable functions  $t_1, \dots, t_d$ . We furthermore define  $\mathbb{M}_p$  to be the mean-value parameter space of  $\mathcal{P}$ , i.e. the set of all  $\mu$  such that  $\mathbb{E}_P[X] = \mu$  for some  $P \in \mathcal{P}$ . For

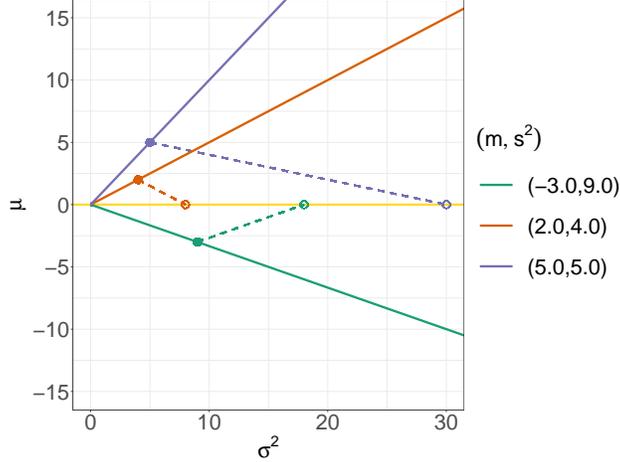


Figure 1: The family  $\mathcal{Q}$  for various  $(m, s^2)$ . The coordinate grid represents the parameters of the full Gaussian family, the horizontal line shows the parameter space of  $\mathcal{P}$ , the sloped lines show the parameters of the distributions in  $\mathcal{Q}$ , and the dashed lines show the projection of  $(m, s^2)$  onto the parameter space of  $\mathcal{P}$ . For example, we may start out with  $Q$  expressing  $U \sim N(m, s^2)$  with  $m = -3.0, s^2 = 9.0$ , represented as the green dot on the green line. Its RIPr onto  $\mathcal{P}$  is the green point on the yellow line. The corresponding family  $\mathcal{Q}$ , constructed in terms of  $Q$  and  $\mathcal{P}$ , is depicted by the green solid line. The theorem implies that the likelihood ratio between any point on the green line and its RIPr onto the yellow line is an e-variable; similarly for the red and blue lines.

any  $\boldsymbol{\mu} \in \mathbb{M}_p$ , we denote by  $P_{\boldsymbol{\mu}}$  the unique element of  $\mathcal{P}$  with  $\mathbb{E}_{P_{\boldsymbol{\mu}}}[X] = \boldsymbol{\mu}$ , so that  $\mathcal{P} = \{P_{\boldsymbol{\mu}} : \boldsymbol{\mu} \in \mathbb{M}_p\}$ . As usual, this parameterization of  $\mathcal{P}$  is referred to as its mean-value parameterization. Furthermore, we use  $\Sigma_p$  to denote the variance function of  $\mathcal{P}$ . That is, for all  $\boldsymbol{\mu} \in \mathbb{M}_p$ ,  $\Sigma_p(\boldsymbol{\mu})$  is the covariance matrix corresponding to  $P_{\boldsymbol{\mu}}$ .

Since  $\mathcal{P}$  is an exponential family, the density of any member of  $\mathcal{P}$  can be written, for each fixed  $\boldsymbol{\mu}^* \in \mathbb{M}_p$ , as

$$p_{\boldsymbol{\beta}; \boldsymbol{\mu}^*}(u) = \frac{1}{Z_p(\boldsymbol{\beta}; \boldsymbol{\mu}^*)} \exp\left(\sum_{j=1}^d \beta_j t_j(u)\right) \cdot p_{\boldsymbol{\mu}^*}(u), \quad (1.1)$$

where  $Z(\boldsymbol{\beta}; \boldsymbol{\mu}^*) = \int \exp(\sum \beta_j t_j(u)) p_{\boldsymbol{\mu}^*}(u) d\nu$ , and  $\boldsymbol{\beta} \in \mathbb{R}^d$  such that  $Z_p(\boldsymbol{\beta}; \boldsymbol{\mu}^*) < \infty$ . Therefore,  $\mathcal{P}$  can also be parameterized as  $\mathcal{P} = \{P_{\boldsymbol{\beta}; \boldsymbol{\mu}^*} : \boldsymbol{\beta} \in \mathbb{B}_{p; \boldsymbol{\mu}^*}\}$ , where  $\mathbb{B}_{p; \boldsymbol{\mu}^*} \subset \mathbb{R}^d$  denotes the canonical parameter space with respect to  $\boldsymbol{\mu}^*$ , i.e. the set of all  $\boldsymbol{\beta}$  for which  $Z_p(\boldsymbol{\beta}; \boldsymbol{\mu}^*) < \infty$ . We use  $\boldsymbol{\beta}_p(\boldsymbol{\mu}'; \boldsymbol{\mu}^*)$  to denote the  $\boldsymbol{\beta} \in \mathbb{B}_{p; \boldsymbol{\mu}^*}$  such that  $\mathbb{E}_{P_{\boldsymbol{\beta}; \boldsymbol{\mu}^*}}[X] = \boldsymbol{\mu}'$  and  $\boldsymbol{\mu}_p(\cdot; \boldsymbol{\mu}^*) = \boldsymbol{\beta}_p^{-1}(\cdot; \boldsymbol{\mu}^*)$  to be its inverse. That is,  $\boldsymbol{\beta}_p(\cdot; \boldsymbol{\mu}^*)$  maps mean-value parameters to corresponding canonical parameters and  $\boldsymbol{\mu}_p(\cdot; \boldsymbol{\mu}^*)$  vice versa. Note that  $p_{\boldsymbol{\mu}^*} = p_{\mathbf{0}; \boldsymbol{\mu}^*}$ , and that we can see from the notation (one versus two subscripts) whether a density is given in the mean- or canonical representation, respectively.

The reason for explicitly denoting the mean  $\boldsymbol{\mu}^*$  of the carrier density, which is unconventional, is that it will be convenient to simultaneously work with different canonical parameterizations, i.e. with respect to a different element of  $\mathbb{M}_p$ , below. These are all linearly related to one another in the sense that for each  $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathbb{M}_p$ , there is a fixed vector  $\boldsymbol{\gamma}$  such that for all  $\boldsymbol{\beta} \in \mathbb{B}_{p; \boldsymbol{\mu}_1}$  it holds that

$p_{\beta;\mu_1} = p_{\beta+\gamma;\mu_2}$ . This can be seen by taking  $\gamma = -\beta_p(\mu_2; \mu_1)$ , since one then has

$$\begin{aligned}
p_{\beta;\mu_1}(u) &= \frac{1}{Z_p(\beta; \mu_1)} \exp\left(\sum_{j=1}^d (\beta_j + \gamma_j) t_j(u)\right) \exp\left(\sum_{j=1}^d -\gamma_j t_j(u)\right) p_{\mu_1}(u) \\
&= \frac{Z_p(-\gamma; \mu_1)}{Z_p(\beta; \mu_1)} \exp\left(\sum_{j=1}^d (\beta_j + \gamma_j) t_j(u)\right) p_{-\gamma;\mu_1}(u) \\
&= \frac{1}{Z_p(\beta + \gamma; \mu_2)} \exp\left(\sum_{j=1}^d (\beta_j + \gamma_j) t_j(u)\right) p_{\mu_2}(u) = p_{\beta+\gamma;\mu_2}(u).
\end{aligned} \tag{1.2}$$

## 1.2 The Composite Alternative Generated by A Simple One

We are mostly concerned with testing the null hypothesis  $\mathcal{P}$  against simple alternative hypotheses of the form  $\{Q\}$  for some distribution  $Q$  on  $\mathcal{U}$ . In particular, we will consider distributions  $Q$  that admit a moment generating function and that have a density  $q$  relative to the underlying measure  $\nu$ . While the former is a strong condition, it holds in many cases of interest. For our analysis, it will be beneficial to define a second exponential family  $\mathcal{Q}$  for  $U$  with distributions  $Q_{\beta;\mu^*}$  and corresponding densities

$$q_{\beta;\mu^*}(u) = \frac{1}{Z_q(\beta; \mu^*)} \cdot \exp\left(\sum_{j=1}^d \beta_j t_j(u)\right) \cdot q(u), \tag{1.3}$$

where  $\mu^*$  is the mean of  $X$  under  $Q$ , and  $Z_q(\beta; \mu^*)$  is the normalizing constant. The notational conventions that we use for  $\mathcal{Q}$  will be completely analogous to that for  $\mathcal{P}$ , e.g.  $\beta_q(\cdot, \mu^*)$ ,  $\mu_q(\cdot, \mu^*)$ ,  $\Sigma_q$ , etc. Since  $Q$  is assumed to have a moment generating function, the canonical domain  $\mathbf{B}_{q,\mu^*}$  is nonempty and contains a neighborhood of  $\mathbf{0}$ . Similarly, the mean-value space  $\mathbf{M}_q$  is also nonempty and contains a neighborhood of  $\mu^*$ . We further have the following: if we take any other  $Q' \in \mathcal{Q}$ , say  $Q' = Q_{\mu'}$  for  $\mu' \in \mathbf{M}_q$ , then the ‘constructed’ family around  $Q'$ , i.e.  $\{q_{\beta;\mu'} : \beta \in \mathbf{B}_{q,\mu'}\}$  coincides with  $\mathcal{Q}$  (as was the case for  $\mathcal{P}$ , by (1.2)).

We may think of the null  $\mathcal{P}$  and the generated family  $\mathcal{Q}$  as two different exponential families that share the same sufficient statistic. Moreover, as we shall see below, there are many examples where their mean-value spaces are equal, that is,  $\mathbf{M}_q = \mathbf{M}_p$ . In this case  $\mathcal{P}$  and  $\mathcal{Q}$  are “matching” pairs: they share the same sufficient statistic as well as the same set of means for this statistic. This is true in many cases that are of practical interest. The most straightforward example is when  $Q$  is originally chosen from some exponential family with the same sufficient statistic and mean-value space as  $\mathcal{P}$ . In this case, the generated family  $\mathcal{Q}$  coincides with the exponential family that  $Q$  was picked from.

This example also highlights that, while the exposition below is focused on simple alternatives, the results are still applicable if one is interested in a composite alternative  $\mathcal{H}_1$ . To this end, take any  $Q \in \mathcal{H}_1$  and use the results below to determine whether a simple e-variable with respect to  $Q$  exists. If one exists for every  $Q$ , an e-variable for the full alternative can easily be constructed either by the method of mixtures or the prequential (sequential plug-in learning) method [Ramdas et al., 2023]. This is particularly easy in cases where the alternative  $\mathcal{Q}$ , that we construct around  $Q$  for our analysis, equals  $\mathcal{H}_1$ , the original alternative of interest, which will be the case in some of our examples. It turns out that it then suffices to check our conditions for a single  $Q \in \mathcal{H}_1$ .

### 1.3 E-variables

We use e-variables to gather evidence against the null hypothesis  $\mathcal{P}$ . An e-variable is a non-negative statistic with expected value bounded by one under the null, i.e. a non-negative statistic  $S(U)$  such that  $\mathbb{E}_P[S(U)] \leq 1$  for all  $P \in \mathcal{P}$ . We give only a brief introduction to e-variables here and refer to e.g. [Grünwald et al., 2024, Ramdas et al., 2023] for detailed discussions. The realization of an e-variable on observed data will be referred to as an e-value, though the two terms are often used interchangeably. Large e-values give evidence against the null hypothesis, since by Markov’s inequality we have that  $Q(S(U) \geq \frac{1}{\alpha}) \leq \alpha$  for any e-variable  $S(U)$  and  $Q \in \mathcal{P}$ . The focus here is on a static setting, where e-variables are computed for a single block of data (i.e. one observation of  $U$ ). However, the main application of e-variables is in anytime-valid settings, where data arrives sequentially and one wants a type-I error guarantee uniformly over time. Indeed, it is well-known that the product of sequentially computed e-variables again gives an e-variable, even if the definition of each subsequent e-variable depends on past e-values, which leads to an easy extension of the methods described here to such anytime-valid settings [Ramdas et al., 2023, Grünwald et al., 2024].

Since large e-values give evidence against the null, we look for e-variables that are, on average, ‘as large as possible’ under the alternative hypothesis. In particular, we study growth-rate optimal (GRO) e-variables, an optimality criterion embraced implicitly or explicitly by most of the e-community [Ramdas et al., 2023]. Grünwald et al. [2024] define the GRO e-variable for single outcome  $U$ , relative to a simple alternative  $\{Q\}$ , to be the e-variable  $S$  that, among all e-variables, maximizes the growth-rate  $\mathbb{E}_{U \sim Q}[\log S(U)]$  (also known as e-power [Wang et al., 2023, Zhang et al., 2023]). Grünwald et al. show that the GRO e-variable is given by:

$$\frac{q(U)}{p_{\leftarrow q}(U)}, \tag{1.4}$$

where  $p_{\leftarrow q}$  denotes the reverse information projection of  $Q$  on the convex hull of the null  $\mathcal{P}$ . The reverse information projection of  $Q$  on  $\text{CONV}(\mathcal{P})$  is defined as the distribution that uniquely achieves  $\inf_{P \in \text{CONV}(\mathcal{P})} D(Q \| P)$ , which is known to exist whenever the latter is finite [Li, 1999, Lardy et al., 2023]. Here,  $D(Q \| P)$  denotes the Kullback-Leibler (KL) divergence between  $Q$  and  $P$ , both defined as distributions for  $U$  [Kullback and Leibler, 1951]. In this article, all reverse information projection will be on  $\text{CONV}(\mathcal{P})$ , so we will not explicitly mention the domain of projection everywhere (i.e. referring to it simply as ‘the reverse information projection of  $Q$ ’). The growth rate achieved by the GRO e-variable is given by

$$\mathbb{E}_Q \left[ \log \frac{q(U)}{p_{\leftarrow q}(U)} \right] = D(Q \| P_{\leftarrow q}) = \inf_{P \in \text{CONV}(\mathcal{P})} D(Q \| P). \tag{1.5}$$

However, due to the fact that, with the exception of the Bernoulli and multinomial models, exponential families are not convex sets of distributions, finding the reverse information projection can be quite challenging [Lardy, 2021, Hao et al., 2024]. In this paper we provide a simple and easily verifiable condition under which

$$\inf_{P \in \text{CONV}(\mathcal{P})} D(Q \| P) = \min_{P \in \mathcal{P}} D(Q \| P), \tag{1.6}$$

that is, the infimum is achieved by an element of  $\mathcal{P}$ , so that the problem greatly simplifies.

In that case, the GRO e-variable simply takes on the form of a likelihood ratio between  $Q$  and a particular member of  $\mathcal{P}$ , i.e.

$$\frac{q(U)}{p(U)}, \tag{1.7}$$

which we will refer to as a simple e-variable relative to  $Q$ . We will frequently use the fact (following from Corollary 1 of [Grünwald et al., 2024, Theorem 1]) that there can be at most one simple e-variable with respect to any fixed alternative, i.e. of the form (1.7). This is captured by the following proposition.

**Proposition 1.** *Fix a probability measure  $Q$  on  $U$ . If there exists a simple e-variable relative to  $Q$ , then it must be the GRO e-variable for testing  $\mathcal{P}$  against alternative  $\{Q\}$ .*

A big advantage of simple e-variables—besides their simplicity—is that their optimality extends beyond the static setting. That is, suppose we were to observe independent copies  $U_1, U_2, \dots$  of the data and assume that a simple e-variable of the form (1.7) exists. As alluded to before, we can measure the total evidence as  $\prod_{i=1}^n q(U_i)/p(U_i)$ , which defines an e-variable for any fixed  $n \in \mathbb{N}$ . Instead of thinking of this as multiplication of individual e-variables, one can think of it as a likelihood ratio of  $U_1, \dots, U_n$ . Proposition 1 then implies that  $\prod_{i=1}^n q(U_i)/p(U_i)$  is the GRO e-variable for testing  $\mathcal{P}$  against  $\{Q\}$  based on  $n$  data points. This statement shows that for any fixed sample size  $n$ , the best e-variable (in the GRO sense of 1.5) is the simple likelihood ratio. Moreover, for applications where the sample size is not fixed beforehand, Koolen and Grünwald [2021, Theorem 12] show that a more flexible statement is also true: if  $\tau$  is any stopping time that is adapted to the data filtration, then  $q(U^\tau)/p(U^\tau)$  is also a maximizer of  $\mathbb{E}[\ln S_\tau]$  over all processes  $S = (S_n)_{n \in \mathbb{N}}$  with  $\mathbb{E}[S_\tau] \leq 1$ . While we will not explicitly consider this type of sequential optimality in the following, it is one of the main motivating factors behind this work.

We assume throughout this paper that, for any considered alternative  $Q$ , there exists a  $\mu^* \in \mathbb{M}_p$  such that  $\mathbb{E}_{X \sim Q}[X] = \mu^*$ . By a standard property of exponential families, the KL divergence from  $Q$  to  $\mathcal{P}$  is then minimized by the element of  $\mathcal{P}$  with the same mean as  $Q$ . If (1.6) holds, then  $P_{\mu^*}$  must therefore be the reverse information projection of  $Q$ . It follows that, if a simple e-variable with respect to  $Q$  exists, then it is given by  $q(U)/p_{\mu^*}(U)$ .

## 2 Existence of Simple Local E-Variables

Here we will show how the family  $\mathcal{Q}$  is related to the question of whether  $q(U)/p_{\mu^*}(U)$  is a *local* GRO e-variable around  $\mu^*$ . We say that a nonnegative statistic  $S(U)$  is a local e-variable around  $\mu^*$  if there exists a connected open subset  $\mathcal{B}'_{\mu^*}$  of  $\mathcal{B}_{p;\mu^*} \cap \mathcal{B}_{q;\mu^*}$  containing  $\mathbf{0}$  such that  $S$  is an e-variable relative to  $\mathcal{P}' = \{P_\beta : \beta \in \mathcal{B}'_{\mu^*}\}$ , i.e.  $\sup_{\beta \in \mathcal{B}'_{\mu^*}} \mathbb{E}_{P_{\beta;\mu^*}}[S] \leq 1$ . If  $S$  also maximizes  $\mathbb{E}_Q[\ln S(U)]$  among all e-variables relative to  $\mathcal{P}'$ , then we say that  $S$  is a local GRO e-variable with respect to  $Q$ . A local (GRO) e-variable may not be an e-variable relative to the full null hypothesis  $\mathcal{P}$ , but it is an e-variable relative to some smaller null hypothesis, restricted to all distributions in the null with mean in a neighborhood of  $\mu^*$ . Investigating when local e-variables exist provides the basic insight on top of which the subsequent, much stronger Theorem 1 about ‘global’ e-variables is built. As stated in Section 1.2, we may view  $\mathcal{P}$  and  $\mathcal{Q}$  as two families with the same sufficient statistic, only differing in their carrier, which for  $\mathcal{P}$  is  $p_{\mu^*} = p_{\mathbf{0};\mu^*}$  and for  $\mathcal{Q}$  is  $q_{\mathbf{0};\mu^*} = q$ . Note that from now on we can and will denote the original  $Q$  by  $Q_{\mu^*}$ .

Define the function  $f(\cdot; \mu^*) : \mathcal{B}_{p;\mu^*} \cap \mathcal{B}_{q;\mu^*} \rightarrow \mathbb{R}$  as

$$f(\beta; \mu^*) := \log \mathbb{E}_{P_{\beta;\mu^*}} \left[ \frac{q_{\mu^*}(U)}{p_{\mu^*}(U)} \right] = \log Z_q(\beta; \mu^*) - \log Z_p(\beta; \mu^*), \quad (2.1)$$

where the equality comes from the fact that we can rewrite the density in the numerator as  $q_{\mu^*}(U) = Z_q(\beta; \mu^*) \exp(\sum_{j=1}^d \beta_j t_j(u))^{-1} q_{\beta;\mu^*}(U)$  and similar for the density in the denominator. It should be clear that the function  $f(\cdot; \mu^*)$  is highly related to the question we are interested

in. Indeed,  $q_{\mu^*}(U)/p_{\mu^*}(U)$  is a local e-variable relative to  $\mathcal{P}' = \{P_{\beta} : \beta \in \mathcal{B}'_{\mu^*}\}$  if and only if  $\sup_{\beta \in \mathcal{B}'_{\mu^*}} f(\beta; \mu^*) \leq 0$ . Equivalently, since  $f(\mathbf{0}; \mu^*) = \mathbf{0}$ , we have that  $q_{\mu^*}/p_{\mu^*}$  is a local e-variable around  $\mu^*$  if and only if there is a local maximum at  $\mathbf{0}$ . To investigate when this happens, a standard result on exponential families gives the following:

$$\nabla f(\beta; \mu^*) = \mathbb{E}_{Q_{\beta; \mu^*}}[X] - \mathbb{E}_{P_{\beta; \mu^*}}[X] \quad (2.2)$$

In particular, it follows that  $\nabla f(\mathbf{0}; \mu^*) = \mu^* - \mu^* = \mathbf{0}$ . Thus,  $q_{\mu^*}/p_{\mu^*}$  is a local e-variable around  $\mu^*$  if and only if the  $d \times d$  Hessian matrix of second partial derivatives of  $f(\cdot; \mu^*)$ , is negative semidefinite in  $\mathbf{0}$ . By (2.1)-(2.2) and using a convex duality property of exponential families, this is equivalent to

$$I_p(\mathbf{0}; \mu^*) - I_q(\mathbf{0}; \mu^*) = \Sigma_p(\mu^*) - \Sigma_q(\mu^*) \text{ is positive semidefinite,}$$

where  $I_p$  and  $I_q$  denote the Fisher information matrix in terms of the canonical parameter spaces of  $\mathcal{P}$  and  $\mathcal{Q}$ , respectively. We have thus proven our first result:

**Proposition 2.**  *$q_{\mu^*}(U)/p_{\mu^*}(U)$  is a local e-variable around  $\mu^*$  (and therefore, by Proposition 1, a GRO local e-variable) if and only if  $\Sigma_p(\mu^*) - \Sigma_q(\mu^*)$  is positive semidefinite.*

The surprising result that follows below essentially adds to this that, if for every  $\mu^* \in \mathcal{M}_q$ ,  $q_{\mu^*}/p_{\mu^*}$  is a local e-variable, then also for every  $\mu^*$ , we have that  $q_{\mu^*}/p_{\mu^*}$  is a full, global e-variable!

### 3 Existence of Global E-Variables (Main Result)

The theorem below gives eight equivalent characterizations of when a global GRO e-variable exists. Not all characterizations are equally intuitive and informative: the simplest ones are Part 1 and 3. To appreciate the more complicated characterizations as well, it is useful to first recall some convex duality properties concerning derivatives of KL divergences with regular exponential families [see e.g. Grünwald, 2007, Section 18.4.3]:

$$\beta_p(\mu; \mu^*) = \nabla_{\mu} D(P_{\mu} \| P_{\mu^*}), \quad (3.1)$$

$$(\Sigma_p^{-1}(\mu))_{ij} = \frac{d^2}{d\mu_i d\mu_j} D(P_{\mu} \| P_{\mu^*}), \quad (3.2)$$

and analogous for  $\mathcal{Q}$ . That is, the gradient of the KL divergence in its first argument at  $\mu$  is given by the canonical parameter vector corresponding to  $\mu$ , and the Hessian is given by the Fisher information, i.e. the inverse covariance matrix.

**Theorem 1.** *Let  $\mathcal{P}$  be a regular exponential family with mean-value parameter space  $\mathcal{M}_p$ . Fix a distribution  $Q$  for  $U$  with  $\mathbb{E}_Q[X] = \mu^*$  for some  $\mu^* \in \mathcal{M}_p \subseteq \mathbb{R}^d$  and consider the corresponding  $\mathcal{Q}$  as defined above. Suppose that  $\mathcal{M}_q$  is convex,  $\mathcal{M}_q \subseteq \mathcal{M}_p$ , and  $\mathcal{B}_{p;\mu} \subseteq \mathcal{B}_{q;\mu}$  for all  $\mu \in \mathcal{M}_q$ . Then the following statements are equivalent:*

1.  $\Sigma_p(\mu) - \Sigma_q(\mu)$  is positive semidefinite for all  $\mu \in \mathcal{M}_q$ .
2.  $(\beta_p(\mu; \mu') - \beta_q(\mu; \mu'))^T \cdot (\mu - \mu') \leq 0$  for all  $\mu, \mu' \in \mathcal{M}_q$ .
3.  $D(Q_{\mu} \| Q_{\mu'}) \geq D(P_{\mu} \| P_{\mu'})$  for all  $\mu, \mu' \in \mathcal{M}_q$ .
4.  $\log Z_p(\beta; \mu) \geq \log Z_q(\beta; \mu)$  for all  $\mu \in \mathcal{M}_q, \beta \in \mathcal{B}_{p;\mu}$ .

5.  $q_{\boldsymbol{\mu}}(U)/p_{\boldsymbol{\mu}}(U)$  is a global e-variable for all  $\boldsymbol{\mu} \in \mathbb{M}_q$ .
6.  $q_{\boldsymbol{\mu}}(U)/p_{\boldsymbol{\mu}}(U)$  is the global GRO e-variable w.r.t.  $Q_{\boldsymbol{\mu}}$  for all  $\boldsymbol{\mu} \in \mathbb{M}_q$ .
7.  $q_{\boldsymbol{\mu}}(U)/p_{\boldsymbol{\mu}}(U)$  is a local e-variable for all  $\boldsymbol{\mu} \in \mathbb{M}_q$ .
8.  $q_{\boldsymbol{\mu}}(U)/p_{\boldsymbol{\mu}}(U)$  is a local GRO e-variable w.r.t.  $Q_{\boldsymbol{\mu}}$  for all  $\boldsymbol{\mu} \in \mathbb{M}_q$ .

Note that the canonical parameter space of a full exponential family is always convex, but the mean-value space need not be [Efron, 2022], so the convexity requirement, while it holds in all examples that we will consider below, is not void. Furthermore, in the one-dimensional case, the first statement simplifies to  $\sigma_p^2(\mu) \geq \sigma_q^2(\mu)$  for all  $\mu \in \mathbb{M}_q$ . Similarly, the second statement reduces to  $\beta_q(\mu; \mu') \geq \beta_p(\mu; \mu')$  for all  $\mu \in \mathbb{M}_q$  such that  $\mu > \mu'$  and  $\beta_q(\mu; \mu') \leq \beta_p(\mu; \mu')$  for all  $\mu \in \mathbb{M}_q$  such that  $\mu < \mu'$  for all  $\mu' \in \mathbb{M}_q$ .

Using standard properties of Loewner ordering, it can be established that  $\Sigma_p(\boldsymbol{\mu}) - \Sigma_q(\boldsymbol{\mu})$  is positive semidefinite if and only if  $\Sigma_q^{-1}(\boldsymbol{\mu}) - \Sigma_p^{-1}(\boldsymbol{\mu})$  is [see e.g. Agrawal, 2018]. Therefore, statement 1 in Theorem 1 can be thought of as a condition on the second derivative of  $D(P_{\boldsymbol{\mu}} \| P_{\boldsymbol{\mu}^*}) - D(Q_{\boldsymbol{\mu}} \| Q_{\boldsymbol{\mu}^*})$ , whereas statement 2 refers to its first derivative, and statement 3 to the difference in KL divergence itself. It is somewhat surprising that signs of differences between the second derivatives and separately signs of differences between the first derivatives are sufficient to determine signs of difference between a function itself.

Finally, we note that it is sometimes easy to check that either  $\mathbb{M}_p = \mathbb{M}_q$  or  $\mathbb{B}_{p;\boldsymbol{\mu}^*} = \mathbb{B}_{q;\boldsymbol{\mu}^*}$ . The following proposition shows that, in the 1-dimensional setting, this is already sufficient to apply the theorem (we do not know whether the analogous result holds in higher dimensions — we suspect it does not):

**Proposition 3.** *Let  $\mathcal{P}$  be a 1-dimensional regular exponential family with mean-value parameter space  $\mathbb{M}_p \subseteq \mathbb{R}$ . Fix a distribution  $Q$  for  $U$  with  $\mathbb{E}_Q[X] = \mu^*$  for some  $\mu^* \in \mathbb{M}_p$  and consider the corresponding  $\mathcal{Q}$  as defined above. Suppose that for all  $\mu \in \mathbb{M}_q$ , we have  $\sigma_p^2(\mu) \geq \sigma_q^2(\mu)$ , i.e. the first condition of Theorem 1 holds. Then:*

1. *If  $\mathbb{M}_q = \mathbb{M}_p$  then for all  $\mu' \in \mathbb{M}_q$ ,  $\mathbb{B}_{p;\mu'} \subseteq \mathbb{B}_{q;\mu'}$ , i.e., Theorem 1 is applicable.*
2. *If for some  $\mu \in \mathbb{M}_q$ , we have that  $\mathbb{B}_{p;\mu} = \mathbb{B}_{q;\mu}$  then  $\mathbb{M}_q \subseteq \mathbb{M}_p$ . Hence if for all  $\mu \in \mathbb{M}_q$ , we have that  $\mathbb{B}_{p;\mu} = \mathbb{B}_{q;\mu}$ , then Theorem 1 is applicable.*

The proof is simple and we only sketch it here: for part 1, draw the graphs of  $\beta_p(\mu; \mu')$  and  $\beta_q(\mu; \mu')$  as functions of  $\mu \in \mathbb{M}_q$ , noting that both functions must take the value 0 at the point  $\mu = \mu'$ . Using that  $1/\sigma_p^2(\mu)$  is the derivative of  $\beta_p(\mu; \mu')$  and similarly for  $\sigma_q^2(\mu)$ , the function  $\beta_q(\mu; \mu')$  must lie above  $\beta_p(\mu; \mu')$  for  $\mu > \mu'$ , and below for  $\mu < \mu'$ . Therefore the co-domain of  $\beta_q$  must include that of  $\beta_p$ . The second part goes similarly, essentially by flipping the above graph of two functions by 90 degrees.

## 4 Examples

In this section we discuss a variety of settings to which Theorem 1 can be applied. In some cases, this gives new insights into whether simple e-variables exist, and in others it simply gives a reinterpretation of existing results. The examples are broadly divided in terms of the curvature of the function  $f(\cdot; \boldsymbol{\mu}^*)$ , as defined in (2.1). Instances where  $f(\cdot; \boldsymbol{\mu}^*)$  is constant will be referred to as having ‘zero curvature’, those with a constant second derivative as having ‘constant curvature’, and ‘nonconstant curvature’ otherwise.

## 4.1 Zero Curvature: Gaussian and Poisson k-sample tests

Hao et al. [2024] provide GRO e-values for  $k$ -sample tests with regular exponential families. In their setting, data arrives in  $k \in \mathbb{N}$  groups, or samples, and they test the hypothesis that all of the data points are distributed according to the same element of some exponential family. That is, let  $U = (Y_1, \dots, Y_k)$  for  $Y_i \in \mathcal{Y}$ , so that  $\mathcal{U} = \mathcal{Y}^k$  for some measurable space  $\mathcal{Y}$ . Furthermore, fix a one-dimensional regular exponential family on  $\mathcal{Y}$ , given in its mean-value parameterization as  $\mathcal{P}_{\text{START}} = \{P_\mu : \mu \in \mathbb{M}_{\text{START}}\}$  with sufficient statistic  $t_{\text{START}}(Y)$ . The composite null hypothesis  $\mathcal{P}$  considered in the k-sample test expresses that  $Y_1, \dots, Y_k \stackrel{\text{i.i.d.}}{\sim} P_\mu$  for some  $\mu \in \mathbb{M}_{\text{START}}$ . On the other hand, the simple alternative  $Q$  that Hao et al. consider is characterized by  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k) \in \mathbb{M}_{\text{START}}^k$ , and expresses that the  $Y_1, \dots, Y_k$  are independent with  $Y_i \sim P_{\mu_i}$  for  $i = 1 \dots k$ . They show that, for the case that  $\mathcal{P}_{\text{START}}$  is either the Gaussian location family or the Poisson family,

$$S(U) := \prod_{i=1}^k \frac{p_{\mu_i}(Y_i)}{p_{\bar{\mu}}(Y_i)}, \text{ with } \bar{\mu} = \frac{1}{k} \sum_{i=1}^k \mu_i,$$

is a simple e-value relative to  $Q$ , and that its expectation is constant as the null varies. That is, for any  $\mu' \in \mathbb{M}_{\text{START}}$ , it holds that

$$\mathbb{E}_{U \sim P_{\mu'} \times \dots \times P_{\mu'}} [S(U)] = 1. \quad (4.1)$$

This finding can now be re-interpreted as an instance of Theorem 1, as we will show in detail for the Poisson family; the analysis for the Gaussian location family is completely analogous. In the Poisson case,  $t_{\text{START}}(Y) = Y$ , so that  $\mathcal{P}$  defines an exponential family on  $\mathcal{U}$  with sufficient statistic  $X = \sum_{i=1}^k Y_i$  and mean-value space  $\mathbb{M}_p = \mathbb{R}^+$ . The latter follows because the sum of Poisson data is itself Poisson distributed with mean equal to the sum of means of the original data. Under the alternative, the mean of the sufficient statistic is given by  $\mu^* := \mathbb{E}_Q[\sum_{i=1}^k Y_i] = \sum_{i=1}^k \mu_i$ , so that the elements of the auxiliary exponential family  $\mathcal{Q}$  as in (1.3) can be written as

$$q_{\beta; \mu^*}(Y_1, \dots, Y_k) = \frac{1}{Z_q(\beta; \mu^*)} \cdot \exp\left(\beta \sum_{i=1}^k Y_i\right) \cdot q(Y_1, \dots, Y_k). \quad (4.2)$$

Note in particular that  $\mathcal{Q}$  is, by construction, a one-dimensional exponential family with sufficient statistic  $\sum_{i=1}^k Y_i$ , which does not equal (yet may be viewed as a subset of) the full  $k$ -dimensional exponential family from which  $Q$  was originally chosen. The normalizing constant  $Z_q(\beta; \mu^*)$  is equal to the moment generating function of  $X$  under  $Q$ , which is given by

$$Z_q(\beta; \mu^*) = \mathbb{E}_Q \left[ \exp\left(\beta \sum_{i=1}^k Y_i\right) \right] = \exp\left(\mu^*(e^\beta - 1)\right).$$

It follows that

$$\mathbb{E}_{Q_{\beta; \mu^*}} \left[ \sum_{i=1}^k Y_i \right] = \frac{d}{d\beta} \log Z_q(\beta; \mu^*) = \mu^* e^\beta,$$

which shows that mean-value space of the alternative is again given by  $\mathbb{M}_q = \mathbb{R}^+$ . Therefore, the assumptions of Theorem 1 are satisfied. The element of  $\mathcal{P}$  with mean  $\mu^*$  is given by  $P_{\bar{\mu}} \times \dots \times P_{\bar{\mu}}$ , so that

$$\frac{q_{\mu^*}(U)}{p_{\mu^*}(U)} = \prod_{i=1}^k \frac{p_{\mu_i}(Y_i)}{p_{\bar{\mu}}(Y_i)}.$$

Under  $P_{\mu^*}$ , the sufficient statistic  $\sum_{i=1}^k Y_i$  has the same distribution as under  $Q_{\mu^*}$ , so that  $Z_p(\beta; \mu^*) = Z_q(\beta; \mu^*)$ . Consequently,  $f(\cdot; \mu^*)$  as in (2.1) is zero, so that its second derivative is zero, and condition 1 of Theorem 1 is verified. It follows that,  $q_{\mu^*}(U)/p_{\mu^*}(U)$  is the global GRO e-variable with respect to  $Q_{\mu^*}$ .

## 4.2 Constant Curvature: Multivariate Gaussian Location

Suppose that  $\mathcal{P}$  is the multivariate Gaussian location family with some given nondegenerate covariance matrix  $\Sigma_p$  and let  $Q$  be any Gaussian distribution with nondegenerate covariance matrix  $\Sigma_q$ . Note that in this case we have that  $X = U$ , i.e. the sufficient statistic is simply given by the original data. The family  $\mathcal{Q}$ , generated from  $Q$  and  $\mathcal{P}$  as in (1.3), is the full Gaussian location family with fixed covariance matrix  $\Sigma_q$ . For both  $\mathcal{P}$  and  $\mathcal{Q}$ , the mean-value space is equal to  $\mathbb{R}^d$ , so that Theorem 1 applies to the pair  $\mathcal{P}$  and  $\mathcal{Q}$ . Furthermore, the covariance functions are constant, since  $\Sigma_p(\boldsymbol{\mu}) = \Sigma_p$  and  $\Sigma_q(\boldsymbol{\mu}) = \Sigma_q$  for all  $\boldsymbol{\mu} \in \mathbb{R}^d$ . It follows that, if  $\Sigma_p - \Sigma_q$  is positive semidefinite, then  $\Sigma_p(\boldsymbol{\mu}) - \Sigma_q(\boldsymbol{\mu})$  is positive semidefinite for all  $\boldsymbol{\mu} \in \mathbb{R}^d$ . In that case, Theorem 1 shows that the simple likelihood ratio  $q_{\boldsymbol{\mu}}/p_{\boldsymbol{\mu}}$  is the GRO e-value w.r.t.  $Q_{\boldsymbol{\mu}}$  for every  $\boldsymbol{\mu} \in \mathbb{R}^d$ . The growth rate is given by

$$\mathbb{E}_Q \left[ \log \frac{q_{\boldsymbol{\mu}}(U)}{p_{\boldsymbol{\mu}}(U)} \right] = D_{\text{GAUSS}}(\Sigma_q \Sigma_p^{-1}),$$

where  $D_{\text{GAUSS}}(B) := \frac{1}{2} (-\log \det(B) - (d - \text{tr}(B)))$ , i.e. the standard formula for the KL divergence between two multivariate Gaussians with the same mean.

In the case that  $\Sigma_p - \Sigma_q$  is negative semidefinite, the simple likelihood ratio does not give an e-value; the GRO e-value for this case can also be derived however and will be reported on in future work.

## 4.3 Nonconstant Curvature

We now discuss two examples with nonconstant curvature where Theorem 1 can be used to show the existence of simple e-variables. The first example concerns the Gaussian scale family, while the second discusses various classes of natural exponential families. Both are univariate in nature; in appendix A we discuss a multivariate example — Poisson regression — where Theorem 1 still applies, but the equivalent conditions are not true, so that it is not clear whether a global simple e-variable exists.

### 4.3.1 More k-Sample Tests

Consider again the  $k$ -sample test setting of Section 4.1. Besides the Gaussian and Poisson case, Hao et al. [2024] identify one more model that gives rise to a  $k$ -sample test with a simple e-value: the case that  $\mathcal{P}_{\text{START}}$  is the Bernoulli model. The difference with the Gaussian location- and Poisson family is that the involved e-value does not have constant expectation 1 here. Nevertheless, this result for the Bernoulli model can also be cast in terms of Theorem 1 using a different argument.

Again,  $\mathcal{P}$  is an exponential family on  $\mathcal{U}$  that states that the  $k$  samples are i.i.d. Bernoulli, which has sufficient statistic  $X = \sum_{i=1}^k Y_i$ . Its mean-value space is given by  $\mathbb{M}_p = (0, k)$ , since the sum of  $k$  i.i.d. Bernoulli random variables with parameter  $\mu$  has a binomial distribution with parameters  $(k, \mu)$ . Under the alternative  $Q$ , the  $k$  samples are independently Bernoulli distributed with means given by  $\boldsymbol{\mu} \in (0, 1)^k$ , in which case the sum has mean  $\mu^* = \sum_{i=1}^k \mu_i$ . When constructing the family

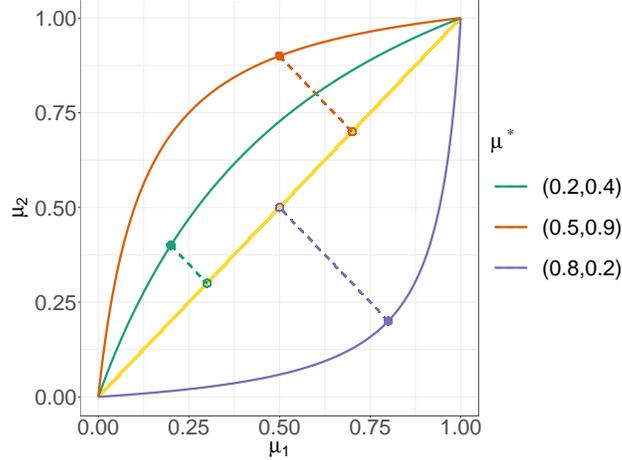


Figure 2: The family  $\mathcal{Q}$  for various  $\boldsymbol{\mu}^*$ . The coordinate grid represents the parameters of the full 2-sample Bernoulli family, the straight line shows the parameter space of  $\mathcal{P}$ , the curved lines show the parameters of the distributions in  $\mathcal{Q}$ , and the dashed lines show the projection of  $\boldsymbol{\mu}^*$  onto the parameter space of  $\mathcal{P}$ .

$\mathcal{Q}$  as in (1.3), it can be verified that  $Q_{\beta, \boldsymbol{\mu}^*}$  is the product of Bernoulli distributions with means

$$\left( \frac{e^\beta \mu_1}{1 - \mu_1 + e^\beta \mu_1}, \dots, \frac{e^\beta \mu_k}{1 - \mu_k + e^\beta \mu_k} \right). \quad (4.3)$$

This family of distributions is illustrated in Figure 2 for different choices of  $\boldsymbol{\mu}^*$ . Seen as a function of  $\beta$ , all entries in (4.3) behave as sigmoid functions, so that the sum takes values in  $(0, k)$ . It follows that the mean-value space of  $\mathcal{Q}$  is given by  $\mathbb{M}_q = (0, k)$ , which equals  $\mathbb{M}_p$ . Furthermore, the normalizing constant  $Z_q(\beta; \boldsymbol{\mu}^*)$  of  $\mathcal{Q}$  must be given by

$$Z_q(\beta; \boldsymbol{\mu}^*) = \prod_{i=1}^k (1 - \mu_i + \mu_i e^\beta).$$

We will now verify that item 4 of Theorem 1 is satisfied by doing a similar construction for arbitrary  $\mu \in (0, k)$ . The element in  $\mathcal{P}$  with mean  $\mu$  corresponds to Bernoulli parameter  $\mu/k$ , so that we have

$$Z_p(\beta; \mu) = \mathbb{E}_{P_{\mu^*}} \left[ \exp \left( \beta \sum_{i=1}^k Y_i \right) \right] = \left( 1 - \frac{\mu}{k} + \frac{\mu}{k} e^\beta \right)^k.$$

Furthermore, there is a corresponding  $\boldsymbol{\mu}' \in (0, 1)^k$  such that  $\sum_{i=1}^k \mu'_i = \mu$  and  $\boldsymbol{\mu}'$  can be written as (4.3) for a specific  $\beta$ . Repeating the reasoning above gives

$$Z_q(\beta; \boldsymbol{\mu}) = \prod_{i=1}^k (1 - \mu'_i + \mu'_i e^\beta).$$

By concavity of the logarithm, it holds that

$$\log Z_p(\beta; \mu) = k \log \left( 1 - \frac{\mu}{k} + \frac{\mu}{k} e^\beta \right) \geq \sum_{i=1}^k \log(1 - \mu'_i + \mu'_i e^\beta) = \log Z_q(\beta; \boldsymbol{\mu}).$$

We can therefore conclude that  $q(U)/p_{\mu^*}(U)$  is the GRO e-variable with respect to  $Q$ .

Hao et al. [2024] investigate several other exponential families for  $k$ -sample testing, such as exponential distributions, Gaussian scale, and beta, but none of these give rise to a simple e-value. Parts 1-4 of Theorem 1 provide some insight into what separates these families from the Gaussian location, Poisson, and Bernoulli.

### 4.3.2 Gaussian Scale Family

Another setting in which Theorem 1 applies is where  $\mathcal{P}$  equals the Gaussian scale family with fixed mean, which we take to be 0 without loss of generality. That is,  $\mathcal{P} = \{P_{\sigma^2} : \sigma^2 \in \mathbb{M}_p\}$  where  $P_{\sigma^2}$  is the normal with mean 0 and variance  $\sigma^2$ , i.e.

$$p_{\sigma^2}(U) = \frac{1}{\sqrt{2\pi\sigma}} \cdot e^{-\frac{1}{2\sigma^2}U^2}. \quad (4.4)$$

This is an exponential family with sufficient statistic  $X = U^2$ , mean-value parameter  $\sigma^2$  and mean-value space given by  $\mathbb{M}_p = \mathbb{R}^+$ . The canonical parameterization of the null relative to any mean-value  $\sigma^2 \in \mathbb{M}_p$  is given by

$$p_{\beta;\sigma^2}(U) = \frac{1}{Z_p(\beta;\sigma^2)} \cdot e^{\beta U^2} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-U^2/(2\sigma^2)} \quad (4.5)$$

with canonical parameter space  $\mathbb{B}_{p;\sigma^2} = (-\infty, 1/(2\sigma^2))$ .

As alternative, we take  $Q$  to be a Gaussian distribution with some fixed mean  $m \neq 0$  and variance  $s^2$ . We abstain from using the Greek alphabet for the alternative to avoid confusion with the mean-value parameters of  $\mathcal{P}$ . The expected value of  $X$  under  $Q$  is given by  $\sigma^{*2} := \mathbb{E}_Q[X] = s^2 + m^2$ . The family  $\mathcal{Q} = \{Q_\beta : \beta \in \mathbb{B}_{q;\sigma^{*2}}\}$  as defined by (1.3) therefore becomes:

$$q_{\beta;\sigma^{*2}}(U) = \frac{1}{Z_q(\beta;\sigma^{*2})} \cdot e^{\beta U^2} \cdot \frac{1}{\sqrt{2\pi s}} \cdot e^{-c(U-m)^2}, \quad (4.6)$$

where  $c = 1/(2s^2)$ , with  $\mathbb{B}_{q;\sigma^{*2}} = (-\infty, c)$ . Comparing (4.5) and the above confirms that  $\mathcal{Q}$  is an exponential family that has the same sufficient statistic, namely  $U^2$ , as  $\mathcal{P}$ , but different carrier.

The normalizing constant  $Z_q$  can be computed using (for example) the moment generating function of the noncentral chi-squared.

$$\begin{aligned} Z_q(\beta;\sigma^{*2}) &= \mathbb{E}_Q \left[ e^{\beta U^2} \right] \\ &= \mathbb{E}_Q \left[ e^{\beta s^2 (\frac{U}{s})^2} \right] \\ &= (1 - 2\beta s^2)^{-1/2} \exp \left( \frac{m^2 \beta}{1 - 2\beta s^2} \right), \end{aligned}$$

where we use that  $(U/s)^2$  has noncentral chi-squared distribution with one degree of freedom and noncentrality parameter  $m^2/s^2$ . Plugging this back in (4.6) shows that  $q_{\beta;\sigma^{*2}}$  is a normal density with mean  $cm/(c - \beta)$  and variance  $1/(2(c - \beta)) = s^2/(1 - 2\beta s^2)$ . This gives

$$\mathbb{E}_{Q_{\beta;\sigma^{*2}}}[U^2] = \frac{2c^2 m^2 - (\beta - c)}{2(\beta - c)^2} \quad (4.7)$$

The mean-value parameter space of  $\mathcal{Q}$  is thus given by  $\mathbb{M}_q = \{\mathbb{E}_{Q_{\beta;\sigma^{*2}}}[U^2], \beta < c\} = \mathbb{R}^+$  which is equal to  $\mathbb{M}_p$ .

Thus, this constructed family does not equal the natural choice of composite alternative that  $Q$  was also chosen from, i.e. the (two-dimensional) set of all Gaussians with arbitrary variance mean unequal to zero. However, it does correspond to a specific one-dimensional subset thereof, as was illustrated in Figure 1 in the introduction.

Now that the assumptions of Theorem 1 are verified, we will work towards using part 1 to conclude that a simple e-variable w.r.t.  $Q$  exists. For each  $\sigma^2 > 0$  we have

$$\text{VAR}_{P_{\sigma^2}}[U^2] = 2\sigma^4 = 2(\mathbf{E}_{P_{\sigma^2}}[U^2])^2 = 2(\mathbf{E}_{Q_{\sigma^2}}[U^2])^2.$$

It is therefore sufficient to check whether, for all  $\sigma^2 > 0$ , it holds that  $\text{VAR}_{Q_{\sigma^2}}[U^2] \leq 2(\mathbf{E}_{Q_{\sigma^2}}[U^2])^2$ . Since there is no more mention of the null hypothesis, it is equivalent to check whether for each  $\beta \in \mathbf{B}_{q;\sigma^{*2}}$  we have

$$\text{VAR}_{Q_{\beta;\sigma^{*2}}}[U^2] \leq 2 \left( \mathbf{E}_{Q_{\beta;\sigma^{*2}}}[U^2] \right)^2.$$

To this end, the variance function in terms of  $\beta$  can be computed as

$$\text{VAR}_{Q_{\beta;\sigma^{*2}}}[U^2] = \frac{d^2}{d\beta^2} \log Z_q(\beta; \sigma^{*2}) = -\frac{4c^2m^2 - (\beta - c)}{2(\beta - c)^3}. \quad (4.8)$$

Comparing this to (4.7) shows that the condition above indeed holds, from which we can conclude that  $q(U)/p_{\sigma^{*2}}(U)$  is an e-value. For this specific case, this was in fact shown directly, without using the construction of an associated family  $\mathcal{Q}$  (but involving a lot of calculus instead) by De Jong [2021].

Finally, note that even though the mean-value parameter spaces of  $\mathcal{P}$  and  $\mathcal{Q}$  are equal, the canonical spaces are not:  $\mathbf{B}_{p;\sigma^{*2}}$  is a proper subset of  $\mathbf{B}_{q;\sigma^{*2}}$ . More generally, for any  $\sigma'^2 > 0$  different from the  $\sigma^{*2}$  we started with, the canonical spaces  $\mathbf{B}_{p;\sigma'^2}$  and  $\mathbf{B}_{q;\sigma'^2}$  both change but remain unequal. Still, Proposition 3 ensures that we will have  $\mathbf{B}_{p;\sigma'^2} \subset \mathbf{B}_{q;\sigma'^2}$ .

### 4.3.3 NEFS and their Variance Functions

In this section, we consider the setting where  $\mathcal{P}$  is a one-dimensional natural exponential family (NEF) and  $Q$  is also an element of an NEF. This setting is particularly suited for the analysis above, because the constructed family  $\mathcal{Q}$  can be seen to equal the NEF that  $Q$  was chosen from. We therefore do not differentiate between the simple or composite alternative in this section. Furthermore, NEFs are fully characterized by the pair  $(\sigma^2(\mu), \mathbf{M})$ , where  $\mathbf{M}$  is the mean-value parameter space and  $\sigma^2(\mu)$  is the variance function as defined before. A wide variety of NEFS and their corresponding variance functions have been studied in the literature [see e.g. Morris, 1982, Letac and Mora, 1990] and this can be used in conjunction with Theorem 1 to quickly check on a case-by-case basis whether any given pair of NEFs provides a simple e-variable.

For example, let  $\mathcal{P} = \{P_{\lambda,r} : \lambda \in \mathbb{R}^+\}$  be the set of Gamma distributions for  $U$  with varying scale parameter  $\lambda$  and fixed shape parameter  $r > 0$ . The sufficient statistic is given by  $X = U$  and its mean under  $P_{\lambda,r}$  equals  $r\lambda$ , so the mean-value parameter space is  $\mathbf{M}_p = \mathbb{R}^+$ . The variance function is given by  $\sigma_p^2(\mu) = \mu^2/r$ . If we set  $Q$  to  $P_{\lambda^*,r'}$  for specific  $\lambda^*, r' \in \mathbb{R}^+$ , then  $\mathcal{Q}$  is the set of Gamma distributions with fixed shape parameter  $r'$ .

Similarly, let  $\mathcal{P}$  be the set of negative binomial distributions with fixed number of successes  $n \in \mathbb{N}$  and let  $Q$  be any Poisson distribution, so that  $\mathcal{Q}$  equals the Poisson family. The variance functions are given by  $\sigma_p^2(\mu) = \mu^2/n + \mu$  and  $\sigma_q^2(\mu) = \mu$  respectively. It is trivially true that  $\sigma_p^2(\mu) \geq \sigma_q^2(\mu)$  for all  $\mu$ , so Theorem 1 reveals that a simple e-variables exists with respect to any element of the Poisson family. More generally, we may look at the Awad-Bar-Lev-Makov (ABM)

class of NEFs [Awad et al., 2022] that are characterized by mean-value space  $\mathbf{M} = \mathbb{R}^+$  and variance function

$$\sigma_s^2(\mu) = \mu \left(1 + \frac{\mu}{s}\right)^r, \quad s > 0, \quad r = 1, 2, \dots$$

This class was proposed as part of a general framework for zero-inflated, over-dispersed alternatives to the Poisson model (which would arise for  $r = 0$ ). The case  $r = 1$  recovers the negative binomial distribution and  $r = 2$  is called the generalized Poisson or Abel distribution. As was the case for the negative binomial distribution, it follows from Theorem 1 that simple e-variables exist for testing any of the ABM NEFs against the Poisson model.

Much more generally, consider the Tweedie-Bar-Lev-Enis class [Bar-Lev, 2020] of NEFs that have mean-value space  $\mathbf{M} = \mathbb{R}^+$  and power variance functions

$$\sigma^2(\mu) = a\mu^\gamma, \quad a > 0, \quad \mu > 0, \quad \gamma \geq 1.$$

We require  $\gamma \geq 1$  because there are no families of this form with  $\gamma \in (0, 1)$  and while there are families in this class with  $\gamma < 0$ , they are not regular and therefore beyond the scope of this paper. The cases  $\gamma = 1$  (Poisson) and  $\gamma = 2$  (Gamma families, with  $a$  depending on the shape parameter) were already encountered above. If we test between two of such families, say  $\mathcal{P}$  with  $\sigma_p^2(\mu) = a_p\mu^{\gamma_p}$  and  $\mathcal{Q}$  with  $\sigma_q^2(\mu) = a_q\mu^{\gamma_q}$  that share the same underlying sample space, there do not exist simple e-variables in general. Indeed, we have that  $\sigma_p^2(\mu) \geq \sigma_q^2(\mu)$  if and only if  $\mu^{\gamma_p - \gamma_q} \geq a_q/a_p$ , which, for certain combinations of parameters, does not hold for all  $\mu \in \mathbf{M}$ . Since this condition might hold for some  $\mu$  but not for others, this suggests that there may be cases where we find local e-variables that are not global.

Let us investigate this for  $(a_p, \gamma_p) = (1, 2)$  and  $(a_q, \gamma_q) = (1/2, 3)$ , which corresponds to the family of exponential distributions and the family of inverse Gaussian distributions with shape parameter  $\lambda := a_q^{-1} = 2$  respectively. In this case, it holds that  $\sigma_p^2(\mu) \geq \sigma_q^2(\mu) \Leftrightarrow \mu \leq a_q^{-1}$ . It follows from the analysis in Section 2 that  $q_\mu(U)/p_\mu(U)$  is a local e-variable for  $\mu \leq a_q^{-1}$ . However, since the condition does not hold for all  $\mu$  we cannot use Proposition 3 (or, equivalently, because, as we will see, the preconditions for Theorem 1 do not hold), this need not necessarily also be a global e-variable. In fact, the expected value under  $\mu' \in \mathbf{M}$  is given by

$$\mathbb{E}_{U \sim P_{\mu'}} \left[ \frac{q_\mu(U)}{p_\mu(U)} \right] = \int_0^\infty \frac{1}{\mu'} \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x} + \frac{x}{\mu} - \frac{x}{\mu'}\right) dx, \quad (4.9)$$

which diverges for  $\mu' \geq (1/\mu - \lambda/(2\mu^2))^{-1}$ . The latter is vacuous for  $\mu \leq \lambda/2$ , which means that for such  $\mu$  we might still get a global e-variable. For  $\mu \in (\lambda/2, \lambda)$ , this shows that we will get a local e-variable that is not a global e-variable. These different regimes are illustrated in Figure 3. For  $\mu > \lambda$ , the lines stop when the integral in (4.9) starts diverging. To see how the potential divergence (for large enough  $\mu'$ , in the regime  $1 < \mu < 2$ ) plays out in terms of the function  $f$  in (2.1), consider for example  $\mu = 3/2$ . Then, as is immediate from the definition of exponential distributions and the inverse Gaussian density with  $\lambda = 2$  we have  $q_{\beta;\mu}(x) \propto \exp((\beta - 4/9)x)h(x)$  with  $h$  the probability density on  $\mathbb{R}^+$  given by  $h(x) = \sqrt{1/(\pi x^3)} \exp(-1/x)$ , whereas  $p_{\beta;\mu} \propto \exp((\beta - 2/3)x)$ . We see that  $\mathbb{B}_{p;\mu} = (-\infty, 6/9)$  whereas  $\mathbb{B}_{q;\mu} = (-\infty, 4/9)$ . Thus, as  $\beta \uparrow 4/9$ , we get that  $\log Z_p(\beta)$  converges to a finite constant whereas  $\log Z_q(\beta) \uparrow \infty$ , so that  $f(\beta, \mu) \rightarrow \infty$ , with  $f$  the function in (2.1), as it should.

## 5 Proof of Theorem 1

The main idea behind the proof of Theorem 1 is the recognition that the distributions  $P_\beta$  and  $Q_\beta$  indexed by the  $\beta$  in the definition of  $f(\beta; \mu^*)$ , i.e. (2.1), are difficult to compare in the sense

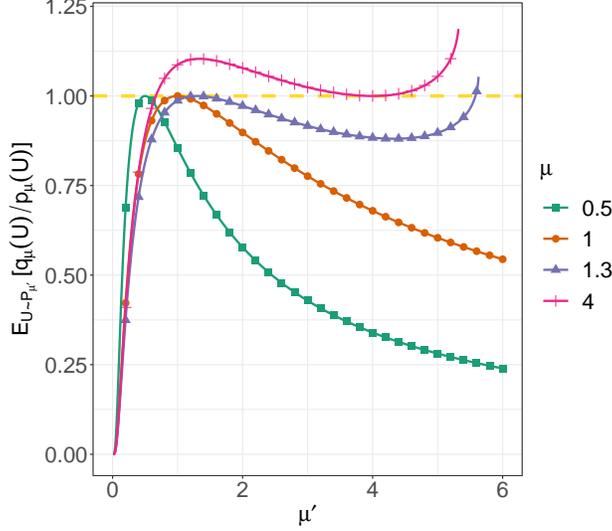


Figure 3: The expected value of  $q_\mu(U)/p_\mu(U)$  under the null  $P_{\mu'}$  for varying  $\mu'$ .

that they do not necessarily have any properties in common. In particular,  $P_\beta$  generally does not achieve  $\min_{P \in \mathcal{P}} D(Q_\beta \| P)$ , so that  $P_\beta$  and  $Q_\beta$  do not have the same mean. This suggests to replace  $f(\beta; \mu^*)$  by a function  $g(\mu; \mu^*)$  on the mean-value parameter space and also to re-express  $f(\beta; \mu^*) \leq 0$ , the condition for being an e-variable, by a condition on  $g$ . In the proof below we establish, using well-known convex duality properties of exponential families, that this can be done with function and condition, respectively, given by:

$$g(\mu; \mu^*) = D(P_\mu \| P_{\mu^*}) - D(Q_\mu \| Q_{\mu^*}), \quad (5.1)$$

$$g(\mu; \mu^*) \leq 0. \quad (5.2)$$

This condition on  $g$  corresponds to item 3 in Theorem 1. The key insight for showing the suitability of  $g$  is the following well-known convex-duality fact about exponential families: for all  $\mu, \mu' \in \mathbb{M}_p$ , all  $\beta \in \mathbb{B}_{p; \mu^*}$ , we have:

$$-\log Z_p(\beta; \mu') = D(P_{\mu_p(\beta; \mu')} \| P_{\mu'}) - \beta^T \mu_p(\beta; \mu') \leq D(P_\mu \| P_{\mu'}) - \beta^T \mu. \quad (5.3)$$

This can be derived as follows:

$$\begin{aligned} D(P_{\mu_p(\beta; \mu')} \| P_{\mu'}) - D(P_\mu \| P_{\mu'}) &= \log \frac{Z_p(\beta_p(\mu; \mu'))}{Z_p(\beta; \mu')} + \beta^T \mu_p(\beta; \mu') - \beta_p(\mu; \mu') \mu \\ &= \log \frac{Z_p(\beta_p(\mu; \mu'))}{Z_p(\beta; \mu')} + \beta^T (\mu_p(\beta; \mu') - \mu) - (\beta_p(\mu; \mu') - \beta)^T \mu \\ &= \beta^T (\mu_p(\beta; \mu') - \mu) - D(P_\mu \| P_{\mu_p(\beta; \mu')}) \\ &\leq \beta^T (\mu_p(\beta; \mu') - \mu). \end{aligned}$$

We now prove the chain of implications in the theorem.

(1)  $\Rightarrow$  (2) Let  $\mu, \mu' \in \mathbb{M}_q$  and denote  $\mu(\alpha) := (1 - \alpha)\mu' + \alpha\mu$ . By assumption of convexity, we have that  $\mu(\alpha) \in \mathbb{M}_q$  for all  $\alpha \in [0, 1]$ . Furthermore, define  $h(\alpha) = (\beta_p(\mu(\alpha); \mu') - \beta_q(\mu(\alpha); \mu'))^T (\mu(\alpha) -$

$\boldsymbol{\mu}'$ ), so that  $h(0) = 0$  and  $h(1) = (\boldsymbol{\beta}_p(\boldsymbol{\mu}; \boldsymbol{\mu}') - \boldsymbol{\beta}_q(\boldsymbol{\mu}; \boldsymbol{\mu}'))^T(\boldsymbol{\mu} - \boldsymbol{\mu}')$ . The derivative of  $h$  is given by

$$\begin{aligned} \frac{d}{d\alpha} h(\alpha) &= \left( \frac{d}{d\alpha} \boldsymbol{\beta}_p(\boldsymbol{\mu}(\alpha); \boldsymbol{\mu}') - \boldsymbol{\beta}_q(\boldsymbol{\mu}(\alpha); \boldsymbol{\mu}') \right)^T (\boldsymbol{\mu}(\alpha) - \boldsymbol{\mu}') \\ &\quad + (\boldsymbol{\beta}_p(\boldsymbol{\mu}(\alpha); \boldsymbol{\mu}') - \boldsymbol{\beta}_q(\boldsymbol{\mu}(\alpha); \boldsymbol{\mu}'))^T \frac{d}{d\alpha} \boldsymbol{\mu}(\alpha). \end{aligned}$$

The chain rule gives

$$\frac{d}{d\alpha} \boldsymbol{\beta}_p(\boldsymbol{\mu}(\alpha); \boldsymbol{\mu}') = \Sigma_p^{-1}(\boldsymbol{\mu}(\alpha))^T (\boldsymbol{\mu} - \boldsymbol{\mu}'),$$

where we use (3.1) and (3.2) together with the fact that the Jacobian of the gradient of a function equals the transpose of its Hessian. The derivative of  $\boldsymbol{\beta}_q(\boldsymbol{\mu}(\alpha); \boldsymbol{\mu}')$  can be found with the same argument, so we see

$$\begin{aligned} \frac{d}{d\alpha} h(\alpha) &= ((\Sigma_p^{-1}(\boldsymbol{\mu}(\alpha)) - \Sigma_q^{-1}(\boldsymbol{\mu}(\alpha)))^T (\boldsymbol{\mu} - \boldsymbol{\mu}'))^T (\boldsymbol{\mu}(\alpha) - \boldsymbol{\mu}') \\ &\quad + (\boldsymbol{\beta}_p(\boldsymbol{\mu}(\alpha); \boldsymbol{\mu}') - \boldsymbol{\beta}_q(\boldsymbol{\mu}(\alpha); \boldsymbol{\mu}'))^T (\boldsymbol{\mu} - \boldsymbol{\mu}') \\ &= \frac{1}{\alpha} (\boldsymbol{\mu}(\alpha) - \boldsymbol{\mu}')^T (\Sigma_p^{-1}(\boldsymbol{\mu}(\alpha)) - \Sigma_q^{-1}(\boldsymbol{\mu}(\alpha))) (\boldsymbol{\mu}(\alpha) - \boldsymbol{\mu}') \\ &\quad + (\boldsymbol{\beta}_p(\boldsymbol{\mu}(\alpha); \boldsymbol{\mu}') - \boldsymbol{\beta}_q(\boldsymbol{\mu}(\alpha); \boldsymbol{\mu}'))^T (\boldsymbol{\mu} - \boldsymbol{\mu}') \\ &= \frac{1}{\alpha} (\boldsymbol{\mu}(\alpha) - \boldsymbol{\mu}')^T (\Sigma_p^{-1}(\boldsymbol{\mu}(\alpha)) - \Sigma_q^{-1}(\boldsymbol{\mu}(\alpha))) (\boldsymbol{\mu}(\alpha) - \boldsymbol{\mu}') + \frac{1}{\alpha} h(\alpha). \end{aligned} \quad (5.4)$$

If  $\Sigma_p(\boldsymbol{\mu}) - \Sigma_q(\boldsymbol{\mu})$  is positive semidefinite for all  $\boldsymbol{\mu}$ , then  $\Sigma_p^{-1}(\boldsymbol{\mu}) - \Sigma_q^{-1}(\boldsymbol{\mu})$  is negative semidefinite (as discussed below the statement of Theorem 1). In this case, the first term in (5.4) is negative and, since  $h(0) = 0$ , the second term is also negative on  $[0, 1]$ . It follows that  $h$  is decreasing when  $\Sigma_p(\boldsymbol{\mu}) - \Sigma_q(\boldsymbol{\mu})$  is positive semidefinite, so that  $(\boldsymbol{\beta}_p(\boldsymbol{\mu}; \boldsymbol{\mu}') - \boldsymbol{\beta}_q(\boldsymbol{\mu}; \boldsymbol{\mu}'))^T(\boldsymbol{\mu} - \boldsymbol{\mu}') \leq 0$ , as was to be shown.

(2)  $\Rightarrow$  (3) We use a similar argument as was used to prove the previous implication, so let  $\boldsymbol{\mu}, \boldsymbol{\mu}' \in \mathbb{M}_q$  and denote  $\boldsymbol{\mu}(\alpha) = (1 - \alpha)\boldsymbol{\mu}' + \alpha\boldsymbol{\mu}$  as before. Define  $h(\alpha) := g(\boldsymbol{\mu}(\alpha); \boldsymbol{\mu}')$ . Using the chain rule of differentiation together with (3.1), we see that the derivative of  $h$  is given by

$$\begin{aligned} \frac{d}{d\alpha} h(\alpha) &= (\boldsymbol{\beta}_p(\boldsymbol{\mu}(\alpha); \boldsymbol{\mu}') - \boldsymbol{\beta}_q(\boldsymbol{\mu}(\alpha); \boldsymbol{\mu}'))^T (\boldsymbol{\mu} - \boldsymbol{\mu}') \\ &= \frac{1}{\alpha} (\boldsymbol{\beta}_p(\boldsymbol{\mu}(\alpha); \boldsymbol{\mu}') - \boldsymbol{\beta}_q(\boldsymbol{\mu}(\alpha); \boldsymbol{\mu}'))^T (\boldsymbol{\mu}(\alpha) - \boldsymbol{\mu}'). \end{aligned}$$

If item (2) holds, then we have that  $\frac{d}{d\alpha} h(\alpha) \leq 0$ . Since  $h(0) = 0$  and  $h(1) = D(P_{\boldsymbol{\mu}} \| P_{\boldsymbol{\mu}'}) - D(Q_{\boldsymbol{\mu}} \| Q_{\boldsymbol{\mu}'})$ , we see that item (2) implies that

$$D(P_{\boldsymbol{\mu}} \| P_{\boldsymbol{\mu}'}) - D(Q_{\boldsymbol{\mu}} \| Q_{\boldsymbol{\mu}'}) \leq 0,$$

as was to be shown.

(3)  $\Rightarrow$  (4) Assume that  $D(P_{\boldsymbol{\mu}} \| P_{\boldsymbol{\mu}'}) - D(Q_{\boldsymbol{\mu}} \| Q_{\boldsymbol{\mu}'}) \leq 0$  for all  $\boldsymbol{\mu}, \boldsymbol{\mu}' \in \mathbb{M}_q$ . Together with (5.3) this gives, for all  $\boldsymbol{\mu}, \boldsymbol{\mu}' \in \mathbb{M}_q$ , all  $\boldsymbol{\beta} \in \mathbb{B}_{p; \boldsymbol{\mu}'}$ :

$$D(P_{\boldsymbol{\mu}_p(\boldsymbol{\beta}; \boldsymbol{\mu}')} \| P_{\boldsymbol{\mu}'}) - \boldsymbol{\beta}^T \boldsymbol{\mu}_p(\boldsymbol{\beta}; \boldsymbol{\mu}') \leq D(P_{\boldsymbol{\mu}} \| P_{\boldsymbol{\mu}'}) - \boldsymbol{\beta}^T \boldsymbol{\mu} \leq D(Q_{\boldsymbol{\mu}} \| Q_{\boldsymbol{\mu}'}) - \boldsymbol{\beta}^T \boldsymbol{\mu}. \quad (5.5)$$

Applying this with  $\boldsymbol{\mu} = \boldsymbol{\mu}_q(\boldsymbol{\beta}; \boldsymbol{\mu}')$  and re-arranging gives

$$-D(P_{\boldsymbol{\mu}_p(\boldsymbol{\beta}; \boldsymbol{\mu}')} \| P_{\boldsymbol{\mu}'} ) + \boldsymbol{\beta}^T \boldsymbol{\mu}_p(\boldsymbol{\beta}; \boldsymbol{\mu}')$$

$$\geq -D(Q_{\boldsymbol{\mu}_q(\boldsymbol{\beta}; \boldsymbol{\mu}')} \| Q_{\boldsymbol{\mu}'} ) + \boldsymbol{\beta}^T \boldsymbol{\mu}_q(\boldsymbol{\beta}; \boldsymbol{\mu}'), \quad (5.6)$$

which, by the equality in key fact (5.3) is equivalent to  $\log Z_p(\boldsymbol{\beta}; \boldsymbol{\mu}') \geq \log Z_q(\boldsymbol{\beta}; \boldsymbol{\mu}')$ , which is what we had to prove.

**Remaining Implications** (4)  $\Rightarrow$  (5) now follows by the equality in (2.1) and the definition of an e-variable. (5)  $\Rightarrow$  (6) follows from proposition 1, (6)  $\Rightarrow$  (7) follows because a global e-variable is automatically also a local one, and (7)  $\Rightarrow$  (8) again follows from proposition 1. Finally, (8)  $\Rightarrow$  (1) has already been established as Proposition 2.  $\square$

## 6 Conclusion and Future Work

We have provided a theorem that, under regularity pre-conditions, provides a general sufficient condition under which there exists a simple e-variable for testing a simple alternative versus a composite regular exponential family null. The characterization was given in terms of several equivalent conditions, the most direct being perhaps the condition ‘ $\Sigma_p(\boldsymbol{\mu}) - \Sigma_q(\boldsymbol{\mu})$  is positive semidefinite for all  $\boldsymbol{\mu} \in \mathbb{M}_q$ ’. A direct follow-up question is: can we construct GRO or close-to-GRO e-variables, in case either the regularity pre-conditions or the positive definiteness condition do *not* hold? Our final example, Section 4.3.3, and in particular Figure 3, indicated that in that case, many things can happen: under some  $\boldsymbol{\mu} \in \mathbb{M}_q$  (green curve),  $q_{\boldsymbol{\mu}}/p_{\boldsymbol{\mu}}$  still gives a global simple e-variable; for other  $\boldsymbol{\mu}$  (blue), it gives a local but not global e-variable; for yet other  $\boldsymbol{\mu}$  (pink), it does not give an e-variable at all.

Nevertheless, it turns out that if the pre-regularity conditions hold and the ‘opposite’ of the positive semidefinite condition holds, i.e. if  $\Sigma_p(\boldsymbol{\mu}) - \Sigma_q(\boldsymbol{\mu})$  is *negative* semidefinite for all  $\boldsymbol{\mu} \in \mathbb{M}_q$ , then there is again sufficient structure to analyze the problem. The GRO e-variable will now be based on a mixture of elements of the null, but the specific mixture will depend on the sample size: we now need to look at i.i.d. repetitions of  $U$  rather than a single outcome  $U$ . We will provide such an analysis in future work.

Another interesting avenue for future work is to extend the analysis to *curved* exponential families [Efron, 2022]. While we do not have any general results in this direction yet, the analysis by Liang [2023] suggests that this may be possible. Liang considers a variation of the Cochran-Mantel-Haenszel test, in which the null hypothesis expresses that the population-weighted *average* effect size over a given set of strata is equal to, or bounded by, some  $\delta$ . This can be rephrased in terms of a curved exponential family null, for which Liang [2023] shows that a local e-variable exists by considering the second derivative of the function  $f(\boldsymbol{\beta}; \boldsymbol{\mu}^*)$  as in (2.1), just like in the present paper but with  $\boldsymbol{\beta}$  representing a particular suitable parameterization rather than the canonical parameterization of an exponential family. The local e-variable is then shown to be a global e-variable by a technique different from the construction of  $\mathcal{Q}$  we use here. Still, the overall derivation is sufficiently similar to suggest that it can be unified with the reasoning underlying Theorem 1.

## References

Akshay Agrawal. Lecture notes on Loewner order, 2018.  
<https://www.akshayagrawal.com/lecture-notes/html/loewner-order.html>.

- Theodore W Anderson and Donald A Darling. A test of goodness of fit. *Journal of the American statistical association*, 49(268):765–769, 1954.
- Yaser Awad, Shaul K Bar-Lev, and Udi Makov. A new class of counting distributions embedded in the Lee–Carter model for mortality projections: A Bayesian approach. *Risks*, 10(6):111, 2022.
- Shaul K Bar-Lev. Independent, though identical results: the class of Tweedie on power variance functions and the class of Bar-Lev and Enis on reproducible natural exponential families. *Int. J. Stat. Probab*, 9(1):30–35, 2020.
- O.E. Barndorff-Nielsen. *Information and Exponential Families in Statistical Theory*. Wiley, Chichester, UK, 1978.
- Lawrence D. Brown. Fundamentals of statistical exponential families with applications in statistical decision theory. *Lecture Notes-Monograph Series*, 9:i–279, 1986. ISSN 07492170. URL <http://www.jstor.org/stable/4355554>.
- Bradley Efron. *Exponential Families in Theory and Practice*. Institute of Mathematical Statistics Textbooks. Cambridge University Press, 2022. doi: 10.1017/9781108773157.
- P. Grünwald, Rianne de Heide, and Wouter Koolen. Safe testing. *Journal of the Royal Statistical Society, Series B*, 2024. with discussion.
- Peter Grünwald. *The minimum description length principle*. MIT press, 2007.
- Yunda Hao, Peter Grünwald, Tyron Lardy, Long Long, and Reuben Adams. E-values for k-sample tests with exponential families. *Sankhya A*, 2024.
- W. Koolen and P. Grünwald. Log-optimal anytime-valid e-values. *International Journal of Approximate Reasoning*, 2021. Festschrift for G. Shafer’s 75th Birthday.
- Solomon Kullback and Richard A Leibler. On information and sufficiency. *The annals of mathematical statistics*, 22(1):79–86, 1951.
- Tyron Lardy. E-values for hypothesis testing with covariates, 2021. Master’s Thesis, Leiden University.
- Tyron Lardy, Peter Grünwald, and Peter Harremoës. Universal reverse information projections and optimal e-statistics. *arXiv preprint arXiv:2306.16646*, 2023.
- Gérard Letac and Marianne Mora. Natural real exponential families with cubic variance functions. *The Annals of Statistics*, 18(1):1–37, 1990.
- Qiang Jonathan Li. *Estimation of mixture models*. Yale University, 1999.
- Haoyi Liang. Stratified safe sequential testing for mean effect, 2023. Master’s Thesis, University of Amsterdam.
- Hubert W Lilliefors. On the Kolmogorov-Smirnov test for normality with mean and variance unknown. *Journal of the American statistical Association*, 62(318):399–402, 1967.
- Carl N Morris. Natural exponential families with quadratic variance functions. *The Annals of Statistics*, pages 65–80, 1982.

Aaditya Ramdas, Peter Grünwald, Vladimir Vovk, and Glenn Shafer. Game-theoretic statistics and safe anytime-valid inference. *Statistical Science*, 2023.

Michael A Stephens. EDF statistics for goodness of fit and some comparisons. *Journal of the American statistical Association*, 69(347):730–737, 1974.

Rosanne Turner, Alexander Ly, and Peter Grünwald. Generic e-variables for exact sequential k-sample tests that allow for optional stopping. *Statistical Planning and Inference*, 230, 2024.

Martijn de Jong. Tests of significance for linear regression using E-values, 2021. Master’s Thesis, Leiden University.

Qiuqi Wang, Ruodu Wang, and Johanna Ziegel. E-backtesting, 2023. arXiv preprint arXiv:200209.00991.

Zhenyuan Zhang, Aaditya Ramdas, and Ruodu Wang. On the existence of powerful p-values and e-values for composite hypotheses, 2023. arXiv preprint arXiv:2304.16539.

## A Nonconstant Curvature: Generalized Linear Models

Consider the setting where data arrives as a block of outcomes together with covariates, i.e.  $U = ((Y_1, \mathbf{X}_1), \dots, (Y_n, \mathbf{X}_n))$  where  $Y_i \in \mathcal{Y}$  for some measurable space  $\mathcal{Y}$  and  $\mathbf{X}_i \in \mathbb{R}^{d+1}$  for all  $i = 1, \dots, n$ . Similar to the k-sample setting, we start with a one-dimensional NEF  $\mathcal{P}_{\text{START}}$  on  $\mathcal{Y}$  with sufficient statistic  $t^{(\text{START})}$  and densities

$$p_\eta^{(\text{START})}(Y) = \frac{1}{Z^{(\text{START})}(\eta)} \exp(\eta \cdot t^{(\text{START})}(Y)) r(Y). \quad (\text{A.1})$$

Here, we use the notational convention that, if densities or distributions are given a superscript, then the subscript indicates their canonical parameterization with respect to the fixed carrier  $r$ . This is necessary, because the null hypothesis will be defined in terms of the canonical parameters. We then construct a  $(d + 1)$ -dimensional exponential family of conditional densities for  $U$  given  $\mathbf{X}^n = (\mathbf{X}_1, \dots, \mathbf{X}_n)$  that can be written as a product of elements of  $\mathcal{P}_{\text{START}}$ , where the corresponding parameters depend on the covariates. That is, we consider densities of the form

$$p_\gamma^{(1)}(Y^n) := \frac{1}{Z^{(1)}(\gamma)} \exp(\gamma^T \cdot t^{(1)}(Y^n)) \cdot r(Y^n), \quad (\text{A.2})$$

where  $t^{(1)}(Y^n) = \sum_{i=1}^n t^{(\text{START})}(Y_i) \mathbf{X}_i$ ,  $Z^{(1)}(\gamma) = \prod_{i=1}^n Z^{(\text{START})}(\gamma^T \mathbf{X}_i)$  and  $\gamma \in \Gamma^{(1)} = \{\gamma' \in \mathbb{R}^{d+1} : Z^{(1)}(\gamma') < \infty\}$ . We will denote  $\mathcal{H}_1 = \{P_\gamma^{(1)} : \gamma \in \Gamma^{(1)}\}$ . Note that we should really write  $p_\gamma^{(1)}(Y^n | \mathbf{X}^n)$ , but since  $\mathbf{X}^n$  will be fixed from now on, we omit it from the notation. Conditional on any  $\mathbf{X}_1, \dots, \mathbf{X}_n$ , this family is again an exponential family with sufficient statistic  $t^{(1)}(Y^n) \in \mathbb{R}^{d+1}$ .

We henceforth concentrate on the case that the  $d + 1$ st component of the covariate is discrete and can take on, say,  $k$  different values  $\mathcal{K} = \{0, 1, \dots, k - 1\}$ . Thus, for each  $1 \leq i \leq n$ , we have  $\mathbf{X}_{i,d+1} \in \mathcal{K}$ . Each  $j \in \mathcal{K}$  stands for a different treatment. In standard cases,  $\mathcal{K} = \{0, 1\}$  with 1 standing for ‘treatment’ (e.g. give a new medication) and 0 for ‘control’ (e.g. give a placebo). The idea is that we want to test whether the conditional distribution of  $Y_i$  (e.g. patient outcome) given  $\mathbf{X}_i$  is affected by the value of the treatment covariate  $\mathbf{X}_{i,d+1}$  or not. To this end, the null hypothesis  $\mathcal{P}$  is defined as the restriction of the family in (A.2) to the set  $\{P_\gamma^{(1)} : \gamma \in \Gamma^{(1)}; \gamma_{d+1} = 0\}$ . Note

that we can reparameterize the null in terms of a vector  $\gamma \in \mathbb{R}^d$  by identifying  $P_\gamma^{(0)} := P_{(\gamma_1, \dots, \gamma_d, 0)}^{(1)}$ , so that we have  $\mathcal{P} = \{P_\gamma^{(0)} : \gamma \in \Gamma^{(0)}\}$ . This gives an exponential family with sufficient statistic  $t^{(0)}(Y^n) = \sum_{i=1}^n \mathbf{X}'_i t^{(\text{START})}(Y_i)$ , where  $\mathbf{X}'_i = (X_{i1}, \dots, X_{id}, 0)$ . On the other hand, the alternative is defined as a fixed element of  $\mathcal{H}_1$ , that is,  $P_{\gamma^*}^{(1)}$  for some  $\gamma^* \in \mathbb{R}^{d+1}$ . We denote its associated mean-value by  $\mu^* := \mathbb{E}_{P_{\gamma^*}^{(1)}}[t(Y^n) \mid \mathbf{X}^n]$ . Then the distribution in  $\mathcal{P}$  that is closest in KL divergence to  $P_{\gamma^*}^{(1)}$  has parameter vector  $\gamma^\circ \in \mathbb{R}^d$  so that  $\mathbb{E}_{P_{\gamma^\circ}^{(0)}}[t(Y^n)] = \mu^*$ .

To use the results in the main body of the text, we denote  $p_{\mu^*} = p_{\gamma^\circ}^{(0)}$  and write  $p_{\beta; \mu^*}$  as in (1.1). Since  $P_{\beta+\beta^\circ}^{(0)} = P_{\beta; \mu^*}$ , the set of distributions  $\{P_{\beta; \mu^*}(U^n) : \beta \in \mathbb{B}_{p; \mu^*}\}$  coincides with  $\mathcal{P}$  as defined before; we only changed the parameterization. Furthermore, we define  $\mathcal{Q}$  as in (1.3) with sufficient statistic  $t^{(0)}$  (i.e. the last component of the  $\mathbf{X}_i$ 's is not used). It is worth to note that  $\mathcal{Q}$  will not be equal to  $\mathcal{H}_1$ , since the latter is  $(d+1)$ -dimensional and  $\mathcal{Q}$  is, by definition,  $d$ -dimensional. Instead,  $\mathcal{Q}$  can be shown to be a proper subset of  $\mathcal{H}_1$ . We now show an example of a specific instance where we can falsify the equivalent statements in Theorem 1, so that this setup does not necessarily lead to simple e-variables.

**Example 1** (Poisson regression). Consider the case above with  $\mathcal{K} = \{0, 1\}$ ,  $n = 2$ , and  $d = 1$ , so that  $\mathcal{P}$  is one-dimensional. Furthermore, we assume that  $X_{1,2} = 0$  and  $X_{2,2} = 1$ , that is, that the data has one point in each group. For convenience, we rename group 0 to  $A$  and group 1 to  $B$ , and denote  $X_A := X_{1,1}$ ,  $X_B := X_{2,1}$ ,  $Y_A := Y_1$ , and  $Y_B := Y_2$ .

In the case of Poisson regression, we have  $\mathcal{Y} = \{0, 1, 2, \dots\}$ ,  $t^{(\text{START})}(Y) = Y$ ,  $\eta = \log \mu$ , where  $\mu = \mathbb{E}_{P_\eta}[t^{(\text{START})}(Y)]$ ,  $\log Z^{(\text{START})}(\eta) = \exp(\eta)$ , and  $r(Y) = 1/(Y!)$ . We therefore get:

$$t^{(1)}(Y_A, Y_B) = \begin{pmatrix} X_A Y_A \\ 0 \end{pmatrix} + \begin{pmatrix} X_B Y_B \\ Y_B \end{pmatrix} = \begin{pmatrix} X_A Y_A + X_B Y_B \\ Y_B \end{pmatrix}, t^{(0)}(Y_A, Y_B) = X_A Y_A + X_B Y_B.$$

Since we condition on  $(X_A, X_B)$ ,  $s^{(1)}$  defined as

$$s^{(1)}(Y_A, Y_B) = \begin{pmatrix} X_A Y_A \\ X_B Y_B \end{pmatrix}$$

is a linear function of  $t^{(1)}$ , so that it is a sufficient statistic of  $\mathcal{H}_1$  as well. We can reparameterize with respect to  $s^{(1)}$ , in which case we get one parameter for the  $A$  group and one parameter for the  $B$  group. Writing  $\gamma^* = (\gamma_A^*, \gamma_B^*)$ , we get

$$\begin{aligned} r(Y_A, Y_B) &= \frac{1}{Y_A!} \cdot \frac{1}{Y_B!} \\ \log Z^{(1)}(\gamma^*) &= \log Z^{(\text{START})}(\gamma_A^* X_A) + \log Z^{(\text{START})}(\gamma_B^* X_B) \\ &= \exp(\gamma_A^* X_A) + \exp(\gamma_B^* X_B), \\ \log Z_q(\beta; \mu^*) &= \log \frac{Z^{(1)}(\gamma^* + \beta)}{Z^{(1)}(\gamma^*)} \\ &= \exp((\beta + \gamma_A^*) X_A) + \exp((\beta + \gamma_B^*) X_B) - \exp(\gamma_A^* X_A) - \exp(\gamma_B^* X_B). \end{aligned}$$

Similarly, for  $\mathcal{P}$ , we get

$$\begin{aligned}
\log Z^{(0)}(\gamma) &= \log Z^{(\text{START})}(\gamma X_A) + \log Z^{(\text{START})}(\gamma X_B) \\
&= \exp(\gamma X_A) + \exp(\gamma X_B), \\
\log Z_p(\beta; \mu^*) &= \log \frac{Z^{(0)}(\beta + \gamma^\circ)}{Z^{(0)}(\gamma^\circ)} \\
&= \exp((\beta + \gamma^\circ) X_A) + \exp((\beta + \gamma^\circ) X_B) - \exp(\gamma^\circ X_A) - \exp(\gamma^\circ X_B).
\end{aligned}$$

To find an expression for  $\gamma^\circ$ , note that  $P_{\beta; \mu^*}$  and  $Q_{\beta; \mu^*}$  must have the same mean at  $\beta = 0$ , hence

$$\left. \frac{d}{d\beta} \log Z_p(\beta; \mu^*) \right|_{\beta=0} = \left. \frac{d}{d\beta} \log Z_q(\beta; \mu^*) \right|_{\beta=0},$$

which gives that  $\gamma^\circ$  is the solution to

$$X_A \exp(\gamma^\circ X_A) + X_B \exp(\gamma^\circ X_B) = X_A \exp(\gamma_A^* X_A) + X_B \exp(\gamma_B^* X_B). \quad (\text{A.3})$$

Suppose now that  $\gamma_A^* > \gamma_B^* > 0$  and  $X_A > X_B > 0$ . Then it can be seen that  $\gamma_A^* > \gamma^\circ > \gamma_B^*$ , so that

$$\begin{aligned}
\sigma_p^2(\mu^*) - \sigma_q^2(\mu^*) &= \left. \frac{d^2}{d\beta^2} \log Z_p(\beta; \mu^*) \right|_{\beta=0} - \left. \frac{d^2}{d\beta^2} \log Z_q(\beta; \mu^*) \right|_{\beta=0} \\
&= X_A^2 \exp(\gamma^\circ X_A) + X_B^2 \exp(\gamma^\circ X_B) - X_A^2 \exp(\gamma_A^* X_A) - X_B^2 \exp(\gamma_B^* X_B) \\
&= X_A^2 (\exp(\gamma^\circ X_A) - \exp(\gamma_A^* X_A)) + X_B^2 (\exp(\gamma^\circ X_B) - \exp(\gamma_B^* X_B)) \\
&= X_A X_B (\exp(\gamma_B^* X_B) - \exp(\gamma^\circ X_B)) + X_B^2 (\exp(\gamma^\circ X_B) - \exp(\gamma_B^* X_B)) \\
&< X_B^2 (\exp(\gamma^\circ X_B) - \exp(\gamma_B^* X_B)) + X_B^2 (\exp(\gamma_B^* X_B) - \exp(\gamma^\circ X_B)) \\
&= 0,
\end{aligned}$$

where in the third equality we substitute (A.3) and for the inequality we use that  $\exp(\gamma_B^* X_B) - \exp(\gamma^\circ X_B) < 0$ . We have thus found that  $\sigma_p^2(\mu^*) < \sigma_q^2(\mu^*)$ , which gives a counter example to statement 1 in Theorem 1. It follows that this line of reasoning cannot generally be used to prove that simple e-variables exist for generalized linear models.