FLOW BY GAUSS CURVATURE TO THE ORLICZ MINKOWSKI PROBLEM FOR q-TORSIONAL RIGIDITY

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Abstract. The Minkowski problem for torsional rigidity (2-torsional rigidity) was firstly studied by Colesanti and Fimiani [12] using variational method. Moreover, Hu, Liu and Ma [20] also studied this problem by method of curvature flow and obtain the existence of smooth solution. In addition, the Minkowski problem for 2-torsional rigidity was also extended to L_p version and Orlicz version.

Recently, Hu and Zhang [23] introduced the concept of Orlicz mixed q-torsional rigidity and obtained Orlicz q-torsional measure through the variational method for $q > 1$. Specially, they established the functional Orlicz Brunn-Minkowski inequality and the functional Orlicz Minkowski inequality.

Motivated by the remarkable work by Hu and Zhang in [23], we can propose the Orlicz Minkowksi problem for q-torsional rigidity, and then confirm the existence of smooth even solutions to the Orlicz Minkowski problem for q-torsional rigidity with $q > 1$ by method of a Gauss curvature flow.

1. Introduction and main results

The classical Minkowski problem argues the existence, uniqueness and regularity of a convex body whose surface area measure is equal to a pre-given Borel measure on the sphere S^{n-1}. If the given measure has a positive continuous density, the Minkowski problem can be seen as the problem of prescribing the Gauss curvature in differential geometry. Minkowski problem and its solution can be traced back to the works of Minkowski [32], other influential works, such as Lewy [26], Nirenberg [33], Pogorelov [34] and Cheng-Yau [10], etc..

In the past 30 years, the Minkowski problem has played an important role in the study of convex geometry, and the research of Minkowski problem has promoted the development of fully nonlinear partial differential equations. The Minkowski problem has produced some variations of it, among which the $L_p(p \in \mathbb{R})$ Minkowski problem is particularly important because the $L_p(p \in \mathbb{R})$ Minkowski problem contains some special versions. Namely: when $p = 1$, it is the classical Minkowski problem; when $p = 0$, this is the famous log-Minkowski problem [3]; when $p = -n$, it is the centro-affine Minkowski problem [40]. The L_p Minkowski problem with $p > 1$ was first proposed and studied by Lutwak [30], whose solution plays a key role in establishing the L_p affine Sobolev inequality [18, 31]. Until 2010, Haberl, Lutwak, Yang and Zhang [17] proposed and studied the even Orlicz Minkowski problem which is a more generalized Minkowksi type problem, and its result contains the classical Minkowski problem and the L_p Minkowski problem.

We know that the different geometric measures correspond to different Minkowski type problems. Some geometric measures with physical backgrounds have been introduced into the Brunn-Minkowski theory, naturally, the related Minkowski type problems have also been gradually studied. For example, prescribing capacitary curvature measures on

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planar convex domains [39]: If given a finite, nonnegative Borel measure $\mu \in S^1$ has centroid at the origin and its supp(μ) does not comprise any pair of antipodal points, then, there is a unique (up to translation) convex, nonempty, open set $\Omega \subset \mathbb{R}^2$ such that $d\mu_q(\Omega, \cdot) = d\mu(\cdot)$, where $\mu_q(\Omega, \cdot)$ is q-capacitary curvature measure of Ω with $q \in (1, 2]$. Inspired by [39], we also study the Minkowski type problem with a physical background in the present paper, namely, the Orlicz Minkowski problem for q-torsional rigidity.

Firstly, we recall the concept of q-torsional rigidity and its related contents. For convenience, let $\mathcal{K}^n$ be the collection of convex bodies in Euclidean space $\mathbb{R}^n$. The set of convex bodies containing the origin in their interiors in $\mathbb{R}^n$, we write $\mathcal{K}^n_o$. Moreover, we let C^2_+ be the class of convex bofies of C^2 if its boundary has the positive Gauss curvature.

Firstly, we recall the torsional rigidity (2-torsional rigidity) of convex body Ω in $\mathbb{R}^n$ is described by (see [11])

$$\frac{1}{T(\Omega)} = \inf\left\{\frac{\int_\Omega |\nabla u|^2 dX}{[\int_\Omega |u| dX]^2} : u \in W_0^{1,2}(\Omega), \int_\Omega |u| dX > 0\right\}.$$

It has been provided that, there exists a unique function u such that

$$T_q(\Omega) = \int_\Omega |\nabla u|^2 dX,$$

where u satisfies the boundary-value problem

$$\begin{cases} \Delta u(X) = -2 & \text{in} \quad \Omega, \\ u(X) = 0, & \text{on} \quad \partial\Omega. \end{cases}$$

Here, Δu is the Laplace operator.

Next, we introduce the q-torsional rigidity [13] with $q > 1$. Obviously, when $q = 2$, it is the 2-torsional rigidity. Let $\Omega \in \mathcal{K}^n$, the q-torsional rigidity $T_q(\Omega)$ is defined by

$$\frac{1}{T_q(\Omega)} = \inf\left\{\frac{\int_\Omega |\nabla u|^q dX}{[\int_\Omega |u| dX]^q} : u \in W_0^{1,q}(\Omega), \int_\Omega |u| dX > 0\right\}. \tag{1.1}$$

The functional defined in (1.1) admints a minimizer $u \in W_0^{1,q}(\Omega)$, and $cu($ for some constant $c)$ is unique positive solution of the following boundary value problem (see [2] or [19])

$$\begin{cases} \Delta_q u = -1 & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where

$$\Delta_q u = \operatorname{div}(|\nabla u|^{q-2} \nabla u)$$

is the q-Laplace operator.

Applying (1.2) with the Gauss-Green formula, we have

$$\int_\Omega |\nabla u|^q dX = \int_\Omega u dX, \tag{1.3}$$

from (1.1) and (1.3), it follows

$$T_q(\Omega) = \frac{(\int_\Omega u dX)^q}{\int_\Omega |\nabla u|^q dX} = \left(\int_\Omega u dX\right)^{q-1} = \left(\int_\Omega |\nabla u|^q dX\right)^{q-1}. \tag{1.4}$$

With the aid of Pohozaev-type identities of [35], The q-torsional rigidity formula (1.4) is given by

$$T_q(\Omega)^{\frac{1}{q-1}} = \frac{q-1}{q+n(q-1)} \int_{S^{n-1}} h(\Omega,\xi) d\mu_q^{tor}(\Omega,\xi) \tag{1.5}$$
$$= \frac{q-1}{q+n(q-1)} \int_{S^{n-1}} h(\Omega,\xi) |\nabla u|^q dS(\Omega,\xi),$$

setting $\widetilde{T}_q(\Omega) = T_q(\Omega)^{\frac{1}{q-1}}$, the q-torsional measure $\mu_q^{tor}(\Omega,\eta)$ is defined by

$$\mu_q^{tor}(\Omega,\eta) = \int_{g^{-1}(\eta)} |\nabla u(X)|^q d\mathcal{H}^{n-1}(X) = \int_\eta |\nabla u(g^{-1}(x))|^q dS(\Omega,x), \tag{1.6}$$

for any Borel set $\eta \subseteq S^{n-1}$. Here, $g : \partial\Omega \to S^{n-1}$ is Gauss map, and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

Recently, Hu and Zhang [23] firstly established the functional form of the Orlicz Brunn-Minkowski theory for the q-torsional rigidity with $q > 1$ by variational formula in the smooth category. Combined the Orlicz Minkowski sum with formula (1.5), they introduced the Orlicz mixed q-torsional rigidity of functional form as follows.

Definition 1.1. [23, Definition 3.4] *Suppose $\varphi \in \overline{\Phi}$, $q > 1$, and $f, g, a\cdot f +_\varphi b\cdot g \in \mathcal{E}$ for $a, b \in I$ (not both zero). Define the Orlicz mixed q-torsional rigidity $\widetilde{T}_{\varphi,q}([f],g)$ by*

$$\widetilde{T}_{\varphi,q}([f],g) = \frac{\gamma\varphi_l'(1)}{\alpha} \frac{d\widetilde{T}_q(f +_\varphi t\cdot g)}{dt}\bigg|_{t=0^+} = \frac{\gamma}{\alpha} \int_{S^{n-1}} \varphi\left(\frac{g(x)}{f(x)}\right) f(x) d\mu_q^{tor}([f],x).$$

Here,

$$\overline{\Phi} = \{\varphi \in C^{2,\alpha}(\mathbb{R}) : \varphi \in \Phi : \Phi \text{ be the class of convex and strictly increasing function}\},$$

$$\mathcal{E} = \{h \in C_+^{2,\alpha}(S^{n-1}) : (h_{ij} + h\delta_{ij}) \text{ is positive definite}\},$$

$I \subset [0,+\infty)$ be a bounded interval and $\frac{\gamma}{\alpha} = \frac{q-1}{n(q-1)+q}$.

Suppose $K, L \in \mathcal{K}_o^n, q > 1$, that are of class $C_+^{2,\alpha}$, analogous to definition 1.1, there also has following definition.

Definition 1.2. [23, Definition 3.8] *Suppose $\varphi \in \overline{\Phi}$, $q > 1$, and $K, L, a\cdot K +_\varphi b\cdot L \in \mathcal{K}_o^n$ that are of class $C_+^{2,\alpha}$ for $a, b \in I$ (not both zero). Then, define the Orlicz mixed q-torsional rigidity $\widetilde{T}_{\varphi,q}(K,L)$ by*

$$\widetilde{T}_{\varphi,q}(K,L) = \frac{\gamma\varphi_l'(1)}{\alpha} \frac{d\widetilde{T}_q(K +_\varphi t\cdot L)}{dt}\bigg|_{t=0^+} = \frac{\gamma}{\alpha} \int_{S^{n-1}} \varphi\left(\frac{h_L(x)}{h_K(x)}\right) h_K(x) d\mu_q^{tor}(K,x).$$

With the help of above variational formula for q-torsional rigidity with respect to Orlicz sum, we not only get the Orlicz mixed q-torsional rigidity of K, L, but also obtain Orlicz q-torsional measure $\mu_{\varphi,q}^{tor}$. Thus, we can propose the normalised Orlicz Minkowski problem for q-torsional rigidity: Let $q > 1$, μ be a finite Borel measure on S^{n-1} and $\varphi : (0,\infty) \to (0,\infty)$ is a fixed continuous function, under what necessary and sufficient conditions, does there exist a unique convex body Ω whose support function is h and positive constant τ so that $\mu = \tau\mu_{\varphi,q}^{tor}$, i.e.

$$d\mu = \tau\varphi(h) d\mu_q^{tor}(\Omega,\cdot). \tag{1.7}$$

Combining (1.6), if the given measure μ is absolutely continuous with respect to the Lebesgue measure and μ has a density function $f : S^{n-1} \to (0,\infty)$ is even and smooth, then, solving problem (1.7) can be equivalently viewed as solving the following Monge-Ampère equation on S^{n-1}:

$$\tau\varphi(h)|\nabla u|^q \det(\nabla_{ij}h + h\delta_{ij}) = f. \tag{1.8}$$

In (1.7), when $q = 2$, the Orlicz Minkowsi problem to the 2-torsional rigidity first developed and proved by Li and Zhu [27]. Moreover, Hu, Liu and Ma [22] combined (1.8) obtained the existence of smooth solution by Gauss curvature flow for this problem.

In the case of $q = 2$, let $\varphi(s) = s$, the Minkowski problem for 2-torsional rigidity was firstly proposed by Colesanti and Fimiani [12] and smooth solution was obtained in [20] by Gauss curvature flow. The same case of q, the Minkowksi problem for 2-torsional rigidity was extended L_p version ($\varphi(s) = s^{1-p}$) by Chen and Dai [9] who proved the existence of solutions for any fixed $p > 1$ and $p \neq n+2$, Hu and Liu [21] for $0 < p < 1$.

If $q > 1$ not only $q = 2$, the Hadamard variational formula for q-torsion rigidity with applications was provided in [24].

As mentioned above, we find that the Orlicz Minkowski problem is an important and generalized Minkowski type problem, it includes the classical Minkowksi problem and L_p Minkowski problem, therefore, it is necessary to study the Orlicz Minkowski problem. In this paper, we will study the Orlicz Minkowski problem for q-torsional rigidity with $q > 1$ and give the existence of even, smooth, uniformly convex solutions for (1.8) by the method of Gauss curvature flow. The Gauss curvature flow was first introduced and studied by Firey [15] to model the shape change of worn stones. Since then, various Gauss curvature flows have been extensively studied, see examples [1, 4, 5, 6, 8, 20, 22, 29] and the references therein.

Let $\partial\Omega_0$ be a smooth, closed, origin symmetric and strictly convex hypersurface in $\mathbb{R}^n$. We construct and consider the long-time existence and convergence of the following Gauss curvature flow which is a family of convex hypersurfaces $\partial\Omega_t$ parameterized by smooth maps $X(\cdot,t) : S^{n-1} \times (0,\infty) \to \mathbb{R}^n$ satisfying the initial value problem

$$\begin{cases} \frac{\partial X(x,t)}{\partial t} = -\lambda(t)f(\nu)\frac{(X\cdot\nu)}{|\nabla u(X,t)|^q\varphi(X\cdot\nu)}\mathcal{K}(x,t)\nu + X(x,t), \\ X(x,0) = X_0(x), \end{cases} \tag{1.9}$$

where $\mathcal{K}(x,t)$ is the Gauss curvature of hypersurface $\partial\Omega_t$, $\nu = x$ is the outer unit normal at $X(x,t)$, $X \cdot \nu$ represents standard inner product of X and ν, and $\lambda(t)$ is defined as follows

$$\lambda(t) = \frac{\int_{S^{n-1}} |\nabla u(X,t)|^q \rho^n d\xi}{\int_{S^{n-1}} \frac{hf}{\varphi} dx},$$

where ρ and h are the radial function and support function of the convex hypersurface $\partial\Omega_t$, respectively.

For the convenience of discussing Gauss curvature flow (1.9) in the following text, we introduce a functional for any $t \geq 0$ as follows:

$$\Gamma(\Omega_t) = \int_{S^{n-1}} f(x)\phi(h(x,t))dx. \tag{1.10}$$

Here, let's assume that $\phi(s) = \int_0^s \frac{1}{\varphi(\zeta)}d\zeta$ exists for all $s > 0$ and $\lim_{s\to\infty}\phi(s) = \infty$, where $h(\cdot,t)$ is the support function of Ω_t.

Combining problem (1.8) with flow (1.9), we establish the following result in this article.

Theorem 1.3. *Suppose $q > 1$, $\varphi : (0,\infty) \to (0,\infty)$ is a fixed continuous function, ϕ satisfy above assumptions, $\partial\Omega_0$ be a smooth, closed, origin symmetric and strictly convex hypersurface in $\mathbb{R}^n$ and f be a positive and even smooth function on S^{n-1}. Then, the flow (1.9) has a unique smooth, origin symmetric uniformly convex solution $\partial\Omega_t = X(S^{n-1},t)$*

for $t \in (0, \infty)$. When $t \to \infty$, there is a subsequence of $\partial\Omega_t$ converges in C^∞ to a smooth, closed, origin symmetric and strictly convex hypersurface Ω_∞ whose support function satisfies (1.8).

This paper is organized as follows. We collect background materials in Section 2. In Section 3, we give the scalar form of flow (1.9) by support function and discuss properties of two important functionals along the flow (1.9). In Section 4, we give the priori estimates for the solution to the flow (1.9). We obtain the convergence of the flow and complete the proof of Theorem 1.3 in Section 5.

2. Preliminaries

In this section, we give a brief review of some relevant notions about convex bodies and recall some basic properties of convex hypersurfaces that readers may refer to [38] and a book of Schneider [36].

2.1. Convex bodies.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\partial\Omega$ be a smooth, closed and strictly convex hypersurface containing the origin in its interior. The support function of convex body Ω enclosed by $\partial\Omega$ is defined by

$$h_\Omega(\xi) = h(\Omega, \xi) = \max\{\xi \cdot y : y \in \Omega\}, \quad \forall \xi \in S^{n-1},$$

and the radial function of Ω with respect to o (origin) $\in \mathbb{R}$ is defined by

$$\rho_\Omega(v) = \rho(\Omega, v) = \max\{c > 0 : cv \in \Omega\}, \quad v \in S^{n-1}.$$

The volume $Vol(\Omega)$ of Ω is defined by

$$Vol(\Omega) = \frac{1}{n}\int_{S^{n-1}} \rho(\Omega, v)^n dv = \frac{1}{n}\int_{S^{n-1}} h(\Omega, \xi) dS(\Omega, \xi). \tag{2.1}$$

For a compact convex subset $\Omega \in \mathcal{K}^n$ and $\xi \in S^{n-1}$, the intersection of a supporting hyperplane with Ω, $H(\Omega, \xi)$ at ξ is given by

$$H(\Omega, \xi) = \{x \in \Omega : x \cdot \xi = h_\Omega(\xi)\}.$$

A boundary point of Ω which only has one supporting hyperplane is called a regular point, otherwise, it is a singular point. The set of singular points is denoted as $\sigma\Omega$, it is well known that $\sigma\Omega$ has spherical Lebesgue measure 0.

For $x \in \partial\Omega \setminus \sigma\Omega$, its Gauss map $g_\Omega : x \in \partial\Omega \setminus \sigma\Omega \to S^{n-1}$ is represented by

$$g_\Omega(x) = \{\xi \in S^{n-1} : x \cdot \xi = h_\Omega(\xi)\}.$$

Correspondingly, for a Borel set $\eta \subset S^{n-1}$, its inverse Gauss map is denoted by g_Ω^{-1},

$$g_\Omega^{-1}(\eta) = \{x \in \partial\Omega : g_\Omega(x) \in \eta\}.$$

Specially, for a convex hypersurface $\partial\Omega$ of class C^2, then, the support function of Ω can be stated as

$$h(\Omega, x) = x \cdot g^{-1}(x) = g(X(x)) \cdot X(x), \quad X(x) \in \partial\Omega.$$

Moreover, the gradient of $h(\Omega, \cdot)$ satisfies

$$\nabla h(\Omega, x) = g^{-1}(x). \tag{2.2}$$

For the Borel set $\eta \subset S^{n-1}$, its surface area measure is defined as

$$S_\Omega(\eta) = \mathcal{H}^n(g_\Omega^{-1}(\eta)),$$

where $\mathcal{H}^n$ is n-dimensional Hausdorff measure. Furthermore, for $\mathcal{H}^n$ almost all $X \in \partial\Omega$,

$$\nabla u(X) = -|\nabla u(X)| g(X) \quad \text{and} \quad |\nabla u| \in L^q(\partial\Omega, \mathcal{H}^n).$$

2.2. **Gauss curvature on Convex hypersurface.** Suppose that Ω is parameterized by the inverse Gauss map $X : S^{n-1} \to \Omega$, that is $X(x) = g_\Omega^{-1}(x)$. Then, the support function h of Ω can be computed by

$$h(x) = x \cdot X(x), \quad x \in S^{n-1}, \tag{2.3}$$

where x is the outer normal of Ω at $X(x)$. Let $\{e_1, e_2, \cdots, e_{n-1}\}$ be an orthonormal frame on S^{n-1}, denote e_{ij} by the standard metric on the sphere S^{n-1}. Differentiating (2.3), there has

$$\nabla_i h = \nabla_i x \cdot X(x) + x \cdot \nabla_i X(x),$$

since $\nabla_i X(x)$ is tangent to Ω at $X(x)$, thus,

$$\nabla_i h = \nabla_i x \cdot X(x).$$

By differentiating (2.3) twice, the second fundamental form A_{ij} of Ω can be computed in terms of the support function,

$$A_{ij} = \nabla_{ij} h + h e_{ij}, \tag{2.4}$$

where $\nabla_{ij} = \nabla_i \nabla_j$ denotes the second order covariant derivative with respect to e_{ij}. The induced metric matrix g_{ij} of Ω can be derived by Weingarten's formula,

$$e_{ij} = \nabla_i x \cdot \nabla_j x = A_{ik} A_{lj} g^{kl}. \tag{2.5}$$

The principal radii of curvature are the eigenvalues of the matrix $b_{ij} = A^{ik} g_{jk}$. When considering a smooth local orthonormal frame on S^{n-1}, by virtues of (2.4) and (2.5), there is

$$b_{ij} = A_{ij} = \nabla_{ij} h + h\delta_{ij}. \tag{2.6}$$

Then, the Gauss curvature $\mathcal{K}(x)$ of $X(x) \in \partial\Omega$ is given by

$$\mathcal{K}(x) = (\det(\nabla_{ij} h + h\delta_{ij}))^{-1}. \tag{2.7}$$

3. Geometric flow and its associated functionals

In this section, we will introduce the geometric flow and its associated functionals for solving the Orlicz Minkowski problem for q-torsional rigidity with $q > 1$. For convenience, the Gauss curvature flow is restated here. Let $\partial\Omega_0$ be a smooth, closed and origin symmetric strictly convex hypersurface in $\mathbb{R}^n$, f be a positive even smooth function on S^{n-1}. We consider the following Gauss curvature flow

$$\begin{cases} \frac{\partial X(x,t)}{\partial t} = -\lambda(t) f(\nu) \frac{(X\cdot\nu)}{|\nabla u(X,t)|^q \varphi(X\cdot\nu)} \mathcal{K}(x,t)\nu + X(x,t), \\ X(x,0) = X_0(x), \end{cases} \tag{3.1}$$

where $\mathcal{K}(x,t)$ is the Gauss curvature of the hypersurface $\partial\Omega_t$ at $X(\cdot,t)$, $\nu = x$ is the unit outer normal vector of $\partial\Omega_t$ at $X(\cdot,t)$, $X \cdot \nu$ represents standard inner product of X and ν, and $\lambda(t)$ is given by

$$\lambda(t) = \frac{\int_{S^{n-1}} |\nabla u(X,t)|^q \rho^n d\xi}{\int_{S^{n-1}} \frac{hf}{\varphi} dx}. \tag{3.2}$$

Taking the scalar product of both sides of the equation and of the initial condition in (3.1) by ν, by means of the definition of support function (2.3) and (2.2) , we describe the flow (1.9) (or (3.1)) with the support function as follows

$$\begin{cases} \frac{\partial h(x,t)}{\partial t} = -\lambda(t) f(x) \frac{h(x,t)}{|\nabla u(\nabla h,t)|^q \varphi(h)} \mathcal{K}(x,t) + h(x,t), \\ h(x,0) = h_0(x). \end{cases} \tag{3.3}$$

Next, we investigate the characteristics of two essential geometric functionals with respect to Eq. (3.3). Firstly, we show the q-torsional rigidity unchanged along the flow (3.1). In fact, the conclusion can be stated as the following lemma.

Lemma 3.1. *For $q>1$, the q-torsional rigidity $T_q(\Omega_t)$ is unchanged with regard to Eq. (3.3), i.e.*

$$T_q(\Omega_t)=T_q(\Omega_0).$$

Proof. Let $h(\cdot,t)$ and $\rho(\cdot,t)$ be the support function and radial function of Ω_t, respectively. $u(X,t)$ be the solution of (1.2) in Ω_t. The proposition 2.5 of [24] tells us that

$$\begin{aligned}\frac{\partial}{\partial t}T_q(\Omega_t)=&\frac{\partial}{\partial t}\left[\left(\frac{q-1}{q+n(q-1)}\int_{S^{n-1}}h(\Omega_t,x)|\nabla u|^q\mathcal{K}^{-1}dx\right)^{q-1}\right]\\=&\frac{(q-1)^2}{q+n(q-1)}T_q^{\frac{q-2}{q-1}}\int_{S^{n-1}}\frac{\partial h(\Omega_t,x)}{\partial t}|\nabla u|^q\mathcal{K}^{-1}dx.\end{aligned}$$

Thus, from (3.2), (3.3) and $\rho^n\mathcal{K}d\xi=hdx$, we have

$$\begin{aligned}\frac{\partial}{\partial t}T_q(\Omega_t)=&\frac{(q-1)^2}{q+n(q-1)}T_q^{\frac{q-2}{q-1}}\int_{S^{n-1}}\left(-\lambda(t)\frac{fh\mathcal{K}}{|\nabla u|^q\varphi}+h\right)|\nabla u|^q\mathcal{K}^{-1}dx\\=&\frac{(q-1)^2}{q+n(q-1)}T_q^{\frac{q-2}{q-1}}\left(-\frac{\int_{S^{n-1}}|\nabla u|^q\rho^n d\xi}{\int_{S^{n-1}}hf/\varphi dx}\int_{S^{n-1}}\frac{fh\mathcal{K}}{|\nabla u|^q\varphi}|\nabla u|^q\mathcal{K}^{-1}dx\right.\\&\left.+\int_{S^{n-1}}h|\nabla u|^q\mathcal{K}^{-1}dx\right)\\=&\frac{(q-1)^2}{q+n(q-1)}T_q^{\frac{q-2}{q-1}}\left(-\int_{S^{n-1}}|\nabla u|^q h\mathcal{K}^{-1}dx+\int_{S^{n-1}}|\nabla u|^q h\mathcal{K}^{-1}dx\right)\\=&0.\end{aligned}$$

This ends the proof of Lemma 3.1. □

The next Lemma will show that the functional (1.10) is non-increasing along the flow (3.1).

Lemma 3.2. *The functional (1.10) is non-increasing along the flow (3.1). Namely, $\frac{\partial}{\partial t}\Gamma(\Omega_t)\le 0$, the equality holds if and only if Ω_t satisfy (1.8).*

Proof. By (1.10), (3.2), (3.3), $\rho^n\mathcal{K}d\xi=hdx$ and the Hölder inequality, we obtain the following result,

$$\begin{aligned}&\frac{\partial}{\partial t}\Gamma(\Omega_t)\\=&\int_{S^{n-1}}f(x)\phi'(h(x,t))\frac{\partial h}{\partial t}dx\\=&\int_{S^{n-1}}\left(-\lambda(t)\frac{f(x)h}{|\nabla u|^q\varphi(h)}\mathcal{K}+h\right)\frac{f(x)}{\varphi(h)}dx\\=&-\lambda(t)\int_{S^{n-1}}\frac{f^2(x)h}{|\nabla u|^q\varphi^2(h)}\mathcal{K}dx+\int_{S^{n-1}}\frac{hf(x)}{\varphi(h)}dx\\=&-\frac{\int_{S^{n-1}}|\nabla u|^q\frac{h}{\mathcal{K}}dx}{\int_{S^{n-1}}\frac{hf}{\varphi(h)}dx}\int_{S^{n-1}}\frac{f^2(x)h}{|\nabla u|^q\varphi^2(h)}\mathcal{K}dx+\int_{S^{n-1}}\frac{hf(x)}{\varphi(h)}dx\\=&\left(\int_{S^{n-1}}\frac{hf}{\varphi(h)}dx\right)^{-1}\left\{-\left[\left(\int_{S^{n-1}}\left(|\nabla u|^{\frac{q}{2}}(\frac{h}{\mathcal{K}})^{\frac{1}{2}}\right)^2dx\right)^{\frac{1}{2}}\left(\int_{S^{n-1}}\left(\frac{f(x)h^{\frac{1}{2}}\mathcal{K}^{\frac{1}{2}}}{|\nabla u|^{\frac{q}{2}}\varphi(h)}\right)^2dx\right)^{\frac{1}{2}}\right]^2\right.\end{aligned}$$

$$+\left(\int_{S^{n-1}}\frac{hf(x)}{\varphi(h)}dx\right)^2\Bigg\}$$
$$\leq\left(\int_{S^{n-1}}\frac{hf(x)}{\varphi(h)}dx\right)^{-1}\Bigg[-\left(\int_{S^{n-1}}|\nabla u|^{\frac{q}{2}}(\frac{h}{\mathcal{K}})^{\frac{1}{2}}\times\frac{f(x)h^{\frac{1}{2}}\mathcal{K}^{\frac{1}{2}}}{|\nabla u|^{\frac{q}{2}}\varphi(h)}dx\right)^2$$
$$+\left(\int_{S^{n-1}}\frac{hf(x)}{\varphi(h)}dx\right)^2\Bigg]=0.$$

By the equality condition of Hölder inequality, we know that the above equality holds if and only if $f=\tau\varphi(h)\mathcal{K}^{-1}|\nabla u|^q$, i.e.,

$$\tau\varphi(h)|\nabla u|^q\det(\nabla_{ij}h+h\delta_{ij})=f.$$

Namely, Ω_t satisfies (1.8) with $\frac{1}{\tau}=\lambda(t)$. □

4. Priori estimates

In this section, we establish the C^0, C^1 and C^2 estimates for the solution to Eq. (3.3). In the following of this paper, we always assume that $\partial\Omega_0$ is a smooth, closed and origin symmetric strictly convex hypersurface in $\mathbb{R}^n$, $h: S^{n-1}\times[0,T)\to\mathbb{R}$ is a smooth even solution to Eq. (3.3) with the initial $h(\cdot,0)$ the support function of $\partial\Omega_0$. Here, T is the maximal time for the existence of smooth even solution to Eq. (3.3).

4.1. **C^0, C^1 estimates.** In order to complete the C^0 estimate, we firstly need to introduce the following Lemmas for convex bodies.

Lemma 4.1. [7, Lemma 2.6] *Let $\Omega\in\mathcal{K}_o^n$, h and ρ be respectively support function and radial function of Ω, and $x_{\max}$ and $\xi_{\min}$ be two points such that $h(x_{\max})=\max_{S^{n-1}}h$ and $\rho(\xi_{\min})=\min_{S^{n-1}}\rho$. Then,*

$$\max_{S^{n-1}}h=\max_{S^{n-1}}\rho\quad\textit{and}\quad\min_{S^{n-1}}h=\min_{S^{n-1}}\rho;$$
$$h(x)\geq x\cdot x_{\max}h(x_{\max}),\quad\forall x\in S^{n-1};$$
$$\rho(\xi)\xi\cdot\xi_{\min}\geq\rho(\xi_{\min}),\quad\forall\xi\in S^{n-1}.$$

Lemma 4.2. *Let $u\in W^{1,q}_{loc}(\Omega)$ be a local weak solution of*

$$\operatorname{div}(|\nabla u|^{q-2}\nabla u)=\psi,\ q>1;\ \psi\in L^{loc}_s(\Omega),$$

$s>q'n$ ($\frac{1}{q'}+\frac{1}{q}=1$). Then, $u\in C^{1+\alpha}_{loc}(\Omega)$. (see [14, Corollary in pp. 830]*)*

Lemma 4.3. *Let $\partial\Omega_t$ be a origin symmetric, smooth convex solution to the flow (3.1) in $\mathbb{R}^n$, $u(X,t)$ be the solution of (1.2) in Ω_t, f and φ satisfy assumptions of Theorem 1.3. Then, there is a positive constant C independent of t such that*

$$\frac{1}{C}\leq h(x,t)\leq C,\quad\forall(x,t)\in S^{n-1}\times[0,T),\tag{4.1}$$
$$\frac{1}{C}\leq\rho(\xi,t)\leq C,\quad\forall(\xi,t)\in S^{n-1}\times[0,T).\tag{4.2}$$

Here, $h(x,t)$ and $\rho(\xi,t)$ are the support function and radial function of Ω_t, respectively.

Proof. We only give proof of (4.1), (4.2) can be obtained by Lemma 4.1 and (4.1).

Firstly, we prove the upper bound of (4.1). At fixed time $t_0\in[0,T)$, assume that the maximum of $h(\cdot,t_0)$ is attained at (x_{t_0},t_0) for $x_{t_0}\in S^{n-1}$. Let

$$\max_{S^{n-1}}h(x,t_0)=h(x_{t_0},t_0),$$

setting

$$h_{\max}(t) = \sup_{t_0\in[0,T)} h(x_{t_0}, t_0).$$

According to the second clause of Lemma 4.1, we have

$$h(x, t_0) \geq h_{\max}(t)x_{t_0} \cdot x, \quad \forall x \in S^{n-1},$$

where h and $h_{\max}(t)$ ($h_{\max}(t) = \max_{S^{n-1}} h(\cdot, t)$) are on the same hypersurface.

Recall the definition of ϕ, we know that ϕ ($\phi' > 0$) is strictly increasing and f is a positive even smooth function on S^{n-1}. Thus, from Lemma 3.2, there is

$$\begin{aligned}
\Gamma(\Omega_0) \geq &\Gamma(\Omega_t) = \int_{S^{n-1}} f(x)\phi(h(x,t))dx \\
&\geq \int_{\{x\in S^{n-1}:x_{t_0}\cdot x\geq\frac{1}{2}\}} f(x)\phi(h(x,t))dx \\
&\geq \int_{\{x\in S^{n-1}:x_{t_0}\cdot x\geq\frac{1}{2}\}} f(x)\phi(h_{max}(t)x_{t_0}\cdot x)dx \\
&\geq \int_{\{x\in S^{n-1}:x_{t_0}\cdot x\geq\frac{1}{2}\}} f(x)\phi\left(\frac{1}{2}h_{max}(t)\right)dx \\
&\geq C\phi\left(\frac{1}{2}h_{max}(t)\right),
\end{aligned}$$

which implies that $\phi(\frac{1}{2}h_{max}(t))$ has uniformly positive upper bound $\frac{\Gamma(\Omega_0)}{C}$. Since ϕ is strictly increasing and $\lim_{s\to\infty}\phi(s) = \infty$, we can know that $h(\cdot, t)$ also has uniformly positive upper bound. Here, C is a positive constant depending only on $\max_{S^{n-1}} h(x, 0)$ and $\max_{S^{n-1}} f(x)$.

That is we have obtained the uniform upper bounds of the convex bodies generated by $\partial\Omega_t = X(S^{n-1}, t)$. Before proving the lower bound of (4.1), let's first explain the following facts.

Combining Lemma 4.2 with Ω_t is convex body, we know that there are positive constants c and C such that

$$c \leq |\nabla u(X(x,t),t)| \leq C, \quad \forall (x,t) \in S^{n-1}\times[0,T),$$

further, according to (2.2), we obtain

$$c \leq |\nabla u(\nabla h(x,t),t)| \leq C, \quad \forall (x,t) \in S^{n-1}\times[0,T). \tag{4.3}$$

In the same time, by virtue of Schauder's theory (see example Chapter 6 in [16]), there is a positive constant $\widetilde{C}$, independent of t, satisfying that

$$|\nabla^k u(\nabla h(x,t),t)| \leq \widetilde{C}, \quad \forall (x,t) \in S^{n-1}\times[0,T), \tag{4.4}$$

for all integer $k \geq 2$.

To prove the lower bound of $h(x, t)$, we use the method of contradiction to discuss. Let's assume that there exists a sequence $\{t_k\} \subset [0, T)$ such that $h(x, t_k)$ is not uniformly bounded away from 0, i.e., $\min_{S^{n-1}} h(x, t_k) \to 0$ as $k \to \infty$. Since f, h_0 is even and Ω_t is a origin-symmetric convex body, thus, $h(x, t)$ is even. On the other hand, making use of the upper bound, by Blaschke-Selection theorem [36], there is a sequence in $\{\Omega_{t_k}\}$, for convenience, which is still denoted by $\{\Omega_{t_k}\}$, such that $\{\Omega_{t_k}\}$ converges to a origin-symmetric convex body $\widetilde{\Omega}$ through the Hausdorff measure as $k \to \infty$, then, we obtain $\min_{S^{n-1}} h(\widetilde{\Omega}, \cdot) = \lim_{k\to\infty} \min_{S^{n-1}} h(\Omega_{t_k}, \cdot) = 0$. Thus, there exist $\widetilde{x}$ such that $h(\widetilde{\Omega}, \widetilde{x}) = 0$ and $h(\widetilde{\Omega}, -\widetilde{x}) = 0$. This implies that $\widetilde{\Omega}$ is contained in a lower-dimensional subspace in $\mathbb{R}^n$. This can lead to $\rho(\xi, t_k) \to 0$ as $k \to \infty$ almost everywhere with respect to the spherical Lebesgue measure. According to bounded convergence theorem and formula (2.1), we can derive

$$Vol(\widetilde{\Omega})=\frac{1}{n}\int_{S^{n-1}}\rho(\xi,t_k)^n dv\to 0\quad \text{as}\quad k\to\infty.$$

However, (4.3) (1.5), (2.1), (4.3), Lemma 3.1 and the conditions for initial hypersurfaces show that

$$\begin{aligned}0<\bar{c}=\widetilde{T}_q(\Omega_0)=\widetilde{T}_q(\widetilde{\Omega})=&\lim_{k\to\infty}\widetilde{T}_q(\Omega_{t_k})\\ =&\lim_{k\to\infty}\frac{q-1}{q+n(q-1)}\int_{S^{n-1}}h(\Omega_{t_k},\xi)|\nabla u|^q dS(\Omega_{t_k},\xi)\\ \leq&\frac{C^q(q-1)}{q+n(q-1)}\int_{S^{n-1}}h(\widetilde{\Omega},\xi)dS(\widetilde{\Omega},\xi)\\ =&\frac{C^q(q-1)}{q+n(q-1)}Vol(\widetilde{\Omega}),\end{aligned}$$

which is a contradiction with $q>1$. It follows that $h(x,t)$ has a uniform lower bound. Therefore, we complete proof of Lemma 4.3. □

Lemma 4.4. *Let $\partial\Omega_t$ be a origin-symmetric, smooth convex solution to the flow (3.1) in $\mathbb{R}^n$, f and φ satisfy assumptions of Theorem 1.3. Then, there is a positive constant C independent of t such that*

$$|\nabla h(x,t)|\leq C,\quad \forall(x,t)\in S^{n-1}\times[0,T),\tag{4.5}$$

and

$$|\nabla \rho(\xi,t)|\leq C,\quad \forall(\xi,t)\in S^{n-1}\times[0,T).\tag{4.6}$$

Proof. The desired results immediately follows from Lemma 4.3 and the following identities (see e.g. [28])

$$h=\frac{\rho^2}{\sqrt{\rho^2+|\nabla\rho|^2}},\qquad \rho^2=h^2+|\nabla h|^2.$$

□

Lemma 4.5. *Under the same conditions as the Lemma 4.3, there always exists a positive constant C independent of t, such that*

$$\frac{1}{C}\leq\lambda(t)\leq C,\quad t\in[0,T).$$

Proof. By the definition of $\lambda(t)$,

$$\lambda(t)=\frac{\int_{S^{n-1}}|\nabla u|^q\rho^n d\xi}{\int_{S^{n-1}}\frac{hf}{\varphi}dx},$$

the conclusion of this result is directly obtained from the (4.1), (4.2) and (4.3). □

4.2. C^2 **estimate.** In this subsection, we establish the upper and lower bounds of principal curvature. This will shows that Eq. (3.3) is uniformly parabolic. The technique used in this proof was first introduced by Tso [37] to derive the upper bound of the Gauss curvature. We begin with completing the following results which will be need in C^2 estimate.

Lemma 4.6. *Let Ω_t be a convex body of C^2_+ in $\mathbb{R}^n$, and $u(X(x,t),t)$ be the solution of (1.2) with $q>1$ in Ω_t, then*

$(i)(\nabla^2 u(X(x,t),t)e_i)\cdot e_j = -\mathcal{K}|\nabla u(X(x,t),t)|c_{ij}(x,t);$

$(ii)(\nabla^2 u(X(x,t),t)e_i)\cdot x = -\mathcal{K}|\nabla u(X(x,t),t)|_j c_{ij}(x,t);$

$(iii)(\nabla^2 u(X(x,t),t)x)\cdot x = \dfrac{1}{q-1}\Big(\mathcal{K}|\nabla u|\mathrm{Tr}(c_{ij}(h_{ij}+h\delta_{ij})) - |\nabla u|^{2-q}\Big).$

Here, e_i and x are orthonormal frame and unite outer normal on S^{n-1}, $\cdot$ is standard inner product and c_{ij} is the cofactor matrix of $(h_{ij}+h\delta_{ij})$ with $\sum_{i,j} c_{ij}(h_{ij}+h\delta_{ij}) = (n-1)\mathcal{K}^{-1}$.

Proof. Similar conclusions have been presented in some References, for example [22], thus, we will briefly state the proofs combining with our problem.

(i) Assume that $h(x,t)$ is the support function of Ω_t for $(x,t)\in S^{n-1}\times(0,\infty)$ and let $\iota=\frac{\partial h}{\partial t}$. Then, $X(x,t)=h_ie_i+hx$, $\frac{\partial X(x,t)}{\partial t}=\dot{X}(x,t)=\frac{\partial}{\partial t}(h_ie_i+hx)=\iota_ie_i+\iota x$. $X_i(X,t)=(h_{ij}+h\delta_{ij})e_j$, let $h_{ij}+h\delta_{ij}=\omega_{ij}$, then, $X_{ij}(x,t)=\omega_{ijk}e_k-\omega_{ij}x$, where ω_{ijk} is the corariant derivatives of ω_{ij}.

From $u(X,t)=0$ on $\partial\Omega_t$, we can not difficult to obtain

$$\nabla u\cdot X_i = 0,$$

and

$$((\nabla^2 u)X_j)X_i + \nabla u X_{ij} = 0.$$

It follows that

$$\omega_{ik}\omega_{jl}(((\nabla^2 u)e_l)\cdot e_k) + \omega_{ij}|\nabla u| = 0. \tag{4.7}$$

Multiplying both sides of (4.7) by c_{ij}, we have

$$c_{ij}\omega_{ik}\omega_{jl}(((\nabla^2 u)e_l)\cdot e_k) + \det(h_{ij}+h\delta_{ij})|\nabla u| = 0.$$

Namely,

$$\delta_{jk}\det(h_{ik}+h\delta_{ik})\omega_{jl}(((\nabla^2 u)e_l)\cdot e_k) + \det(h_{ij}+h\delta_{ij})|\nabla u| = 0.$$

It yields

$$\omega_{ij}(((\nabla^2 u)e_i)\cdot e_j) + |\nabla u| = 0,$$

then,

$$c_{ij}\omega_{ij}(((\nabla^2 u)e_i)\cdot e_j) + c_{ij}|\nabla u| = 0,$$

i.e.,

$$\mathcal{K}^{-1}(((\nabla^2 u)e_i)\cdot e_j) + c_{ij}|\nabla u| = 0,$$

thus,

$$((\nabla^2 u)e_i)\cdot e_j = -c_{ij}\mathcal{K}|\nabla u|.$$

This gives proof of (i).

(ii) Recall that

$$|\nabla u(X(x,t),t)| = -\nabla u(X(x,t),t)\cdot x,$$

taking the covariant of both sides for above formula, we obtain

$$|\nabla u|_j = -\nabla u\cdot e_j - (\nabla^2 u)X_j\cdot x = -\omega_{ij}((\nabla^2 u)e_i\cdot x). \tag{4.8}$$

Multiplying both sides of (4.8) by c_{lj} and combining

$$c_{lj}\omega_{ij} = \delta_{li}\det(h_{ij}+h\delta_{ij}).$$

We conclude that

$$c_{ij}|\nabla u|_j = -\det(h_{ij}+h\delta_{ij})(\nabla^2 u)e_i \cdot x.$$

Hence,

$$((\nabla^2 u)e_i)\cdot x = -\mathcal{K}c_{ij}|\nabla u|_j.$$

This proves (ii).

(iii) From (1.2), we know that

$$-1 = \mathrm{div}(|\nabla u|^{q-2}\nabla u) = |\nabla u|^{q-2}(\Delta u + \frac{q-2}{|\nabla u|^2}(\nabla^2 u\nabla u)\cdot\nabla u),$$

then,

$$\frac{q-2}{|\nabla u|^2}(\nabla^2 u\nabla u)\cdot\nabla u = -\Delta u - |\nabla u|^{2-q},$$

further,

$$\begin{aligned}
&(q-2)((\nabla^2 u)x)\cdot x\\
&= -\Delta u - |\nabla u|^{2-q}\\
&= -\mathrm{Tr}(\nabla^2 u) - |\nabla u|^{2-q}\\
&= -\sum_i((\nabla^2 u)e_i)\cdot e_j - ((\nabla^2 u)x)\cdot x - |\nabla u|^{2-q}\\
&= \mathcal{K}|\nabla u|\mathrm{Tr}(c_{ij}(h_{ij}+h\delta_{ij})) - ((\nabla^2 u)x)\cdot x - |\nabla u|^{2-q},
\end{aligned}$$

hence,

$$(q-1)((\nabla^2 u)x)\cdot x = \mathcal{K}|\nabla u|\mathrm{Tr}(c_{ij}(h_{ij}+h\delta_{ij})) - |\nabla u|^{2-q},$$

consequently,

$$((\nabla^2 u)x)\cdot x = \frac{1}{q-1}\Big(\mathcal{K}|\nabla u|\mathrm{Tr}(c_{ij}(h_{ij}+h\delta_{ij})) - |\nabla u|^{2-q}\Big).$$

This completes proof of the (iii). $\square$

By Lemma 4.3 and Lemma 4.4, if h is a smooth even solution of Eq. (3.3) on $S^{n-1}\times[0,T)$ (T is the maximal time for the existence of smooth even solution) and f,φ satisfying assumptions of Theorem 1.3, then, along the flow for $[0,T)$, $\nabla h + hx$, and h are smooth functions whose ranges are within some bounded domain $\Omega_{[0,T)}$ and bounded interval $I_{[0,T)}$, respectively. Here, $\Omega_{[0,T)}$ and $I_{[0,T)}$ depend only on the upper and lower bounds of h on $[0,T)$.

Lemma 4.7. *Let $\partial\Omega_t$ be a origin-symmetric, smooth convex solution to the flow (3.1) in $\mathbb{R}^n$, f and φ be as Theorem 1.3 and $q>1$, there is a positive constant C depending on $\|f\|_{C^0(S^{n-1})}$, $\|\varphi\|_{C^1(I_{[0,T)})}$, $\|\varphi\|_{C^2(I_{[0,T)})}$, $\|h\|_{C^1(S^{n-1}\times[0,T)}$ and $\|\lambda\|_{C^0(S^{n-1}\times[0,T)}$, where $\|s\|$ represent* $\max s$ *and* $\min s$, *such that the principal curvatures κ_i of $\partial\Omega_t$, $i=1,\cdots,n-1$, are bounded from above and below, satisfying*

$$\frac{1}{C}\le\kappa_i(x,t)\le C,\quad \forall(x,t)\in S^{n-1}\times[0,T).$$

Proof. The proof is divided into two parts: in the first part, we derive an upper bound for the Gauss curvature $\mathcal{K}(x,t)$; in the second part, we give an estimate of bound above for the principal radii $b_{ij}=h_{ij}+h\delta_{ij}$.

Step 1: Prove $\mathcal{K}\le C$.

Firstly, we construct the following auxiliary function,

$$\Theta(x,t)=\frac{\lambda(t)(\varphi|\nabla u|^q)^{-1}\mathcal{K}f(x)h-h}{h-\varepsilon_0}\equiv\frac{-h_t}{h-\varepsilon_0},$$

where

$$\varepsilon_0=\frac{1}{2}\min_{S^{n-1}\times[0,T)}h(x,t)>0,\quad h_t=\frac{\partial h}{\partial t}.$$

For any fixed $t\in[0,T)$, we assume that $\Theta(x_{t_0},t_0)=\max_{S^{n-1}}\Theta(x,t)$. Then at (x_{t_0},t_0), we have

$$0=\nabla_i\Theta=\frac{-h_{ti}}{h-\varepsilon_0}+\frac{h_th_i}{(h-\varepsilon_0)^2},\tag{4.9}$$

and from (4.9), at (x_{t_0},t_0), we also get

$$\begin{aligned}0\geq\nabla_{ii}\Theta=&\frac{-h_{tii}}{h-\varepsilon_0}+\frac{h_{ti}h_i}{(h-\varepsilon_0)^2}+\frac{h_{ti}h_i+h_th_{ii}}{(h-\varepsilon_0)^2}-\frac{h_th_i(2(h-\varepsilon_0)h_i)}{(h-\varepsilon_0)^4}\\=&\frac{-h_{tii}}{h-\varepsilon_0}+\frac{2h_{ti}h_i+h_th_{ii}}{(h-\varepsilon_0)^2}-\frac{2h_th_ih_i}{(h-\varepsilon_0)^3}\\=&\frac{-h_{tii}}{h-\varepsilon_0}+\frac{h_th_{ii}}{(h-\varepsilon_0)^2}+\frac{2h_{ti}h_i(h-\varepsilon_0)-2h_th_ih_i}{(h-\varepsilon_0)^3}\\=&\frac{-h_{tii}}{h-\varepsilon_0}+\frac{h_th_{ii}}{(h-\varepsilon_0)^2}.\end{aligned}\tag{4.10}$$

From (4.10), we obtain

$$-h_{tii}\leq\frac{-h_th_{ii}}{h-\varepsilon_0},$$

hence,

$$\begin{aligned}-h_{tii}-h_t\delta_{ii}\leq&\frac{-h_th_{ii}}{h-\varepsilon_0}-h_t\delta_{ii}=\frac{-h_t}{h-\varepsilon_0}(h_{ii}+(h-\varepsilon_0)\delta_{ii})\\=&\Theta(h_{ii}+h\delta_{ii}-\epsilon_0\delta_{ii})=\Theta(b_{ii}-\varepsilon_0\delta_{ii}).\end{aligned}\tag{4.11}$$

At (x_{t_0},t_0), we also have

$$\begin{aligned}\frac{\partial}{\partial t}\Theta=&\frac{-h_{tt}}{h-\epsilon_0}+\frac{h_t^2}{(h-\epsilon_0)^2}\\=&\frac{f}{h-\epsilon_0}\left[\frac{\partial(\lambda(t)(\varphi|\nabla u|^q)^{-1}h)}{\partial t}\mathcal{K}+\lambda(t)(\varphi|\nabla u|^q)^{-1}h\frac{\partial(\det(\nabla^2h+hI))^{-1}}{\partial t}\right]+\Theta+\Theta^2,\end{aligned}\tag{4.12}$$

where

$$\frac{\partial}{\partial t}((\varphi|\nabla u|^q)^{-1}h)=-\varphi^{-2}\varphi'\frac{\partial h}{\partial t}|\nabla u|^{-q}h-q|\nabla u|^{-(q+1)}\frac{\partial}{\partial t}|\nabla u|\varphi^{-1}h+(\varphi|\nabla u|^q)^{-1}\frac{\partial h}{\partial t},$$

and φ' denotes $\frac{\partial\varphi(s)}{\partial s}$.

According to Lemma 5.3 of [22], it shows that

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla u|=&-(\nabla^2u)x\cdot\left(\frac{\partial h_i}{\partial t}\right)e_i-\left(\frac{\partial h}{\partial t}\right)(\nabla^2u)x\cdot x-(|\nabla u|^{-1}\nabla u\nabla^2u\cdot x)\left(\frac{\partial h}{\partial t}\right)-|\nabla u|\left(\frac{\partial h}{\partial t}\right)\\=&-(\nabla^2u)x\cdot\Big(-\Theta_i(h-\epsilon_0)-\Theta h_i\Big)e_i\\&+\Theta(h-\epsilon_0)\Big((\nabla^2u)x\cdot x+|\nabla u|^{-1}\nabla u\nabla^2u\cdot x+|\nabla u|\Big)\end{aligned}$$

$$\leq\Theta(x_{t_0},t_0)\bigg((\nabla^2 u)xh_ie_i+h((\nabla^2 u)x\cdot x+|\nabla u|^{-1}\nabla u\nabla^2 u\cdot x+|\nabla u|)\bigg),$$

then,

$$\begin{aligned}&\frac{\partial}{\partial t}((\varphi|\nabla u|^q)^{-1}h) \qquad (4.13)\\ \leq&-\varphi^{-2}\varphi'|\nabla u|^{-q}h(-\Theta(x_{t_0},t_0)(h-\epsilon_0)\\ &-q|\nabla u|^{-(q+1)}\varphi^{-1}h\Theta(x_{t_0},t_0)\bigg((\nabla^2 u)xh_ie_i+h((\nabla^2 u)x\cdot x+|\nabla u|^{-1}\nabla u\nabla^2 u\cdot x+|\nabla u|)\bigg)\\ &-(\varphi|\nabla u|^q)^{-1}\Theta(x_{t_0},t_0)(h-\varepsilon_0).\end{aligned}$$

Thus, combining (4.3) with Lemma 4.6, and dropping some negative terms in (4.13), we have

$$\frac{\partial}{\partial t}((\varphi|\nabla u|^q)^{-1}h)\leq-\varphi^{-2}\varphi'|\nabla u|^{-q}h\times(-\Theta(x_{t_0},t_0))(h-\epsilon_0)\leq C_1\Theta(x_{t_0},t_0).$$

And from (3.2), we know that

$$\begin{aligned}\frac{\partial}{\partial t}(\lambda(t))=&\frac{\partial}{\partial t}\left(\frac{\int_{S^{n-1}}|\nabla u|^q\rho^n d\xi}{\int_{S^{n-1}}hf/\varphi dx}\right)\\ =&\frac{\frac{\partial}{\partial t}\left(\int_{S^{n-1}}|\nabla u|^q\rho^n d\xi\right)}{\int_{S^{n-1}}hf/\varphi dx}-\frac{\frac{\partial}{\partial t}\left(\int_{S^{n-1}}hf/\varphi dx\right)\left(\int_{S^{n-1}}|\nabla u|^q\rho^n d\xi\right)}{\left(\int_{S^{n-1}}hf/\varphi dx\right)^2}.\end{aligned}$$

By $\rho^2=h^2+|\nabla h|^2$ and (x_{t_0},t_0) is a maximum of Θ, we get

$$\frac{\partial\rho}{\partial t}=\rho^{-1}(hh_t+\sum h_kh_{kt})=\rho^{-1}\Theta(\varepsilon_0h-\rho^2)\leq\rho^{-1}\Theta(x_{t_0},t_0)(\varepsilon_0h-\rho^2).$$

For $q>1$, one can obtain form (4.3) and Lemma 4.6

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\int_{S^{n-1}}|\nabla u|^q\rho^n d\xi\right)\\ =&\int_{S^{n-1}}\left(\frac{\partial}{\partial t}|\nabla u|^q\right)\rho^n d\xi+\int_{S^{n-1}}|\nabla u|^q\left(\frac{\partial}{\partial t}\rho^n\right)d\xi\\ =&q\int_{S^{n-1}}|\nabla u|^{q-1}\frac{\partial}{\partial t}|\nabla u|\rho^n d\xi+n\int_{S^{n-1}}|\nabla u|^q\rho^{n-1}\left(\frac{\partial\rho}{\partial t}\right)d\xi\\ \leq&q\Theta(x_{t_0},t_0)\bigg((\nabla^2 u)xh_ie_i+h((\nabla^2 u)x\cdot x+|\nabla u|^{-1}\nabla u\nabla^2 u\cdot x+|\nabla u|)\bigg)\int_{S^{n-1}}|\nabla u|^{q-1}\rho^n d\xi\\ &+n\rho^{-1}\Theta(x_{t_0},t_0)(\varepsilon_0h-\rho^2)\int_{S^{n-1}}|\nabla u|^q\rho^{n-1}d\xi\\ \leq&C_2\Theta(x_{t_0},t_0),\end{aligned}$$

and

$$\begin{aligned}-\frac{\partial}{\partial t}\left(\int_{S^{n-1}}hf/\varphi dx\right)=&-\int_{S^{n-1}}\frac{\frac{\partial h}{\partial t}f}{\varphi}+\frac{hf\varphi'\frac{\partial h}{\partial t}}{\varphi^2}dx\\ =&\int_{S^{n-1}}\left(\frac{f}{\varphi}+\frac{hf\varphi'}{\varphi^2}\right)\Theta(h-\varepsilon_0)dx\\ \leq&\int_{S^{n-1}}\left(\frac{fh}{\varphi}+\frac{h^2f\varphi'}{\varphi^2}\right)dx\Theta(x_{t_0},t_0)\end{aligned}$$

$$\leq C_3\Theta(x_{t_0}, t_0).$$

Hence,

$$\frac{\partial}{\partial t}(\lambda(t)) \leq C_4\Theta(x_{t_0}, t_0).$$

We use (2.7), (4.11) and recall $b_{ij} = \nabla_{ij}h + h\delta_{ij}$ may give

$$\begin{aligned}
\frac{\partial(\det(\nabla^2 h + hI))^{-1}}{\partial t} =& -(\det(\nabla^2 h + hI))^{-2}\frac{\partial(\det(\nabla^2 h + hI))}{\partial b_{ij}}\frac{\partial(\nabla^2 h + hI)}{\partial t} \\
=& -(\det(\nabla^2 h + hI))^{-2}\frac{\partial(\det(\nabla^2 h + hI))}{\partial b_{ij}}(h_{tij+h_t\delta_{ij}}) \\
\leq& (\det(\nabla^2 h + hI))^{-2}\frac{\partial(\det(\nabla^2 h + hI))}{\partial b_{ij}}\Theta(b_{ij} - \varepsilon_0\delta_{ij}) \\
\leq& \mathcal{K}\Theta((n-1) - \varepsilon_0(n-1)\mathcal{K}^{\frac{1}{n-1}}).
\end{aligned}$$

Therefore, we have following conclusion with (4.12) at (x_{t_0}, t_0),

$$\frac{\partial}{\partial t}\Theta \leq \frac{1}{h-\varepsilon_0}\left(C_5\Theta^2 + f\lambda h(\varphi|\nabla u|^q)^{-1}\mathcal{K}\Theta((n-1) - \varepsilon_0(n-1)\mathcal{K}^{\frac{1}{n-1}})\right) + \Theta + \Theta^2. \tag{4.14}$$

According to construction of Θ and the previous estimate, we easily obtain

$$\frac{1}{C_6}\mathcal{K} \leq \Theta \leq C_6\mathcal{K}.$$

Then, it is clear that $\Theta(x_{t_0}, t_0)$ is sufficiently large, (4.14) implies that

$$\begin{aligned}
\frac{\partial}{\partial t}\Theta \leq& \frac{1}{h-\varepsilon_0}\left(C_5\Theta^2 + f\lambda h(\varphi|\nabla u|^q)^{-1}C_6\Theta^2((n-1) - \varepsilon_0(n-1)(C_6\Theta)^{\frac{1}{n-1}})\right) + \Theta + \Theta^2 \\
\leq& \frac{1}{h-\varepsilon_0}\Theta^2\left(C_5 + [f\lambda h(\varphi|\nabla u|^q)^{-1}C_6(n-1)] - [f\lambda h(\varphi|\nabla u|^q)^{-1}C_6^{\frac{n}{n-1}}(n-1)]\varepsilon_0\Theta^{\frac{1}{n-1}} + 2\right) \\
=& \frac{[f\lambda h(\varphi|\nabla u|^q)^{-1}C_6^{\frac{n}{n-1}}(n-1)]}{h-\varepsilon_0}\Theta^2\left(\frac{C_5 + [f\lambda h(\varphi|\nabla u|^q)^{-1}C_6(n-1)] + 2}{[f\lambda h(\varphi|\nabla u|^q)^{-1}C_6^{\frac{n}{n-1}}(n-1)]} - \varepsilon_0\Theta^{\frac{1}{n-1}}\right) \\
\leq& C_7\Theta^2(C_8 - \varepsilon_0\Theta^{\frac{1}{n-1}}) < 0,
\end{aligned}$$

since C_7 and C_8 depend on $\|f\|_{C^0(S^{n-1})}$, $\|\varphi\|_{C^1(I_{[0,T)})}$, $\|\varphi\|_{C^2(I_{[0,T)})}$, $\|h\|_{C^1(S^{n-1}\times[0,T)}$, $\|\lambda\|_{C^0(S^{n-1}\times[0,T)}$ and $|\nabla u|$. Consequently, above ODE tells us that

$$\Theta(x_{t_0}, t_0) \leq C,$$

and for any (x, t),

$$\mathcal{K}(x,t) = \frac{(h-\varepsilon_0)\Theta(x,t) + h}{f(x)h(\varphi|\nabla u|^q)^{-1}\lambda} \leq \frac{(h-\varepsilon_0)\Theta(x_{t_0}, t_0) + h}{f(x)h(\varphi|\nabla u|^q)^{-1}\lambda} \leq C.$$

Step 2: Prove $\kappa_i \geq \frac{1}{C}$.

We consider the auxiliary function as follows

$$F(x,t) = \log\beta_{\max}(\{b_{ij}\}) - A\log h + B|\nabla h|^2,$$

where A, B are positive constants which will be chosen later, and $\beta_{\max}(\{b_{ij}\})$ denotes the maximal eigenvalue of $\{b_{ij}\}$; for convenience, we write $\{b^{ij}\}$ for $\{b_{ij}\}^{-1}$.

For every fixed $t \in [0, T)$, suppose $\max_{S^{n-1}} \digamma(x,t)$ is attained at point $x_0 \in S^{n-1}$. By a rotation of coordinates, we may assume

$$\{b_{ij}(x_0,t)\} \text{ is diagonal,} \quad \text{and} \quad \beta_{\max}(\{b_{ij}\}(x_0,t)) = b_{11}(x_0,t).$$

Hence, in order to show $\kappa_i \geq \frac{1}{C}$, that is to prove $b_{11} \leq C$. By means of the above assumption, we transform $\digamma(x,t)$ into the following form,

$$\widetilde{\digamma}(x,t) = \log b_{11} - A \log h + B|\nabla h|^2.$$

Utilizing again the above assumption, for any fixed $t \in [0,T)$, $\widetilde{\digamma}(x,t)$ has a local maximum at (x_0,t), thus, we have at (x_0,t),

$$\begin{aligned} 0 = \nabla_i \widetilde{\digamma} =& b^{11}\nabla_i b_{11} - A\frac{h_i}{h} + 2B\sum h_k h_{ki} \\ =& b^{11}(h_{i11} + h_1\delta_{i1}) - A\frac{h_i}{h} + 2Bh_i h_{ii}, \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} 0 \geq& \nabla_{ii}\widetilde{\digamma} \\ =& \nabla_i b^{11}(h_{i11} + h_1\delta_{i1}) + b^{11}[\nabla_i(h_{i11} + h_1\delta_{i1})] - A\left(\frac{h_{ii}}{h} - \frac{h_i^2}{h^2}\right) + 2B(\sum h_k h_{kii} + h_{ii}^2) \\ =& -(b_{11})^{-2}\nabla_i b_{11}(h_{i11} + h_1\delta_{i1}) + b^{11}(\nabla_{ii}b_{11}) - A\left(\frac{h_{ii}}{h} - \frac{h_i^2}{h^2}\right) + 2B(\sum h_k h_{kii} + h_{ii}^2) \\ =& b^{11}\nabla_{ii}b_{11} - (b^{11})^2(\nabla_i b_{11})^2 - A\left(\frac{h_{ii}}{h} - \frac{h_i^2}{h^2}\right) + 2B(\sum h_k h_{kii} + h_{ii}^2). \end{aligned}$$

At (x_0,t), we also have

$$\begin{aligned} \frac{\partial}{\partial t}\widetilde{\digamma} =& \frac{1}{b_{11}}\frac{\partial b_{11}}{\partial t} - A\frac{h_t}{h} + 2B\sum h_k h_{kt} \\ =& b^{11}\frac{\partial}{\partial t}(h_{11} + h\delta_{11}) - A\frac{h_t}{h} + 2B\sum h_k h_{kt} \\ =& b^{11}(h_{11t} + h_t) - A\frac{h_t}{h} + 2B\sum h_k h_{kt}. \end{aligned}$$

From Eq. (3.3) and (2.7), we know that

$$\begin{aligned} \log(h - h_t) =& \log(h + \lambda(\varphi|\nabla u|^q)^{-1}\mathcal{K}hf - h) \\ =& \log\mathcal{K} + \log(\lambda(\varphi|\nabla u|^q)^{-1}hf) \\ =& -\log(\det(\nabla^2 h + hI)) + \log(\lambda(\varphi|\nabla u|^q)^{-1}hf). \end{aligned} \tag{4.16}$$

Let

$$\chi(x,t) = \log(\lambda(\varphi|\nabla u|^q)^{-1}hf).$$

Differentiating (4.16) once and twice, we respectively get

$$\begin{aligned} \frac{h_k - h_{kt}}{h - h_t} =& -\sum b^{ij}\nabla_k b_{ij} + \nabla_k\chi \\ =& -\sum b^{ii}(h_{kii} + h_i\delta_{ik}) + \nabla_k\chi, \end{aligned}$$

and

$$\begin{aligned} \frac{h_{11} - h_{11t}}{h - h_t} - \frac{(h_1 - h_{1t})^2}{(h - h_t)^2} =& -\left(-\sum(b^{ii})^2(\nabla_i b_{ii})^2 + b^{ii}\nabla_{ii}b_{ii}\right) + \nabla_{11}\chi \\ =& -\sum b^{ii}\nabla_{11}b_{ii} + \sum b^{ii}b^{jj}(\nabla_1 b_{ij})^2 + \nabla_{11}\chi. \end{aligned}$$

By the Ricci identity, we have

$$\nabla_{11} b_{ii} = \nabla_{ii} b_{11} - b_{11} + b_{ii}.$$

Thus, we can derive

$$\begin{aligned}
\frac{\frac{\partial}{\partial t}\widetilde{F}}{h-h_t} =& b^{11}\left(\frac{h_{11t}+h_t}{h-h_t}\right) - A\frac{h_t}{h(h-h_t)} + \frac{2B\sum h_k h_{kt}}{h-h_t} \\
=& b^{11}\left(\frac{h_{11t}-h_{11}}{h-h_t} + \frac{h_{11}+h-h+h_t}{h-h_t}\right) - A\frac{1}{h}\frac{h_t-h+h}{h-h_t} + \frac{2B\sum h_k h_{kt}}{h-h_t} \\
=& b^{11}\left(-\frac{(h_1-h_{1t})^2}{(h-h_t)^2} + \sum b^{ii}\nabla_{11}b_{ii} - \sum b^{ii}b^{jj}(\nabla_1 b_{ij})^2 - \nabla_{11}\chi \right.\\
&\left. + \frac{h_{11}+h-(h-h_t)}{h-h_t}\right) - \frac{A}{h}\left(\frac{-(h-h_t)+h}{h-h_t}\right) + \frac{2B\sum h_k h_{kt}}{h-h_t} \\
=& b^{11}\left(-\frac{(h_1-h_{1t})^2}{(h-h_t)^2} + \sum b^{ii}\nabla_{11}b_{ii} - \sum b^{ii}b^{jj}(\nabla_1 b_{ij})^2 - \nabla_{11}\chi \right) \\
& + b^{11}\left(\frac{h_{11}+h}{h-h_t} - 1\right) + \frac{A}{h} - \frac{A}{h-h_t} + \frac{2B\sum h_k h_{kt}}{h-h_t} \\
=& b^{11}\left(-\frac{(h_1-h_{1t})^2}{(h-h_t)^2} + \sum b^{ii}\nabla_{11}b_{ii} - \sum b^{ii}b^{jj}(\nabla_1 b_{ij})^2 - \nabla_{11}\chi \right) + \frac{1-A}{h-h_t} \\
& - b^{11} + \frac{A}{h} + \frac{2B\sum h_k h_{kt}}{h-h_t} \\
\leq& b^{11}\left(\sum b^{ii}(\nabla_{ii}b_{11} - b_{11} + b_{ii}) - \sum b^{ii}b^{jj}(\nabla_1 b_{ij})^2 \right) - b^{11}\nabla_{11}\chi + \frac{1-A}{h-h_t} \\
& + \frac{A}{h} + \frac{2B\sum h_k h_{kt}}{h-h_t} \\
\leq& \sum b^{ii}\left[(b^{11})^2(\nabla_i b_{11})^2 + A\left(\frac{h_{ii}}{h} - \frac{h_i^2}{h^2}\right) - 2B(\sum h_k h_{kii} + h_{ii}^2) \right] \\
& - b^{11}\sum b^{ii}b^{jj}(\nabla_1 b_{ij})^2 - b^{11}\nabla_{11}\chi + \frac{1-A}{h-h_t} + \frac{A}{h} + \frac{2B\sum h_k h_{kt}}{h-h_t} \\
\leq& \sum b^{ii}\left[A\left(\frac{h_{ii}+h-h}{h} - \frac{h_i^2}{h^2}\right)\right] + 2B\sum h_k\left(-\sum b^{ii}h_{kii} + \frac{h_{kt}}{h-h_t}\right) \\
& - 2B\sum b^{ii}(b_{ii}-h)^2 - b^{11}\nabla_{11}\chi + \frac{1-A}{h-h_t} + \frac{A}{h} \\
\leq& \sum b^{ii}\left[A\left(\frac{b_{ii}}{h} - 1\right)\right] + 2B\sum h_k\left(\frac{h_k}{h-h_t} + b^{kk}h_k - \nabla_k\chi\right) \\
& - 2B\sum b^{ii}(b_{ii}^2 - 2b_{ii}h) - b^{11}\nabla_{11}\chi + \frac{1-A}{h-h_t} + \frac{A}{h} \\
\leq& -2B\sum h_k\nabla_k\chi - b^{11}\nabla_{11}\chi + (2B|\nabla h| - A)\sum b^{ii} - 2B\sum b_{ii} \\
& + 4B(n-1)h + \frac{2B|\nabla h|^2 + 1 - A}{h-h_t} + \frac{nA}{h}.
\end{aligned}$$

Recall

$$\chi(x,t) = \log(\lambda(\varphi|\nabla u|^q)^{-1}hf) = \log\lambda - \log\varphi - q\log|\nabla u| + \log h + \log f,$$

since λ is a constant factor, we have $\lambda_k = 0$. Consequently, we may obtain following form by $\chi(x,t)$ and (4.15),

$$
\begin{aligned}
&-2B\sum h_k\nabla_k\chi - b^{11}\nabla_{11}\chi\\
=&-2B\sum h_k\left(\frac{f_k}{f}+\frac{h_k}{h}-q\frac{(|\nabla u|)_k}{|\nabla u|}-\frac{\varphi' h_k}{\varphi}\right)-b^{11}\nabla_{11}\chi\\
=&-2B\sum h_k\left(\frac{f_k}{f}+\frac{h_k}{h}-q\frac{(|\nabla u|)_k}{|\nabla u|}-\frac{\varphi' h_k}{\varphi}\right)\\
&-b^{11}\left(\frac{ff_{11}-f_1^2}{f^2}+\frac{hh_{11}-h_1^2}{h^2}-q\frac{|\nabla u|(|\nabla u|)_{11}-(|\nabla u|)_1^2}{(|\nabla u|)^2}-\frac{\varphi'' h_1^2+\varphi' h_{11}}{\varphi}+\frac{(\varphi' h_1)^2}{\varphi^2}\right)\\
\le&C_9B+C_{10}b^{11}+2qB\sum h_k\frac{(|\nabla u|)_k}{|\nabla u|}+b^{11}\frac{h(b_{11}-h)}{h^2}\\
&+qb^{11}\frac{|\nabla u|(|\nabla u|)_{11}-(|\nabla u|)_1^2}{(|\nabla u|)^2}+b^{11}\left(\frac{\varphi'' h_1^2+\varphi' h_{11}}{\varphi}+\frac{(\varphi' h_1)^2}{\varphi^2}\right),
\end{aligned}
$$

where $\varphi''=\frac{\partial^2\varphi(s)}{\partial s^2}$,

$$
b^{11}\left(\frac{\varphi'' h_1^2+\varphi' h_{11}}{\varphi}+\frac{(\varphi' h_1)^2}{\varphi^2}\right)=b^{11}\left(\frac{\varphi'' h_1^2+\varphi'(b_{11}-h)}{\varphi}+\frac{(\varphi' h_1)^2}{\varphi^2}\right)\le C_{11}b^{11}.
$$

Recall that

$$
|\nabla u(X,t)|=-\nabla u(X,t)\cdot x,
$$

taking the covariant derivative above equality, we get

$$
(|\nabla u|)_k=-b_{ik}((\nabla^2u)e_i\cdot x),
$$

further,

$$
\begin{aligned}
(|\nabla u|)_{11}=&-b_{i11}((\nabla^2u)e_i\cdot x)-b_{j1}b_{i1}((\nabla^3u)e_je_i\cdot x)\\
&+b_{i1}((\nabla^2u)x\cdot x)-b_{i1}((\nabla^2u)e_i\cdot e_1).
\end{aligned}
$$

Thus, combing (4.3) with Lemma 4.6, we get

$$
2qB\sum h_k\frac{(|\nabla u|)_k}{|\nabla u|}=2qB\sum h_k\frac{-b_{ik}((\nabla^2u)e_i\cdot x)}{|\nabla u|}\le C_{12}Bb_{11}.
$$

From (4.15), we obtain

$$
b^{11}b_{i11}=A\frac{h_i}{h}+2Bh_ih_{ii}=A\frac{h_i}{h}+2Bh_i(b_{ii}-h\delta_{ii}),
$$

therefore, from (4.3), (4.4) and Lemma 4.6, we get

$$
qb^{11}\frac{|\nabla u|(|\nabla u|)_{11}-(|\nabla u|)_1^2}{(|\nabla u|)^2}\le C_{13}Bb_{11}.
$$

It follows that

$$
\frac{\frac{\partial}{\partial t}\widetilde{F}}{h-h_t}\le C_{14}Bb_{11}+C_{15}b^{11}+(2B|\nabla h|-A)\sum b^{ii}-2B\sum b_{ii}+4B(n-1)h+\frac{nA}{h}<0,
$$

provided $b_{11}>>1$ and if we choose $A>>B$. We obtain

$$
\widetilde{F}(x_0,t)\le C,
$$

hence,

$$
F(x_0,t)=\widetilde{F}(x_0,t)\le C.
$$

This tells us the principal radii are bounded from above, or equivalently $\kappa_i \geq \frac{1}{C}$. □

5. The convergence of the flow

With the help of priori estimates in the section 4, the long-time existence and asymptotic behaviour of the flow (1.9) (or (3.1)) are obtained, we also can complete proof of Theorem 1.3.

Proof of the Theorem 1.3. Since Eq. (3.3) is parabolic, we can get its short time existence. Let T be the maximal time such that $h(\cdot,t)$ is a positive, smooth, even and strictly convex solution to Eq. (3.3) for all $t \in [0,T)$. Lemma 4.3-4.6 enable us to apply Lemma 4.7 to Eq. (3.3), thus, we can deduce a uniformly upper and lower bounds for the biggest eigenvalue of $\{(h_{ij}+h\delta_{ij})(x,t)\}$. This implies

$$C^{-1}I \leq (h_{ij}+h\delta_{ij})(x,t) \leq CI, \quad \forall (x,t) \in S^{n-1}\times[0,T),$$

where $C>0$ independents on t. This shows that Eq. (3.3) is uniformly parabolic. Estimates for higher derivatives follows from the standard regularity theory of uniformly parabolic equations Krylov [25]. Hence, we obtain the long time existence and regularity of solutions for the flow (1.9) (or (3.1)). Moreover, we obtain

$$\|h\|_{C^{l,m}_{x,t}(S^{n-1}\times[0,T))} \leq C_{l,m},$$

for some $C_{l,m}$ (l,m are nonnegative integers pairs) independent of t, then $T=\infty$. Using parabolic comparison principle, we can attain the uniqueness of the smooth even solution $h(\cdot,t)$ of Eq. (3.3).

By the monotonicity of Γ in Lemma 3.2, there is a constant $C>0$ independent of t, such that

$$|\Gamma(X(\cdot,t))| \leq C, \qquad \forall t \in [0,\infty). \tag{5.1}$$

By the Lemma 3.2, we obtain

$$\lim_{t\to\infty}\Gamma(X(\cdot,t)) - \Gamma(X(\cdot,0)) = -\int_0^\infty \left|\frac{d}{dt}\Gamma(X(\cdot,t))\right|dt. \tag{5.2}$$

From (5.1), the left hand side of (5.2) is bounded below by $-2C$, therefore, there is a sequence $t_j\to\infty$ such that

$$\frac{d}{dt}\Gamma(X(\cdot,t_j)) \to 0 \quad \text{as} \quad t_j\to\infty,$$

using Lemma 3.2, Lemma 4.3 and Lemma 4.4 again, above equation implies $h(\cdot,t)$ converges to a positive, even and uniformly convex function $h_\infty \in C^\infty(S^{n-1})$ which satisfies (1.8) with τ given by

$$\frac{1}{\tau} = \lim_{t_j\to\infty}\lambda(t_j).$$

This completes the proof of Theorem 1.3. □

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