

# ON THE HODGE STRUCTURES OF GLOBAL SMOOTHINGS OF NORMAL CROSSING VARIETIES

KUAN-WEN CHEN

ABSTRACT. Let  $f : X \rightarrow \Delta$  be a one-parameter semistable degeneration of  $m$ -dimensional compact complex manifolds. Assume that each component of the central fiber  $X_0$  is Kähler. Then, we provide a criterion for a general fiber to satisfy the  $\partial\bar{\partial}$ -lemma and a formula to compute the Hodge index on the middle cohomology of the general fiber in terms of the topological conditions/invariants on the central fiber.

We apply our theorem to several examples, including the global smoothing of  $m$ -fold ODPs, Hashimoto-Sano's non-Kähler Calabi-Yau threefolds [23], and Sano's non-Kähler Calabi-Yau  $m$ -folds [39].

To deal with the last example, we also prove a Lefschetz-type theorem for the cohomology of the fiber product of two Lefschetz fibrations over  $\mathbb{P}^1$  with disjoint critical locus.

## 1 Introduction

The classical  $\partial\bar{\partial}$ -lemma for compact Kähler manifolds states that if we are given a smooth differential form  $\alpha$  on a compact Kähler manifold which is  $\partial$ -closed,  $\bar{\partial}$ -closed and  $d$ -exact, then  $\alpha$  is  $\partial\bar{\partial}$ -exact. It is natural to ask when the  $\partial\bar{\partial}$ -lemma holds beyond the category of compact Kähler manifolds. It is well-known that if  $M$  is a compact complex manifold such that the Hodge to de Rham spectral sequence degenerates at the  $E_1$  page, then the following three conditions are equivalent (cf. [13, (4.3.1)], [14, (5.21)], [4, Lemma 13.6]):

- (1)  $M$  satisfies the  $\partial\bar{\partial}$ -lemma,
- (2) the Hodge filtration  $F^\bullet H^d(X)$  is  $d$ -opposed to its complex conjugate  $\bar{F}^\bullet H^d(M)$  for all  $d$ ,
- (3) For all  $d$ , there is a Hodge decomposition

$$H^d(M) = \bigoplus_p H^{p,d-p}(M),$$

where  $H^{p,d-p}(M) = F^p H^d(M) \cap \bar{F}^{d-p} H^d(M)$ .

Here, we recall that if  $V$  is a complex vector space and  $'F^\bullet$  and  $''F^\bullet$  are two decreasing filtrations on  $V$ , then  $'F^\bullet$  and  $''F^\bullet$  are said to be  $d$ -opposed if, for every  $p$ ,

$$'F^p \oplus ''F^{d-p+1} \cong V.$$

In particular since the degeneration of Hodge to de Rham spectral sequence and the  $d$ -opposedness of the Hodge filtration with its complex conjugate are both open conditions, the  $\partial\bar{\partial}$ -lemma is an open condition.

It is also well-known that if a compact complex manifold  $M$  is dominated either by a compact Kähler manifold or by a compact complex manifold of the same dimension as

$M$  that satisfies the  $\partial\bar{\partial}$ -lemma, then  $M$  satisfies the  $\partial\bar{\partial}$ -lemma (cf. [14, Theorem 5.22] and the argument used in [47, Lemma 7.28, Remark 7.29]). In particular, any Moishezon manifold and any compact complex manifold of Fujiki class  $\mathcal{C}$  (i.e., bimeromorphic to a compact Kähler manifold) satisfy the  $\partial\bar{\partial}$ -lemma. Conversely, if  $M$  is a compact complex manifold that satisfies the  $\partial\bar{\partial}$ -lemma and  $Z$  is a closed submanifold of  $M$  that satisfies the  $\partial\bar{\partial}$ -lemma, then the blowup  $Bl_Z M$  of  $M$  along  $Z$  also satisfies the  $\partial\bar{\partial}$ -lemma ([43],[36],[49],[3]). Thus, the  $\partial\bar{\partial}$ -lemma is a bimeromorphic invariant in dimension less or equal to three, and also, if  $M$  is a compact complex analytic space with at worst isolated singularities, then one resolution of singularities of  $M$  satisfies the  $\partial\bar{\partial}$ -lemma if and only if every resolution of singularities of  $M$  satisfies the  $\partial\bar{\partial}$ -lemma. But in general, whether the  $\partial\bar{\partial}$ -lemma is a bimeromorphic invariant is still an open conjecture.

There are several examples of compact complex manifolds that satisfy the  $\partial\bar{\partial}$ -lemma but not of Fujiki class  $\mathcal{C}$ . First, since the  $\partial\bar{\partial}$ -lemma is an open condition under deformation but being of Fujiki class  $\mathcal{C}$  isn't ([7],[29]), the small deformation of some Moishezon twistor spaces produce threefold examples. Second, some examples are found among solvmanifolds ([1],[2],[24]). Third, [3] constructs a simply connected threefold example. Fourth, Friedman [16] proved that the  $\partial\bar{\partial}$ -lemma holds for general nearby global smoothings of Calabi-Yau threefolds with ordinary double points (i.e. general (nearby) Clemens manifolds) and later Li [31] proved that Friedman's result holds without the general assumption and showed that the Hodge structure on the middle cohomology of any (nearby) Clemens manifold together with the cup product is polarized Hodge structure. Fifth, [25] constructs some other simply connected threefold examples as well as some interesting examples of compact complex manifolds that don't satisfy the  $\partial\bar{\partial}$ -lemma.

This paper aims to study the general phenomenon behind the existence of polarized Hodge structures on the middle cohomology of the (nearby) Clemens manifolds. Recall that Clemens manifolds are constructed by first contracting several rational curves on a projective Calabi-Yau threefold to ordinary double points and then taking the global smoothing of the ordinary double points on such a singular compact complex analytic space. Thus, we can propose the following general question: if  $M$  is a singular compact complex analytic space that admits a Kähler resolution of singularities as well as a global smoothing, when does the global smoothing satisfy the  $\partial\bar{\partial}$ -lemma? Also, if this is the case, can we compute the Hodge index on the middle cohomology of the global smoothing?

Consider a one-parameter degeneration of  $m$ -dimensional compact complex manifolds  $f : X \rightarrow \Delta$ , where  $X$  is a connected complex manifold,  $\Delta$  is the unit disk, and  $f$  is a proper holomorphic map that is smooth over the punctured disk  $\Delta^*$ . By the semistable reduction theorem, we may assume  $E := f^{-1}(0)$  is a simple normal crossing divisor, all of whose components are reduced. Suppose, moreover, that the irreducible components of  $E$  are all Kähler or, more generally, each  $k$ -fold intersection of the irreducible components of  $E$  satisfies the  $\partial\bar{\partial}$ -lemma. Then Steenbrink (cf. [33, Chap 11]) constructs a limiting mixed Hodge structure on  $H^d(X_\infty, \mathbb{C})$  for all  $d$ , where  $X_\infty$  is the canonical fiber. We will denote it by  $(H_{\text{lim}}^d, W^{St}, F_{\text{lim}})$ . Steenbrink shows that in this case, the monodromy operator  $T$  on  $H^d(X_t, \mathbb{C})$  induced by counter-clockwise loop is unipotent. Moreover, if we consider the natural extension of  $T$  to  $H_{\text{lim}}^d$ , still denoted by  $T$ , then by

Steenbrink's construction, the nilpotent operator  $N := \log(T)$  is a morphism of mixed Hodge structure of type  $(-1, -1)$ , i.e.  $N$  satisfies  $NW_r^{St} \subset W_{r-2}^{St}$  and  $NF_{\lim}^p \subset F_{\lim}^{p-1}$ . Furthermore, as a consequence of the theory, the Hodge to de Rham spectral sequence on the general fiber  $X_q$  degenerates at  $E_1$  for  $|q|$  sufficiently small.

On the other hand, a well-known theorem of Schmid ([41]) is that the degeneration of a one-parameter integral polarized variation of Hodge structure of weight  $d$  forms a polarized mixed Hodge structure  $(H, W, F)$  (see Definition 2.1), where  $W = W(N, d)$  is the monodromy weight filtration of  $N$  centered at  $d$ .

This paper aims to deal with the converse situation of Schmid's theorem above in the geometric setup. Consider the following two situations:

- (A) For the limiting mixed Hodge structure  $(H_{\lim}^d, W^{St}, F_{\lim})$  on  $H^d(X_{\infty}, \mathbb{C})$ , we assume that  $W^{St} = W(N, d)$ .
- (B) Suppose we are in the case  $d = m$  and that  $W^{St} = W(N, m)$  on  $H^m(X_{\infty}, \mathbb{C})$ , i.e. (A) holds. Let  $Q(\bullet, \bullet)$  denote the natural cup product on  $H^m(X_{\infty}, \mathbb{C}) \cong H^m(X_q, \mathbb{C})$ . (If there is no ambiguity, we will denote the natural cup product on any compact complex manifold by  $Q(\bullet, \bullet)$ .) Write  $S(\bullet, \bullet) = \epsilon(m)Q(\bullet, \bullet)$ , where  $\epsilon(a) := \frac{a(a-1)}{2}$ . Let  $C$  denote the Weil operator, which is defined as  $(\sqrt{-1})^{p-r}$  on the  $(p, r)$ -component of a Hodge structure of weight  $p+r$ . Since (A) holds, for all  $l \geq 0$ , the Hermitian form

$$(u, v) \mapsto S(Cu, (-N)^l \bar{v})$$

is nongenerate on  $\text{Gr}_{m+l}^{W^{St}} H_{\lim}^m$  (cf. Remark 3.2 for why we use  $-N$  instead of  $N$ ). Let  $(s_+^{p, m+l-p}, s_-^{p, m+l-p})$  be the signature of this Hermitian form on the  $(p, m+l-p)$ -component of the Hodge structure (of weight  $d+l$ ) on the primitive part

$$P_{m+l} = \ker(N^{l+1} : \text{Gr}_{m+l}^{W^{St}} H_{\lim}^m \rightarrow \text{Gr}_{m-l-2}^{W^{St}} H_{\lim}^m).$$

Given an integer  $n$ , set  $n_+ = \max\{n, 0\}$ . Since (A) holds, for all  $l$  with  $-m \leq l \leq m$ , there is an  $N$ -Lefschetz decomposition:

$$\text{Gr}_{m+l}^{W^{St}} H_{\lim}^m = \bigoplus_{r \geq (-l)_+} N^r P_{m+l+r}.$$

Thus for  $0 \leq p \leq m+l$ , define the invariants

$$\mathbf{S}_+^{p, m+l-p} = \sum_{r \geq (-l)_+} s_+^{p+r, m+l-p+r}$$

and

$$\mathbf{S}_-^{p, m+l-p} = \sum_{r \geq (-l)_+} s_-^{p+r, m+l-p+r},$$

which corresponds to (the  $N$ -Lefschetz decomposition of) the  $(p, m+l-p)$ -component of the Hodge structure on  $\text{Gr}_{m+l}^{W^{St}} H_{\lim}^m$ . Note that  $\mathbf{S}_+^{p, m+l-p} + \mathbf{S}_-^{p, m+l-p}$  is equal to the dimension of the  $(p, m+l-p)$ -component of the Hodge structure on  $\text{Gr}_{m+l}^{W^{St}} H_{\lim}^m$ , but when  $l \geq 0$ ,  $(\mathbf{S}_+^{p, m+l-p}, \mathbf{S}_-^{p, m+l-p})$  is not the signature of the Hermitian form  $S(C\bullet, (-N)^l \bar{\bullet})$  on the  $(p, m+l-p)$ -component of the Hodge structure on  $\text{Gr}_{m+l}^{W^{St}} H_{\lim}^m$  since  $S(CNu, v) + S(u, Nv) = 0$  for all  $u$  and  $v$ .

Then, the main theorem of the article is:

**Theorem 1.1.** *Let  $f : X \rightarrow \Delta$  be a one-parameter semi-stable degeneration of  $m$ -dimensional compact complex manifolds such that all irreducible components of the central fiber  $X_0$  are Kähler or, more generally, each  $k$ -fold intersection of the irreducible components of  $X_0$  satisfies the  $\partial\bar{\partial}$ -lemma. Let  $(H_{\text{lim}}^d, W^{\text{St}}, F_{\text{lim}})$  be the limiting mixed Hodge structure on  $H^d(X_\infty, \mathbb{C})$ . Then*

- (1) *In the situation (A), the Hodge filtration  $F_q^\bullet$  of  $H^d(X_q, \mathbb{C})$  on the nearby fiber  $X_q := f^{-1}(q)$  is  $d$ -opposed to its complex conjugate for  $|q|$  sufficiently small.*
- (2) *In the situation (B), the signature of  $S(C\bullet, \bar{\bullet})$  on the  $(p, m-p)$ -component of the Hodge structure on  $H^m(X_q, \mathbb{C})$  is*

$$\left( \sum_{k=0}^m \mathbf{s}_+^{p,k}, \sum_{k=0}^m \mathbf{s}_-^{p,k} \right)$$

*for  $|q|$  sufficiently small. In particular, if  $(H_{\text{lim}}^m, W^{\text{St}}, F_{\text{lim}})$  is a polarized mixed Hodge structure with respect to the cup product  $S(\bullet, \bullet)$  (i.e.,  $s_-^{p,r} = 0$  for all  $p$  and  $r$ ), then the middle cohomology  $H^m(X_q, \mathbb{C})$  together with the cup product  $S(\bullet, \bullet)$  is a polarized Hodge structure for  $|q|$  sufficiently small.*

**Remark 1.2.** (1) The theorem above describes the general phenomenon behind Li's computation for Clemens's manifolds in [31]. We will see later that the leading term in his computation corresponds to the nilpotent orbit, and the nearby Hodge filtration approximates the nilpotent orbit in a certain way (sufficiently fast) so that it still underlines a polarized Hodge structure.

(2) By Steenbrink's theory, the assumption  $W^{\text{St}} = W(N, d)$  in Situation (A) is only a topological condition on the central fiber  $X_0$ . Indeed, this topological condition only involves the Gysin morphisms and the restriction morphisms of the strata of  $X_0$ . Also, the computation of the Hodge index  $(s_+^{p,q}, s_-^{p,q})$  in Situation (B) can be reduced to the computation of the Hodge index on the middle cohomology of the strata of  $X_0$  by Fujisawa's theory ([19], or see Proposition 3.5 and Proposition 3.6). This will be elaborated in Section 3. Hence, our theorem provides a criterion of the  $\partial\bar{\partial}$ -lemma on the general fiber and a formula to compute the Hodge index on the middle cohomology of the general fiber in terms of the singularity of the singular fiber.

The first consequence of Theorem 1.1 is based on a theorem of Guillén-Navarro Aznar. Suppose that  $X_0$  satisfies a strong Kähler type condition: there exists a class in  $H^2(X_0, \mathbb{R})$  that restricts to a Kähler class on each irreducible component. Then Guillén-Navarro Aznar ([22], or cf. [33, §11.3]) shows that there is a polarized Hodge-Lefschetz module structure on  $W^{\text{St}}E_1$ , and thus Steenbrink's weight filtration  $W^{\text{St}}$  and the monodromy weight filtration  $W(N, d)$  coincides on  $H^d(X_\infty, \mathbb{C})$  for all  $d$ . (The coincidence of  $W^{\text{St}}$  and  $W(N, d)$  for a projective degeneration is first proved by Saito [38, 4.2.5 Remarque] and Usui [46, (A.1)] independently.) Therefore, combined with Theorem 1.1 and Fujisawa's theory ([19], or see Proposition 3.5 and Proposition 3.6), we can prove the following theorem:

**Theorem 1.3** (Corollary 3.9). *Let  $f : X \rightarrow \Delta$  be a semistable degeneration of  $m$ -dimensional compact complex manifolds. Suppose that there exists a class  $L \in H^2(X_0, \mathbb{R})$*

which restricts to a Kähler class on each component of the normal crossing complex analytic space  $X_0$ . Then

- (1) The general fiber  $X_q$  satisfies the  $\partial\bar{\partial}$ -lemma for  $|q|$  sufficiently small.
- (2) If we let  $h_q^{a,b} = \dim_{\mathbb{C}} H^b(X_q, \Omega_{X_q}^a)$ , then the signature of  $S(C\bullet, \bar{\bullet})$  on the  $(p, m-p)$ -component of the Hodge structure on  $H^m(X_q, \mathbb{C})$  is

$$\left( \sum_{r \geq 0} (h_q^{p-2r, m-p-2r} - h_q^{p-2r-1, m-p-2r-1}), \sum_{r \geq 0} (h_q^{p-2r-1, m-p-2r-1} - h_q^{p-2r-2, m-p-2r-2}) \right)$$

for  $|q|$  small.

- (3) In particular, when  $m$  is even, then the general fiber  $X_q$  satisfies the Hodge index theorem for compact Kähler manifolds for  $|q|$  small: the signature of the cup product  $Q(\bullet, \bullet)$  on  $H^m(X_q, \mathbb{R})$  is  $\sum_{a,b \geq 0} (-1)^a h_q^{a,b}$  for  $|q|$  small.

**Remark 1.4.** Roughly speaking, the theorem above says that under such a strong Kähler type condition on the central fiber, the general fiber behaves like a Kähler manifold. Indeed, if  $M$  is a  $m$ -dimensional compact Kähler manifold, then from the Lefschetz decomposition on the middle cohomology, we have

$$H^{p, m-p}(M) = \bigoplus_{r \geq 0} L^r H_{\text{prim}}^{p-r, m-p-r}(M),$$

where  $H_{\text{prim}}^{a,b}(M)$  denotes the  $(a, b)$ -component of the primitive cohomology of  $M$  and  $L$  is the Lefschetz operator. Denote the dimension of  $H^{a,b}(M)$  and  $H_{\text{prim}}^{a,b}(M)$  by  $h^{a,b}$  and  $h_{\text{prim}}^{a,b}$ . Then the signature of  $S(C\bullet, \bar{\bullet})$  on  $H^{p, m-p}(M)$  is

$$\begin{aligned} & \left( \sum_{r \geq 0} h_{\text{prim}}^{p-2r, m-p-2r}, \sum_{r \geq 0} h_{\text{prim}}^{p-2r-1, m-p-2r-1} \right) \\ &= \left( \sum_{r \geq 0} (h^{p-2r, m-p-2r} - h^{p-2r-1, m-p-2r-1}), \sum_{r \geq 0} (h^{p-2r-1, m-p-2r-1} - h^{p-2r-2, m-p-2r-2}) \right), \end{aligned}$$

which is the same as the formula in (2) above. It is interesting to ask whether the general fibers are Kähler manifolds if there are Kähler forms on all irreducible components of  $X_0$  that agree on all intersections.

Now, we consider a  $m$ -dimensional compact complex analytic space  $X_0$  with at most ordinary double points, which admits a resolution of singularities that satisfies the  $\partial\bar{\partial}$ -lemma. Let  $\{x_i | 1 \leq i \leq l\}$  be the set of ordinary double points on  $X_0$  and  $\tilde{X}_0$  be the blowup of  $X_0$  at all ordinary double points. The exceptional divisors  $Q_i$  in  $\tilde{X}_0$  over ordinary points  $x_i$  are  $m-1$ -dimensional quadrics. If  $m$  is odd, each  $Q_i$  is an even-dimensional quadric with standard  $\frac{m-1}{2}$ -dimensional planes whose homology classes will be denoted by  $A_i$  and  $B_i$ . Define the number

$$R = \dim_{\mathbb{C}} \ker \left( \bigoplus_{i=1}^l H_{\text{prim}}^{m-1}(Q_i) \xrightarrow{-\gamma} H^{m+1}(\tilde{X}_0) \right),$$

where  $\gamma$  is the Gysin morphism. The number  $R$  represents the number of linear relations of  $A_i - B_i$  in  $H_{m-1}(\tilde{X}_0)$ . If  $m$  is even, consider the kernel of the restriction map

$$V^m := \ker \left( H^m(\tilde{X}_0) \rightarrow \bigoplus_{i=1}^l H^m(Q_i) \right),$$

which has dimension  $h^m(\tilde{X}_0) - l$  if  $X_0$  admits a global smoothing. Then, by applying Theorem 1.1 to this situation, we get the following theorem, which generalizes the theorem of Friedman [16] and Li [31] for Clemens manifolds:

**Theorem 1.5** (Theorem 4.1). *Let  $f : X \rightarrow \Delta$  be a one-parameter degeneration of  $m$ -dimensional compact complex manifolds such that the central fiber  $X_0$  has at worst ordinary double points. Suppose that  $X_0$  admits a resolution of singularities that satisfies the  $\partial\bar{\partial}$ -lemma. Then the nearby fiber  $X_q$  satisfies the  $\partial\bar{\partial}$ -lemma for  $|q|$  small.*

Let  $(h_+^{k,m-k}(\tilde{X}_0), h_-^{k,m-k}(\tilde{X}_0))$  be the signature of  $S(C\bullet, \bar{\bullet})$  on  $H^{k,m-k}(\tilde{X}_0)$ . When  $m$  is even, let  $(\hat{h}_+^{m/2,m/2}(\tilde{X}_0), \hat{h}_-^{m/2,m/2}(\tilde{X}_0))$  be the signature of  $S(C\bullet, \bar{\bullet})$  on the  $(m/2, m/2)$  part of  $V^m$ .

(1) *If  $m$  is odd, then the signature of  $S(C\bullet, \bar{\bullet})$  on  $H^{k,m-k}(X_q)$  is*

$$\begin{cases} (h_+^{k,m-k}(\tilde{X}_0), h_-^{k,m-k}(\tilde{X}_0)), & k \neq \frac{m+1}{2}, \frac{m-1}{2} \\ (h_+^{k,m-k}(\tilde{X}_0) + R, h_-^{k,m-k}(\tilde{X}_0)), & k = \frac{m+1}{2} \text{ or } \frac{m-1}{2} \end{cases}$$

for  $|q|$  sufficiently small.

(2) *If  $m$  is even, then the signature of  $S(C\bullet, \bar{\bullet})$  on  $H^{k,m-k}(X_q)$  is*

$$\begin{cases} (h_+^{k,m-k}(\tilde{X}_0), h_-^{k,m-k}(\tilde{X}_0)), & k \neq \frac{m}{2} \\ (\hat{h}_+^{m/2,m/2}(\tilde{X}_0) + l, \hat{h}_-^{m/2,m/2}(\tilde{X}_0)), & k = \frac{m}{2} \end{cases}$$

for  $|q|$  sufficiently small.

In particular, if  $H^m(\tilde{X}_0, \mathbb{C})$  together with the cup product  $S(\bullet, \bullet)$  is a polarized Hodge structure, then  $H^m(X_q, \mathbb{C})$  together with the cup product  $S(\bullet, \bullet)$  is a polarized Hodge structure as well for  $|q|$  sufficiently small.

Recently, there have been several new classes of non-Kähler Calabi-Yau manifolds (which are not even of Fujiki class  $\mathcal{C}$ ) constructed by the global smoothing of  $d$ -semistable SNC Calabi-Yau varieties. The critical ingredient is to apply the log deformation theory of the  $d$ -semistable SNC Calabi-Yau variety developed by Namikawa-Kawamata [27, Theorem 4.2] (or see [10, Corollary 7.4] for some recent developments). Then, by choosing suitable  $d$ -semistable SNC Calabi-Yau varieties and applying the theory of Namikawa-Kawamata, Hashimoto-Sano [23] construct non-Kähler Calabi-Yau threefolds  $X_3(a)$ , whose  $b_2$  tends to infinity as  $a$  tends to infinity. Later, by using the same method, Lee [30] constructs an example of non-Kähler Calabi-Yau fourfold, Sano [39] constructs non-Kähler Calabi-Yau  $m$ -folds  $X_m(a)$  for all  $m \geq 4$ , whose  $b_2$  tends to infinity as  $a$  tends to infinity, and also Sano [40] gives another example of non-Kähler Calabi-Yau threefolds with arbitrarily large  $b_2$ .

If we apply our Theorem 1.1 to their examples, then we have:

**Theorem 1.6** (Theorem 4.5). *The non-Kähler Calabi-Yau threefolds  $X_3(a)$  constructed by Hashimoto-Sano [23, Theorem 1.1] satisfy the  $\partial\bar{\partial}$ -lemma. Moreover,  $H^3(X_3(a), \mathbb{C})$  together with the cup product  $S(\bullet, \bullet)$  is a polarized Hodge structure.*

**Theorem 1.7** (Theorem 4.7). *The non-Kähler Calabi-Yau  $m$ -folds  $X_m(a)$  for  $m \geq 4$  constructed by Sano [39, Theorem 1.1] satisfy the  $\partial\bar{\partial}$ -lemma. Moreover, the Hodge index on the middle cohomology  $H^m(X_m(a))$  is as follows:*

(1) *If  $m \equiv 3 \pmod{4}$ , then  $H^m(X_m(a))$  together with the cup product  $S(\bullet, \bullet)$  is a polarized Hodge structure.*

(2) *If  $m \equiv 1 \pmod{4}$ , then the signature of  $S(C\bullet, \bar{\bullet})$  on  $H^{k, m-k}(X_m(a))$  is*

$$\begin{cases} (h^{k, m-k}(X_m(a)), 0), & k \neq \frac{m+1}{2}, \frac{m-1}{2} \\ (h^{k, m-k}(X_m(a)) - 9(27a^2 - 2a + 5) - a - 6, 9(27a^2 - 2a + 5) + a + 6), & k = \frac{m+1}{2} \text{ or } \frac{m-1}{2} \end{cases} .$$

(3) *If  $m = 4$ , then the signature of  $S(C\bullet, \bar{\bullet})$  on  $H^{k, m-k}(X_m(a))$  is*

$$\begin{cases} (h^{k, m-k}(X_m(a)), 0), & k \neq \frac{m}{2} \\ (h^{k, m-k}(X_m(a)) - a - 1, a + 1), & k = \frac{m}{2} \end{cases} .$$

(4) *If  $m$  is even and  $m \geq 6$ , then the signature of  $S(C\bullet, \bar{\bullet})$  on  $H^{k, m-k}(X_m(a))$  is*

$$\begin{cases} (h^{k, m-k}(X_m(a)), 0), & k \neq \frac{m}{2} \\ (h^{k, m-k}(X_m(a)) - a - 2, a + 2), & k = \frac{m}{2} \end{cases} .$$

Since the construction of Sano's non-Kähler Calabi-Yau  $m$ -folds relies on the projective Calabi-Yau manifolds of Schoen's type (fibered product type), the difficulty in applying Theorem 1.1 to this case is to understand the cohomology of projective Calabi-Yau manifolds constructed by the fibered product of two anti-canonical Lefschetz fibrations. Throughout the paper, when we refer to a Lefschetz fibration, we mean a Lefschetz fibration that arises from the blowup of the base locus of a Lefschetz pencil, which is contained in an ample linear system. We prove the following Lefschetz-type theorem for the fibered product of two Lefschetz fibrations over  $\mathbb{P}^1$  with disjoint critical locus:

**Theorem 1.8** (Theorem 5.1). *Let  $\tilde{X}_i \xrightarrow{f_i} \mathbb{P}^1$  ( $i = 1, 2$ ) be two Lefschetz fibrations and  $\Delta(f_i) \subset \mathbb{P}^1$  be the critical locus of  $f_i$ . Write  $m_i = \dim_{\mathbb{C}} \tilde{X}_i$ . Assume that  $\Delta(f_1)$  and  $\Delta(f_2)$  are disjoint. Then the canonical morphism of graded rings*

$$H^\bullet(\tilde{X}_1, \mathbb{Q}) \otimes_{H^\bullet(\mathbb{P}^1, \mathbb{Q})} H^\bullet(\tilde{X}_1, \mathbb{Q}) \rightarrow H^\bullet(\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2, \mathbb{Q})$$

*is an isomorphism in degree  $\leq m_1 + m_2 - 2$  and is injective in degree  $m_1 + m_2 - 1$ .*

A brief description of the contents is as follows: in Section 2, we first recall some basic results of the degeneration of abstract variation of Hodge structure and prove an abstract version of Theorem 1.1 by assuming two technical estimates on nilpotent orbits. In Section 3, we review Steenbrink's theory and Fujisawa's theory on limiting mixed Hodge structures and show that Theorem 1.1 follows from the abstract version we prove in Section 2. In Section 4, we apply Theorem 1.1 to several examples, including the global smoothings of ODPs, the global smoothings of Calabi-Yau threefolds with an  $O_{16}$  singularity, Hashimoto-Sano's non-Kähler Calabi-Yau threefolds [23], Sano's non-Kähler Calabi-Yau  $m$ -folds [39]. In Section 5, we prove the Lefschetz-type theorem

for the fibered product of two Lefschetz fibrations over  $\mathbb{P}^1$  with disjoint critical locus. Finally, in Section 6, we derive the unpolarized and generalized version of the converse of nilpotent orbits theorem that we use in the proof of Theorem 1.1.

**Remark 1.9.** Theorem 1.6 and some special cases of Theorem 1.1 and Theorem 1.3 for the threefold degenerations with  $N^2 = 0$  have also been obtained by Tsung-Ju Lee [45] independently, where the finite distance degenerations of non-Kähler Calabi-Yau threefolds that satisfy the  $\partial\bar{\partial}$ -lemma are studied.

**Acknowledgments.** The author would like to thank Hung Jiang, Po-Sheng Wu, Alex Xu, and Professor Taro Fujisawa for inspiring discussions. In the communication with Professor Chin-Lung Wang, the author was informed that Tsung-Ju Lee also obtained similar results in this direction independently. After communicating with Tsung-Ju Lee, the author wants to thank him for his suggestions on the earlier version of the article. Lastly, the author wants to thank Professor Robert Friedman for his numerous suggestions, explanations, clarifications, and the time he spent editing this paper.

**Notation:** For every  $a \in \mathbb{Z}$ , let  $\epsilon(a) := \frac{a(a-1)}{2}$ . We have

$$\begin{aligned}\epsilon(a+1) &= (-1)^a \epsilon(a) = \epsilon(-a), & \epsilon(a+2) &= -\epsilon(a), \\ \epsilon(a+b) &= (-1)^{ab} \epsilon(a)\epsilon(b).\end{aligned}$$

## 2 Abstract Variation of Hodge Filtration

Let  $\mathcal{H}$  be a holomorphic vector bundle on the punctured disc  $\Delta^*$  with an integrable connection  $\nabla$ . Consider the canonical extension  $\tilde{\mathcal{H}}$  of  $(\mathcal{H}, \nabla)$  of  $\mathcal{H}$  to the unit disc  $\Delta$  so that  $\nabla$  extends to a regular-singular connection on  $\Delta$ . Let  $T$  be the monodromy automorphism of the local system of horizontal sections  $\mathcal{H}^\nabla$  determined by a counter-clockwise loop around 0, i.e. at each stalk of  $\mathcal{H}^\nabla$  over  $\Delta^*$ ,  $T$  is obtained by the inverse of the pullback along the counter-clockwise loop by parallel translation. The convention for  $T$  here is compatible with the convention for the Picard-Lefschetz monodromy  $T$  in the next section. Assume that  $T$  is unipotent and let  $N = -\log(T)$ . Then by [12], we can write  $\tilde{\mathcal{H}} = H \otimes \mathcal{O}_\Delta$ , where  $H$  is the  $\mathbb{C}$ -vector space of multivalued horizontal sections of  $\mathcal{H}$ , and the fiber  $\tilde{\mathcal{H}}_0$  of  $\tilde{\mathcal{H}}$  over 0 is identified with  $H$  up to the action of the unipotent group  $\{\exp(\lambda N) : \lambda \in \mathbb{C}\}$ . The fiber of  $\mathcal{H}$  at any point is identified with  $H$  modulo the action of  $\{\exp(kN) : k \in \mathbb{Z}\}$ . The connection  $\nabla$  is associated to the connection 1-form  $\frac{N}{2\pi\sqrt{-1}} \frac{dq}{q}$ , where  $q$  is the coordinate of  $\Delta^*$ .

Write

$$q = e^{2\pi\sqrt{-1}z}, \quad z = \frac{\log q}{2\pi\sqrt{-1}},$$

where  $z$  is the uniformizer of  $\Delta^*$  and  $\varphi : \mathfrak{H} \rightarrow \Delta^*$  the corresponding uniformization. A holomorphic section  $v$  of  $\mathcal{H}$ , viewed as a holomorphic section of the trivial bundle  $\varphi^*\mathcal{H} = H \otimes \mathcal{O}_\mathfrak{H}$  with the invariance property  $v(z-1) = Tv(z)$ , extends to a holomorphic section of  $\tilde{\mathcal{H}}$  if and only if the section  $\exp(-zN)\varphi^*v$ , viewed as a holomorphic section of  $\tilde{\mathcal{H}}$ , extends to a (single-valued) holomorphic function from  $\Delta$  to  $H$ .

We first recall some definitions in the context of the degeneration of Hodge structures.

**Definition 2.1.** Let  $H$  be a finite-dimensional complex vector space, and  $N$  a nilpotent endomorphism of  $H$  defined over  $\mathbb{R}$ ,

- (1) The monodromy weight filtration of  $N$  centered at  $d$  is the unique increasing filtration  $W = W(N, d)$  of  $H$  such that  $N(W_i) \subset W_{i-2}$  for  $i \geq 2$  and the map

$$N^l : \mathrm{Gr}_{d+l}^W H \rightarrow \mathrm{Gr}_{d-l}^W H$$

is an isomorphism for all  $l$ .

- (2) ([9, Definition 2.26]) Suppose we are given a non-degenerate bilinear form  $S$  on  $H$  defined over  $\mathbb{R}$  which is  $(-1)^d$ -symmetric:  $S(v, u) = (-1)^d S(u, v)$ , and such that  $N$  is an infinitesimal isometry:  $S(u, Nv) + S(Nu, v) = 0$ . Then a mixed Hodge structure  $(H, W, F)$  on  $H$  is polarized by  $N$  centered at  $d$  if
- (a)  $N^{d+1} = 0$ ;
  - (b)  $W = W(N, d)$ ;
  - (c)  $S(F^p, F^{d-p+1}) = 0$ ;
  - (d)  $NF^p \subset F^{p-1}$ ;
  - (e) the Hodge structure (of weight  $d+l$ ) on the primitive part

$$P_{d+l} = \ker(N^{l+1} : \mathrm{Gr}_{d+l}^W H \rightarrow \mathrm{Gr}_{d-l-2}^W H)$$

is polarized by the form  $S(\bullet, N^l \bullet)$ .

A well-known consequence of Schmid's nilpotent orbit theorem and  $SL_2$  orbit theorem (cf. [41]) is that if  $\mathcal{H}$  underlies an integral polarized variation of Hodge structure of weight  $d$ , then Hodge bundles  $\mathcal{F}^p$  extends to holomorphic subbundles  $\tilde{\mathcal{F}}^p$  of  $\tilde{\mathcal{H}}$  so that the triple  $(\tilde{\mathcal{H}}_0, W(N, d), \tilde{\mathcal{F}}_0)$  is a polarized mixed Hodge structure, where  $W(N, d)$  is the monodromy weight filtration of  $N$  centered at  $d$ .

This section aims to deal with the converse situation of Schmid's theorem. We are interested in the following two situations:

- (A') We are given a mixed Hodge structure  $(H, W, F)$  defined over  $\mathbb{R}$  and a nilpotent endomorphism  $N$  on  $H$  such that
  - (a)  $N$  is a morphism of mixed Hodge structure of type  $(-1, -1)$
  - (b)  $W = W(N, d)$  is the weight filtration of  $N$  centered at  $d$ .
- (B') Besides the assumption in (A'), we are given a non-degenerate bilinear form  $S$  on  $H$  defined over  $\mathbb{R}$  which satisfies
  - (a)  $S(v, u) = (-1)^d S(u, v)$ ,
  - (b)  $S(u, Nv) + S(Nu, v) = 0$ ,
  - (c)  $S(F^p, F^{d-p+1}) = 0$  for all  $p$ .

In the situation (A'), since  $W = W(N, d)$ , we have

$$\dim_{\mathbb{C}} \mathrm{Gr}_F^p H = \dim_{\mathbb{C}} \mathrm{Gr}_F^{d-p} H,$$

and hence

$$\dim F^p + \dim \bar{F}^{d+1-p} = \dim H$$

for all  $p$ . Also, in the situation (B'), we have  $S(W_a H, W_b H) = 0$  for  $a+b \leq 2d-1$ . Thus,  $S(\bullet, N^l \bullet)$  defines a non-degenerate pairing on  $\mathrm{Gr}_{d+l}^W H$ . Moreover, by the first Hodge-Riemann condition on  $F$ ,  $S(\bullet, N^l \bullet)$  is non-degenerate on the  $(p, d+l-p)$ -component of the Hodge structure on  $\mathrm{Gr}_{d+l}^W H$  (of weight  $d+l$ ) for all  $p$  and  $l$ . Note that  $S(\bullet, N^l \bullet)$  is

also non-degenerate on the  $(p, d + l - p)$ -component of the Hodge structure (of weight  $d + l$ ) on the primitive part

$$P_{d+l} := \ker(N^{l+1} : \mathrm{Gr}_{d+l}^W H \rightarrow \mathrm{Gr}_{d-l-2}^W H)$$

for all  $p$  and  $l$ . Let  $(s_+^{p,d+l-p}, s_-^{p,d+l-p})$  denote the signature of  $S(C\bullet, N^{l\bar{\bullet}})$  on the  $(p, d + l - p)$ -component of  $P_{d+l}$ , where  $C$  is the Weil operator defined by  $(\sqrt{-1})^{p-q}$  on the  $(p, q)$ -component of a Hodge structure of weight  $p + q$ . Since  $S(u, Nv) + S(Nu, v) = 0$ , the signature of  $S(C\bullet, N^{l\bar{\bullet}})$  on the  $(p, d + l - p)$ -component of the Hodge structure on  $\mathrm{Gr}_{d+l}^W H$  is

$$\left( \sum_{r \geq 0} s_+^{p+2r, d+l-p+2r} + \sum_{r \geq 0} s_-^{p+2r+1, d+l-p+2r+1}, \sum_{r \geq 0} s_-^{p+2r, d+l-p+2r} + \sum_{r \geq 0} s_+^{p+2r+1, d+l-p+2r+1} \right).$$

Given an integer  $n$ , set  $n_+ = \max\{n, 0\}$ . In the situation  $(\mathbf{B}')$ , for  $0 \leq p \leq d + l$ , define the invariants

$$\mathbf{S}_+^{p, m+l-p} = \sum_{r \geq (-l)_+} s_+^{p+r, d+l-p+r}$$

and

$$\mathbf{S}_-^{p, d+l-p} = \sum_{r \geq (-l)_+} s_-^{p+r, d+l-p+r},$$

which corresponds to (the  $N$ -Lefschetz decomposition of) the  $(p, d + l - p)$ -component of the Hodge structure on  $\mathrm{Gr}_{d+l}^W H$ .

Our main theorem is:

**Theorem 2.2.** *Suppose we are given a decreasing filtration  $\mathcal{F}^\bullet$  of holomorphic subbundles of  $\mathcal{H}$ . Let  $\tilde{\mathcal{F}}^\bullet$  be the corresponding holomorphic subbundles of  $\tilde{\mathcal{H}}|_{\Delta^*}$  under canonical extension. We suppose that  $\tilde{\mathcal{F}}^\bullet$  extend to holomorphic subbundles of  $\tilde{\mathcal{H}}$ , still denoted by  $\tilde{\mathcal{F}}^\bullet$ . Consider the fiber  $\tilde{\mathcal{F}}_0^\bullet$  of  $\tilde{\mathcal{F}}^\bullet$  over 0.*

- (1) *If the triple  $(\tilde{\mathcal{H}}_0, W(N, d), \tilde{\mathcal{F}}_0)$  satisfies the situation  $(A')$ , then the fiber  $\mathcal{F}_q^\bullet$  of  $\mathcal{F}^\bullet$  is  $d$ -opposed to its complex conjugate for  $|q|$  small. In other words,  $\mathcal{F}_q^\bullet$  induces a pure Hodge structure on  $\mathcal{H}_q$  of weight  $d$  for  $|q|$  small.*
- (2) *Suppose moreover that we are given a non-degenerate bilinear form  $S$  on  $H$  defined over  $\mathbb{R}$  which makes  $(\tilde{\mathcal{H}}_0, W(N, d), \tilde{\mathcal{F}}_0)$  satisfy the situation  $(\mathbf{B}')$ . We also replace the assumption (c) of the situation  $(\mathbf{B}')$  by a stronger first Hodge-Riemann bilinear relation:  $S(\mathcal{F}^p, \mathcal{F}^{d-p+1}) = 0$  for all  $p$ .*

*Then the signature of  $S(C\bullet, \bar{\bullet})$  on the  $(p, d-p)$ -component of the Hodge structure  $\mathcal{H}_q$  becomes*

$$\left( \sum_{k=0}^d \mathbf{S}_+^{p,k}, \sum_{k=0}^d \mathbf{S}_-^{p,k} \right) = \left( \sum_{\substack{l \geq p-d, \\ r \geq (-l)_+}} s_+^{p+r, d+l-p+r}, \sum_{\substack{l \geq p-d, \\ r \geq (-l)_+}} s_-^{p+r, d+l-p+r} \right)$$

for  $|q|$  sufficiently small.

*In particular, if  $(\tilde{\mathcal{H}}_0, W(N, d), \tilde{\mathcal{F}}_0)$  is a polarized mixed Hodge structure centered at  $d$ , (i.e.  $s_-^{p, d+l-p} = 0$  for all  $p$  and  $l$ ), then  $\mathcal{F}_q^\bullet$  induces a polarized Hodge structure of weight  $d$  on  $\mathcal{H}_q$  for  $|q|$  small.*

**Remark 2.3.**

- (1) The condition that  $S(\mathcal{F}^p, \mathcal{F}^{d-p+1}) = 0$  for all  $p$  implies the assumption (c) in the situation  $(\mathbf{B}')$  since

$$S(\exp(zN)v, \exp(zN)v') = S(v, v')$$

for all vectors  $v$  and  $v'$  in  $H$ .

- (2) In the theorem, it is surprising that we do not assume that the filtration  $\mathcal{F}^\bullet$  is horizontal:  $\nabla_q \mathcal{F}^p \subset \mathcal{F}^{p-1}$ , while it is a natural condition in the geometric setting.

The idea to prove Theorem 2.2 is to compare our Hodge filtration  $\mathcal{F}_q$  with the "nilpotent orbit." To be precise, the following two theorems are the unpolarized version and the generalized version of the converse of nilpotent orbit theorem of Cattani-Kaplan-Schmid ([9], Corollary 3.13), which will serve as the leading terms of the estimates on  $\mathcal{F}_q$  in the proof of our main theorem. We will prove the following two theorems in the last section of the paper.

**Theorem 2.4.** *For each mixed Hodge structure  $(H, W, F)$  in the situation  $(\mathbf{A}')$ , the filtration  $\exp(zN)F^\bullet$  is  $d$ -opposed to its complex conjugate for  $\text{Im}(z)$  sufficiently large.*

**Theorem 2.5.** *For each mixed Hodge structure  $(H, W, F)$  and non-degenerate bilinear form  $S$  on  $H$  in the situation  $(\mathbf{B}')$ ,*

- (1) *The Hodge filtration  $F^\bullet$  can be refined to a filtration  $\hat{F}^\bullet$  such that*
- (a)  *$\dim_{\mathbb{C}} \text{Gr}_F^k H = 1$  for all  $k$ ,*
  - (b) *The restriction of the Hermitian form  $(\sqrt{-1})^d S(\bullet, \bar{\bullet})$  on  $\exp(zN)\hat{F}^k$  is non-degenerate for  $\text{Im}(z)$  sufficiently large and for all  $k$ .*
- (2) *The signature of the Hermitian form  $S(C\bullet, \bar{\bullet})$  on  $\exp(zN)F^p \cap \overline{\exp(zN)F^{d-p}}$  becomes*

$$\left( \sum_{k=0}^d \mathbf{s}_+^{p,k}, \sum_{k=0}^d \mathbf{s}_-^{p,k} \right) = \left( \sum_{\substack{l \geq p-d, \\ r \geq (-l)_+}} s_+^{p+r, d+l-p+r}, \sum_{\substack{l \geq p-d, \\ r \geq (-l)_+}} s_-^{p+r, d+l-p+r} \right)$$

for  $\text{Im}(z)$  sufficiently large.

*In particular, if  $(H, W, F)$  is a polarized mixed Hodge structure centered at  $d$ , (i.e.  $s_-^{p, d+l-p} = 0$  for all  $p$  and  $l$ ), then  $\exp(zN)F^\bullet$  is a nilpotent orbit.*

**Remark 2.6.** Theorem 2.4 and Theorem 2.5 are generalizations of the converse of nilpotent orbit theorem of Cattani-Kaplan-Schmid [9, Corollary 3.13]. It is possible to follow their methods to prove the first theorem. More precisely, if we consider the classifying space  $D$  of unpolarized Hodge structures according to the assumption in Theorem 2.4, then  $D$  is still a homogeneous space of the form  $G(\mathbb{R})/K$ , where  $G(\mathbb{R})$  is a real Lie group and  $K$  is a subgroup of  $G(\mathbb{R})$ , at the expense that  $K$  might be non-compact. Since  $D$  is still a real open subset of its compact dual  $D^\vee$ , we can follow the line of Cattani-Kaplan [8, Proposition 2.18] to deal with the case of the mixed Hodge structure split over  $\mathbb{R}$  and then use the same methods as in [9, Corollary 3.13] to reduce the general case to the case of the mixed Hodge structure split over  $\mathbb{R}$ . In the last section of the paper, we will prove the two theorems above by direct combinatorial calculation.

Now, we can prove the main theorem of this section:

*Proof that Theorem 2.4 + Theorem 2.5  $\Rightarrow$  Theorem 2.2.* First, since  $\tilde{\mathcal{F}}^\bullet$  is the corresponding filtration of  $\mathcal{F}^\bullet$  under canonical extension, by Deligne's construction, this means  $\exp(-zN)\varphi^*\mathcal{F}^\bullet$  viewed as the filtration on  $\tilde{\mathcal{H}}|_{\Delta^*}$  coincides with  $\tilde{\mathcal{F}}^\bullet|_{\Delta^*}$ .

To prove Theorem 2.2 (1), it suffices to show that  $(\varphi^*\mathcal{F}^k)_z \oplus \overline{(\varphi^*\mathcal{F}^{d-k+1})_z} = (\varphi^*\mathcal{H})_z$  for all  $0 \leq k \leq d$  and  $\text{Im}(z)$  sufficiently large. We may assume  $0 \leq \text{Re}(z) \leq 1$  and write  $z = a + it$ . We choose a basis  $\{v_i^k\}_{i \in I^k}$  of  $\tilde{\mathcal{F}}_0^k$  for every  $0 \leq k \leq d$ , and possibly by shrinking  $\Delta$ , we can extend  $v_i^k$  to a holomorphic section  $v_i^k(q)$  of  $\tilde{\mathcal{F}}^k$  so that  $\{v_i^k(q)\}_{i \in I^k}$  is a basis of  $\tilde{\mathcal{F}}_q^k$ . Then  $(\varphi^*\mathcal{F}^k)_z$  has a basis of the form

$$u_i^k(z) = e^{zN} v_i^k(q).$$

We have  $|q| = O(e^{-2\pi t})$  and  $|z| = O(t)$  since  $0 \leq \text{Re}(z) \leq 1$ . And since  $N^{d+1} = 0$ , we have

$$e^{zN} v_i^k(q) = e^{zN} (v_i^k + O(e^{-2\pi t})) = e^{zN} v_i^k + O(t^d e^{-2\pi t}).$$

Therefore, the top wedge of the basis of  $(\varphi^*\mathcal{F}^k)_z$  and  $\overline{(\varphi^*\mathcal{F}^{d-k+1})_z}$  would be

$$\begin{aligned} \Phi^k &:= \left( \bigwedge_{i \in I^k} u_i^k(z) \right) \wedge \left( \bigwedge_{j \in I^{d-k+1}} \overline{u_j^{d-k+1}(z)} \right) \\ &= \left( \bigwedge_{i \in I^k} e^{zN} v_i^k + O(t^d e^{-2\pi t}) \right) \wedge \left( \bigwedge_{j \in I^{d-k+1}} e^{\bar{z}N} \overline{v_j^{d-k+1}} + O(t^d e^{-2\pi t}) \right) \\ &= \left( \bigwedge_{i \in I^k} e^{zN} v_i^k \right) \wedge \left( \bigwedge_{j \in I^{d-k+1}} e^{\bar{z}N} \overline{v_j^{d-k+1}} \right) + O(t^{d(\dim_{\mathbb{C}} H)} e^{-2\pi t}) \end{aligned}$$

By our assumption,  $(\tilde{\mathcal{H}}_0, W(N, d), \tilde{\mathcal{F}}_0)$  is a mixed Hodge structure. So by Theorem 2.4,  $\exp(zN)\tilde{\mathcal{F}}_0^\bullet$  is  $d$ -opposed to its complex conjugate for  $\text{Im}(z)$  sufficiently large. Therefore, the leading term

$$\left( \bigwedge_{i \in I^k} e^{zN} v_i^k \right) \wedge \left( \bigwedge_{j \in I^{d-k+1}} e^{\bar{z}N} \overline{v_j^{d-k+1}} \right)$$

is a nonvanishing top form for  $t$  sufficiently large and we can rewrite it as

$$C_k(a, t) \left( \bigwedge_{i \in I^0} v_i^0 \right)$$

where  $C_k(a, t)$  is a polynomial in  $a$  and  $t$  and is nonzero for  $t$  sufficiently large. Then we must have  $|C_k(a, t)| \geq C_k t^{L_k}$  for  $t$  sufficiently large, for some  $C_k > 0$  and nonnegative integer  $L_k$ . So we can conclude that  $\Phi^k$  is nonvanishing for  $t$  sufficiently large, and hence  $(\varphi^*\mathcal{F}^k)_z$  is  $d$ -opposed to its complex conjugate for  $t$  large.

To prove Theorem 2.2 (2), we need to show that  $S$  has the corresponding signature on  $(\varphi^*\mathcal{F}^k)_z \cap \overline{(\varphi^*\mathcal{F}^{d-k})_z}$ . Due to the  $d$ -opposedness condition that we have just proved

and the first Hodge-Riemann bilinear relation, it is equivalent to show that for  $0 \leq k \leq d$ , the signature of the Hermitian form  $(\sqrt{-1})^d S(\bullet, \bar{\bullet})$  on  $(\varphi^* \mathcal{F}^k)_z$  is

$$(2.1) \quad \sum_{j=k}^d (-1)^{d-j} \left( \sum_{\substack{l \geq j-d, \\ r \geq (-l)_+}} s_+^{j+r, d+l-j+r} - \sum_{\substack{l \geq j-d, \\ r \geq (-l)_+}} s_-^{j+r, d+l-j+r} \right)$$

for  $t$  sufficiently large. Here, by abuse of terminology, given a non-degenerate bilinear form  $S$  (or Hermitian form) with signature  $(s_+, s_-)$ , we call the number  $s_+ - s_-$  the signature of  $S$  as well.

Replace the vectors  $v_i^k$  with the basis compatible with the refined filtration from Theorem 2.4. We have

$$\begin{aligned} & (\sqrt{-1})^d S(u_i^k(z), \overline{u_{i'}^k(z)}) \\ &= (\sqrt{-1})^d S(e^{zN} v_i^k + O(t^d e^{-2\pi t}), \overline{e^{zN} v_{i'}^k + O(t^d e^{-2\pi t})}) \\ &= (\sqrt{-1})^d S(e^{zN} v_i^k, \overline{e^{zN} v_{i'}^k}) + O(t^{2d} e^{-2\pi t}). \end{aligned}$$

Since the mixed Hodge structure  $(\tilde{\mathcal{H}}_0, W(N, d), \tilde{\mathcal{F}}_0)$  and  $S(\bullet, \bullet)$  satisfy all of the assumptions in Theorem 2.5, the Hermitian form  $(\sqrt{-1})^d S(\bullet, \bar{\bullet})$  has signature as in (2.1) on the filtration  $\exp(zN)\tilde{\mathcal{F}}_0^k$ . Thus the leading term

$$\left( (\sqrt{-1})^d S(e^{zN} v_i^k, \overline{e^{zN} v_{i'}^k}) \right)_{i, i'}$$

of the matrix  $\left( (\sqrt{-1})^d S(u_i^k(z), \overline{u_{i'}^k(z)}) \right)_{i, i'}$  is a nondegenerate Hermitian matrix with signature the same as (2.1).

We recall the following easy linear algebra fact:

**Lemma 2.7.** *Suppose that  $A$  is a non-degenerate Hermitian matrix such that the leading principal  $l \times l$  submatrices are non-degenerate for all  $l$ . Then  $A$  is  $*$ -congruent to the diagonal matrix  $D = \text{diag}(M_0, M_1/M_0, M_2/M_1, \dots)$ , where  $M_l$  is the leading principal  $l \times l$  minor of  $A$ . (In other words, there exists an invertible matrix  $P$  such that  $A = P^* D P$ .)*

*Proof.* Given a Hermitian matrix with  $A = \begin{pmatrix} G & B \\ B^* & C \end{pmatrix}$  with  $G$  invertible, we can perform the block diagonalization

$$\begin{pmatrix} G & B \\ B^* & C \end{pmatrix} = \begin{pmatrix} I & 0 \\ B^* G^{-1} & I \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & C - B^* G^{-1} B \end{pmatrix} \begin{pmatrix} I & G^{-1} B \\ 0 & I \end{pmatrix}.$$

Inductively performing this operation to  $A$  with respect to the leading principal submatrices, one gets the diagonal matrix  $D = \text{diag}(M_0, M_1/M_0, M_2/M_1, \dots)$ .  $\square$

If  $A$  is the matrix  $\left( (\sqrt{-1})^d S(e^{zN} v_i^k, \overline{e^{zN} v_{i'}^k}) \right)_{i, i'}$ , then by our choice of basis, the leading principal  $l \times l$  submatrices are non-degenerate. In particular, if we let  $P_l(a, t)$  be the leading principal  $l \times l$  minor of  $A$ , then  $P_l(a, t)$  is a polynomial in  $a$  and  $t$  which must be nonvanishing for  $t$  sufficiently large. Therefore, we have  $|P_l(a, t)| \geq C_l t^{K_l}$  for  $t$  sufficiently large, for some  $C_l > 0$  and nonnegative integer  $K_l$ .

Let  $B = \left( (\sqrt{-1})^d S(u_i^k(z), \overline{u_{i'}^k(z)}) \right)_{i,i'}$ . By our computation, we can see that the upper left  $l \times l$  minor  $M_l$  of  $B$  is of the form

$$P_l(a, t) + O(t^{n_l} e^{-2\pi t}),$$

where  $n_l$  is a non-negative integer. Hence,  $M_l$  is nonzero for  $t$  sufficiently large, and the leading principal  $l \times l$  submatrices of  $B$  are also non-degenerate. Now we have

$$M_l^{-1} = \frac{1}{P_l(a, t)} \left( 1 + \sum_{j \geq 0} (-P_l(a, t) O(t^{n_l} e^{-2\pi t}))^j \right) = \frac{1}{P_l(a, t)} + O(t^{n_l} e^{-2\pi t})$$

and

$$\frac{M_{l+1}}{M_l} = \frac{P_{l+1}(a, t)}{P_l(a, t)} + O(t^{n'_l} e^{-2\pi t})$$

for some nonnegative integer  $n'_l$ . Therefore,  $M_{l+1}/M_l$  must have the same sign as  $P_{l+1}(a, t)/P_l(a, t)$  for  $t$  sufficiently large. So we can conclude that the Hermitian matrix  $\left( (\sqrt{-1})^d S(u_i^k(z), \overline{u_{i'}^k(z)}) \right)_{i,i'}$  also has signature as in (2.1) for  $t$  sufficiently large. The proof is complete.  $\square$

**Remark 2.8.** In section 6, we will show that  $C_k(a, t) = C_k t^{L_k} + O(t^{L_k-1})$  with  $C_k \neq 0$  and  $L_k \geq 0$ ,  $P_{l+1}(a, t)/P_l(a, t) = C'_l t^{L'_l} + O(t^{L'_l-1})$  with  $C'_l \neq 0$  and  $k - d \leq L'_l \leq d$ . The asymptotic orders  $L_k$  and  $L'_l$  are explicitly computable (cf. Remark 6.4).

### 3 Geometric Variation of Hodge Filtration

Suppose we are given a one-parameter degeneration of  $m$ -dimensional compact complex manifolds  $f : X \rightarrow \Delta$ , where  $X$  is a connected complex manifold,  $\Delta$  is the unit disk and  $f$  is a proper holomorphic map that is smooth over the punctured disk  $\Delta^*$  and  $E := f^{-1}(0) = X_0$  is a simple normal crossing divisor all of whose components are reduced. Suppose, moreover, that the irreducible components of  $E$  are all Kähler or, more generally, each  $k$ -fold intersection of  $E$  satisfies the  $\partial\bar{\partial}$ -lemma. The total space is homotopy equivalent to  $E$  by a fiber preserving retraction  $r : X \rightarrow E$ . So the composition of the inclusion map  $i_t : X_t \hookrightarrow X$  and the retraction gives a specialization map  $r_t : X_t \rightarrow E$ . The complex  $\psi_f \underline{\mathbb{Z}}_X := (Rr_t)_* i_t^* \underline{\mathbb{Z}}_X$  is the complex of the nearby cocycle so that  $\mathbb{H}^d(E, \psi_f \underline{\mathbb{Z}}_X) = H^d(X_\infty)$  for all  $d \geq 0$ , where  $X_\infty := X \times_{\Delta^*} \mathfrak{H}$  is the canonical fiber.

We first recall and summarize Steenbrink's construction of limiting mixed Hodge structures. Consider the relative log de Rham complex

$$\Omega_{X/\Delta}^\bullet(\log E) := \Omega_X^\bullet(\log E) / f^* \Omega_\Delta^1(\log 0) \wedge \Omega_X^{\bullet-1}(\log E).$$

**Theorem 3.1.** ([42], or see [33, §11.2.4-§11.2.7])

- (1) *The hypercohomology  $\mathbb{H}^d(E, \Omega_{X/\Delta}^\bullet(\log E) \otimes \mathcal{O}_E)$  is isomorphic to the cohomology  $H^d(X_\infty; \mathbb{C})$ , where  $X_\infty := X \times_{\Delta^*} \mathfrak{H}$  is the canonical fiber. The sheaf  $\tilde{\mathcal{H}}^d = \mathbb{R}^d f_* \Omega_{X/\Delta}^\bullet(\log E)$  is locally free and satisfies:  $\tilde{\mathcal{H}}^d|_{\Delta^*}$  is isomorphic to the holomorphic vector bundle  $\mathcal{H}^d := (R^d(f|_{f^{-1}(\Delta^*)})_* \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\Delta^*}$ , and  $\tilde{\mathcal{H}}^d$  is Deligne's canonical extension of  $\mathcal{H}^d$ .*

(2) There exists a marked mixed Hodge complex of sheaves on  $E$

$$\psi_f^{\text{Hdg}} = (\psi_f \underline{\mathbb{Z}}_X, (s(C^{\bullet, \bullet}), W^{\text{St}}), (s(A^{\bullet, \bullet}), W^{\text{St}}, F_{\text{lim}})),$$

which defines the limiting mixed Hodge structure over  $\mathbb{Q}$  on  $H^d(X_\infty)$  so that the nilpotent endomorphism  $N = \log(T)$  acts as a morphism of mixed Hodge structure of type  $(-1, -1)$ . In particular, the weight spectral sequence

$$W^{\text{St}} E_1^{-r, d+r} = \bigoplus_{k \geq 0, -r} H^{d-r-2k}(E(2k+r+1))(-r-k) \Rightarrow H^d(X_\infty, \mathbb{Q})$$

degenerates at  $E_2$

(3) The spectral sequence with  $E_1$  page

$$E_1^{p,q} = H^q(E; \Omega_{X/\Delta}^p(\log E) \otimes \mathcal{O}_E) \Rightarrow H^{p+q}(X_\infty; \mathbb{C})$$

degenerates at  $E_1$  and the corresponding filtration on  $H^{p+q}(X_\infty; \mathbb{C})$  is the Hodge filtration. Moreover, possibly after shrinking  $\Delta$ , the spectral sequence with  $E_1$  page

$$E_1^{p,q} = R^q f_* \Omega_{X/\Delta}^p(\log E) \Rightarrow \mathbb{R}^{p+q} f_* \Omega_{X/\Delta}^\bullet(\log E) = \tilde{\mathcal{H}}^{p+q}$$

degenerates at  $E_1$ . Thus, for  $t \in \Delta^*$ , the Hodge to de Rham spectral sequence for  $X_t$  degenerates at  $E_1$  and the coherent sheaves  $R^q f_* \Omega_{X/\Delta}^p(\log E)$  are locally free.

**Remark 3.2.** It should be noticed that the nilpotent operator  $N$  under Steenbrink's setup is different from the nilpotent operator  $N$  in the abstract setup (that is, the nilpotent operator  $N$  of Schmid [41], Cattani-Kaplan-Schmid [9], and the previous section) by a *negative* sign. The reason is that under the abstract setup, we use the inverse of the monodromy operator to trivialize the local system on the upper half-plane, which leads to a negative sign on the nilpotent operator. For some more elaboration on this difference, we can refer to [20, 6.27, 6.28].

For later use, we describe Steenbrink's weight filtration more precisely here. Let  $E_i$  be irreducible components of  $E$ . For each  $l \geq 1$  and each  $I = (i_1, \dots, i_l)$ , we denote  $E_I$  the intersection  $E_{i_1} \cap \dots \cap E_{i_l}$  and  $E(l)$  the disjoint union of  $E_I$  with  $|I| = l$ . We have natural inclusions  $\rho_j^I : E_I \rightarrow E_J$  where  $J = (i_1, \dots, \hat{i}_j, \dots, i_l)$  and  $\rho_l^k : \bigoplus_{|I|=l} \rho_j^I : E(l) \rightarrow E(l-1)$ .

We denote  $\gamma_l$  the alternating sum of the Gysin morphisms:

$$\gamma_l = \bigoplus_{j=1}^l (-1)^{j-1} (\rho_j^r)^* : H^k(E(l)) \rightarrow H^{k+2}(E(l-1))(1)$$

and  $\theta_l$  the alternating sum of the pullback morphism:

$$\theta_l = \bigoplus_{j=1}^l (-1)^{j-1} (\rho_j^{m+1})^* : H^k(E(l)) \rightarrow H^k(E(l+1)).$$

The  $E_1$  page of the weight spectral sequence is

$$W^{\text{St}} E_1^{-r, d+r} = \bigoplus_{k \geq 0, -r} H^{d-r-2k}(E(2k+r+1))(-r-k)$$

and the differential  $d_1$  corresponds to the map  $-\bigoplus_{k \geq 0, -r} \gamma_{r+2k+1} + \bigoplus_{k \geq 0, -r} \theta_{r+2k+1}$  on  $W^{St} E_1^{-r, d+r}$  (which will be denoted by  $-\gamma + \theta$  for simplicity):

$$\bigoplus_{k \geq 0, -r} H^{d-r-2k}(E(2k+r+1))(-r-k) \rightarrow \bigoplus_{k \geq 0, -r+1} H^{d-r-2k+2}(E(2k+r))(-r-k+1).$$

For all  $r \geq 0$ , note that

$$\begin{aligned} W^{St} E_1^{r, d-r} &= \bigoplus_{k \geq 0, r} H^{d+r-2k}(E(2k-r+1))(r-k) \\ &= \bigoplus_{l \geq 0, -r} H^{d-r-2l}(E(2l+r+1))(-l) = W^{St} E_1^{-r, d+r}(r) \end{aligned}$$

by taking  $l = k - r$ . Let  $I$  denote the identity map  $W^{St} E_1^{-r, d+r} \rightarrow W^{St} E_1^{r, d-r}(-r)$  above. Then the morphism  $N^r$  from  $W^{St} E_2^{-r, d+r} = \text{Gr}_{d+r}^{W^{St}} H^d(X_\infty)$  to  $W^{St} E_2^{r, d-r} = \text{Gr}_{d-r}^{W^{St}} H^d(X_\infty)$  is induced by the map  $(2\pi\sqrt{-1})^r I$ . Thus, to check that  $W^{St} = W(-N, d)$  on  $H^d(X_\infty)$ , it suffices to show that the isomorphism  $(2\pi\sqrt{-1})^r I$  from  $W^{St} E_1^{-r, d+r}$  to  $W^{St} E_1^{r, d-r}$  also induces an isomorphism at the  $E_2$  page for every  $r \geq 0$ .

Now, we turn to the product structure on the limiting mixed Hodge structure. Since the nearby cocycle complex is quasi-isomorphic to the relative log de Rham complex, there is a natural cup product defined on  $H^\bullet(X_\infty)$ , which will be denoted by  $Q(\bullet, \bullet)$ . More precisely, from the product structure on  $\Omega_{X/\Delta}^\bullet(\log(E))$ , we have a natural product morphism

$$\mathbb{R}^i f_* \Omega_{X/\Delta}^\bullet(\log(E)) \otimes \mathbb{R}^j f_* \Omega_{X/\Delta}^\bullet(\log(E)) \rightarrow \mathbb{R}^{i+j} f_* \Omega_{X/\Delta}^\bullet(\log(E)).$$

Also, we have a relative trace morphism

$$\mathbb{R}^{2m} f_* \Omega_{X/\Delta}^\bullet(\log(E)) = R^m f_* \omega_{X/\Delta} \rightarrow \mathcal{O}_\Delta,$$

where  $\omega_{X/\Delta}$  is the relative dualizing sheaf of  $f$ . The cup product

$$Q: H^\bullet(X_\infty, \mathbb{C}) \otimes H^{2m-\bullet}(X_\infty, \mathbb{C}) \rightarrow \mathbb{C}$$

is identified with the fiber over zero of the (relative) cup product

$$\mathbb{R}^\bullet f_* \Omega_{X/\Delta}^\bullet(\log(E)) \otimes \mathbb{R}^{2m-\bullet} f_* \Omega_{X/\Delta}^\bullet(\log(E)) \rightarrow \mathcal{O}_\Delta.$$

**Remark 3.3.** On a  $m$ -dimensional compact complex manifold  $Z$ , the natural trace map in the duality theory is different from the integration map by a sign of  $(-1)^m$ . To be precise, if we let

$$\int_Z : H^{2m}(Z, \mathbb{C}) \rightarrow \mathbb{C}$$

be the integration map on  $Z$ , then the natural trace map in the duality theory is  $(-1)^m \int_Z$  or  $\frac{(-1)^m}{(2\pi\sqrt{-1})^m} \int_Z$  depending on whether we fix a choice of  $\sqrt{-1}$  or not. We refer to [6] and [34] for the absolute and relative duality theory on complex spaces and [11] for the clarification of the signs. In our case, we would normalize the relative trace map so that the restriction of

$$R^m f_* \omega_{X/\Delta} \rightarrow \mathcal{O}_\Delta$$

to the general fiber  $X_t$  is just the integration map

$$\int_{X_t} : H^{2m}(X_t, \mathbb{C}) \rightarrow \mathbb{C}.$$

Since Steenbrink's limiting mixed Hodge complex  $s(A^{\bullet,\bullet})$  doesn't carry a natural product structure, it is not clear from the definition how Steenbrink's weight filtration  $W^{St}$  behaves under the cup product. To study the behavior of  $W^{St}$  under the cup product, Fujisawa [19] constructs another mixed Hodge complex of sheaves  $(K_{\mathbb{Q}}, K_{\mathbb{C}}, W, F)$  which carries a natural product structure and admits a quasi-isomorphism of bigraded complexes

$$(s(C^{\bullet,\bullet}), s(A^{\bullet,\bullet}), W^{St}, F) \rightarrow (K_{\mathbb{Q}}, K_{\mathbb{C}}, W, F).$$

Hence, there is a product morphism

$$\mathbb{H}^i(E, K_{\mathbb{C}}) \otimes \mathbb{H}^j(E, K_{\mathbb{C}}) \rightarrow \mathbb{H}^{i+j}(E, K_{\mathbb{C}}).$$

Fujisawa also constructs a trace morphism

$$\mathrm{Tr} : \mathbb{H}^{2m}(E, K_{\mathbb{C}}) \rightarrow \mathbb{C}$$

and then defines the cup product pairing

$$Q_K : \mathbb{H}^{\bullet}(E, K_{\mathbb{C}}) \otimes \mathbb{H}^{2m-\bullet}(E, K_{\mathbb{C}}) \rightarrow \mathbb{C}$$

in [ibid., Definition 7.11 and Definition 7.13].

It is not stated in [19] that the cup product  $Q_K$  defined in [ibid., Definition 7.13] coincides with the one we describe above. In [21], Fujisawa-Nakayama checked the compatibility of the cup product  $Q_K$  defined in [19] and the cup product on the Kato-Nakayama space  $E_{\infty}^{\log}$  of  $E$ . Using similar tricks, we can check the compatibility of the cup product  $Q_K$  defined in [19] with the cup product  $Q$  described above. We sketch the proof of this compatibility below:

**Lemma 3.4.** *The cup product  $Q_K$  defined in [19] is compatible with the cup product  $Q$  described above.*

*Proof.* It is not hard to see that the product morphisms of  $Q_K$  and  $\Omega_{X/\Delta}^{\bullet}(\log(E)) \otimes \mathcal{O}_E$  are compatible. More precisely, this is explained in the last paragraph of the proof of [21, Proposition 4.6].

The nontrivial part is to show that the trace morphism defined in [19, Definition 7.11] coincides with the one induced by the dualizing sheaf  $\omega_E$  on  $E$ . To see this, let  $E_{\mathrm{sing}}$  be the subspace consisting of the singular points of  $E$  and let  $U = E - E_{\mathrm{sing}}$ . The trick is that since  $H^m(E_{\mathrm{sing}}, \omega_E) = 0$ , there is a surjective morphism

$$H_c^m(U, \omega_E) \twoheadrightarrow H^m(E, \omega_E) \xrightarrow{\mathrm{Tr}} \mathbb{C},$$

where  $H_c^m(U, \omega_E)$  is the cohomology with compact support of  $\omega_E$  on  $U$ . Thus, the trace morphism is determined by the composite  $H_c^m(U, \omega_E) \rightarrow \mathbb{C}$ , which is simply the map induced by integration on  $U$  because the trace morphism in the duality theory is defined by gluing the trace morphisms on a Stein open cover (cf. [6, p.262-263]). Therefore, it suffices to check that the composite of the trace morphism

$$\mathrm{Tr} : \mathbb{H}^{2m}(E, K_{\mathbb{C}}) \rightarrow \mathbb{C}$$

defined in [19, Definition 7.11] with the morphism

$$H_c^m(U, \omega_U) \rightarrow H_c^{2m}(U, \mathbb{C}) \rightarrow H_c^{2m}(U, W_0 K_{\mathbb{C}}) \rightarrow \mathbb{H}_c^{2m}(U, K_{\mathbb{C}}) \rightarrow \mathbb{H}^{2m}(E, K_{\mathbb{C}})$$

becomes integration on  $U$ . This is shown in [21, Proposition 4.7].  $\square$

As a result, by Fujisawa [19], the cup product  $Q$  satisfies the following properties:

**Proposition 3.5.** *Suppose that  $x \in H^d(X_\infty, \mathbb{Q})$  and  $y \in H^{2m-d}(X_\infty, \mathbb{Q})$  and let  $N$  denote the nilpotent operator. Then*

- (1)  $Q(y, x) = (-1)^d Q(x, y)$ .
- (2)  $Q(Nx, y) + Q(x, Ny) = 0$
- (3) For all  $p$ ,  $Q(F^p H^d(X_\infty, \mathbb{C}), F^{m-p+1} H^{2m-d}(X_\infty, \mathbb{C})) = 0$
- (4)  $Q(W_a^{St} H^d(X_\infty, \mathbb{Q}), W_b^{St} H^{2m-d}(X_\infty, \mathbb{Q})) = 0$  for  $a + b \leq 2m - 1$ .

By Proposition 3.5 (4), the cup product induces the morphism

$$Gr_{d+r}^{W^{St}} H^d(X_\infty, \mathbb{Q}) \otimes_{\mathbb{Q}} Gr_{2m-d-r}^{W^{St}} H^{2m-d}(X_\infty, \mathbb{Q}) \rightarrow \mathbb{Q}$$

for each  $d$  and  $r$ , still denoted by  $Q$ . Consider  $\langle \bullet, \bullet \rangle = \epsilon(d - 2m)Q(\bullet, \bullet)$ . To calculate  $\langle \bullet, \bullet \rangle$ , we introduce another linear pairing  $\psi$  as follows: note that  $\langle \bullet, \bullet \rangle$  is a map

$${}_{W^{St}}E_2^{-r, d+r} \otimes_{\mathbb{C}} {}_{W^{St}}E_2^{r, 2m-d-r} \rightarrow \mathbb{Q}.$$

At the  ${}_{W^{St}}E_1$  page, we have

$${}_{W^{St}}E_1^{-r, d+r} = \bigoplus_{k \geq 0, -r} H^{d-r-2k}(E(2k+r+1))(-r-k)$$

and

$${}_{W^{St}}E_1^{r, 2m-d-r} = \bigoplus_{l \geq 0, r} H^{2m-d+r-2l}(E(2l-r+1))(r-l) = \bigoplus_{k \geq 0, -r} H^{2m-d-r-2k}(E(2k+r+1))(-k)$$

by taking  $k = l - r$ . Consider the linear mapping

$$\psi : {}_{W^{St}}E_1^{-r, d+r} \otimes_{\mathbb{C}} {}_{W^{St}}E_1^{r, 2m-d-r} \rightarrow \mathbb{Q}(-m),$$

defined by

$$\psi(\eta, \xi) = \frac{\epsilon(r+d-2m)}{(2\pi\sqrt{-1})^{m-2k-r}} \int_{E(2k+r+1)} \eta \wedge \xi$$

for  $\eta \in H^{d-r-2k}(E(2k+r+1))(-r-k)$  and  $\xi \in H^{2m-d-r-2k}(E(2k+r+1))(-k)$  and

$$\psi(H^{d-r-2k_1}(E(2k_1+r+1))(-r-k_1), H^{2m-d-r-2k_2}(E(2k_2+r+1))(-k_2)) = 0$$

for  $k_1 \neq k_2$ . Then Fujisawa [19] shows that  $\langle \bullet, \bullet \rangle$  can be calculated in the following way:

**Proposition 3.6.** *We have*

$$\langle \tilde{\eta}, \tilde{\xi} \rangle = (2\pi\sqrt{-1})^m \psi(\eta, \xi),$$

where  $\eta, \xi$  are any representative of  $\tilde{\eta}, \tilde{\xi}$  at the  ${}_{W^{St}}E_1$  page.

*Proof of Proposition 3.5 and Proposition 3.6.* The statements (1), (2), (3), (4) in Proposition 3.5 are proved in [19, §7]. Proposition 3.6 follows from [19, Lemma 6.13] and the computation in the proof of [ibid., Theorem 8.11]  $\square$

**Remark 3.7.** The linear pairing  $\psi$  is first defined by Guillén-Navarro Aznar [22, (3.4)] to construct the polarized Hodge-Lefschetz module structure under some strong Kähler type condition on the central fiber (or see [33, §11.3.2]). Fujisawa [19] studies the behavior of the weight filtration under the cup product structure and shows that such a linear pairing  $\psi$  comes geometrically from the cup product structure on the limiting mixed Hodge structure.

**Lemma 3.8.** (cf. [16, Lemma 1.4]) *Let  $X$  be a compact complex manifold of dimension  $m$  for which the Hodge-de Rham spectral sequence degenerates at  $E_1$ . Then, the middle cohomology  $H^m(X, \mathbb{C})$  satisfies the first Hodge-Riemann condition with respect to the cup product.*

Now we can prove Theorem 1.1:

*Proof of Theorem 1.1.* By Theorem 3.1, there are holomorphic subbundles  $\tilde{\mathcal{F}}^\bullet$  of  $\tilde{\mathcal{H}}^d$  that correspond to the Hodge subbundles  $\mathcal{F}^\bullet$  of  $\mathcal{H}^d$  under Deligne's canonical extension. As a result, by taking  $S(\bullet, \bullet) = \epsilon(m)Q(\bullet, \bullet)$  and using Theorem 2.2, Proposition 3.5 and Lemma 3.8, Theorem 1.1 follows.  $\square$

As a consequence, if we assume some strong Kähler type condition on the central fiber, then the general fiber satisfies the  $\partial\bar{\partial}$ -lemma, and the Hodge index on the middle cohomology of general fiber is the same as compact Kähler manifolds:

**Corollary 3.9.** *Let  $f : X \rightarrow \Delta$  be a semistable degeneration of  $m$ -dimensional compact complex manifolds. Suppose that there exists a class  $L \in H^2(X_0, \mathbb{R})$  which restricts to a Kähler class on each component of the normal crossing complex analytic space  $X_0$ . Then the general fiber  $X_q$  satisfies the  $\partial\bar{\partial}$ -lemma for  $|q|$  sufficiently small. Moreover, if we let  $h_q^{a,b} = \dim_{\mathbb{C}} H^b(X_q, \Omega_{X_q}^a)$ , then the signature of  $S(C\bullet, \bar{\bullet})$  on the  $(p, m-p)$ -component of the Hodge structure on  $H^m(X_q, \mathbb{C})$  is*

$$\left( \sum_{r \geq 0} (h_q^{p-2r, m-p-2r} - h_q^{p-2r-1, m-p-2r-1}), \sum_{r \geq 0} (h_q^{p-2r-1, m-p-2r-1} - h_q^{p-2r-2, m-p-2r-2}) \right)$$

for  $|q|$  small.

*In particular, when  $m$  is even, the general fiber  $X_q$  satisfies the Hodge index theorem for compact Kähler manifolds: the signature of the cup product  $Q(\bullet, \bullet)$  on  $H^m(X_q, \mathbb{R})$  is  $\sum_{a,b \geq 0} (-1)^a h_q^{a,b}$  for  $|q|$  small.*

*Proof.* By [22] (or cf. [33, §11.3.2]), under the assumption that there exists a class  $L \in H^2(X_0, \mathbb{R})$  which restricts to a Kähler class on each component of the normal crossing complex analytic space  $X_0$ , there is a polarized Hodge-Lefschetz module structure on  ${}_{Wst}E_1^{-r, q+r} \otimes \mathbb{R}$ . As a consequence, we have  $W^{St} = W(N, d)$  on  $H^d(X_\infty, \mathbb{Q})$  for all  $d \geq 0$  (cf. [33, Theorem 11.40]). Hence, by Theorem 1.1,  $X_q$  satisfies the  $\partial\bar{\partial}$ -lemma for  $|q|$  small.

On the other hand, Proposition 3.6 asserts that the cup product on  $H^m(X_\infty)$  restricts to the pairing  $\psi$  on grading pieces, which coincides with the one used in [22] to define the polarized Hodge-Lefschetz module structure on  ${}_{Wst}E_1^{-r, d+r} \otimes \mathbb{R}$ . So, we can compute the Hodge index via the Hodge-Lefschetz decomposition. To be precise, for  $p+s \geq d$ , let  $P_d^{p,s}$  be the  $(p, s)$ -component of the Hodge structure of the  $N$ -primitive space

$$\ker \left( N^{p+s-d+1} : \mathrm{Gr}_{p+s}^{W^{St}} H^d(X_\infty, \mathbb{C}) \rightarrow \mathrm{Gr}_{2d-p-s-2}^{W^{St}} H^d(X_\infty, \mathbb{C}) \right),$$

and for  $d \leq m$ , we define the  $L$ -primitive subspace of  $P_d^{p,s}$  by

$$(P_d^{p,s})_0 := P_d^{p,s} \cap \ker(L^{m-d+1}).$$

Then  $P_m^{p,s}$  has a Lefschetz decomposition:

$$P_m^{p,s} = \bigoplus_{r \geq 0} L^r (P_{m-2r}^{p-r,s-r})_0.$$

We claim that the Hermitian form  $S(C\bullet, (-N)^{p+s-m}\bar{\bullet})$  on  $L^r (P_{m-2r}^{p-r,s-r})_0$  is positive definite if  $r$  is even and is negative definite if  $r$  is odd. To see this, let  $x \in (P_{m-2r}^{p-r,s-r})_0$ , and then we have

$$\begin{aligned} & (-1)^r S(CL^r x, (-N)^{p+s-m} L^r \bar{x}) \\ &= (-1)^r (-1)^m \langle CL^r x, (-N)^{p+s-m} L^r \bar{x} \rangle \\ &= (-1)^r (-1)^m (2\pi\sqrt{-1})^m \psi(CL^r x, (-N)^{p+s-m} L^r \bar{x}) \\ &= (-1)^r (-1)^m (2\pi\sqrt{-1})^m (-1)^{p+s} \psi(L^r x, C(-N)^{p+s-m} L^r \bar{x}) \\ &= (-1)^r (-1)^m (2\pi\sqrt{-1})^m (-1)^{p+s} (-1)^r \psi(x, L^r C(-N)^{p+s-m} L^r \bar{x}) \\ &= (2\pi\sqrt{-1})^m \psi(x, CN^{p+s-m} L^{2r} \bar{x}) > 0. \end{aligned}$$

The last inequality follows from the definition of polarized Hodge-Lefschetz module in [22, (4.3)].

As a result, the signature of  $S(C\bullet, (-N)^{p+s-m}\bar{\bullet})$  on  $P_m^{p,s}$  is

$$\left( \sum_{r \geq 0} \dim_{\mathbb{C}}(P_{m-4r}^{p-2r,s-2r})_0, \sum_{r \geq 0} \dim_{\mathbb{C}}(P_{m-4r-2}^{p-2r-1,s-2r-1})_0 \right).$$

Then by Theorem 1.1, for  $|q|$  small, the signature of  $S(C\bullet, \bar{\bullet})$  on the  $(p, m-p)$ -component of the Hodge structure on  $H^m(X_q, \mathbb{C})$  is

$$\left( \sum_{\substack{l \geq p-m, \\ r \geq (-l)_+}} \sum_{s \geq 0} \dim_{\mathbb{C}}(P_{m-4s}^{p+r-2s, m+l-p+r-2s})_0, \sum_{\substack{l \geq p-m, \\ r \geq (-l)_+}} \sum_{s \geq 0} \dim_{\mathbb{C}}(P_{m-4s-2}^{p+r-2s-1, m+l-p+r-2s-1})_0 \right).$$

Since for all  $a \geq 0$  and  $l \geq p-m$ , we have the decomposition

$$\mathrm{Gr}_F^{p-a} \mathrm{Gr}_{m+l-2a}^{W^{St}} H^{m-2a}(X_\infty, \mathbb{C}) = \bigoplus_{r \geq (-l)_+} N^r P_{m-2a}^{p+r-a, m+l-p+r-a},$$

we can see that

$$\begin{aligned} & \sum_{\substack{l \geq p-m, \\ r \geq (-l)_+}} \dim_{\mathbb{C}}(P_{m-2a}^{p+r-a, m+l-p+r-a})_0 \\ &= \sum_{l \geq p-m} \dim_{\mathbb{C}} \mathrm{Gr}_F^{p-a} \left( \mathrm{Gr}_{m+l-2a}^{W^{St}} H^{m-2a}(X_\infty, \mathbb{C}) \right)_0 \\ &= \dim_{\mathbb{C}} \mathrm{Gr}_F^{p-a} H^{m-2a}(X_\infty, \mathbb{C}) - \dim_{\mathbb{C}} \mathrm{Gr}_F^{p-a-1} H^{m-2a-2}(X_\infty, \mathbb{C}) \\ &= h_q^{p-a, m-p-a} - h_q^{p-a-1, m-p-a-1} \end{aligned}$$

for  $|q|$  small. Hence, by summing, the signature of  $S(C\bullet, \bar{\bullet})$  on the  $(p, m-p)$ -component of the Hodge structure on  $H^m(X_q, \mathbb{C})$  is

$$\left( \sum_{r \geq 0} (h_q^{p-2r, m-p-2r} - h_q^{p-2r-1, m-p-2r-1}), \sum_{r \geq 0} (h_q^{p-2r-1, m-p-2r-1} - h_q^{p-2r-2, m-p-2r-2}) \right)$$

for  $|q|$  small. The last statement in the theorem is a standard consequence of the signature  $S(C\bullet, \bar{\bullet})$  on the middle cohomology we just computed. The only thing to

notice is that, although we don't have the Hard Lefschetz theorem on the general fiber  $X_q$ , we still have  $h_q^{p,r} = h_q^{m-p,m-r}$  for  $|q|$  small since

$$h_q^{p,r} = \dim_{\mathbb{C}} \mathrm{Gr}_F^p H^{p+r}(X_{\infty}, \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Gr}_F^{m-p} H^{2m-p-r}(X_{\infty}, \mathbb{C}) = h_q^{m-p,m-r}$$

for  $|q|$  small by the existence of  $L$  on the central fiber. □

## 4 Examples

### 4.1 A non-example for degeneration of non-Kähler surfaces

In [28], Kodaira constructs an example of the degeneration of Hopf surfaces into a rationally ruled surface with an ordinary double curve. To be precise, for all  $m \geq 1$ , there exists a one-parameter complex analytic family  $X \rightarrow \Delta$  such that  $X_t$  is a Hopf surface for  $0 \neq t \in \Delta$  and  $X_0$  can be identified with the quotient of a Hirzebruch surface  $F_m$  by gluing the zero section and the infinity section together in the normal crossing way, from which the gluing becomes an ordinary double curve in  $X_0$ . In this case,  $H^1(X_t)$  is one dimensional and hence doesn't have Hodge decomposition. We can check that Steenbrink weight filtration  $W^{St}$  and the monodromy weight filtration  $W(N, 1)$  are different on  $H^1(X_{\infty}, \mathbb{C})$ , which gives a non-example to Theorem 1.1 (1).

To see this, we first describe the semistable model of  $X \rightarrow \Delta$ . Let  $Y$  be the blowup of  $X$  along the ordinary double curve. Consider the double covering of  $\Delta$  defined by  $q \mapsto q^2$ , and let  $\tilde{Y} \rightarrow \Delta$  be the pullback family. Then  $\tilde{Y}_0$  is normal crossings, all of whose components are reduced.  $\tilde{Y}_0$  can also be described as follows: Consider two copies  $S_1, S_2$  of the Hirzebruch surface  $F_m$  and let  $(D_0)_i$  and  $(D_{\infty})_i$  be the zero section and the infinity section of  $S_i$  for  $i = 1, 2$ . Here, we have  $(D_0)_i^2 = -m$  and  $(D_{\infty})_i^2 = m$ . Then

$$\tilde{Y}_0 \cong S_1 \sqcup S_2 / \sim,$$

where the equivalence is by identifying  $(D_0)_1$  with  $(D_{\infty})_2$  and  $(D_0)_2$  with  $(D_{\infty})_1$  via some automorphism of  $\mathbb{P}^1$ . We will denote  $D_1 = (D_0)_1 = (D_{\infty})_2$  and  $D_2 = (D_0)_2 = (D_{\infty})_1$  as curves in  $\tilde{Y}_0$ .

Now we can compute Steenbrink's limiting mixed Hodge structure associated to the family  $\tilde{Y} \rightarrow \Delta$ . Note that  ${}_{W_{St}}E_1^{-r,d+r} = 0$  for  $|r| \geq 2$  and any  $d$ . Also, we have

$$Gr_0^{W^{St}} H_{\lim}^1 = {}_{W_{St}}E_2^{1,0} = \mathrm{coker} \left( H^0(S_1) \oplus H^0(S_2) \xrightarrow{\theta} H^0(D_1) \oplus H^0(D_2) \right) \cong \mathbb{Q},$$

$$Gr_1^{W^{St}} H_{\lim}^1 = {}_{W_{St}}E_2^{0,1} = \mathrm{ker} \left( H^1(S_1) \oplus H^1(S_2) \xrightarrow{\theta} H^1(D_1) \oplus H^1(D_2) \right) = 0,$$

and

$$Gr_2^{W^{St}} H_{\lim}^1 = {}_{W_{St}}E_2^{-1,2} = \mathrm{ker} \left( H^0(D_1)(-1) \oplus H^0(D_2)(-1) \xrightarrow{-\gamma} H^2(S_1) \oplus H^2(S_2) \right) = 0.$$

Thus  $W^{St}$  is different from the monodromy weight filtration  $W(N, 1)$  on  $H_{\lim}^1$ .

## 4.2 Ordinary Double Points

In this section, we consider the global smoothing of a  $m$ -dimensional compact complex analytic space  $X_0$  with at most ordinary double points. In this case, we assume that  $X_0$  admits a resolution of singularities that satisfies the  $\partial\bar{\partial}$ -lemma. This condition is equivalent to that every resolution of singularities of  $X_0$  satisfies the  $\partial\bar{\partial}$ -lemma since  $X_0$  has only isolated singularities. Let  $\{x_i | 1 \leq i \leq l\}$  be the set of ordinary double points on  $X_0$  and  $\tilde{X}_0$  be the blowup of  $X_0$  at all ordinary double points. The exceptional divisors  $Q_i$  in  $\tilde{X}_0$  over ordinary points  $x_i$  are  $m-1$ -dimensional quadrics. If  $m$  is odd, each  $Q_i$  is an even-dimensional quadric with standard  $\frac{m-1}{2}$ -dimensional planes whose homology classes will be denoted by  $A_i$  and  $B_i$ . Define the number

$$R = \dim_{\mathbb{C}} \ker \left( \bigoplus_{i=1}^l H_{\text{prim}}^{m-1}(Q_i) \xrightarrow{-\gamma} H^{m+1}(\tilde{X}_0) \right),$$

where  $\gamma$  is the Gysin morphism. The number  $R$  represents the number of linear relations of  $A_i - B_i$  in  $H_{m-1}(\tilde{X}_0)$ . If  $m$  is even, consider the kernel of the restriction map

$$V^m := \ker \left( H^m(\tilde{X}_0) \rightarrow \bigoplus_{i=1}^l H^m(Q_i) \right),$$

which has dimension  $h^m(\tilde{X}_0) - l$  if  $X_0$  admits a global smoothing.

As a generalization of [16] and [31] for the existence of polarized Hodge structure on the middle cohomology of Clemens threefolds, we have the following theorem:

**Theorem 4.1.** *Let  $f : X \rightarrow \Delta$  be a one-parameter degeneration of  $m$ -dimensional compact complex manifolds such that the central fiber  $X_0$  has at worst ordinary double points. Suppose that  $X_0$  admits a resolution of singularities that satisfies the  $\partial\bar{\partial}$ -lemma. Then the nearby fiber  $X_q$  satisfies the  $\partial\bar{\partial}$ -lemma for  $|q|$  small.*

Let  $(h_+^{k,m-k}(\tilde{X}_0), h_-^{k,m-k}(\tilde{X}_0))$  be the signature of  $S(C\bullet, \bar{\bullet})$  on  $H^{k,m-k}(\tilde{X}_0)$ . When  $m$  is even, let  $(\hat{h}_+^{m/2,m/2}(\tilde{X}_0), \hat{h}_-^{m/2,m/2}(\tilde{X}_0))$  be the signature of  $S(C\bullet, \bar{\bullet})$  on the  $(m/2, m/2)$  part of  $V^m$ .

(1) *If  $m$  is odd, then the signature of  $S(C\bullet, \bar{\bullet})$  on  $H^{k,m-k}(X_q)$  is*

$$\begin{cases} (h_+^{k,m-k}(\tilde{X}_0), h_-^{k,m-k}(\tilde{X}_0)), & k \neq \frac{m+1}{2}, \frac{m-1}{2} \\ (h_+^{k,m-k}(\tilde{X}_0) + R, h_-^{k,m-k}(\tilde{X}_0)), & k = \frac{m+1}{2} \text{ or } \frac{m-1}{2} \end{cases}$$

for  $|q|$  sufficiently small.

(2) *If  $m$  is even, then the signature of  $S(C\bullet, \bar{\bullet})$  on  $H^{k,m-k}(X_q)$  is*

$$\begin{cases} (h_+^{k,m-k}(\tilde{X}_0), h_-^{k,m-k}(\tilde{X}_0)), & k \neq \frac{m}{2} \\ (\hat{h}_+^{m/2,m/2}(\tilde{X}_0) + l, \hat{h}_-^{m/2,m/2}(\tilde{X}_0)), & k = \frac{m}{2} \end{cases}$$

for  $|q|$  sufficiently small.

In particular, if  $H^m(\tilde{X}_0, \mathbb{C})$  together with the cup product  $S(\bullet, \bullet)$  is a polarized Hodge structure, then  $H^m(X_q, \mathbb{C})$  together with the cup product  $S(\bullet, \bullet)$  is a polarized Hodge structure as well for  $|q|$  sufficiently small.

*Proof.* The computation below is similar to the proof of Theorem 2.12 in [16] for Calabi-Yau threefolds with ODPs. We first describe a semistable model of the family. Let  $\{x_i | 1 \leq i \leq l\}$  be the set of ordinary double points on  $X_0$  and let  $Y$  be the blowup of  $X$  along  $\{x_i | 1 \leq i \leq l\}$ . Consider the double covering of  $\Delta$  defined by  $q \mapsto q^2$ , and let  $\tilde{Y} \rightarrow \Delta$  be the pullback family. Then  $\tilde{Y}_0$  is normal crossings, all of whose components are reduced.  $\tilde{Y}_0$  can also be described as follows: Let  $\tilde{X}_0$  be the blowup of  $X_0$  along the ordinary double points  $\{x_i\}$ . Then  $\tilde{X}_0$  satisfies the  $\partial\bar{\partial}$ -lemma by our assumption. The preimage of each  $x_i$  in  $\tilde{X}_0$  is a quadric  $(m-1)$ -fold  $Q_i$  in  $\mathbb{P}^m$ . Let  $E_i$  be a quadric  $m$ -fold in  $\mathbb{P}^{m+1}$ . Then

$$\tilde{Y}_0 \cong \tilde{X}_0 \sqcup \coprod_{i \in I} E_i / \sim,$$

where the equivalence is by identifying each  $Q_i$  with some hyperplane section in  $E_i$ .

Consider Steenbrink's limiting Hodge structure associated to the family  $\tilde{Y} \rightarrow \Delta$ . By Theorem 2.2, to show that  $X_q$  satisfies the  $\partial\bar{\partial}$ -lemma, we need to prove that  $W^{St}$  coincides with  $W(N, d)$  on the cohomology  $H^d(X_\infty)$  for all  $d$ .

Note that  ${}_{WSt}E_1^{-r, d+r} = 0$  for  $|r| \geq 2$  and any  $d$ . Thus, by our discussion in the previous section, to show that  $W^{St} = W(N, d)$  on  $H^d(X_\infty)$ , it suffices to show that

$${}_{WSt}E_2^{-1, d+1} = \ker \left( \bigoplus_{1 \leq i \leq l} H^{d-1}(Q_i)(-1) \xrightarrow{-\gamma} H^{d+1}(\tilde{X}_0) \oplus \bigoplus_{1 \leq i \leq l} H^{d+1}(E_i) \right)$$

is isomorphic via the map induced by  $(2\pi\sqrt{-1})I$  to

$${}_{WSt}E_2^{1, d-1} = \text{coker} \left( H^{d-1}(\tilde{X}_0) \oplus \bigoplus_{1 \leq i \leq l} H^{d-1}(E_i) \xrightarrow{\theta} \bigoplus_{1 \leq i \leq l} H^{d-1}(Q_i) \right).$$

By the Lefschetz hyperplane theorem, the Gysin morphism  $H^{d-1}(Q_i)(-1) \rightarrow H^{d+1}(E_i)$  is injective unless  $d = m$  and the restriction map  $H^{d+1}(E_i) \rightarrow H^{d+1}(Q_i)$  is surjective unless  $d = m$ . Therefore, the assertion is obvious for  $d \neq m$ .

Now, we consider the case  $d = m$ . If  $m$  is even, then each  $Q_i$  is an odd-dimensional quadric, and hence the middle cohomology  $H^{m-1}(Q_i)$  is zero. So, the assertion is trivial. If  $m$  is odd, then  $Q_i$  is an even-dimensional quadric with standard  $\frac{m-1}{2}$ -dimensional planes whose homology classes are denoted by  $A_i$  and  $B_i$ . Note that the restriction map  $H^{m-1}(E_i) \xrightarrow{\theta} H^{m-1}(Q_i)$  sends the  $(\frac{m-1}{2})$ -th intersection of the hyperplane class  $[Q_i]$  to  $A_i + B_i$  and the kernel of the Gysin map  $H^{m-1}(Q_i)(-1) \xrightarrow{-\gamma} H^{m+1}(E_i)$  consists of multiples of the class  $A_i - B_i$ . Thus we may identify  ${}_{WSt}E_2^{-1, m+1}$  as

$$\left\{ (m_1, \dots, m_l) \in \mathbb{Q}(-1)^l \mid \sum_{i=1}^l m_i(A_i - B_i) = 0 \text{ in } H^{m+1}(\tilde{X}_0) \right\}$$

and  ${}_{WSt}E_2^{1, m-1}$  as the quotient of  $\mathbb{Q}^l$  by the subspace

$$\left\{ (\xi \cdot (A_1 - B_1), \dots, \xi \cdot (A_l - B_l)) \mid \xi \in H^{m-1}(\tilde{X}_0) \right\}.$$

The two subspaces are orthogonal under the standard inner product, so the map  ${}_{WSt}E_2^{-1, m+1} \rightarrow {}_{WSt}E_2^{1, m-1}$  is injective. Note also that  ${}_{WSt}E_2^{-1, m+1}$  and  ${}_{WSt}E_2^{1, m-1}$  are dual vector spaces since the Gysin map can be identified with the pushforward morphism of homology, which is dual to the pullback morphism of cohomology. In particular, they

have the same dimension, and the surjectivity also follows. In conclusion, we have shown that  ${}^{St}W = W(N, m)$  on the middle cohomology  $H^m(X_\infty)$ , and hence by Theorem 1.1, the middle cohomology of the nearby fiber  $H^m(X_q)$  admits Hodge decomposition for  $|q|$  sufficiently small.

To compute the Hodge index on the middle cohomology of the general fiber, by Theorem 1.1, it suffices to compute the signature of  $S(C\bullet, (-N)^l\bullet)$  on the grading pieces  $Gr_{m+l}^{W^{St}} H_{\text{lim}}^m$  of the limiting mixed Hodge structure on the middle cohomology for each  $l$ . This can be computed explicitly using Proposition 3.6. First, suppose that  $m$  is odd. Then

$$Gr_m^{W^{St}} H_{\text{lim}}^m = {}_{W^{St}}E_2^{0,m} = \frac{\ker\left(H^m(\tilde{X}_0) \oplus \bigoplus_{i=1}^l H^m(E_i) \xrightarrow{\theta} \bigoplus_{i=1}^l H^m(Q_i)\right)}{\text{im}\left(\bigoplus_{i=1}^l H^{m-2}(Q_i)(-1) \xrightarrow{-\gamma} H^m(\tilde{X}_0) \oplus \bigoplus_{i=1}^l H^m(E_i)\right)},$$

which is just  $H^m(\tilde{X}_0)$  since  $m$  is odd. Proposition 3.6 implies that the Hermitian form  $S(C\bullet, \bar{\bullet})$  on  $Gr_m^{W^{St}} H^m(X_\infty)$  restricts to the Hermitian form  $S(C\bullet, \bar{\bullet})$  on  $H^m(\tilde{X}_0)$ , which has signature  $(h_+^{k,m-k}(\tilde{X}_0), h_-^{k,m-k}(\tilde{X}_0))$  on  $H^{k,m-k}(\tilde{X}_0)$  by our assumption. On the other hand, let  $\eta = \sum_{i=1}^l \frac{m_i}{2\pi\sqrt{-1}}(A_i - B_i)$  be an element of  ${}_{W^{St}}E_2^{-1,m+1}$ , by Proposition 3.6 again, we have

$$\begin{aligned} (\sqrt{-1})^{\frac{m+1}{2} - \frac{m+1}{2}} S(\eta, -N\eta) &= \epsilon(m)Q(\eta, -N\eta) \\ &= (-1)\epsilon(m)\epsilon(-m)\epsilon(1-m) \sum_{i=1}^l m_i^2 \langle A_i - B_i, A_i - B_i \rangle_{H^{m-1}(Q_i)} \\ &= \sum_{i=1}^l m_i^2 \epsilon(m-1) \langle A_i - B_i, A_i - B_i \rangle_{H^{m-1}(Q_i)} > 0. \end{aligned}$$

Here,  $\langle \bullet, \bullet \rangle_{H^{m-1}(Q_i)}$  denotes the cup product on  $H^{m-1}(Q_i)$  and the last inequality follows from the fact that  $A_i - B_i$  is a primitive class in  $H^{m-1}(Q_i)$ . As a result, the signature of  $S(C\bullet, (-N)\bullet)$  on  $Gr_{F_{\text{lim}}}^k Gr_{m+1}^{W^{St}} H_{\text{lim}}^m$  is  $(R, 0)$  for  $k = \frac{m+1}{2}$  and is  $(0, 0)$  for the other  $k$ . Then by Theorem 1.1, the signature of  $S(C\bullet, \bar{\bullet})$  on  $H^m(X_q)$  is

$$\begin{cases} \left( h_+^{k,m-k}(\tilde{X}_0), h_-^{k,m-k}(\tilde{X}_0) \right), & k \neq \frac{m+1}{2}, \frac{m-1}{2}, \\ \left( h_+^{k,m-k}(\tilde{X}_0) + R, h_-^{k,m-k}(\tilde{X}_0) \right), & k = \frac{m+1}{2} \text{ or } \frac{m-1}{2} \end{cases}$$

for  $|q|$  sufficiently small.

The positivity of  $S(\eta, -N\eta)$  above also follows from the Picard Lefschetz formula. In fact, regard  $\eta$  as an element of  $H^m(X_\infty, \mathbb{Q})$ , if we let  $\delta_i$  be the vanishing cycle associated to each ODP  $x_i$ , then we must have

$$T(\eta) = \eta + 2 \sum_{i=1}^l \epsilon(m+2) \langle \eta, \delta_i \rangle_{H^m(X_t)} \delta_i,$$

where the multiplicity 2 reflects the 2 to 1 base change in the construction of the semistable family. So combined with the fact that  $N\eta \neq 0$ , we have

$$\epsilon(m)Q(\eta, -N\eta) = -\epsilon(m)\epsilon(m+2) \sum_{i=1}^l 2(\langle \eta, \delta_i \rangle_{H^m(X_q)})^2 > 0.$$

Next, we discuss the case of  $m$  being even. In this case, we still have

$$Gr_m^{W^{St}} H_{\lim}^m = {}_{W^{St}}E_2^{0,m} = \frac{\ker\left(H^m(\tilde{X}_0) \oplus \bigoplus_{i=1}^l H^m(E_i) \xrightarrow{\theta} \bigoplus_{i=1}^l H^m(Q_i)\right)}{\operatorname{im}\left(\bigoplus_{i=1}^l H^{m-2}(Q_i)(-1) \xrightarrow{-\gamma} H^m(\tilde{X}_0) \oplus \bigoplus_{i=1}^l H^m(E_i)\right)}.$$

It suffices to show that every class of  ${}_{W^{St}}E_2^{0,m}$  has a representative in  $V^m \oplus H^m(E_i)_{\text{prim}}$ . To see this, if we let  $A_i$  and  $B_i$  be the homology class of the standard  $\frac{m}{2}$ -dimensional planes on  $E_i$ , then the image of the Gysin morphism  $H^{m-2}(Q_i) \rightarrow H^m(E_i)$  is generated by  $A_i + B_i$ , and the primitive part  $H^m(E_i)_{\text{prim}}$  is generated by  $A_i - B_i$ . So by modulo the image of Gysin morphism, each class in  ${}_{W^{St}}E_2^{0,m}$  has a representative in  $V^m \oplus H^m(E_i)_{\text{prim}}$ . This finishes the proof.  $\square$

**Remark 4.2.** We can see that the argument above doesn't rely on any topological constraints in deformation theory to ensure the existence of smoothing. For Calabi-Yau threefolds with ODPs, by the work of Clemens, Friedman [15], Tian [44], Kawamata [26], Ran [35], a necessary and sufficient condition for the existence of smoothing is that there must be some linear relation  $\sum_{i=1}^l m_i(A_i - B_i) = 0$  in  $H^2(\tilde{X}_0)$  with  $m_i \neq 0$  for all  $i$ . In higher dimension, Rollenske-Thomas [37] shows that a necessary condition for an odd-dimensional Calabi-Yau with ODPs to have a smoothing is that there must be some non-linear relation between  $A_i - B_i$  in  $H^{m-1}(\tilde{X}_0)$ . It is possible that for more complicated singularities than ODPs, the topological constraints in deformation theory should come in to ensure that the middle limiting cohomology is endowed with a polarized mixed Hodge structure. (See the next example for some elaboration.)

### 4.3 $O_{16}$ Singularities (Cones over Cubic Surfaces)

In this section, we consider Calabi-Yau threefolds with an  $O_{16}$  singularity. Before we state our result, we review some general deformation theory of Calabi-Yau threefolds with isolated rational singularities. The general reference is [18, Section 5].

In the following, let  $X$  be a canonical Calabi-Yau threefold in the sense of [18, Definition 5.1] such that  $H^1(X, \mathcal{O}_X) = 0$ . Let  $Z$  be the singular locus of  $X$  and  $\pi: \hat{X} \rightarrow X$  a good equivariant resolution of  $X$  so that  $E = \pi^{-1}(Z)$  is a divisor with simple normal crossing. For each  $x \in Z$ , let  $E_x = \pi^{-1}(x)$  and  $\hat{X}_x$  be the germ of  $\hat{X}$  around  $E_x$ . Consider the following spaces:

$$S := \bigoplus_{x \in Z} \ker(H_{E_x}^4(\hat{X}_x) \rightarrow H^4(E_x)),$$

$$T := \ker(S \rightarrow H^4(\hat{X})),$$

$$K' := \bigoplus_{x \in Z} \ker(H_{E_x}^2(\Omega_{\hat{X}_x}^2) \rightarrow H_{E_x}^4(\hat{X}_x)),$$

$$K := \ker(H_E^2(\hat{X}, \Omega_{\hat{X}}^2) \rightarrow H^0(R^2\pi_*\Omega_{\hat{X}}^2)) \cong K' \oplus S.$$

Then by [18, (5.1) and Lemma 5.6], there exists a commutative diagram of exact sequences

$$\begin{array}{ccccc}
\mathbb{T}_X^1 & \longrightarrow & K' \oplus T & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
H^0(X, T_X^1) & \longrightarrow & K & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
H^2(X, T_X^0) & \longrightarrow & H^2(\widehat{X}, \Omega_{\widehat{X}}^2) & \longrightarrow & H^0(R^2\pi_*\Omega_{\widehat{X}}^2)
\end{array}$$

We define the *defect*  $\sigma(X)$  of  $X$  to be  $\dim_{\mathbb{C}} \text{im}(S \rightarrow H^4(\widehat{X})) = \dim_{\mathbb{C}} S - \dim_{\mathbb{C}} T$ . By the above diagram, we can see that if the restriction of global deformations  $\mathbb{T}_X^1$  to local deformations  $H^0(X, T_X^1)$  is surjective, then  $T \cong S$ , or equivalently  $\sigma(X) = 0$ .

**Remark 4.3.** The invariant  $\sigma(X)$  was introduced by Kawamata [26, p.27] for a normal projective variety  $X$  to measure the failure of  $\mathbb{Q}$ -factoriality on  $X$ . If  $X$  is a compact complex threefold with only isolated rational singularities, then the defect  $\sigma(X)$  also measures the failure of Poincaré duality on  $X$  (cf. [32, Section 3] and [17, Remark 2.2]).

Now, if  $X$  is a quintic threefold with an  $O_{16}$  singularity, it is easy to see that any deformation of the  $O_{16}$  singularity can be extended to a global embedded deformation of  $X$ , and hence  $\sigma(X) = 0$ . As a result, it is natural to expect that a Calabi-Yau threefold with an  $O_{16}$  singularity and with "good deformation theory" has no defect. Then we have the following theorem:

**Theorem 4.4.** *Let  $f : X \rightarrow \Delta$  be a one-parameter degeneration of 3-dimensional compact complex manifolds such that the central fiber  $X_0$  is a generalized Calabi-Yau threefold with an  $O_{16}$  singularity. Suppose, moreover, that  $X_0$  admits a Kähler resolution of singularities and the defect  $\sigma(X_0)$  is zero. Then the nearby fiber  $X_q$  satisfies the  $\partial\bar{\partial}$ -lemma for  $|q|$  small.*

*Moreover, let  $\tilde{X}_0$  be the blowup of  $X_0$  at the  $O_{16}$  singularity. If  $H^3(\tilde{X}_0, \mathbb{C})$  together with the cup product  $S(\bullet, \bullet)$  is a polarized Hodge structure, then  $H^3(X_q, \mathbb{C})$  together with the cup product  $S(\bullet, \bullet)$  is a polarized Hodge structure for  $|q|$  sufficiently small.*

*Proof.* We describe a semistable model of the family first. Let  $Y$  be the blowup of  $X$  along the  $O_{16}$  singularity. Consider the 3 to 1 covering of  $\Delta$  defined by  $q \mapsto q^3$ , and let  $\tilde{Y} \rightarrow \Delta$  be the pullback family. Then  $\tilde{Y}_0$  is normal crossings, all of whose components are reduced. We can also describe  $\tilde{Y}_0$  as following: let  $\tilde{X}_0$  be the blowup of  $X_0$  along the  $O_{16}$  singularity. Then  $\tilde{X}_0$  is a compact complex manifold of Fujiki class  $\mathcal{C}$ , and hence satisfies the  $\partial\bar{\partial}$ -lemma. The preimage of the  $O_{16}$  singularity in  $\tilde{X}_0$  is a cubic surface  $Q$  in  $\mathbb{P}^3$ . Let  $E$  be a cubic 3-fold in  $\mathbb{P}^4$ . Then

$$\tilde{Y}_0 \cong \tilde{X}_0 \sqcup E / \sim,$$

where the equivalence is by identifying  $Q$  with some hyperplane section in  $E$ .

Consider Steenbrink's limiting Hodge structure associated to the family  $\tilde{Y} \rightarrow \Delta$ . As in the previous example, the mixed Hodge structure on  $H^d(X_\infty)$  is pure unless  $d = 3$  by the Lefschetz hyperplane theorem. Thus it suffice to show that  $W^{St} = W(N, 3)$  on middle cohomology  $H^3(X_\infty)$ , or equivalently

$$W^{St} E_2^{-1,4} = \ker \left( H^2(Q)(-1) \xrightarrow{-\gamma} H^4(\tilde{X}_0) \oplus H^4(E) \right)$$

is isomorphic via the map induced by  $(2\pi\sqrt{-1})I$  to

$${}_{Wst}E_2^{1,2} = \text{coker} \left( H^2(\tilde{X}_0) \oplus H^2(E) \xrightarrow{\theta} H^2(Q) \right).$$

We can see that  ${}_{Wst}E_2^{-1,4} = \ker \left( H^2(Q)_{\text{prim}}(-1) \xrightarrow{-\gamma} H^4(\tilde{X}_0) \right)$  consists of the linear relations in  $X$  between the classes of difference of lines in  $Q$ . Thus, unlike the previous example, since  $H^2(Q)_{\text{prim}}$  is of 6 dimension, it is not sufficient for us to get the isomorphism of  ${}_{Wst}E_2^{-1,4}$  and  ${}_{Wst}E_2^{1,2}$  without further assumption on  $X_0$ . Note that by the Thom isomorphism theorem, we have

$$S := \ker(H_Q^4(\tilde{X}_0) \rightarrow H^4(Q)) \cong \ker(H^2(Q) \rightarrow H^4(Q)) = H^2(Q)_{\text{prim}}.$$

By our assumption, the defect  $\sigma(X_0)$  is zero, and hence  ${}_{Wst}E_2^{-1,4} = H^2(Q)_{\text{prim}}(-1)$ . Also, by duality,

$${}_{Wst}E_2^{1,2} = \text{coker} \left( H^2(E) \xrightarrow{\theta} H^2(Q) \right),$$

which is isomorphic to  $H^2(Q)_{\text{prim}}$  via the identity map. Then by Theorem 1.1,  $X_q$  satisfies the  $\partial\bar{\partial}$ -lemma for  $|q|$  small.

The computation about polarization is also similar to the previous example. First, since  ${}_{Wst}E_2^{-1,4} = H^2(Q)_{\text{prim}}(-1)$ ,  $Gr_4^{Wst} H_{\text{lim}}^3$  is polarized by  $S(C\bullet, \overline{(-N)\bullet})$ . Also,

$$Gr_3^{Wst} H_{\text{lim}}^3 = {}_{Wst}E_2^{0,3} = H^3(\tilde{X}_0)$$

is polarized by  $S(C\bullet, \bar{\bullet})$  by assumption. Hence, Theorem 1.1 implies  $H^3(X_q, \mathbb{C})$  together with the cup product  $S$  is a polarized Hodge structure for  $|q|$  small.  $\square$

#### 4.4 Non-Kähler Calabi-Yau Threefolds with Arbitrarily Large $b_2$

As a consequence of Theorem 1.1, we have the following theorem:

**Theorem 4.5.** *The non-Kähler Calabi-Yau threefolds  $X_3(a)$  constructed by Hashimoto-Sano [23, Theorem 1.1] satisfy the  $\partial\bar{\partial}$ -lemma. Moreover,  $H^3(X_3(a), \mathbb{C})$  together with the cup product  $S(\bullet, \bullet)$  is a polarized Hodge structure.*

*Proof.* We sketch the construction of Hashimoto-Sano first. Let

$$\mathbb{P}(3) := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

and  $S$  be a very general  $(2, 2, 2)$  hypersurface of  $\mathbb{P}(3)$ . Then  $S$  is a  $K_3$  surface, and by the Noether-Lefschetz theorem, we have  $\text{Pic}(S) = \bigoplus_{i=1}^3 \mathbb{Z}e_i$ , which is generated by the three fiber class of the elliptic fibrations  $S \rightarrow \mathbb{P}^1$  given by projection to each  $\mathbb{P}^1$ . By [23, §3.1], for all positive integer  $a$ , there is an automorphism  $\iota^a \in \text{Aut}(S)$  such that

$$(\iota^a)^* = \begin{pmatrix} 1 & 4a^2 - 2a & 4a^2 + 2a \\ 0 & 1 - 2a & -2a \\ 0 & 2a & 1 + 2a \end{pmatrix} \in \text{Aut}(\mathbb{Z}^3) \cong \text{Aut}(\text{Pic}(S)).$$

The  $d$ -semistable SNC Calabi-Yau variety  $X_0$  is constructed as follows. Let  $X'_1$  be the blowup of  $\mathbb{P}(3)$  along  $f_1, \dots, f_a \in |\mathcal{O}_S(1, 0, 0)|$  and  $X_1$  be the blowup of  $X'_1$  along the strict transform of the general member  $C_a \in |\mathcal{O}_S(16a^2 - a + 4, 4 - 8a, 4 + 8a)|$ . Let  $X_2 := \mathbb{P}(3)$ ,  $S_1 := S \subseteq X_1$  and  $S_2 := S \subseteq X_1$ . Then  $X_0$  is constructed by gluing  $X_1$  and  $X_2$  along  $S_1$

and  $S_2$  via the automorphism  $\iota^a : S_1 \rightarrow S_2$ . By the theorem of Namikawa-Kawamata [27, Theorem 4.2],  $X_0$  admits a global smoothing  $f : X \rightarrow \Delta$  where the central fiber is isomorphic to  $X_0$ . Then  $X_3(a)$  is defined as the general fiber  $X_q$  for  $|q|$  small.

We need to calculate the limiting mixed Hodge structure associated to  $f : X \rightarrow \Delta$ . As in the previous examples, the mixed Hodge structure on  $H^d(X_\infty, \mathbb{C})$  is pure unless  $d = 3$  by the Lefschetz hyperplane theorem. Also, we have

$${}_{Wst}E_2^{-1,4} = \ker \left( H^2(S_1)(-1) \xrightarrow{-\gamma} H^4(X_1) \oplus H^4(X_2) \right)$$

and

$${}_{Wst}E_2^{1,2} = \text{coker} \left( H^2(X_1) \oplus H^2(X_2) \xrightarrow{\theta} H^2(S_1) \right).$$

We need to show that the two spaces above are isomorphic via the map induced by  $(2\pi\sqrt{-1})I$ . Note that the image of the restriction map  $H^2(X_1) \oplus H^2(X_2) \xrightarrow{\theta} H^2(S_1)$  is simply  $\text{Pic}(S) \subseteq H^2(S)$ . Since the composition map  $H^2(X_2) \rightarrow H^2(S_2) \rightarrow H^4(X_2)$  of the restriction morphism and the Gysin morphism coincides with the cup product with  $c_1(N_{S_2/X_2})$ , the image of  $\text{Pic}(S_1)$  under  $-\gamma$  is three-dimensional. By this and the fact that  ${}_{Wst}E_2^{-1,4}$  and  ${}_{Wst}E_2^{1,2}$  have the same dimension, we see that  ${}_{Wst}E_2^{-1,4}$  and  ${}_{Wst}E_2^{1,2}$  are isomorphic via the map induced by  $(2\pi\sqrt{-1})I$ . Hence,  $W^{St} = W(N, 3)$  on  $H^3(X_\infty, \mathbb{C})$ , and then by Theorem 1.1,  $X(a)$  satisfies the  $\partial\bar{\partial}$ -lemma.

Next, we show that the mixed Hodge structure on  $H^3(X_\infty, \mathbb{C})$  together with the cup product  $S$  is a polarized Hodge structure. We have

$$Gr_3^{W^{St}} H^3(X_\infty) = {}_{Wst}E^{0,3} \cong H^1(C_a, \mathbb{Q})(-1) \oplus \bigoplus_{i=1}^a H^1(f_i, \mathbb{Q})(-1),$$

which, together with the cup product  $S$ , is a polarized Hodge structure. (Note that the cup product  $\epsilon(3)\langle \bullet, \bullet \rangle_{X_1}$  restricts to  $\epsilon(1)\langle \bullet, \bullet \rangle_{C_a}$  on  $H^1(C_a)$  by the projection formula and it holds for  $H^1(f_i)$  similarly.) On the other hand, note that

$$\ker \left( H^2(S_1)(-1) \xrightarrow{-\gamma} H^4(X_2) \right)$$

consists of the primitive class with respect to the ample line bundle  $(\iota^a)^* N_{S_2/X_2}$ , i.e. it consists of those  $\eta \in H^2(S_1)(-1)$  such that  $\eta \wedge c_1((\iota^a)^* N_{S_2/X_2}) = 0$ , where  $N_{S_2/X_2}$  is the normal bundle of  $S_2$  in  $X_2$ . Therefore, we see that  ${}_{Wst}E_2^{-1,4} \subseteq H^2(S_1)_{\text{prim}}(-1)$  and hence  ${}_{Wst}E_2^{-1,4}$  is polarized by  $S(C\bullet, (-N)\bullet)$  by Proposition 3.6. By Theorem 1.1, we can conclude that  $H^3(X_3(a))$  together with the cup product  $S(\bullet, \bullet)$  is a polarized Hodge structure. □

**Remark 4.6.** By a similar computation, one can show that another class of the non-Kähler Calabi-Yau threefolds with arbitrarily large  $b_2$  constructed by Sano [40] satisfies the  $\partial\bar{\partial}$ -lemma as well, and the middle cohomology together with the cup product is a polarized Hodge structure. Thus, it is interesting to ask if there exist non-Kähler Calabi-Yau threefolds that don't satisfy the  $\partial\bar{\partial}$ -lemma or satisfy the  $\partial\bar{\partial}$ -lemma but whose middle cohomology, together with the cup product  $S$ , isn't a polarized Hodge structure.

#### 4.5 Non Kähler Calabi-Yau $m$ -folds with Arbitrarily Large $b_2$

We turn to the non-Kähler Calabi-Yau  $m$ -folds with arbitrarily large  $b_2$  for  $m \geq 4$  constructed by Sano [39, Theorem 1.1]. Our main theorem is:

**Theorem 4.7.** *The non-Kähler Calabi-Yau  $m$ -folds  $X_m(a)$  with arbitrarily large  $b_2$  for  $m \geq 4$  constructed by Sano [39, Theorem 1.1] satisfy the  $\partial\bar{\partial}$ -lemma. Moreover, the Hodge index on the middle cohomology  $H^m(X_m(a))$  is as follows:*

- (1) *If  $m \equiv 3 \pmod{4}$ , the Hodge structure on  $H^m(X_m(a))$  together with the cup product  $S(\bullet, \bullet)$  is a polarized Hodge structure.*
- (2) *If  $m \equiv 1 \pmod{4}$ , the signature of  $S(C\bullet, \bar{\bullet})$  on  $H^{k, m-k}(X_m(a))$  is*

$$\left\{ \begin{array}{ll} (h^{k, m-k}(X_m(a)), 0), & k \neq \frac{m+1}{2}, \frac{m-1}{2} \\ (h^{k, m-k}(X_m(a)) - 9(27a^2 - 2a + 5) - a - 6, 9(27a^2 - 2a + 5) + a + 6), & k = \frac{m+1}{2} \text{ or } \frac{m-1}{2} \end{array} \right. .$$
- (3) *If  $m = 4$ , the signature of  $S(C\bullet, \bar{\bullet})$  on  $H^{k, m-k}(X_m(a))$  is*

$$\left\{ \begin{array}{ll} (h^{k, m-k}(X_m(a)), 0), & k \neq \frac{m}{2} \\ (h^{k, m-k}(X_m(a)) - a - 1, a + 1), & k = \frac{m}{2} \end{array} \right. .$$
- (4) *If  $m$  is even and  $m \geq 6$ , the signature of  $S(C\bullet, \bar{\bullet})$  on  $H^{k, m-k}(X_m(a))$  is*

$$\left\{ \begin{array}{ll} (h^{k, m-k}(X_m(a)), 0), & k \neq \frac{m}{2} \\ (h^{k, m-k}(X_m(a)) - a - 2, a + 2), & k = \frac{m}{2} \end{array} \right. .$$

Before we prove the theorem, we first sketch the construction of non-Kähler Calabi-Yau  $m$ -folds (not even of Fujiki class  $\mathcal{C}$ ) constructed by Sano [39]. The construction relies on the projective Calabi-Yau manifolds of Schoen-type. Let  $S \subset \mathbb{P}^2 \times \mathbb{P}^1$  be a general hypersurface of bidegree  $(3, 1)$  (i.e. a general rational elliptic surface) and  $T \subset \mathbb{P}^1 \times \mathbb{P}^{m-2}$  be a general hypersurface of bidegree  $(1, m-1)$ . Then the fiber product  $S \times_{\mathbb{P}^1} T$  is a projective Calabi-Yau manifold.  $S$  can be regarded as the blowup of  $\mathbb{P}^2$  at 9 points. Let  $H$  be the pullback of  $\mathcal{O}_{\mathbb{P}^2}(1)$  on  $S$  and  $h$  the cohomology class of  $H$ . Also let  $E_i$ ,  $i = 1, \dots, 9$ , be the exceptional divisors and  $e_i$ ,  $i = 1, \dots, 9$ , the cohomology class of  $E_i$ . Then  $H^2(S)$  is generated by  $h, e_1, \dots, e_9$ . By using the quadratic transformation on  $\mathbb{P}^2$ , [39, Proposition 2.6] shows that for all positive integer  $a$ , there exists an automorphism  $\phi_a : S \rightarrow S$  over  $\mathbb{P}^1$  such that

$$\phi_a^* H = (27a^2 + 1)H - (9a^2 - 3a)(E_1 + E_2 + E_3) - 9a^2(E_4 + E_5 + E_6) - (9a^2 + 3a)(E_7 + E_8 + E_9).$$

Let  $Y_i = \mathbb{P}^2 \times T$  for  $i = 1, 2$  and  $D_i := (S \times \mathbb{P}^{m-2}) \cap Y_i = S \times_{\mathbb{P}^1} T$ . Write  $p_S : S \times_{\mathbb{P}^1} T \rightarrow S$  the projection map. Consider  $f_i \in |-K_S|$ ,  $i = 1, \dots, a$ , and  $F_i := p_S^{-1}(f_i)$ ,  $i = 1, \dots, a$ .  $F_i$  is simply the general smooth fiber of the fibration  $S \times_{\mathbb{P}^1} T \xrightarrow{g} \mathbb{P}^1$ . Also, consider

$$C_a \in |3H + 3\phi_a^* H + (a-2)K_S|,$$

which is a smooth and irreducible member of an ample and free linear system as shown by [39, Proposition 2.6 (iii)]. By the proof of [39, Proposition 3.6],  $C_a$  is a curve of genus  $g(C_a) = 9(27a^2 - 2a + 5) - 1$  and the map  $C_a \xrightarrow{g'} \mathbb{P}^1$  induced by the elliptic fibration  $S \xrightarrow{g'} \mathbb{P}^1$  is of degree 18. Let  $\Gamma_a = p_S^{-1}(C_a) \cong C_a \times_{\mathbb{P}^1} T$ . Then we let  $\tilde{Y}_1$  be the blowup of  $Y_1$  at  $F_1, \dots, F_m$  and  $X_1$  be the blowup of  $\tilde{Y}_1$  at the strict transform of  $\Gamma_a$ . We also regard  $D_1$  as its strict transform; hence, we have  $D_1 \subset X_1$ . Let  $X_2 = Y_2 = \mathbb{P}^2 \times T$ . The

$d$ -semistable simple normal crossing Calabi-Yau variety  $X_0 = X_1 \sqcup X_2 / \sim$  is obtained by gluing  $D_1$  and  $D_2$  via

$$X_1 \supseteq S \times_{\mathbb{P}^1} T \xrightarrow{\phi_m \times id} S \times_{\mathbb{P}^1} T \subseteq X_2.$$

By the theorem of Namikawa-Kawamata [27, Theorem 4.2],  $X_0$  admits a global smoothing  $f : X \rightarrow \Delta$  where the central fiber is isomorphic to  $X_0$ . Then  $X_m(a)$  is defined as the general fiber  $X_q$  for  $|q|$  small.

The main difficulty in applying Theorem 1.1 to  $X(a)$  is to understand the cohomology ring of  $S \times_{\mathbb{P}^1} T$ . We will prove in the next section that the canonical morphism

$$H^\bullet(S) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(T) \rightarrow H^\bullet(S \times_{\mathbb{P}^1} T)$$

induces isomorphism in degree  $\leq m - 2$  and is injective in degree  $m - 1$ . More generally, we will prove that this type of Lefschetz property holds for the fiber product of two Lefschetz fibrations with disjoint critical locus (Theorem 5.1).

We recall the following lemma about the Hodge structure on the blowup of a Kähler manifold for later use.

**Lemma 4.8.** ([47, Theorem 7.31])

Let  $X$  be a compact complex manifold and  $Z$  a compact complex submanifold of codimension  $r$ . Let  $\pi : \tilde{X} \rightarrow X$  be the blowup of  $X$  along  $Z$  and  $E$  the exceptional divisor. Denote  $\iota : Z \rightarrow X$  and  $\iota' : E \rightarrow \tilde{X}$  the inclusion maps. Then

(1) For each  $k$ , there is a pushout diagram of cohomology

$$\begin{array}{ccc} H_Z^k(X) & \longrightarrow & H^k(X) \\ \downarrow & & \downarrow \pi^* \\ H_E^k(\tilde{X}) & \longrightarrow & H^k(\tilde{X}), \end{array}$$

where the vertical maps are induced by the pullback morphism.

(2) If we identify  $H_Z^k(X) \cong H^{k-2r}(Z)$  and  $H_E^k(\tilde{X}) \cong H^{k-2}(E)$  via the Thom isomorphism, then the pushout diagram above becomes

$$\begin{array}{ccc} H^{k-2r}(Z) & \xrightarrow{\iota_!} & H^k(X) \\ \downarrow \Phi & & \downarrow \pi^* \\ H^{k-2}(E) & \xrightarrow{\iota'_!} & H^k(\tilde{X}). \end{array}$$

The horizontal maps are the Gysin morphisms and  $\Phi$  can be described as follows: for  $\alpha \in H^{k-2r}(Z)$ ,

$$\Phi(\alpha) = (\pi|_E^* \alpha) \cup e(Q),$$

where  $Q$  is the universal quotient bundle of  $\pi|_E^* N_{Z/X}$  on  $E \cong \mathbb{P}(N_{Z/X})$ , i.e.  $Q$  comes from the short exact sequence

$$0 \rightarrow \mathcal{O}_E(-1) \rightarrow \pi|_E^* N_{Z/X} \rightarrow Q \rightarrow 0,$$

and  $e(Q)$  is the Euler class of  $Q$ .

(3) Let  $h = c_1(\mathcal{O}_E(1)) \in H^2(E)$ . If  $X$  is Kähler, then the pushout diagram above induces the isomorphism of Hodge structures:

$$H^k(X) \oplus \bigoplus_{i=0}^{r-2} H^{k-2i-2}(Z) \xrightarrow{\pi^* + \sum_i u_i^! \circ (\cup h^i) \circ \pi|_E^*} H^k(\tilde{X}).$$

*Proof.* For clarity, we explain why the map  $\Phi$  can be described as in (2) above. By [5, Chapter I, §6], the local cohomology  $H_Z^k(X) = H^k(N_{Z/X}, N_{Z/X} - Z)$  can be identified with the compact vertical cohomology  $H_{cv}^k(N_{Z/X})$ , which is defined by using the complex of smooth differential forms on  $N_{Z/X}$  with compact support in the vertical direction. This identification follows from the fact that the local cohomology  $H_Z^k(X)$  is isomorphic to the cohomology of the Thom space of the normal bundle  $N_{Z/X}$  and that the Thom space is simply the one-point compactification of  $N_{Z/X}$ . Under this identification, the Thom isomorphism  $H^{k-2r}(Z) \rightarrow H_{cv}^k(N_{Z/X})$  is given by the composition of the cup product with the Thom class  $Th(N_{Z/X}) \in H_{cv}^{2r}(N_{Z/X})$  and the pullback morphism. Since  $\pi|_E^* N_{Z/X} \cong \mathcal{O}_E(-1) \oplus Q$  as smooth vector bundles, by [5, Chapter I, Proposition 6.19], we have the equality of Thom classes

$$Th(\pi|_E^* N_{Z/X}) = p_1^* Th(\mathcal{O}_E(-1)) \cup p_2^* Th(Q),$$

where  $p_i$  are the projection maps  $\pi|_E^* N_{Z/X} \rightarrow \mathcal{O}_E(-1)$  and  $\pi|_E^* N_{Z/X} \rightarrow Q$ . We have the commutative diagram

$$\begin{array}{ccc} H^{k-2r}(Z) & \xrightarrow{\cup Th(N_{Z/X})} & H_{cv}^k(N_{Z/X}) \\ \downarrow \Phi & & \downarrow g^* \\ H^{k-2}(E) & \xrightarrow{\cup Th(\mathcal{O}_E(-1))} & H_{cv}^k(\mathcal{O}_E(-1)), \end{array}$$

where the horizontal maps are the Thom isomorphisms and the vertical map on the right comes from the pullback along the map  $g : \mathcal{O}_E(-1) \hookrightarrow \pi|_E^* N_{Z/X} \rightarrow N_{Z/X}$ . For all  $\alpha \in H^{k-2r}(Z)$ , we have

$$\begin{aligned} g^*(\alpha \cup Th(N_{Z/X})) &= \pi^* \alpha \cup (Th(\pi|_E^* N_{Z/X})|_{\mathcal{O}_E(-1)}) \\ &= \pi|_E^* \alpha \cup (p_2^* Th(Q))|_{\mathcal{O}_E(-1)} \cup Th(\mathcal{O}_E(-1)). \end{aligned}$$

Also, We have the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_E(-1) & \longrightarrow & \pi|_E^* N_{Z/X} = \mathcal{O}_E(-1) \oplus Q \\ \downarrow & & \downarrow \\ E & \longrightarrow & Q, \end{array}$$

where the vertical maps are projection maps, the horizontal map on the top is the inclusion map of vector bundles, and the horizontal map on the bottom is the embedding as the zero section. Thus,  $(p_2^* Th(Q))|_{\mathcal{O}_E(-1)}$  is the same as (the pullback) of the restriction of  $Th(Q)$  to  $E$ . It follows from [5, Chapter I, Proposition 6.41] that  $Th(Q)|_E = e(Q)$ . Hence,  $\Phi(\alpha) = \pi|_E^* \alpha \cup e(Q)$ . This finishes the proof.  $\square$

In our situation, we need the following lemma, which describes the restriction morphism and Gysin morphism between the blowup and the strict transform.

**Lemma 4.9.** *Let  $Z \xrightarrow{\iota_2} Y \xrightarrow{\iota_1} X$  be the inclusion of compact complex manifolds of dimension  $m-2$ ,  $m-1$ ,  $m$  respectively. Consider  $\tilde{X} = \text{Bl}_Z X$ ,  $Y \cong \tilde{Y} = \text{Bl}_Z Y$  and the inclusion  $\tilde{Y} \xrightarrow{\tilde{\iota}_1} \tilde{X}$ . Then for all  $k$  the restriction map  $H^k(\tilde{X}) \xrightarrow{\tilde{\iota}_1^*} H^k(\tilde{Y})$  can be identified with*

$$H^k(X) \oplus H^{k-2}(Z) \xrightarrow{(\iota_1^*, (\iota_2)_!)} H^k(Y),$$

where  $\iota_1^*$  is the restriction map and  $(\iota_2)_!$  is the Gysin morphism.

Dually, for all  $k$  the Gysin morphism  $H^k(\tilde{Y}) \xrightarrow{(\tilde{\iota}_1)_!} H^{k+2}(\tilde{X})$  can be identified with

$$H^k(Y) \xrightarrow{((\iota_1)_!, -\iota_2^*)} H^{k+2}(X) \oplus H^k(Z),$$

where  $(\iota_1)_!$  is the Gysin morphism and  $\iota_2^*$  is the restriction map.

*Proof.* Let  $E$  denote the exceptional divisor of the blowup morphism  $\tilde{X} \xrightarrow{f} X$ . By Lemma 4.8 (1), the cohomology  $H^k(\tilde{X})$  is computed by the pushout diagram

$$\begin{array}{ccc} H_Z^k(X) & \longrightarrow & H^k(X) \\ \downarrow & & \downarrow f^* \\ H_E^k(\tilde{X}) & \longrightarrow & H^k(\tilde{X}). \end{array}$$

Note that the composition map  $H^k(X) \xrightarrow{f^*} H^k(\tilde{X}) \xrightarrow{\tilde{\iota}_1^*} H^k(\tilde{Y})$  is just the restriction map  $H^k(X) \xrightarrow{\iota_1^*} H^k(Y)$ . Also, the composition map  $H_E^k(\tilde{X}) \rightarrow H^k(\tilde{X}) \xrightarrow{\tilde{\iota}_1^*} H^k(\tilde{Y})$  factors as  $H_E^k(\tilde{X}) \rightarrow H_{E \cap \tilde{Y}}^k(\tilde{Y}) \rightarrow H^k(\tilde{Y})$ . The last map  $H_{E \cap \tilde{Y}}^k(\tilde{Y}) \rightarrow H^k(\tilde{Y})$  is isomorphic to  $H_Z^k(Y) \rightarrow H^k(Y)$ , which is simply the Gysin morphism  $H^{k-2}(Z) \rightarrow H^k(Y)$  by the Thom isomorphism theorem. Hence, the first assertion follows.

The second dual statement follows from taking the Poincaré duality of the first statement. The only thing to notice is the negative sign of the restriction map, which comes from the isomorphism  $\mathcal{O}_{\tilde{X}}(E)|_E = \mathcal{O}_E(-1)$ .  $\square$

Now, we are ready to prove Theorem 4.7.

*Proof of Theorem 4.7.* As before, in order to prove the  $\partial\bar{\partial}$ -lemma on  $X_m(a)$  by Theorem 1.1, we need to show that

$$W_{St} E_2^{-1, k+1} = \ker \left( H^{k-1}(S \times_{\mathbb{P}^1} T)(-1) \xrightarrow{-\gamma} H^{k+1}(X_1) \oplus H^{k+1}(X_2) \right)$$

and

$$W_{St} E_2^{1, k-1} = \text{coker} \left( H^{k-1}(X_1) \oplus H^{k-1}(X_2) \xrightarrow{\theta} H^{k-1}(S \times_{\mathbb{P}^1} T) \right).$$

are isomorphic via the map induced by  $(2\pi\sqrt{-1})I$  for all  $k \geq 0$ . By duality, it suffices to verify it for  $k \leq m$ .

We denote the cohomology class of pullback of  $\mathcal{O}_{\mathbb{P}^2}(1)$  on  $S$  by  $h$ , and the cohomology class of the pullback of  $\mathcal{O}_{\mathbb{P}^1}(1)$  on  $S$  by  $f'$ . We know that

$$f' = [-K_S] = 3h - \sum_{i=1}^9 e_i.$$

We also denote the cohomology class of pullback of  $\mathcal{O}_{\mathbb{P}^{m-2}}(1)$  on  $T$  by  $\eta$ , and the cohomology class of the pullback of  $\mathcal{O}_{\mathbb{P}^1}(1)$  on  $T$  by  $f''$ . Then the ring  $H^\bullet(S) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(T)$  is simply the quotient of  $H^\bullet(S) \otimes_{\mathbb{C}} H^\bullet(T)$  by identifying the class  $f'$  with  $f''$ . We will denote this class in the quotient by  $f$ .

We first consider the case  $k < m$ . In this case, since the canonical morphism

$$H^\bullet(S) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(T) \rightarrow H^\bullet(S \times_{\mathbb{P}^1} T)$$

induces isomorphism in degree  $\leq m-2$  (cf. Theorem 5.1), We see that  $H^{k-1}(S \times_{\mathbb{P}^1} T)$  is generated by

$$\left\{ \begin{array}{ll} \eta^{\frac{k-1}{2}}, H^2(S) \cup \eta^{\frac{k-3}{2}}, h^2 \cup \eta^{\frac{k-5}{2}}, & \text{if } k \text{ is odd, } k \neq m-1 \\ H_{\text{prim}}^{m-2}(T), \eta^{\frac{k-1}{2}}, H^2(S) \cup \eta^{\frac{k-3}{2}}, h^2 \cup \eta^{\frac{k-5}{2}}, & \text{if } k \text{ is odd, } k = m-1 \\ 0, & \text{if } k \text{ is even, } k \neq m-1 \\ H^{m-2}(T), & \text{if } k \text{ is even, } k = m-1 \end{array} \right.$$

Also, if we denote  $\iota : S \times_{\mathbb{P}^1} T \rightarrow \mathbb{P}^2 \times T$  the inclusion map, then it is not hard to see that the Gysin morphism

$$H^{k-1}(S \times_{\mathbb{P}^1} T) \xrightarrow{\iota!} H^{k+1}(\mathbb{P}^2 \times T)$$

coincides with the map

$$H^\bullet(S) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(T) \rightarrow H^\bullet(\mathbb{P}^2 \times \mathbb{P}^1) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(T) = H^\bullet(\mathbb{P}^2 \times T),$$

from degree  $k-1$  part to  $k+1$  part, induced by the Gysin morphism associated to the inclusion  $S \rightarrow \mathbb{P}^2 \times \mathbb{P}^1$ . Since

$$\ker(H^2(S) \rightarrow H^4(\mathbb{P}^2 \times \mathbb{P}^1)) = \langle e_j - e_i \mid i \neq j \rangle = \langle h, f' \rangle^\perp,$$

one sees that

$$\begin{aligned} & \ker \left( H^{k-1}(S \times_{\mathbb{P}^1} T) \xrightarrow{(\iota_!, -\iota_! \circ ((\phi_a^{-1})^*))} H^{k+1}(\mathbb{P}^2 \times T) \oplus H^{k+1}(\mathbb{P}^2 \times T) \right) \\ &= \left\{ \begin{array}{ll} \langle h, \phi_a^* h, f \rangle^\perp \cup \eta^{\frac{k-3}{2}}, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even} \end{array} \right. \end{aligned}$$

Moreover, since for  $i = 1, \dots, a$ , the cohomology class of the divisor  $F_i$  is  $-f$ , and the cohomology class of the divisor  $\Gamma_a$  is  $3h + 3\phi_a^* h + (a-2)(-f)$ , one sees that when  $k$  is odd,

$$\langle h, \phi_a^* h, f \rangle^\perp \cup \eta^{\frac{k-3}{2}} \subseteq \ker(H^{k-1}(S \times_{\mathbb{P}^1} T) \rightarrow H^{k-1}(\Gamma_a) \oplus \bigoplus_{i=1}^a H^{k-1}(F_i)).$$

Thus, we conclude by Lemma 4.9 that

$$\ker(H^{k-1}(S \times_{\mathbb{P}^1} T) \xrightarrow{-\gamma} H^{k+1}(X_1) \oplus H^{k+1}(X_2))$$

$$= \begin{cases} \langle h, \phi_a^* h, f \rangle^\perp \cup \eta^{\frac{k-3}{2}}, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even} \end{cases}.$$

On the other hand, since the restriction morphism

$$H^{k-1}(\mathbb{P}^2 \times T) \xrightarrow{\iota^*} H^{k-1}(S \times_{\mathbb{P}^1} T)$$

coincides with the map

$$H^\bullet(\mathbb{P}^2 \times T) = H^\bullet(\mathbb{P}^2 \times \mathbb{P}^1) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(T) \rightarrow H^\bullet(S) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(T)$$

from degree  $k-1$  part to  $k-1$  part, induced by the restriction morphism associated to the inclusion  $S \rightarrow \mathbb{P}^2 \times \mathbb{P}^1$ , we can compute that

$$\text{im} \left( H^{k-1}(\mathbb{P}^2 \times T) \oplus H^{k-1}(\mathbb{P}^2 \times T) \xrightarrow{(-\iota^*, (\phi_a^*) \circ (\iota^*))} H^{k-1}(S \times_{\mathbb{P}^1} T) \right)$$

is generated by

$$\begin{cases} \eta^{\frac{k-1}{2}}, \langle h, \phi_a^* h, f \rangle \cup \eta^{\frac{k-3}{2}}, h^2 \cup \eta^{\frac{k-5}{2}}, & \text{if } k \text{ is odd, } k \neq m-1 \\ H_{\text{prim}}^{m-2}(T), \eta^{\frac{k-1}{2}}, \langle h, \phi_a^* h, f \rangle \cup \eta^{\frac{k-3}{2}}, h^2 \cup \eta^{\frac{k-5}{2}}, & \text{if } k \text{ is odd, } k = m-1 \\ 0, & \text{if } k \text{ is even, } k \neq m-1 \\ H^{m-2}(T), & \text{if } k \text{ is even, } k = m-1 \end{cases}.$$

Also, since  $\Gamma_a = C_a \times_{\mathbb{P}^1} T$  is a smooth divisor of  $C_a \times \mathbb{P}^{m-2}$  corresponding to the ample line bundle

$$(\pi_{C_a}^* (g')^* \mathcal{O}_{\mathbb{P}^1}(1)) \otimes (\pi_{\mathbb{P}^{m-2}}^* \mathcal{O}_{\mathbb{P}^{m-2}}(1)),$$

it follows from the Lefschetz hyperplane theorem that the following diagram

$$\begin{array}{ccc} H^\bullet(C_a) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(\mathbb{P}^1 \times \mathbb{P}^{m-2}) & \xlongequal{\quad} & H^\bullet(C_a \times \mathbb{P}^{m-2}) \\ \downarrow & & \downarrow \\ H^\bullet(C_a) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(T) & \longrightarrow & H^\bullet(C_a \times_{\mathbb{P}^1} T) \end{array}$$

induces isomorphism in degree  $\leq m-3$ , and the two vertical arrows are injective in degree  $m-2$ . Indeed, the horizontal arrow at the bottom is also injective in degree  $m-2$ . To see this, we can compute the composition of the restriction map and the Gysin morphism:

$$H^{m-2}(C_a \times T) \rightarrow H^{m-2}(C_a \times_{\mathbb{P}^1} T) \rightarrow H^m(C_a \times T),$$

which coincides with the cup product with  $c_1((g')^* \mathcal{O}_{\mathbb{P}^1}(1)) + f$ . Then we find that the kernel of the composition is generated by

$$\begin{cases} H^1(C_a) \cup f \cup \eta^{\frac{m-5}{2}}, & \text{if } m \text{ is odd} \\ c_1((g')^* \mathcal{O}_{\mathbb{P}^1}(1)) \cup f \cup \eta^{\frac{m-6}{2}}, (c_1((g')^* \mathcal{O}_{\mathbb{P}^1}(1)) - f) \cup \eta^{\frac{m-4}{2}}, & \text{if } m \text{ is even} \end{cases},$$

which are simply the generators of the degree  $m-2$  part of the kernel of the quotient map

$$H^\bullet(C_a) \otimes_{\mathbb{C}} H^\bullet(T) \rightarrow H^\bullet(C_a) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(T).$$

In particular, it is not hard to see that the Gysin morphism

$$H^{k-3}(C_a \times_{\mathbb{P}^1} T) \rightarrow H^{k-1}(S \times_{\mathbb{P}^1} T)$$

coincides with the map

$$H^\bullet(C_a) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(T) \rightarrow H^\bullet(S) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(T),$$

from degree  $k-3$  part to  $k-1$  part, induced by the Gysin morphism associated to the inclusion  $C_a \rightarrow S$ . As a result, the image of the Gysin morphism is given by

$$\begin{aligned} & \text{im}\left(H^{k-3}(C_a \times_{\mathbb{P}^1} T) \rightarrow H^{k-1}(S \times_{\mathbb{P}^1} T)\right) \\ &= \begin{cases} \text{Span}\left\{(3h + 3\phi_a^*h + (a-2)(-f)) \cup \eta^{\frac{k-3}{2}}, h^2 \cup \eta^{\frac{k-5}{2}}\right\}, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even} \end{cases}. \end{aligned}$$

Also, write  $F_t = S_t \times T_t$  for some general  $t \in \mathbb{P}^1$  and  $\iota_t : S_t \times T_t \rightarrow S \times_{\mathbb{P}^1} T$  the inclusion map, where  $S_t$  (resp.  $T_t$ ) is the fiber of the Lefschetz fibration  $S \xrightarrow{g_1} \mathbb{P}^1$  (resp.  $T \xrightarrow{g_2} \mathbb{P}^1$ ) at  $t$ . Then by the theorem of the fixed part (cf. [33, Theorem 4.23]), for all  $l \geq 0$ , the image of the restriction map

$$\text{im}\left(H^l(S \times_{\mathbb{P}^1} T) \xrightarrow{\iota_t^*} H^l(S_t \times T_t)\right)$$

coincides with the cohomology classes which are invariant under the monodromy action. Since the critical locus of the two Lefschetz fibrations  $g_1$  and  $g_2$  are disjoint (by the generality of  $S$  and  $T$ ), the monodromy of  $S \times_{\mathbb{P}^1} T \xrightarrow{g} \mathbb{P}^1$  acts on

$$H^l(S_t \times T_t) = \bigoplus_{i+j=l} H^i(S_t) \otimes H^j(T_t)$$

by the product action of the monodromy group of  $g_1$  and  $g_2$ . Hence, we see that

$$\begin{aligned} & \text{im}\left(H^l(S \times_{\mathbb{P}^1} T) \xrightarrow{\iota_t^*} H^l(S_t \times T_t)\right) \\ &= \bigoplus_{i+j=l} \text{im}\left(H^i(S) \rightarrow H^i(S_t)\right) \otimes \text{im}\left(H^j(T) \rightarrow H^j(T_t)\right) \\ &= \bigoplus_{i+j=l} H_{\text{fixed}}^i(S_t) \otimes H_{\text{fixed}}^j(T_t), \end{aligned}$$

where  $H_{\text{fixed}}^i(S_t)$  and  $H_{\text{fixed}}^j(T_t)$  are the fixed cohomology of the Lefschetz fibrations  $g_1$  and  $g_2$  (cf. [33, (C-11) and Theorem C.23]). This vector space has dimension

$$\begin{cases} 0, & \text{if } l \text{ is odd} \\ 1, & \text{if } l = 0, 2m-4 \\ 2, & \text{if } l \text{ is even, } 0 < l < 2m-4 \end{cases}.$$

Hence, by the Poincaré duality, the image of the Gysin morphism

$$\text{im}\left(H^l(S_t \times T_t) \xrightarrow{\iota_t!} H^{l+2}(S \times_{\mathbb{P}^1} T)\right)$$

has the same dimension as above. In particular, for all  $l \geq 0$ ,

$$\begin{aligned} & \text{im}\left(H^l(S_t \times T_t) \xrightarrow{\iota_t!} H^{l+2}(S \times_{\mathbb{P}^1} T)\right) \\ &= \text{im}\left(H^l(S \times_{\mathbb{P}^1} T) \xrightarrow{\iota_t^*} H^l(S_t \times T_t) \xrightarrow{\iota_t!} H^{l+2}(S \times_{\mathbb{P}^1} T)\right) \\ &= \text{im}\left(H^l(S \times_{\mathbb{P}^1} T) \xrightarrow{\cup f} H^{l+2}(S \times_{\mathbb{P}^1} T)\right). \end{aligned}$$

It follows that the image of the Gysin morphism is given by

$$\begin{aligned} & \text{im} \left( H^{k-3}(F_i) \rightarrow H^{k-1}(S \times_{\mathbb{P}^1} T) \right) \\ &= \begin{cases} \text{Span} \left\{ f \cup \eta^{\frac{k-3}{2}}, h^2 \cup \eta^{\frac{k-5}{2}} \right\}, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even} \end{cases}. \end{aligned}$$

Thus, we conclude by Lemma 4.9 that

$$\begin{aligned} & \text{im} \left( H^{k-1}(X_1) \oplus H^{k-1}(X_2) \xrightarrow{\theta} H^{k-1}(S \times_{\mathbb{P}^1} T) \right) \\ &= \text{im} \left( H^{k-1}(\mathbb{P}^2 \times T) \oplus H^{k-1}(\mathbb{P}^2 \times T) \xrightarrow{(-\iota^*, (\phi_a^*) \circ (\iota^*))} H^{k-1}(S \times_{\mathbb{P}^1} T) \right) \end{aligned}$$

and there is an orthogonal splitting

$$H^{k-1}(S \times_{\mathbb{P}^1} T) = \ker(-\gamma) \oplus \text{im}(\theta).$$

This proves that  ${}_{WSt}E_2^{-1, k+1}$  and  ${}_{WSt}E_2^{1, k-1}$  are isomorphic via the map induced by  $(2\pi\sqrt{-1})I$  for  $k < m$ .

Now we turn to the case  $k = m$ . In this case, due to the dimension reason, it suffices to show that the map

$$H^{m-1}(S \times_{\mathbb{P}^1} T) \xrightarrow{-\gamma} H^{m+1}(X_1) \oplus H^{m+1}(X_2)$$

is injective on the subspace

$$\text{im} \left( H^{m-1}(X_1) \oplus H^{m-1}(X_2) \xrightarrow{\theta} H^{m-1}(S \times_{\mathbb{P}^1} T) \right).$$

Note that the restriction morphism

$$H^{m-1}(\mathbb{P}^2 \times T) \xrightarrow{\iota^*} H^{m-1}(S \times_{\mathbb{P}^1} T),$$

the Gysin morphism

$$H^{m-3}(F_i) \rightarrow H^{m-1}(S \times_{\mathbb{P}^1} T)$$

and

$$H^{m-3}(\Gamma_a) \rightarrow H^{m-1}(S \times_{\mathbb{P}^1} T)$$

factor through the degree  $m-1$  part of

$$H^\bullet(S) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(T).$$

As a consequence, the argument above shows that

$$\text{im} \left( H^{m-1}(X_1) \oplus H^{m-1}(X_2) \xrightarrow{\theta} H^{m-1}(S \times_{\mathbb{P}^1} T) \right)$$

is generated by

$$\begin{cases} \eta^{\frac{m-1}{2}}, \langle h, \phi_a^* h, f \rangle \cup \eta^{\frac{m-3}{2}}, h^2 \cup \eta^{\frac{m-5}{2}}, & \text{if } m \text{ is odd} \\ 0, & \text{if } m \text{ is even} \end{cases}.$$

On the other hand, the argument above also shows that the restriction of  $-\gamma$  on the degree  $m-1$  part of

$$H^\bullet(S) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(T)$$

has kernel

$$\begin{cases} \langle h, \phi_a^* h, f \rangle^\perp \cup \eta^{\frac{m-3}{2}}, & \text{if } m \text{ is odd} \\ 0, & \text{if } m \text{ is even} \end{cases}.$$

The assertion is immediate. In conclusion, we have proved that  $W^{St} = W(N)$  on the limiting cohomology, and hence by Theorem 1.1,  $X(a)$  satisfies the  $\partial\bar{\partial}$ -lemma.

Now, we turn to the Hodge index on the middle cohomology of  $X(a)$ . We first consider the case that  $m$  is odd. Then the preceding paragraph shows that we have an orthogonal splitting

$$H^{m-1}(S \times_{\mathbb{P}^1} T) = \text{im}(\theta) \oplus \left( \langle h, \phi_a^* h, f \rangle^\perp \cup \eta^{\frac{m-3}{2}} \right) \oplus W^\perp$$

and

$${}_{W^{St}}E_2^{-1,m+1} = \left( \langle h, \phi_a^* h, f \rangle^\perp \cup \eta^{\frac{m-3}{2}} \right) \oplus W^\perp,$$

where  $W^\perp$  is the orthogonal complement of the degree  $m-1$  part of  $H^\bullet(S) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(T)$  in  $H^{m-1}(S \times_{\mathbb{P}^1} T)$ . If we let  $L$  be the Lefschetz operator on  $S \times_{\mathbb{P}^1} T$  associated to the Kähler form  $h + f + \eta$ , then  $LH^{m-3}(S \times_{\mathbb{P}^1} T)$  is contained in the  $m-1$  part of  $H^\bullet(S) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(T)$ , and hence  $W^\perp \subseteq H_{\text{prim}}^{m-1}(S \times_{\mathbb{P}^1} T)$ . Also, we have

$$\langle h, \phi_a^* h, f \rangle^\perp \cup \eta^{\frac{m-3}{2}} = L^{\frac{m-3}{2}} \left( \langle h, \phi_a^* h, f \rangle^\perp \right),$$

and  $\langle h, \phi_a^* h, f \rangle^\perp \subseteq H_{\text{prim}}^2(S \times_{\mathbb{P}^1} T)$ . This shows that the index of  $S(C_\bullet, \overline{(-N)_\bullet})$  is positive on  ${}_{W^{St}}E_2^{-1,m+1}$  if  $m \equiv 3 \pmod{4}$ , and there are 7 dimensional negative classes in the  $(\frac{m+1}{2}, \frac{m+1}{2})$  part of  ${}_{W^{St}}E_2^{-1,m+1}$  if  $m \equiv 1 \pmod{4}$ . Also,

$${}_{W^{St}}E_2^{0,m} = \frac{\ker \left( H^m(X_1) \oplus H^m(X_2) \xrightarrow{\theta} H^m(S \times_{\mathbb{P}^1} T) \right)}{\text{im} \left( H^{m-2}(S \times_{\mathbb{P}^1} T) \xrightarrow{-\gamma} H^m(X_1) \oplus H^m(X_2) \right)}.$$

Note that since  $m$  is odd, the Gysin map  $H^{m-2}(S \times_{\mathbb{P}^1} T) \rightarrow H^m(\mathbb{P}^2 \times T)$  can be identified with the isomorphism  $H^0(S) \otimes H^{m-2}(T) \rightarrow H^2(\mathbb{P}^2) \otimes H^{m-2}(T)$  given by the cup product with  $3h$ . Let  $L$  denote any Lefschetz operator on  $F_i$  or  $\Gamma_a$ . Then we have the Lefschetz decomposition

$$H^{m-2}(\Gamma_a) = H_{\text{prim}}^{m-2}(\Gamma_a) \oplus L^{\frac{m-3}{2}} H^1(\Gamma_a),$$

where

$$H^1(\Gamma_a) \cong H^1(C_a) \otimes H^0(\mathbb{P}^{m-2}).$$

Write

$$H_{\text{prim}}^{m-2}(\Gamma_a) = H^0(C_a) \otimes H^{m-2}(T) \oplus U,$$

where  $U$  is the orthogonal complement of  $H^0(C_a) \otimes H^{m-2}(T)$ . Since  $F_i \cong S_t \times T_t$ , where  $S_t$  is an elliptic curve and  $T_t$  is a hypersurface of  $\mathbb{P}^{m-2}$  of degree  $m-1$ , we have the Lefschetz decomposition

$$H^{m-2}(F_i) = H_{\text{prim}}^{m-2}(F_i) \oplus L^{\frac{m-3}{2}} H^1(F_i),$$

where

$$H^1(F_i) \cong H^1(S_t) \otimes H^0(\mathbb{P}^{m-2}).$$

From this, one easily sees that

$${}_{Wst}E_2^{0,m} = U' \oplus U \oplus L^{\frac{m-3}{2}} H^1(\Gamma_a) \oplus \bigoplus_{i=1}^a H^{m-2}(F_i),$$

where

$$U' \subset H^2(\mathbb{P}^2) \otimes H^{m-2}(T) \oplus H^0(C_a) \otimes H^{m-2}(T) \subset H^{m-2}(X_1)$$

is the subspace

$$\{(6h \cup a, a) \mid a \in H^{m-2}(T)\}.$$

This shows that the index of  $S(C_\bullet, \bar{\bullet})$  is positive on  ${}_{Wst}E_2^{0,m}$  if  $m \equiv 3 \pmod{4}$ , and there are  $g(C_a) + a$  dimensional negative classes in the  $(\frac{m-1}{2}, \frac{m+1}{2})$  part and  $(\frac{m+1}{2}, \frac{m-1}{2})$  part of  ${}_{Wst}E_2^{0,m}$  if  $m \equiv 1 \pmod{4}$ . By summing  $g(C_a) + a$  and 7 and applying Theorem 1.1, we get the index formula in Theorem 4.7 (1) and (2).

If  $m$  is even, then

$${}_{Wst}E_2^{-1,m+1} = H^{m-1}(S \times_{\mathbb{P}^1} T) = H_{\text{prim}}^{m-1}(S \times_{\mathbb{P}^1} T).$$

So  $S(C_\bullet, (-N)\bar{\bullet})$  is positive on  ${}_{Wst}E_2^{-1,m+1}$ . On the other hand, write

$$H^m(X_1) \oplus H^m(X_2) = H^m(\mathbb{P}^2 \times T) \oplus H^{m-2}(\Gamma_a) \oplus \left( \bigoplus_{i=1}^a H^{m-2}(F_i) \right) \oplus H^m(X_2)$$

and define the vanishing cohomology of  $H^{m-2}(T)$  as the kernel of the Gysin morphism:

$$H_{\text{van}}^{m-2}(T) := \ker(H^{m-2}(T) \rightarrow H^m(\mathbb{P}^1 \times \mathbb{P}^{m-2}))$$

Then it is elementary to compute the orthogonal decomposition

$${}_{Wst}E_2^{0,m} \cong U_1 \oplus U_2 \oplus U_3 \oplus U_4 \oplus U_5,$$

where  $U_1$  is the orthogonal complement of  $H^0(C_a) \otimes H^{m-2}(T)$  in  $H^{m-2}(\Gamma_a)$ ;

$$U_2 = \bigoplus_{i=1}^a H_{\text{prim}}^{m-2}(F_i);$$

$$U_3 \subset H^2(\mathbb{P}^2) \otimes H^{m-2}(T) \oplus H^0(C_a) \otimes H^{m-2}(T) \subset H^m(X_1)$$

is the subspace

$$\{(6h \cup \alpha, \alpha) \mid \alpha \in H_{\text{van}}^{m-2}(T)\};$$

$$U_4 \subset \bigoplus_{i=1}^a H^{m-2}(F_i)$$

is the subspace

$$\{\phi_j^* \eta^{\frac{m-2}{2}} - \phi_k^* \eta^{\frac{m-2}{2}}, \phi_j^* h \eta^{\frac{m-4}{2}} - \phi_k^* h \eta^{\frac{m-4}{2}} \mid j \neq k \text{ with } \phi_j : F_j \hookrightarrow S \times_{\mathbb{P}^1} T \text{ the embedding}\};$$

$$U_5 \subset H^m(\mathbb{P}^2 \times T) \oplus (H^0(C_a) \otimes H^{m-2}(T)) \oplus H^m(X_2)$$

is the six dimensional subspace

$$\text{Span}_{\mathbb{C}}\{(\eta^{\frac{m}{2}}, 0, \eta^{\frac{m}{2}}), (f \cup \eta^{\frac{m-2}{2}}, 0, f \cup \eta^{\frac{m-2}{2}}), ((3h-f) \cup h \cup \eta^{\frac{m-4}{2}}, 0, 0), \\ (0, 0, (3h-f) \cup h \cup \eta^{\frac{m-4}{2}}), (h^2 \cup f \cup \eta^{\frac{m-6}{2}}), ((3h-(m-2)f) \cup \eta^{\frac{m-2}{2}}, \eta^{\frac{m-2}{2}}, -3h \cup \eta^{\frac{m-2}{2}})\}$$

if  $m \geq 6$  and is the four-dimensional subspace

$$\text{Span}_{\mathbb{C}}\{(\eta^{\frac{m}{2}}, 0, \eta^{\frac{m}{2}}), ((3h-f) \cup h \cup \eta^{\frac{m-4}{2}}, 0, 0), (0, 0, (3h-f) \cup h \cup \eta^{\frac{m-4}{2}}),$$

$$\left( (3h - (m-2)f) \cup \eta^{\frac{m-2}{2}}, \eta^{\frac{m-2}{2}}, -3h \cup \eta^{\frac{m-2}{2}} \right)$$

if  $m = 4$ . Furthermore,  $S(C\bullet, \bar{\bullet})$  is positive on  $U_1 \oplus U_2 \oplus U_3$ , has index  $(m-1, m-1)$  on  $U_4$ , has index  $(3, 3)$  on  $U_5$  if  $m \geq 6$ , and has index  $(2, 2)$  on  $U_5$  if  $m = 4$ . By summing all the indexes and applying Theorem 1.1, we get the index formula in Theorem 4.7 (3) and (4).  $\square$

## 5 Cohomology Ring of Fiber Product of Lefschetz Fibrations

For  $i = 1, 2$ , let  $X_i$  be a projective manifold of dimension  $m_i$  and  $\{Y_{i,t}\}_{t \in \mathbb{P}^1}$  be a Lefschetz pencil on  $X_i$ , where  $Y_{i,t}$  are sections of some ample linear system  $|\mathcal{L}_i|$ . Let  $B_i$  be the base locus of the pencil and denote  $\iota_i : B_i \hookrightarrow X_i$  the inclusion map. Let

$$\tilde{X}_i := \text{Bl}_{B_i} X_i \xrightarrow{f_i} \mathbb{P}^1$$

be the associated Lefschetz fibration and  $\Delta(f_i) \subset \mathbb{P}^1$  be the critical locus of  $f_i$ . We also let  $E_i$  be the exceptional divisor of the birational morphism  $\tilde{X}_i \xrightarrow{F_i} X_i$  and write  $\tilde{\iota}_i : E_i \hookrightarrow X_i$  the inclusion map. Note that

$$E_i \cong \mathbb{P}(N_{Z_i/X_i}) = \mathbb{P}(\iota_i^* \mathcal{L}_i \oplus \iota_i^* \mathcal{L}_i).$$

Let  $Y_i := Y_{i,t}$  for some general  $t$  in  $\mathbb{P}^1$ . Recall that by the Lefschetz hyperplane theorem, the restriction map  $H^k(X_i) \rightarrow H^k(Y_i)$  is an isomorphism for  $k \leq m_i - 2$ , is injective for  $k = m_i - 1$ , and is surjective for  $k > m_i - 1$ . Consider the fixed cohomology of  $H^{m_i-1}(Y_i)$  defined by

$$H_{\text{fixed}}^{m_i-1}(Y_i) = \text{im} \left( H^{m_i-1}(X_i) \xrightarrow{\iota_i'^*} H^{m_i-1}(Y_i) \right)$$

and the vanishing cohomology of  $Y_i$  defined by

$$H_{\text{van}}^{m_i-1}(Y_i) = \ker \left( H^{m_i-1}(Y_i) \xrightarrow{\iota_{i!}} H^{m_i+1}(X_i) \right),$$

where  $\iota_i' : Y_i \hookrightarrow X_i$  is the inclusion and  $\iota_{i!}$  is the associated Gysin morphism. Then we have an orthogonal decomposition

$$H^{m_i-1}(Y_i) = H_{\text{fixed}}^{m_i-1}(Y_i) \oplus H_{\text{van}}^{m_i-1}(Y_i).$$

Similarly, since  $B_i$  is a complete intersection of two sections of an ample line bundle, by the Lefschetz hyperplane theorem, the restriction map  $H^k(X_i) \rightarrow H^k(B_i)$  is an isomorphism for  $k \leq m_i - 3$ , is injective for  $k = m_i - 2$  and is surjective for  $k > m_i - 2$ . We can similarly define the fixed cohomology of  $H^{m_i-2}(B_i)$  by

$$H_{\text{fixed}}^{m_i-2}(B_i) = \text{im} \left( H^{m_i-2}(X_i) \xrightarrow{\iota_i'^*} H^{m_i-2}(B_i) \right)$$

and the vanishing cohomology of  $B_i$  by

$$H_{\text{van}}^{m_i-2}(B_i) = \ker \left( H^{m_i-2}(B_i) \xrightarrow{\iota_{i!}} H^{m_i+2}(X_i) \right),$$

where  $\iota_{i!}$  is the associated Gysin morphism. Then we sill have the orthogonal decomposition

$$H^{m_i-2}(B_i) = H_{\text{fixed}}^{m_i-2}(B_i) \oplus H_{\text{van}}^{m_i-2}(B_i).$$

We are only interested in the case that  $\Delta(f_1)$  and  $\Delta(f_2)$  are disjoint. In such case, it is easy to see that the fiber product  $\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2$  is a smooth projective manifold. The goal of the section is to prove the following Lefschetz-type theorem:

**Theorem 5.1.** *Assume that  $\Delta(f_1)$  and  $\Delta(f_2)$  are disjoint. Then the canonical morphism of graded rings*

$$H^\bullet(\tilde{X}_1, \mathbb{Q}) \otimes_{H^\bullet(\mathbb{P}^1, \mathbb{Q})} H^\bullet(\tilde{X}_1, \mathbb{Q}) \rightarrow H^\bullet(\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2, \mathbb{Q})$$

is an isomorphism in degree  $\leq m_1 + m_2 - 2$  and is injective in degree  $m_1 + m_2 - 1$ .

*Proof.*<sup>1</sup> We have a natural map  $\tilde{X}_1 \times \tilde{X}_2 \xrightarrow{F} X_1 \times X_2$  induced by the birational morphism  $\tilde{X}_i \xrightarrow{F_i} X_i$ . Consider the image of  $\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2$  under  $F$ , denoted by  $W$ . Note that if the Lefschetz pencil  $\{Y_{i,t}\}_{t \in \mathbb{P}^1}$  is defined by the secitons  $\sigma_{i,t} = \sigma_{i,0} + t\sigma_{i,\infty}$  ( $t \in \mathbb{P}^1$ ) of  $|\mathcal{L}_i|$  on  $X_i$ , then we have

$$W = V(\sigma_{1,0}\sigma_{2,\infty} - \sigma_{2,0}\sigma_{1,\infty}) \subset X_1 \times X_2.$$

Also,  $W$  is only singular at

$$B_1 \times B_2 = V(\sigma_{1,0}, \sigma_{1,\infty}, \sigma_{2,0}, \sigma_{2,\infty}) \subset X_1 \times X_2,$$

and locally at any point  $p$  on  $B_1 \times B_2$ , the germ  $(W, p)$  is isomorphic to the product of the germ of a threefold ordinary double point and  $(B_1 \times B_2, p)$ . The fiber product  $\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2$  can be regarded as the small resolution of  $W$  with exceptional set  $E$ , where  $E$  is a  $\mathbb{P}^1$  bundle on  $Z := B_1 \times B_2$  isomorphic to  $\mathbb{P}(\pi_1'^* \iota_1^* \mathcal{L}_1 \oplus \pi_2'^* \iota_2^* \mathcal{L}_2)$ . Here,  $\pi_i': B_1 \times B_2 \rightarrow B_i$  is the natural projection map. Thus, we have the discriminant square

$$\begin{array}{ccc} E & \xrightarrow{\tilde{\iota}} & \tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2 \\ \downarrow F|_E & & \downarrow F|_W \\ Z & \xrightarrow{\iota} & W \end{array}$$

and the discriminant square has a natural inclusion into the discriminant square formed by  $F$ :

$$\begin{array}{ccc} E' := E_1 \times \tilde{X}_2 \cup \tilde{X}_1 \times E_2 & \xrightarrow{\tilde{\iota}'} & \tilde{X}_1 \times \tilde{X}_2 \\ \downarrow F|_{E'} & & \downarrow F \\ Z' := B_1 \times X_2 \cup X_1 \times B_2 & \xrightarrow{\iota'} & X_1 \times X_2. \end{array}$$

Note that the two components of  $E'$  intersect at  $E_1 \times E_2$  and  $E$  can be regarded as the diagonal subset of this intersection. Then, by the Mayer-Vietoris sequence for the discriminant square (cf. [33, Corollary-Definition 5.37]), we have a commutative diagram of long exact sequences of mixed Hodge structures:

$$\begin{array}{ccccccc} \dots & \longrightarrow & H^k(X_1 \times X_2) & \longrightarrow & H^k(\tilde{X}_1 \times \tilde{X}_2) \oplus H^k(Z') & \longrightarrow & H^k(E') \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & H^k(W) & \longrightarrow & H^k(\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2) \oplus H^k(Z) & \longrightarrow & H^k(E) \longrightarrow \dots \end{array}$$

<sup>1</sup>The proof of the theorem is inspired by Kloosterman's answer on Mathoverflow: <https://mathoverflow.net/questions/68625/hodge-diamond-of-a-calabi-yau-fourfold/70052#70052>

Since  $E'$  is compact and  $X_1 \times X_2$  is smooth, the connecting homomorphism  $H^{k-1}(E') \rightarrow H^k(X_1 \times X_2)$  is zero for all  $k$ , and hence the exact sequence on the top forms a short exact sequence for all  $k$ . Also, since  $W$  is a section of the ample line bundle  $\pi_1^* \mathcal{L}_1 \otimes \pi_2^* \mathcal{L}_2$  on  $X_1 \times X_2$ , where  $\pi_i : X_1 \times X_2 \rightarrow X_i$  is the natural projection, by the Lefschetz hyperplane theorem (cf. [33, Theorem C.15]), the restriction map

$$H^k(X_1 \times X_2) \rightarrow H^k(W)$$

is an isomorphism if  $k \leq m_1 + m_2 - 2$  and is injective for  $k = m_1 + m_2 - 1$ . In particular, the mixed Hodge structure on  $H^k(W)$  is pure for  $k \leq m_1 + m_2 - 2$ , and hence the exact sequence on the bottom is injective on the left for  $k \leq m_1 + m_2 - 2$  and is surjective on the right for  $l \leq m_1 + m_2 - 3$ .

We can compute the cohomology  $H^k(E')$  and  $H^k(Z')$  by using the Mayer-Vietoris sequence associated to the two components of  $E'$  and  $Z'$ . By comparing the two Mayer-Vietoris sequences, it is not hard to see that

$$H^k(E') = H^k(Z') \oplus H^{k-2}(B_1 \times X_2) \oplus H^{k-2}(X_1 \times B_2) \oplus H^{k-4}(B_1 \times B_2).$$

If we write  $H^k(E) = H^k(Z) \oplus H^{k-2}(Z)$ , then the restriction map  $H^k(E') \rightarrow H^k(E)$  can be identified as follows: we have the restriction map  $H^k(Z') \rightarrow H^k(Z)$  and  $H^k(Z')$  is mapped to zero in the summand  $H^{k-2}(Z)$ ;

$$H^{k-2}(B_1 \times X_2) \oplus H^{k-2}(X_1 \times B_2)$$

is mapped to zero in the summand  $H^k(Z)$  and is mapped to  $H^{k-2}(B_1 \times B_2) = H^{k-2}(Z)$  by the restriction map;  $H^{k-4}(B_1 \times B_2)$  is mapped to  $H^k(Z) \oplus H^{k-2}(Z)$  via the map

$$\alpha \mapsto (-\alpha \cup c_1(\pi_1^* \mathcal{L}_1) \cup c_1(\pi_2^* \mathcal{L}_2), -\alpha \cup (c_1(\pi_1^* \mathcal{L}_1) + c_1(\pi_2^* \mathcal{L}_2)))$$

by using Grothendieck's definition of Chern classes. We can then easily see that the image of  $H^{k-4}(B_1 \times B_2)$  is contained in the image of the first three direct summands of  $H^k(E')$  for all  $k$ . Hence,  $\ker(H^k(E') \rightarrow H^k(E))$  has dimension

$$\begin{aligned} & \dim_{\mathbb{C}} \ker(H^k(Z') \rightarrow H^k(Z)) + \dim_{\mathbb{C}} H^{k-4}(B_1 \times B_2) \\ & + \dim_{\mathbb{C}} \ker(H^{k-2}(B_1 \times X_2) \oplus H^{k-2}(X_1 \times B_2) \rightarrow H^{k-2}(B_1 \times B_2)) \end{aligned}$$

for all  $k$ . Note also that

$$\operatorname{coker}(H^k(E') \rightarrow H^k(E)) = \operatorname{coker}(H^k(Z') \rightarrow H^k(Z))$$

for  $k \neq m_1 + m_2 - 2$ . Also,

$$\begin{aligned} \operatorname{coker}(H^{m_1+m_2-4}(E') \rightarrow H^{m_1+m_2-4}(E)) &= \operatorname{coker}(H^{m_1+m_2-4}(Z') \rightarrow H^{m_1+m_2-4}(Z)) \\ & \oplus H_{\text{van}}^{m_1-2}(B_1) \otimes H_{\text{van}}^{m_2-2}(B_2). \end{aligned}$$

Now coming back to cohomology of the fibered product  $H^k(\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2)$ , we would like to prove the following dimension bounds:

**Claim:** The degree  $k$  part of the kernel of the quotient map

$$H^\bullet(\tilde{X}_1 \times \tilde{X}_2) = H^\bullet(\tilde{X}_1) \otimes_{\mathbb{C}} H^\bullet(\tilde{X}_2) \rightarrow H^\bullet(\tilde{X}_1) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(\tilde{X}_2)$$

has dimension greater than or equal to

$$\begin{aligned} & \dim_{\mathbb{C}} H^{k-4}(B_1 \times B_2) \\ & + \dim_{\mathbb{C}} \ker(H^{k-2}(B_1 \times X_2) \oplus H^{k-2}(X_1 \times B_2) \rightarrow H^{k-2}(B_1 \times B_2)) \end{aligned}$$

for  $k \leq m_1 + m_2 - 1$ .

Assuming the claim, then by the snake lemma, the canonical map

$$H^\bullet(\tilde{X}_1) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(\tilde{X}_1) \rightarrow H^\bullet(\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2)$$

is injective for  $k \leq m_1 + m_2 - 1$  and is surjective for  $k \leq m_1 + m_2 - 3$ .

To prove our claim, we first note the following lemma:

**Lemma 5.2.** *Let  $f_i : \tilde{X}_i \rightarrow \mathbb{P}^1$ ,  $F_i : \tilde{X}_i \rightarrow X_i$ ,  $\mathcal{L}_i$  and  $E_i$  be as above. Then*

$$f_i^* \mathcal{O}_{\mathbb{P}^1}(1) \cong F_i^* \mathcal{L}_i \otimes \mathcal{O}(-E_i).$$

*Proof.* Indeed, from the birational morphism  $\tilde{X}_i \xrightarrow{F} X_i$ , we have the canonical bundle formula  $K_{\tilde{X}_i} \cong F_i^* K_{X_i} \otimes \mathcal{O}(E_i)$ , and if we regard  $\tilde{X}_i$  as the hypersurface of  $\mathbb{P}^1 \times X_i$  defined by a section of the line bundle  $f_i^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes F_i^* \mathcal{L}_i$  (where by abuse of notation we denote projection map on  $\mathbb{P}^1 \times X_i$  by  $f_i$  and  $F_i$  as well), then by adjunction we have

$$K_{\tilde{X}_i} \cong f_i^* \mathcal{O}_{\mathbb{P}^1}(-2) \otimes F_i^* K_{X_i} \otimes f_i^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes F_i^* \mathcal{L}_i.$$

Thus by the comparison of the two expressions of  $K_{\tilde{X}_i}$ , the expression of  $f_i^* \mathcal{O}_{\mathbb{P}^1}(1)$  follows.  $\square$

Then we have the following lemma:

**Lemma 5.3.** *Let  $f_i : \tilde{X}_i \rightarrow \mathbb{P}^1$ ,  $F_i : \tilde{X}_i \rightarrow X_i$ ,  $\mathcal{L}_i$ ,  $E_i$  and  $B_i$  be as above. If we write  $H^k(\tilde{X}_i) = H^k(X_i) \oplus H^{k-2}(B_i)$  for all  $k$ , then the morphism*

$$H^k(\tilde{X}_i) \xrightarrow{\cup_{c_1(f_i^* \mathcal{O}_{\mathbb{P}^1}(1))}} H^{k+2}(\tilde{X}_i)$$

is given by

$$(\alpha, \beta) \mapsto (\alpha \cup c_1(\mathcal{L}_i) + \iota_{i!} \beta, \iota_i^* \alpha - \beta \cup c_1(\iota_i^* \mathcal{L}_i)).$$

*Proof.* Note that by Lemma 4.8, the Hodge structure on  $H^k(\tilde{X}_i)$  can be computed by the pushout diagram

$$\begin{array}{ccc} H^{k-4}(B_i) & \xrightarrow{\iota_{i!}} & H^k(X_i) \\ \downarrow \Psi_i & & \downarrow F_i^* \\ H^{k-2}(E_i) & \xrightarrow{\tilde{\iota}_{i!}} & H^k(\tilde{X}_i), \end{array}$$

where the horizontal maps are Gysin morphisms, and  $\Psi_i$  is the cup product with the Euler class of the universal quotient bundle of  $F_i|_{E_i}^* N_{B_i/X_i}$  on  $E_i \cong \mathbb{P}(N_{B_i/X_i})$ . In our case, if we write  $H^{k-2}(E_i) = H^{k-2}(B_i) \oplus (H^{k-4}(B_i) \cup h)$ , where  $h = c_1(\mathcal{O}_{E_i}(1))$ , then we have

$$\Psi(\alpha) = (\alpha \cup (2c_1(\iota_i^* \mathcal{L}_i)), \alpha \cup h).$$

By this and the projection formula, we can compute that for a  $k$ -form  $F_i^* \alpha + \tilde{\iota}_{i!} F_i|_{E_i}^* \beta \in H^k(\tilde{X}_i)$  with  $\alpha \in H^k(X_i)$  and  $\beta \in H^{k-2}(B_i)$ , we have

$$\begin{aligned} (F_i^* \alpha + \tilde{\iota}_{i!} F_i|_{E_i}^* \beta) \cup [-E_i] &= \tilde{\iota}_{i!} (-\tilde{\iota}_i^* F_i^* \alpha + F_i|_{E_i}^* \beta \cup h) \\ &= \tilde{\iota}_{i!} (F_i|_{E_i}^* \iota_i^* \alpha + \Psi_i(\beta) - 2F_i|_{E_i}^* (\beta \cup c_1(\iota_i^* \mathcal{L}_i))) \\ &= F_i^* \iota_{i!} \beta + \tilde{\iota}_{i!} F_i|_{E_i}^* (\iota_i^* \alpha - 2\beta \cup c_1(\iota_i^* \mathcal{L}_i)). \end{aligned}$$

In other words, if we write  $H^k(\tilde{X}_i) = H^k(X_i) \oplus H^{k-2}(B_i)$  for all  $k$ , then the morphism

$$H^k(\tilde{X}_i) \xrightarrow{\cup[-E_i]} H^{k+2}(\tilde{X}_i)$$

is given by

$$(\alpha, \beta) \mapsto (\iota_{i!}\beta, \iota_i^*\alpha - 2\beta \cup c_1(\iota_i^*\mathcal{L}_i)).$$

As a result, by using the isomorphism

$$f_i^*\mathcal{O}_{\mathbb{P}^1}(1) \cong F_i^*\mathcal{L}_i \otimes \mathcal{O}(-E_i)$$

in Lemma 5.2, we can see that the morphism

$$H^k(\tilde{X}_i) \xrightarrow{\cup c_1(f_i^*\mathcal{O}_{\mathbb{P}^1}(1))} H^{k+2}(\tilde{X}_i)$$

is given by

$$(\alpha, \beta) \mapsto (\alpha \cup c_1(\mathcal{L}_i) + \iota_{i!}\beta, \iota_i^*\alpha - \beta \cup c_1(\iota_i^*\mathcal{L}_i)).$$

□

If we write

$$H^k(\tilde{X}_1 \times \tilde{X}_2) = H^k(X_1 \times X_2) \oplus H^{k-2}(B_1 \times X_2) \oplus H^{k-2}(X_1 \times B_2) \oplus H^{k-4}(B_1 \times B_2),$$

then by applying Lemma 5.3, we can see that the degree  $k$  part of the kernel of the quotient map

$$H^\bullet(\tilde{X}_1 \times \tilde{X}_2) \rightarrow H^\bullet(\tilde{X}_1) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(\tilde{X}_2)$$

is generated by  $(M_1, M_2, M_3, M_4)$ , where

$$M_1 = (\alpha \cup c_1(\mathcal{L}_1) + \iota_{1!}\beta) \cup \gamma - \alpha \cup (\gamma \cup c_1(\mathcal{L}_2) + \iota_{2!}\delta),$$

$$M_2 = (\iota_1^*\alpha - \beta \cup c_1(\iota_1^*\mathcal{L}_1)) \cup \gamma - \beta \cup (\gamma \cup c_1(\mathcal{L}_2) + \iota_{2!}\delta),$$

$$M_3 = (\alpha \cup c_1(\mathcal{L}_1) + \iota_{1!}\beta) \cup \delta - \alpha \cup (\iota_2^*\gamma - \delta \cup c_1(\iota_2^*\mathcal{L}_2)),$$

$$M_4 = (\iota_1^*\alpha - \beta \cup c_1(\iota_1^*\mathcal{L}_1)) \cup \delta - \beta \cup (\iota_2^*\gamma - \delta \cup c_1(\iota_2^*\mathcal{L}_2)),$$

and  $\alpha \in H^l(X_1)$ ,  $\beta \in H^{l-2}(B_1)$ ,  $\gamma \in H^{k-l-2}(X_2)$ ,  $\delta \in H^{k-l-4}(B_2)$  with  $0 \leq l \leq k$ . If we put  $\beta = \delta = 0$ , then

$$(M_1, M_2, M_3, M_4) = (\alpha \cup c_1(\mathcal{L}_i) \cup \gamma - \alpha \cup \gamma \cup c_1(\mathcal{L}_2), \iota_1^*\alpha \cup \gamma, -\alpha \cup \iota_2^*\gamma, 0).$$

Take the projection of it to  $(M_2, M_3)$  factor, and we can see that the subspace generated by these elements has dimension greater than or equal to

$$\dim_{\mathbb{C}} \ker(H^{k-2}(B_1 \times X_2) \oplus H^{k-2}(X_1 \times B_2) \rightarrow H^{k-2}(B_1 \times B_2)).$$

Similarly, if we put either  $\beta = \gamma = 0$  or  $\alpha = \delta = 0$  and project them to the  $M_4$  factor, then these elements generate the subspace

$$\text{im}(H^{k-4}(X_1 \times B_2) \oplus H^{k-4}(B_1 \times X_2) \rightarrow H^{k-4}(B_1 \times B_2))$$

in the  $H^{k-4}(B_1 \times B_2)$  factor. So the dimension of the subspace generated by  $(M_1, M_2, M_3, M_4)$  with either  $\beta = \gamma = 0$  or  $\alpha = \delta = 0$  is greater than or equal to

$$\dim_{\mathbb{C}} H^{k-4}(B_1 \times B_2)$$

for  $k \neq m_1 + m_2$ . This proves the claim.

The surjectivity in  $k = m_1 + m_2 - 2$  doesn't follow from this picture. Instead, we will use the Leray spectral sequence to compute the dimension of  $H^{m_1+m_2-2}(X_1 \times_{\mathbb{P}^1} X_2)$  and show that it is equal to the dimension of degree  $m_1 + m_2 - 2$  part of

$$H^\bullet(\tilde{X}_1) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(\tilde{X}_1).$$

We will prove the following lemma, where  $h^k$ ,  $h_{\text{van}}^k$  means the  $\mathbb{C}$ -dimension of the cohomology  $H^k$ ,  $H_{\text{van}}^k$  respectively:

**Lemma 5.4.** *Let  $d_i$  be the cardinality of  $\Delta(f_i)$ . Then*

- (1)  $h^{m_i}(\tilde{X}_i) = d_i + h^{m_i}(Y_i) + h^{m_i-2}(Y_i) - 2h_{\text{van}}^{m_i-1}(Y_i)$ ,
- (2) *The following formula holds:*

$$\begin{aligned} h^{m_1+m_2-2}(\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2) &= h^{m_1+m_2-2}(Y_1 \times Y_2) - h_{\text{van}}^{m_1-1}(Y_1)h^{m_2-1}(Y_2) \\ &\quad - h^{m_1-1}(Y_1)h_{\text{van}}^{m_2-1}(Y_2) + h_{\text{van}}^{m_1-1}(Y_1)h_{\text{van}}^{m_2-1}(Y_2) \\ &\quad + h^{m_1+m_2-4}(Y_1 \times Y_2) - h_{\text{van}}^{m_1-1}(Y_1)h^{m_2-3}(Y_2) \\ &\quad - h^{m_1-3}(Y_1)h_{\text{van}}^{m_2-1}(Y_2) + d_1h^{m_2-2}(Y_2) + d_2h^{m_1-1}(Y_1) \\ &\quad - 2(h_{\text{van}}^{m_1-1}(Y_1)h^{m_2-2}(Y_2) + h^{m_1-2}(Y_1)h_{\text{van}}^{m_2-1}(Y_2)) \end{aligned}$$

Assuming the lemma, note also that the degree  $m_1 + m_2 - 2$  part of

$$H^\bullet(\tilde{X}_1) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(\tilde{X}_1)$$

has dimension

$$(5.1) \quad \begin{aligned} &h^{m_1+m_2-2}(\tilde{X}_1 \times \tilde{X}_2) - h^{m_1+m_2-6}(B_1 \times B_2) - h^{m_1+m_2-4}(B_1 \times X_2) \\ &- h^{m_1+m_2-4}(X_1 \times B_2) + h^{m_1+m_2-4}(B_1 \times B_2) - h_{\text{van}}^{m_1-2}(B_1)h_{\text{van}}^{m_2-2}(B_2). \end{aligned}$$

Thus, by canceling  $d_i$  of Lemma 5.4 (2) using Lemma 5.4 (1), it is pretty elementary to verify that  $h^{m_1+m_2-2}(\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2)$  is equal to the formula (5.1) by the Lefschetz hyperplane theorem, the Künneth formula and the Poincaré duality.  $\square$

*Proof of Lemma 5.4.* We will apply the method discussed in [33, 4.5.4]. We first consider the Lefschetz fibration  $\tilde{X}_i \xrightarrow{f_i} \mathbb{P}^1$  and prove the formula (1). In this case, the Leray spectral sequence for  $f_i$  degenerates at  $E_2$  (cf. [33, Theorem 4.24]). Moreover, the Lefschetz fibration satisfies the local invariant cycle property ([33, Corollary C.21]): the adjunction homomorphism

$$a_k : R^k f_{i*} \underline{\mathbb{Z}}_{X_i} \rightarrow j_{i*} j_i^* R^k f_{i*} \underline{\mathbb{Z}}_{X_i}$$

is surjective for all  $k$ , where  $j_i : U_i := \mathbb{P}^1 - \Delta(f_i) \hookrightarrow \mathbb{P}^1$  is the inclusion map.

We need to divide the computation into two cases.

**Case I:** The first case (the generic case) is that  $H_{\text{van}}^{m_i-1}(Y_i)$  is non-zero, or equivalently, all of the vanishing cycles  $\delta_x$  for  $x \in \Delta(f_i)$  are non-zero in  $H^{m_i-1}(Y_i)$  since the vanishing cycles are conjugate under monodromy. In this case,  $R^k f_{i*} \underline{\mathbb{Z}}_{X_i}$  is locally free for  $k \neq m_i - 1$  and the adjunction homomorphism  $a_{m_i-1}$  is an isomorphism ([33, Lemma C.13]). Therefore, the Leray spectral sequence for  $f_i$  reads

$$\begin{aligned} E_2^{0, m_i}(f_i) &= H^{m_i}(Y_i, \mathbb{C}), \\ E_2^{1, m_i-1}(f_i) &= H^1(\mathbb{P}^1, j_{i*} j_i^* R^k f_{i*} \underline{\mathbb{C}}_{X_i}), \end{aligned}$$

$$E_2^{2,m_i-2}(f_i) = H^{m_i-2}(Y_i, \mathbb{C}).$$

Let  $\mathcal{L}$  be the local system  $j_i^* R^{m_i-1} f_{i*} \underline{\mathbb{C}}_{X_i}$  on  $U_i$ . We can calculate  $E_2^{1,m_i-1}(f_i)$  by the exact sequence

$$0 \rightarrow j_{i!} \mathcal{L} \rightarrow j_{i*} \mathcal{L} \rightarrow \bigoplus_{x \in \Delta(f_i)} \mathcal{L}_x^T \rightarrow 0,$$

where  $\Delta_x^*$  is a small punctured disk centered at  $x$ ,  $\mathcal{L}_x$  is the restriction of  $\mathcal{L}$  on  $\Delta_x^*$ , and  $\mathcal{L}_x^T$  is the subspace of invariants in  $\mathcal{L}_x$  under the local monodromy  $T$ . By taking the long exact sequence of the cohomology, we have

$$0 \rightarrow H^0(U_i, \mathcal{L}) \rightarrow \bigoplus_{x \in \Delta(f_i)} \mathcal{L}_x^T \rightarrow H_c^1(U_i, \mathcal{L}) \rightarrow H^1(\mathbb{P}^1, j_{i*} \mathcal{L}) \rightarrow 0.$$

By the theorem of the fixed part ([33, Theorem 4.23]),

$$H^0(U_i, \mathcal{L}) = \text{im}(H^{m_i-1}(\tilde{X}_i) \rightarrow H^{m_i-1}(Y_i)) = \text{im}(H^{m_i-1}(X_i) \rightarrow H^{m_i-1}(Y_i)).$$

The last equality follows from Lemma 4.9 since the Gysin morphism  $H^{m_i-3}(B_i) \rightarrow H^{m_i-1}(Y_i)$  can be identified via the Lefschetz hyperplane theorem with the composition  $H^{m_i-3}(X_i) \rightarrow H^{m_i-1}(X_i) \rightarrow H^{m_i-1}(Y_i)$ , where the first map is the cup product with the Lefschetz operator and the second map is the restriction map. Hence, we see that

$$h^0(U, \mathcal{L}) = h^{m_i-1}(Y_i) - h_{\text{van}}^{m_i-1}(Y_i).$$

Also, since  $\mathcal{L}_x^T$  is generated by the class in  $H^{m_i-1}(Y_i)$  which is orthogonal to the vanishing cycle  $\delta_x$ ,  $\mathcal{L}_x^T$  has dimension  $h^{m_i-1}(Y_i) - 1$  for all  $x \in \Delta(f_i)$ . On the other hand,  $h_c^1(U_i, \mathcal{L}) = h^1(U_i, \mathcal{L}^\vee)$  by the Poincaré duality and we have

$$H^k(U_i, \mathcal{L}^\vee) = \text{Ext}_{\mathbb{Z}[\pi_1(U_i)]}^k(\mathbb{Z}, \mathcal{L}^\vee)$$

for all  $k$ . Since  $U_i$  is homotopic to  $\bigvee_{i=1}^{d_i-1} S^1$ ,  $\mathbb{Z}[\pi_1(U_i)]$  is the group ring of a free group of  $d_i - 1$  generators. There is a natural two-term free resolution of  $\mathbb{Z}$  as a  $\mathbb{Z}[\pi_1(U_i)]$  module (cf. [48, Corollary 6.2.7]):

$$0 \rightarrow \mathbb{Z}[\pi_1(U_i)]^{d_i-1} \rightarrow \mathbb{Z}[\pi_1(U_i)] \rightarrow \mathbb{Z} \rightarrow 0.$$

Therefore, by the isomorphism  $H^0(U_i, \mathcal{L}^\vee) \cong H^0(U_i, \mathcal{L})$ , we have

$$h^1(U_i, \mathcal{L}^\vee) - h^0(U_i, \mathcal{L}) = (d_i - 2)\text{rank}(\mathcal{L}).$$

As a consequence,

$$\begin{aligned} h^1(\mathbb{P}^1, j_{i*} \mathcal{L}) &= h^1(U_i, \mathcal{L}^\vee) + h^0(U_i, \mathcal{L}) - \dim \left( \bigoplus_{x \in \Delta(f_i)} \mathcal{L}_x^T \right) \\ &= 2h^0(U_i, \mathcal{L}) + (d_i - 2)\text{rank}(\mathcal{L}) - d_i(\text{rank} \mathcal{L} - 1) \\ &= d_i - 2h_{\text{van}}^{m_i-1}(Y_i). \end{aligned}$$

By summing the dimension of  $E^{0,m_i}(f_i)$ ,  $E^{1,m_i-1}(f_i)$  and  $E^{2,m_i-2}(f_i)$ , we get the formula in Lemma 5.4 (1).

**Case II:** The second case we need to consider is the special case that  $H_{\text{van}}^{m_i-1}(Y_i)$  is zero, i.e., the restriction map  $H^{m_i-1}(X_i) \rightarrow H^{m_i-1}(Y_i)$  is an isomorphism. In this

case,  $R^k f_{i*} \underline{\mathbb{C}}_{X_i}$  is locally constant for  $k \neq m_i$ , and for  $k = m_i$ , we have the short exact sequence

$$0 \rightarrow \bigoplus_{x \in \Delta(f_i)} H^{m_i-1}(F_x) \rightarrow R^{m_i} f_{i*} \underline{\mathbb{C}}_{X_i} \rightarrow j_{i*} j_i^* R^{m_i} f_{i*} \underline{\mathbb{C}}_{X_i} \rightarrow 0,$$

where  $H^{m_i-1}(F_x)$  is the skyscraper sheaf (supported on  $x$ ) of the cohomology of the Milnor fiber of  $f_i$  around  $x$ . Also,  $j_{i*} j_i^* R^{m_i} f_{i*} \underline{\mathbb{C}}_{X_i}$  is a locally constant sheaf with stalks isomorphic to  $H^{m_i}(Y_i)$ . Therefore, the Leray spectral sequence for  $f_i$  reads

$$\begin{aligned} E_2^{0,m_i}(f_i) &= H^0(\mathbb{P}^1, R^{m_i} f_{i*} \underline{\mathbb{C}}_{X_i}), \\ E_2^{1,m_i-1}(f_i) &= 0, \\ E_2^{2,m_i-2}(f_i) &= H^{m_i-2}(Y_i, \mathbb{C}), \end{aligned}$$

and we have

$$h^0(\mathbb{P}^1, R^{m_i} f_{i*} \underline{\mathbb{C}}_{X_i}) = d_i + h^{m_i}(Y_i, \mathbb{C}).$$

The formula in Lemma 5.4 (1) follows.

Now we turn to the fibration  $f : \tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2 \rightarrow \mathbb{P}^1$  and prove the formula in Lemma 5.4 (2). Write  $\Delta(f) = \Delta(f_1) \cup \Delta(f_2)$ . We want to carry out the same computation as in the Lefschetz fibration case for  $f$ . Since  $\Delta(f_1)$  and  $\Delta(f_2)$  are disjoint and the local monodromy action on  $H^k(Y_1 \times Y_2) = \bigoplus_{i+j=k} H^i(Y_1) \otimes H^j(Y_2)$  is induced by product group action of local monodromy group action on  $H^i(Y_1)$  and  $H^j(Y_2)$ , we still have the local invariant cycle property for  $f$ : the adjunction homomorphism

$$a_k : R^k f_* \underline{\mathbb{Z}}_{\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2} \rightarrow j_* j^* R^k f_* \underline{\mathbb{Z}}_{\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2}$$

is surjective for all  $k$ , where  $j : U := \mathbb{P}^1 - \Delta(f) \hookrightarrow \mathbb{P}^1$  is the inclusion map.

We divide the computation into three cases: neither of  $Y_i$  has vanishing cohomology, exactly one of  $Y_i$  has vanishing cohomology, and both of  $Y_i$  have vanishing cohomology.

**Case I' (the generic case):**  $H_{\text{van}}^{m_i}(Y_i) \neq 0$  for  $i = 1, 2$ .

For  $x \in \Delta(f_1)$ , let  $\Delta_x$  be a small punctured disc centered at  $x$  and  $B_x$  be a small ball around the singularity at  $f_1^{-1}(x)$ . Let  $B_x \cap f_1^{-1}(\Delta_x - \{x\}) \xrightarrow{f_1} \Delta_x - \{x\}$  denote the Milnor fibration around the singularity at  $f_1^{-1}(x)$  and  $F_x$  the Milnor fiber. By the Künneth formula and the excision theorem, for all  $t \in \Delta_x - \{x\}$ , we have

$$\begin{aligned} H^{k+1}(f^{-1}\Delta_x, f^{-1}(t)) &= H^{k+1}(f_1^{-1}\Delta_x \times Y_2, Y_1 \times Y_2) \\ &\cong \bigoplus_{i+j=k+1} H^i(f_1^{-1}\Delta_x, Y_1) \otimes H^j(Y_2) \\ &\cong \bigoplus_{i+j=k+1} H^i(B_x, F_x) \otimes H^j(Y_2) \\ &\cong H^{m_1-1}(F_x) \otimes H^{k+1-m_1}(Y_2). \end{aligned}$$

Therefore, the long exact sequence of the cohomology for the pair  $(f^{-1}\Delta_x, f^{-1}(t))$  gives

$$0 \rightarrow H^k(f^{-1}\Delta_x) \rightarrow H^k(Y_1 \times Y_2) \rightarrow H^{m_1-1}(F_x) \otimes H^{k+1-m_1}(Y_2) \rightarrow 0.$$

The surjectivity on the right (and the injectivity on the left) follows from the assumption that the vanishing cycles for  $f_1$  survive globally. A similar short exact sequence holds

for  $x \in \Delta(f_2)$ . In particular, the adjunction homomorphism

$$a_k : R^k f_* \underline{\mathbb{C}}_{\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2} \rightarrow j_* j^* R^k f_* \underline{\mathbb{C}}_{\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2}$$

is an isomorphism for all  $k$ . For all  $k$ , let  $\mathcal{L}^k$  be the local system  $j^* R^k f_* \underline{\mathbb{C}}_{\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2}$  on  $U$ . We have the short exact sequence

$$0 \rightarrow j_! \mathcal{L}^k \rightarrow j_* \mathcal{L}^k \rightarrow \bigoplus_{x \in \Delta(f)} (\mathcal{L}_x^k)^T \rightarrow 0,$$

where  $\Delta_x^*$  is a small punctured disk centered at  $x$ ,  $\mathcal{L}_x^k$  is the restriction of  $\mathcal{L}^k$  on  $\Delta_x^*$ , and  $(\mathcal{L}_x^k)^T$  is the subspace of invariants in  $\mathcal{L}_x^k$  under the local monodromy  $T$ . For each  $k$ ,

$$H^0(\mathbb{P}^1, j_* \mathcal{L}^k) = H^k(Y_1 \times Y_2)^{\pi_1(U)} = \bigoplus_{i+j=k} H^i(Y_1)^{\pi_1(U_1)} \otimes H^j(Y_2)^{\pi_1(U_2)},$$

and for  $i = 1, 2$ ,

$$H^l(Y_i)^{\pi_1(U_i)} = \begin{cases} H^l(Y_i), & l \neq m_i - 1 \\ \text{im}(H^l(X_i) \rightarrow H^l(Y_i)), & l = m_i - 1. \end{cases}$$

Also,  $H^2(\mathbb{P}^1, j_* \mathcal{L}^k) = H^2(\mathbb{P}^1, j_! \mathcal{L}^k) = H_c^2(U, \mathcal{L})$  is dual to  $H^0(U, \mathcal{L}^\vee) = H^0(\mathbb{P}^1, j_* \mathcal{L})$ . So we have

$$h^2(\mathbb{P}^1, j_* \mathcal{L}^k) = h^0(\mathbb{P}^1, j_* \mathcal{L}^k).$$

On the other hand, we have the exact sequence

$$0 \rightarrow H^0(U, \mathcal{L}^k) \rightarrow \bigoplus_{x \in \Delta(f)} (\mathcal{L}_x^k)^T \rightarrow H_c^1(U, \mathcal{L}^k) \rightarrow H^1(\mathbb{P}^1, j_* \mathcal{L}^k) \rightarrow 0.$$

Note that

$$\dim(\mathcal{L}_x^k)^T = \begin{cases} h^k(Y_1 \times Y_2) - h^{k+1-m_1}(Y_2), & x \in \Delta(f_1) \\ h^k(Y_1 \times Y_2) - h^{k+1-m_2}(Y_1), & x \in \Delta(f_2) \end{cases},$$

and similar to the Lefschetz fibration case, we have

$$h_c^1(U, \mathcal{L}^k) - h^0(U, \mathcal{L}^k) = (d_1 + d_2 - 2) \text{rank} \mathcal{L}^k.$$

As a result,

$$\begin{aligned} h^1(\mathbb{P}^1, j_* \mathcal{L}^k) &= h_c^1(U, \mathcal{L}^k) + h^0(U, \mathcal{L}^k) - \dim \left( \bigoplus_{x \in \Delta(f)} (\mathcal{L}_x^k)^T \right) \\ &= 2h^0(U, \mathcal{L}^k) + (d_1 + d_2 - 2) \text{rank} \mathcal{L}^k \\ &\quad - d_1 (\text{rank} \mathcal{L}^k - h^{k+1-m_1}(Y_2)) - d_2 (\text{rank} \mathcal{L}^k - h^{k+1-m_2}(Y_1)) \\ &= 2h^0(U, \mathcal{L}^k) + d_1 h^{k+1-m_1}(Y_2) + d_2 h^{k+1-m_2}(Y_1) - 2h^k(Y_1 \times Y_2) \end{aligned}$$

It follows that

$$\begin{aligned} \dim_{\mathbb{C}} E_2^{0, m_1+m_2-2}(f) &= h^0(\mathbb{P}^1, j_* \mathcal{L}^{m_1+m_2-2}) \\ &= h^{m_1+m_2-2}(Y_1 \times Y_2) - h_{\text{van}}^{m_1-1}(Y_1) h^{m_2-1}(Y_2) \\ &\quad - h^{m_1-1}(Y_1) h_{\text{van}}^{m_2-1}(Y_2) + h_{\text{van}}^{m_1-1}(Y_1) h_{\text{van}}^{m_2-1}(Y_2), \end{aligned}$$

$$\begin{aligned}
\dim_{\mathbb{C}} E_2^{1, m_1+m_2-3}(f) &= h^1(\mathbb{P}^1, j_* \mathcal{L}^{m_1+m_2-3}) \\
&= d_1 h^{m_2-2}(Y_2) + d_2 h^{m_1-2}(Y_1) \\
&\quad - 2(h_{\text{van}}^{m_1-1}(Y_1) h^{m_2-2}(Y_2) + h^{m_1-2}(Y_1) h_{\text{van}}^{m_2-1}(Y_2)),
\end{aligned}$$

and

$$\begin{aligned}
\dim_{\mathbb{C}} E_2^{2, m_1+m_2-4}(f) &= h^0(\mathbb{P}^1, j_* \mathcal{L}^{m_1+m_2-4}) \\
&= h^{m_1+m_2-4}(Y_1 \times Y_2) - h_{\text{van}}^{m_1-1}(Y_1) h^{m_2-3}(Y_2) \\
&\quad - h^{m_1-3}(Y_1) h_{\text{van}}^{m_2-1}(Y_2).
\end{aligned}$$

We get the formula in Lemma 5.4 (2) by summing the three dimensions.

**Case II':**  $H_{\text{van}}^{m_1-1}(Y_1) = 0$  and  $H_{\text{van}}^{m_2-1}(Y_2) \neq 0$ .

In this case, for  $x \in \Delta(f_2)$ , the same local computation of  $R^k f_* \underline{\mathbb{C}}_{\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2}$  as in the previous case holds. For  $x \in \Delta(f_1)$ , since the cohomology class of the vanishing cycle  $\delta_x$  is zero, we have instead the short exact sequence

$$0 \rightarrow H^{m_1-1}(F_x) \otimes H^{k-m_1}(Y_2) \rightarrow H^k(f^{-1}\Delta_x) \rightarrow H^k(Y_1 \times Y_2) \rightarrow 0.$$

Thus we have a short exact sequence

$$0 \rightarrow \bigoplus_{x \in \Delta(f_1)} H^{m_1-1}(F_x) \otimes H^{k-m_1}(Y_2) \rightarrow R^k f_* \underline{\mathbb{C}}_{\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2} \rightarrow j^* j_* R^k f_* \underline{\mathbb{C}}_{\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2} \rightarrow 0,$$

where  $H^{m_1-1}(F_x) \otimes H^{k-m_1}(Y_2)$  is regarded as a skyscraper sheaf supported on  $x$ . Also, with the notation in the previous case, we have instead

$$\dim(\mathcal{L}_x^k)^T = \begin{cases} h^k(Y_1 \times Y_2), & x \in \Delta(f_1) \\ h^k(Y_1 \times Y_2) - h^{k+1-m_2}(Y_1), & x \in \Delta(f_2) \end{cases},$$

As a result,

$$\begin{aligned}
\dim_{\mathbb{C}} E_2^{0, m_1+m_2-2}(f) &= h^0(\mathbb{P}^1, j_* \mathcal{L}^{m_1+m_2-2}) + \dim \left( \bigoplus_{x \in \Delta(f_1)} H^{m_2-2}(Y_2) \right) \\
&= h^{m_1+m_2-2}(Y_1 \times Y_2) - h^{m_1-1}(Y_1) h_{\text{van}}^{m_2-1}(Y_2) + d_1 h^{m_2-2}(Y_2),
\end{aligned}$$

$$\begin{aligned}
\dim_{\mathbb{C}} E_2^{1, m_1+m_2-3}(f) &= h^1(\mathbb{P}^1, j_* \mathcal{L}^{m_1+m_2-3}) \\
&= d_2 h^{m_1-2}(Y_1) - 2h^{m_1-2}(Y_1) h_{\text{van}}^{m_2-1}(Y_2),
\end{aligned}$$

and

$$\begin{aligned}
\dim_{\mathbb{C}} E_2^{2, m_1+m_2-4}(f) &= h^0(\mathbb{P}^1, j_* \mathcal{L}^{m_1+m_2-4}) \\
&= h^{m_1+m_2-4}(Y_1 \times Y_2) - h^{m_1-3}(Y_1) h_{\text{van}}^{m_2-1}(Y_2).
\end{aligned}$$

We still get the formula in Lemma 5.4 (2) by summing the three dimensions.

**Case III':**  $H_{\text{van}}^{m_1-1}(Y_1) = H_{\text{van}}^{m_2-1}(Y_2) = 0$ .

In this case, we see that  $j^* j_* R^k f_* \underline{\mathbb{C}}_{\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2}$  is a locally constant sheaf and the kernel of the adjunction homomorphism  $a_k$  is

$$\bigoplus_{x \in \Delta(f_1)} H^{m_1-1}(F_x) \otimes H^{k-m_1}(Y_2) \oplus \bigoplus_{x \in \Delta(f_2)} H^{k-m_2}(Y_1) \times H^{m_2-1}(F_x).$$

Hence,

$$\dim_{\mathbb{C}} E_2^{0, m_1+m_2-2}(f) = h^{m_1+m_2-2}(Y_1 \times Y_2) + d_1 h^{m_2-2}(Y_2) + d_2 h^{m_1-2}(Y_1),$$

$$\dim_{\mathbb{C}} E_2^{1, m_1+m_2-3}(f) = 0,$$

and

$$\dim_{\mathbb{C}} E_2^{2, m_1+m_2-4}(f) = h^0(\mathbb{P}^1, j_* \mathcal{L}^{m_1+m_2-4}) = h^{m_1+m_2-4}(Y_1 \times Y_2).$$

We still get the formula in Lemma 5.4 (2) by summing the three dimensions.  $\square$

**Remark 5.5.** Apply a similar computation, we can prove that the difference of

$$h^{m_1+m_2-1}(\tilde{X}_1 \times_{\mathbb{P}^1} \tilde{X}_2)$$

and the dimension of degree  $m_1 + m_2 - 1$  part of

$$H^\bullet(\tilde{X}_1) \otimes_{H^\bullet(\mathbb{P}^1)} H^\bullet(\tilde{X}_1)$$

is

$$\begin{aligned} & (h^{m_1}(\tilde{X}_1) - 2h^{m_1-2}(X_1)) h_{\text{van}}^{m_2-1}(Y_2) + h_{\text{van}}^{m_1-1}(Y_1) (h^{m_2}(\tilde{X}_2) - 2h^{m_2-2}(X_2)) \\ & + h_{\text{van}}^{m_1-2}(B_1) (h^{m_2-1}(X_2) - h^{m_2-3}(X_2)) + (h^{m_1-1}(X_1) - h^{m_1-3}(X_1)) h_{\text{van}}^{m_2-2}(B_2) \\ & + 2h_{\text{van}}^{m_1-1}(Y_1) h_{\text{van}}^{m_2-1}(Y_2). \end{aligned}$$

Therefore, the difference can be nonzero even if both  $Y_1$  and  $Y_2$  have no vanishing cohomology.

## 6 Proof of Theorem 2.4 and 2.5

Recall that in Section 2, we consider the following two situations:

- (A') We are given a mixed Hodge structure  $(H, W, F)$  defined over  $\mathbb{R}$  and a nilpotent endomorphism  $N$  on  $H$  such that
  - (a)  $N$  is a morphism of mixed Hodge structure of type  $(-1, -1)$ ,
  - (b)  $W = W(N, d)$  is the weight filtration of  $N$  centered at  $d$ .
- (B') Besides the assumption in (1), we are given a non-degenerate bilinear form  $S$  on  $H$  defined over  $\mathbb{R}$ , which satisfies
  - (a)  $S(v, u) = (-1)^d S(u, v)$ ,
  - (b)  $S(u, Nv) + S(Nu, v) = 0$ ,
  - (c)  $S(F^p, F^{d-p+1}) = 0$  for all  $p$ .

In the situation (B'), since  $W = W(N, d)$ , we must have  $S(W_a, W_b) = 0$  for  $a+b \leq 2d-1$ . Therefore, for  $p+q \geq d$ , we can define  $(s_+^{p,q}, s_-^{p,q})$  to be the signature of  $S(C\bullet, N^{p+q-d}\bullet)$  on the  $(p, q)$ -component of the Hodge structure (of weight  $p+q$ ) on the primitive part

$$P_{p+q} = \ker(N^{p+q-d+1} : \text{Gr}_{p+q}^W H \rightarrow \text{Gr}_{2d-p-q-2}^W H),$$

where  $C$  is the Weil operator.

Consider Deligne's weak splitting of real mixed Hodge structure (cf. [33, Lemma-Definition 3.4]):

$$I^{p,q} := F^p \cap W_{p+q} \cap (\overline{F^q} \cap W_{p+q} + \sum_{j \geq 2} \overline{F^{q-j+1}} \cap W_{p+q-j}).$$

Then, in the situation (A'), this splitting satisfies the following conditions:

- (1)  $W_l = \bigoplus_{p+q \leq l} I^{p,q}$ ,  $F^r = \bigoplus_{p \geq r} I^{p,q}$ .
- (2)  $I^{p,q} \equiv \overline{I^{q,p}} \pmod{\bigoplus_{r \leq p-1, s \leq q-1} I^{r,s}}$ .
- (3)  $NI^{p,q} \subset I^{p-1, q-1}$ .

In the situation  $(\mathbf{B}')$ , Deligne's splitting also satisfies

- (4)  $S(I^{p,q}, I^{r,s}) = 0$  unless  $r = d - p$ ,  $s = d - q$ .
- (5) If we define the primitive part of  $I^{p,q}$  to be

$$I_{\text{prim}}^{p,q} := I^{p,q} \cap \ker N^{p+q-d+1},$$

then  $(\sqrt{-1})^{p-q} S(\bullet, N^{p+q-d}\bullet)$  has signature  $(s_+^{p,q}, s_-^{p,q})$  on  $I_{\text{prim}}^{p,q}$ .

(1), (2), (3), and (5) follow directly from the definition of Deligne's splitting. Let's prove (4) here:

*Proof of (4).* Since  $S(W_a, W_b) = 0$  for  $a+b \leq 2d-1$ , we have  $S(I^{p,q}, I^{r,s}) = 0$  if  $r+s+p+q \leq 2d-1$ . Also, since  $S(F^a, F^b) = 0$  for  $a+b \geq d+1$  by assumption, we have  $S(I^{p,q}, I^{r,s}) = 0$  if  $r+p \geq d+1$ . Now, suppose that  $r+p \leq d$ , we claim that  $S(I^{p,q}, I^{r,s}) = 0$  for  $q+s \geq d+1$  too. Were this proved, the only nonzero possibility is  $r = d-p$  and  $s = d-q$ . To see this claim, write

$$I^{p,q} := F^p \cap W_{p+q} \cap (\overline{F^q} \cap W_{p+q} + \sum_{j \geq 2} \overline{F^{q-j+1}} \cap W_{p+q-j})$$

and

$$I^{r,s} := F^r \cap W_{r+s} \cap (\overline{F^s} \cap W_{r+s} + \sum_{i \geq 2} \overline{F^{s-i+1}} \cap W_{r+s-i}).$$

Again, since  $S(\overline{F^a}, \overline{F^b}) = 0$  for  $a+b \geq d+1$  and  $S(W_a, W_b) = 0$  for  $a+b \leq 2d-1$ , we have

$$S(\overline{F^{q-j+1}} \cap W_{p+q-j}, \overline{F^{s-i+1}} \cap W_{r+s-i}) = 0$$

if either  $q-j+s-i \geq d-1$  or  $p+q+r+s-i-j \leq 2d-1$ ; or equivalently

$$S(\overline{F^{q-j+1}} \cap W_{p+q-j}, \overline{F^{s-i+1}} \cap W_{r+s-i})$$

is nonzero only if  $p+r \geq d+2$ , which is empty under our assumption. Similarly, we can see that

$$S(\overline{F^q} \cap W_{p+q}, \overline{F^{s-i+1}} \cap W_{r+s-i})$$

and

$$S(\overline{F^{q-j+1}} \cap W_{p+q-j}, \overline{F^s} \cap W_{r+s})$$

are nonzero only if  $p+r \geq d+1$ , which is also empty. Finally, we have

$$S(\overline{F^q} \cap W_{p+q}, \overline{F^s} \cap W_{r+s}) = 0$$

for  $q+s \geq d+1$ . The claim is proved.  $\square$

The key point of our calculation is the following combinatorial lemma:

**Lemma 6.1.** *Let  $u \in V$  be a vector and  $N$  a nilpotent operator on  $V$ . Suppose that  $N^{n+1}u = 0$  but  $N^n u \neq 0$ . Then for all  $0 \leq k \leq n+1$  we have*

$$\begin{aligned} & (e^{zN} u \wedge e^{zN} Nu \wedge \dots \wedge e^{zN} N^{n-k} u) \wedge (e^{\bar{z}N} u \wedge e^{\bar{z}N} Nu \wedge \dots \wedge e^{\bar{z}N} N^{k-1} u) \\ &= \frac{C_{n-k+1, k}}{((n-k+1)k)!} (\bar{z} - z)^{(n-k+1)k} u \wedge Nu \wedge \dots \wedge N^n u, \end{aligned}$$

where  $C_{n-k+1,k}$  is the number of standard Young tableaux of the shape  $k \times (n-k+1)$  squares.

*Proof.* Let

$$\Phi = (e^{zN}u \wedge e^{zN}Nu \wedge \dots \wedge e^{zN}N^{n-k}u) \wedge (e^{\bar{z}N}u \wedge e^{\bar{z}N}Nu \wedge \dots \wedge e^{\bar{z}N}N^{k-1}u).$$

We first show that  $\Phi$  is a homogeneous polynomial in  $z$  and  $\bar{z}$  of degree  $(n-k+1)k$ . To see this, note that the difference of the degree in  $z$  (resp,  $\bar{z}$ ) and the degree in  $N$  in each term of  $e^{zN}N^l u$  (resp.  $e^{\bar{z}N}N^l u$ ) is  $l$ , so the difference of the degree in  $z$ ,  $\bar{z}$  and the degree in  $N$  in  $\Phi$  will be  $\frac{(n-k+1)(n-k)}{2} + \frac{k(k-1)}{2}$ . Also,  $\Phi$  is a polynomial in  $z$  and  $\bar{z}$  over  $u \wedge Nu \wedge \dots \wedge N^n u$ , and the degree of  $u \wedge Nu \wedge \dots \wedge N^n u$  in  $N$  is  $\frac{(n+1)n}{2}$ . So the degree of  $\Phi$  in  $z$  and  $\bar{z}$  should be

$$\frac{(n+1)n}{2} - \frac{(n-k+1)(n-k)}{2} - \frac{k(k-1)}{2} = (n-k+1)k.$$

Next we compute the limit  $\lim_{\bar{z} \rightarrow z} \frac{\Phi}{(\bar{z}-z)^{k(n-k+1)}}$ . It suffices to show that this limit is finite and has a coefficient equal to  $\frac{C_{n-k+1,k}}{((n-k+1)k)!}$ . In fact, since this limit is in indefinite form, we may apply the L'hospital rule (with respect to partial derivative in  $\bar{z}$ ). Observe that at each time we take the partial derivative to  $\Phi$  with respect to  $\bar{z}$ , we replace one  $e^{\bar{z}N}N^l u$  by  $e^{\bar{z}N}N^{l+1}u$ , so the partial derivative will still be an indefinite form until  $k(n-k+1)$ -th partial derivative. In this case, we have already replaced  $e^{\bar{z}N}u \wedge e^{\bar{z}N}Nu \wedge \dots \wedge e^{\bar{z}N}N^{k-1}u$  by  $e^{\bar{z}N}N^{n-k+1}u \wedge e^{\bar{z}N}N^{n-k+2}u \wedge \dots \wedge e^{\bar{z}N}N^n u$ , and the coefficient is just  $\frac{1}{((n-k+1)k)!}$  times the number of the ways of moving  $\{1, \dots, k-1\}$  to  $\{n-k+1, \dots, n\}$  by replacing some element  $l$  with  $l+1$  but without repetition of elements. We can identify this combinatorial number as the number of standard Young tableaux of the shape  $k \times (n-k+1)$  squares. The lemma is proved.  $\square$

**Corollary 6.2.** Consider the  $(n+1) \times (n+1)$  upper triangular matrix  $A(x) = A_{0,n+1}(x)$  which is defined using the Taylor expansion of  $e^x$  by

$$A_{0,n+1}(x) = \begin{pmatrix} 1 & \frac{x}{1!} & \dots & \frac{x^{n-1}}{(n-1)!} & \frac{x^n}{n!} \\ & 1 & \frac{x}{1!} & \dots & \frac{x^{n-1}}{(n-1)!} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & \frac{x}{1!} \\ & & & & 1 \end{pmatrix}.$$

For  $0 \leq k \leq n+1$ , let  $A_{n-k+1,k}(x)$  be the upper right  $k \times k$  submatrix of  $A(x)$ , i.e.

$$A_{n-k+1,k}(x) = \begin{pmatrix} \frac{x^{n-k+1}}{(n-k+1)!} & \frac{x^{n-k+2}}{(n-k+2)!} & \dots & \frac{x^{n-1}}{(n-1)!} & \frac{x^n}{n!} \\ \frac{x^{n-k+2}}{(n-k+2)!} & \frac{x^{n-k+3}}{(n-k+3)!} & \dots & \frac{x^{n-2}}{(n-2)!} & \frac{x^{n-1}}{(n-1)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & \dots & \frac{x^{n-k+1}}{(n-k+1)!} & \frac{x^{n-k+2}}{(n-k+2)!} \\ & & & \dots & \frac{x^{n-k+1}}{(n-k+1)!} \end{pmatrix}.$$

Then the preceding lemma says that

$$\det(A_{n-k+1,k}(x)) = \frac{C_{n-k+1,k}}{((n-k+1)k)!} x^{(n-k+1)k},$$

where  $C_{n-k+1,k}$  is the number of standard Young tableaux of the shape  $k \times (n-k+1)$  squares.

*Proof of Theorem 2.4.* For  $p+q \geq d$ , We choose a basis  $\{u_i^{p,q} | i \in J_{\text{prim}}^{p,q}\}$  of  $I_{\text{prim}}^{p,q}$  for each  $p, q$ , where  $J_{\text{prim}}^{p,q}$  is an (ordered) indexed set, such that  $J_{\text{prim}}^{p,q} = J_{\text{prim}}^{q,p}$  and  $\overline{u_i^{p,q}} = u_i^{q,p} + s_i^{q,p}$  with  $s_i^{q,p} \in W_{q+p-2}$ . Using the weight filtration of  $N$ , we may extend this basis to a basis  $\{u_i^{p,q} | i \in J^{p,q}\}$  of  $I^{p,q}$ , where  $J^{p,q} = \coprod_{j \geq 0} J_{\text{prim}}^{p+q-j, j}$  for  $p+q \geq d$  and  $J^{p,q} = \coprod_{j \geq d-p-q} J_{\text{prim}}^{p+q-j, j}$  for  $p+q < d$ , such that

$$N^l u_i^{p,q} = u_i^{p-l, q-l}$$

for all  $i \in J_{\text{prim}}^{p,q}$  and  $0 \leq l \leq p+q-d$ , and such that

$$\overline{u_i^{p,q}} = u_i^{q,p} + s_i^{q,p}, \quad s_i^{q,p} \in W_{q+p-2}$$

for all  $p, q$  and  $i \in J^{p,q}$ .

We would like to show that  $\exp(zN)F^k \oplus \overline{\exp(zN)F^{d-k+1}} = H$  for  $\text{Im}(z)$  sufficiently large. Without loss of generality, we may assume  $k \leq d-k+1$ . Also, since

$$\exp((z+1)N)F^k \oplus \overline{\exp((z+1)N)F^{d-k+1}} = \exp(N) \left( \exp(zN)F^k \oplus \overline{\exp(zN)F^{d-k+1}} \right),$$

we may and do assume  $z$  lies in a domain with bounded real parts. It suffices to show that

$$\frac{(\bigwedge_{p \geq k, i \in J^{p,q}} e^{zN} u_i^{p,q}) \wedge (\bigwedge_{p \geq d-k+1, i \in J^{p,q}} \overline{e^{zN} u_i^{p,q}})}{\bigwedge_{p, q \geq 0, i \in J^{p,q}} u_i^{p,q}} \neq 0$$

for  $\text{Im}(z)$  sufficiently large.

Let  $u_i^{p,q} \in I_{\text{prim}}^{p,q}$  with  $i \in J_{\text{prim}}^{p,q}$  and  $p \geq k$ ,  $q \geq d-k+1$ , we first compute

$$\begin{aligned} \Phi_i^{p,q} &:= \left( \left( \bigwedge_{0 \leq l \leq p-k} e^{zN} u_i^{p-l, q-l} \right) \wedge \left( \bigwedge_{0 \leq l \leq q-d+k-1} e^{\bar{z}N} \overline{u_i^{q-l, p-l}} \right) \right) \\ &= \left( \bigwedge_{0 \leq l \leq p-k} e^{zN} N^l u_i^{p,q} \right) \wedge \left( \bigwedge_{0 \leq l \leq q-d+k-1} e^{\bar{z}N} (N^l u_i^{p,q} + s_i^{p-l, q-l}) \right) \end{aligned}$$

Then by the preceding lemma, we can write

$$\Phi_i^{p,q} = \Psi_i^{p,q} + \Xi_i^{p,q},$$

where

$$\Psi_i^{p,q} = \frac{C_{p-k+1, q-d+k}}{((p-k+1)(q-d+k))!} (\bar{z}-z)^{(p-k+1)(q-d+k)} u_i^{p,q} \wedge u_i^{p-1, q-1} \wedge \dots \wedge u_i^{d-q, d-p},$$

and

$$\Xi_i^{p,q} = \left( \bigwedge_{0 \leq l \leq p-k} e^{zN} u_i^{p-l, q-l} \right) \wedge \left( \sum_{0 \leq j \leq q-d+k-1} e^{\bar{z}N} s_i^{p,q} \wedge \dots \wedge e^{\bar{z}N} s_i^{p-j, q-j} \wedge \dots \right)$$

$$e^{\bar{z}N} \overline{u_i^{p-j-1, q-j-1}} \wedge \dots \wedge e^{\bar{z}N} \overline{u_i^{d-k+1, p-q+d-k+1}})$$

On the other hand, for  $u_i^{p,q} \in I_{\text{prim}}^{p,q}$  with  $i \in J_{\text{prim}}^{p,q}$  and  $p \geq k$ ,  $q < d - k + 1$ . We have

$$\Phi_i^{p,q} := \bigwedge_{0 \leq l \leq p+q-d} e^{zN} N^l u_i^{p,q} = u_i^{p,q} \wedge u_i^{p-1, q-1} \wedge \dots \wedge u_i^{d-q, d-p}.$$

If the mixed Hodge structure splits over  $\mathbb{R}$ , i.e.  $s_i^{p,q} = 0$  for all  $i, p, q$ , then  $\Xi_i^{p,q} = 0$  too. Then we have

$$\begin{aligned} & \frac{(\bigwedge_{p \geq k, i \in J_{p,q}} e^{zN} u_i^{p,q}) \wedge (\bigwedge_{p \geq d-k+1, i \in J_{p,q}} \overline{e^{zN} u_i^{p,q}})}{\bigwedge_{p, q \geq 0, i \in J_{p,q}} u_i^{p,q}} \\ &= C(\bar{z} - z)^{\sum_{p \geq k, q \geq d-k+1} |J_{\text{prim}}^{p,q}| (p-k+1)(q-d+k)}, \quad C \neq 0, \end{aligned}$$

which is nonzero for  $\text{Im}(z) \neq 0$  and the theorem is proved.

In the general case, we need to estimate the remainder term arising from  $\Xi_i^{p,q}$ . We can decompose  $\Xi_i^{p,q}$  into a sum of terms  $\Xi_{i,j_i}^{p,q}$  such that each  $\Xi_{i,j_i}^{p,q}$  is a wedge product of some of the basis  $u_{i'}^{p',q'}$  of  $H$  with polynomial (in  $\bar{z}$  and  $z$ ) coefficient. Consider two numbers associated to  $\Xi_{i,j_i}^{p,q}$ . The first number is the degree difference

$$\sigma(\Xi_{i,j_i}^{p,q}) := \deg_{z, \bar{z}}(\Xi_{i,j_i}^{p,q}) - \deg_{z, \bar{z}}(\Psi_i^{p,q}),$$

and the second number is the weight difference

$$\tau(\Xi_{i,j_i}^{p,q}) := \text{wt}(\Xi_{i,j_i}^{p,q}) - \text{wt}(\Psi_i^{p,q}),$$

where the weight  $\text{wt}$  of a wedge of basis  $u_{i'}^{p',q'}$  is defined as the sum of all  $p' + q'$ . The key observation is that

$$\tau + 2\sigma \leq -1.$$

Roughly speaking, this is because when the degree in  $z$ ,  $\bar{z}$  grows by one, the weight will decrease by at least two due to the nilpotent operator  $N$ , and the  $-1$  comes from the terms  $s_i^{p',q'}$ .

Let's make an explicit computation. First by Lemma 6.1, we know

$$\text{wt}(\Psi_i^{p,q}) = d(p+q-d+1)$$

and

$$\deg_{z, \bar{z}}(\Psi_i^{p,q}) = (p-k+1)(q-d+k).$$

Note that in the expression of  $\Xi_i^{p,q}$ , we have  $\deg_N(\Xi_{i,j_i}^{p,r}) = \deg_{z, \bar{z}}(\Xi_{i,j_i}^{p,q})$ , where  $\deg_N(\Xi_{i,j_i}^{p,q})$  denotes the degree of  $N$  of the term forming by the wedge of some  $N^l u_i^{p',q'}$ ,  $N^l s_i^{p',r'}$  that  $\Xi_{i,j_i}^{p,r}$  comes from. Also, we have

$$\begin{aligned} \text{wt}(\Xi_{i,j_i}^{p,q}) &\leq \sum_{0 \leq l \leq p-k} \text{wt}(u_i^{p-l, q-l}) + \sum_{0 \leq l \leq q-d+k-1} \text{wt}(u_i^{p-l, q-l}) - 2\deg_N(\Xi_{i,j_i}^{p,q}) - 1 \\ &= (k+q)(p-k+1) + (p+d-k+1)(q-d+1) - 2\deg_N(\Xi_{i,j_i}^{p,q}) - 1. \end{aligned}$$

Here,  $-1$  appears due to the existence of  $s_i^{p',q'}$ , and that  $s_i^{p',r'}$  has weight at most  $p'+q'-1$ . Therefore, we have

$$\begin{aligned} \tau(\Xi_{i,j_i}^{p,q}) + 2\sigma(\Xi_{i,j_i}^{p,q}) &\leq (k+r)(p-k+1) + (p+d-k+1)(r-d+k) - 1 \\ &\quad - d(p+r-d+1) - 2(p-k+1)(r-d+k) = -1. \end{aligned}$$

Finally, note that the terms in the top wedge formed by the wedge of some  $\Xi_{i,j_i}^{p,q}$  and some  $\Psi_i^{p,q}$  with at least one  $\Xi_{i,j_i}^{p,q}$  must satisfy

$$\sum \tau(\Xi_{i,j_i}^{p,q}) = 0.$$

Using the inequalities

$$\tau(\Xi_{i,j_i}^{p,q}) + 2\sigma(\Xi_{i,j_i}^{p,q}) \leq -1,$$

it implies that

$$\sum \sigma(\Xi_{i,j_i}^{p,q}) \leq -1.$$

So these terms have degree in  $z$ ,  $\bar{z}$  strictly less than the top wedge of  $\Psi_i^{p,r}$ . Therefore, if we write  $z = a + \sqrt{-1}t$ , we can conclude that

$$\begin{aligned} & \frac{(\bigwedge_{p \geq k, i \in J^{p,q}} e^{zN} u_i^{p,q}) \wedge (\bigwedge_{p \geq d-k+1, i \in J^{p,q}} \overline{e^{zN} u_i^{p,q}})}{\bigwedge_{p,q \geq 0, i \in J^{p,q}} u_i^{p,q}} \\ &= C(\bar{z} - z)^{\sum_{p \geq k, q \geq d-k+1} |J_{\text{prim}}^{p,q}|(p-k+1)(q-d+k)} + O(|t|^{-1 + \sum_{p \geq k, q \geq d-k+1} |J_{\text{prim}}^{p,q}|(p-k+1)(q-d+k)}), \end{aligned}$$

where  $C \neq 0$  and  $z$  lies in a domain with bounded real parts by our assumption.

This proves the theorem.  $\square$

*Proof of Theorem 2.5.* We adopt the notation in the proof of Theorem 2.4. In this case, since

$$S(e^{(z+1)N} v, \overline{e^{(z+1)N} u}) = S(e^{zN} v, \overline{e^{zN} u}),$$

we may and do assume  $z$  lies in a domain with bounded real parts.

We choose the basis  $u_i^{p,q}$  of  $I_{\text{prim}}^{p,q}$  again so that the Hermitian form  $S(C\bullet, N^{p+q-d}\bullet)$  on  $I_{\text{prim}}^{p,q}$  is diagonalized under this basis. We also make the basis satisfies

$$\overline{u_i^{p,q}} = u_i^{q,p} + s_i^{q,p}, \quad s_i^{q,p} \in W_{p+q+2},$$

which is possible by the condition (4) of Deligne's splitting. In this case, we also separate  $J_{\text{prim}}^{p,q}$  into the disjoint union of  $J_{\text{prim},+}^{p,q}$  and  $J_{\text{prim},-}^{p,q}$ , where  $u_i^{p,q}$  is a positive (resp. negative) eigenvector of  $S(C\bullet, N^{p+q-d}\bullet)$  for  $i \in J_{\text{prim},+}^{p,q}$  (resp.  $J_{\text{prim},-}^{p,q}$ ). By assumption,  $|J_{\text{prim},\pm}^{p,q}| = s_{\pm}^{p,q}$ .

Consider the basis  $\{N^r u_i^{p,q} \mid 0 \leq p, q \leq d, i \in J_{\text{prim}}^{p,q}, 0 \leq r \leq p+q-d\}$ . We give a well-ordering on this indexed set  $(p, q, i, r)$  by  $(p, q, i, r) > (p', q', i', r')$  if

- (1)  $p - r > p' - r'$ ,
- (2)  $p - r = p' - r'$  and  $q - r > q' - r'$ ,
- (3)  $p - r = p' - r'$  and  $q - r = q' - r'$  and  $r > r'$ ,
- (4)  $p - r = p' - r'$  and  $q - r = q' - r'$  and  $r = r'$  and  $i > i'$ .

Here, the well-ordering on  $i \in J_{\text{prim}}^{p,q}$  is chosen arbitrarily. We will prove that the decreasing filtration  $\hat{F}^\bullet$  constructed using the well-ordered basis above satisfies the desired property in Theorem 2.5 (1).

On the other hand, we need to show that the signature of the Hermitian form  $S(C\bullet, \bar{\bullet})$  on  $\exp(zN)F^k \cap \overline{\exp(zN)F^{d-k}}$  becomes

$$\left( \sum_{\substack{l \geq k-d, \\ r \geq (-l)_+}} s_+^{k+r, d+l-k+r}, \sum_{\substack{l \geq k-d, \\ r \geq (-l)_+}} s_-^{k+r, d+l-k+r} \right)$$

for  $\text{Im}(z)$  sufficiently large.

Besides the opposedness condition proved in Theorem 2.4, we notice that  $\exp(zN)F^\bullet$  satisfies the first Hodge-Riemann bilinear relation:

$$S(\exp(zN)F^k, \exp(zN)F^{d-k+1}) = S(F^k, F^{d-k+1}) = 0,$$

where we use the first Hodge-Riemann bilinear relation for  $F^\bullet$  and the fact that

$$S(e^{zN}u, e^{zN}v) = S(u, v)$$

for all  $u, v$ .

As a result, it suffices to prove that for  $0 \leq k \leq d$ , the signature of the Hermitian form  $(\sqrt{-1})^d S(\bullet, \bar{\bullet})$  on  $\exp(zN)F^k$  is

$$(6.1) \quad \sum_{j=k}^d (-1)^{d-j} \left( \sum_{\substack{l \geq j-d, \\ r \geq (-l)_+}} s_+^{j+r, d+l-j+r} - \sum_{\substack{l \geq j-d, \\ r \geq (-l)_+}} s_-^{j+r, d+l-j+r} \right)$$

for  $\text{Im}(z)$  sufficiently large.

Let's start with the simplest case first. For all  $p, q, k$  with  $d - q \leq k \leq p$  and any  $i \in J_{\text{prim}, \pm}^{p, q}$ , consider the subspace

$$V_{k, i}^{p, q} := \bigoplus_{0 \leq l \leq p-k} N^l (\mathbb{C}u_i^{p, q})$$

of  $F^k$ . Write  $z = a + \sqrt{-1}t$ . We have

$$\begin{aligned} & S(e^{zN} N^r u_i^{p, q}, e^{\bar{z}N} N^s \overline{u_i^{p, q}}) = (-1)^r S(u_i^{p, q}, e^{(\bar{z}-z)N} N^{r+s} \overline{u_i^{p, q}}) \\ & = \begin{cases} (-1)^r \frac{(\bar{z}-z)^{p+q-d-r-s}}{(p+q-d-r-s)!} S(u_i^{p, q}, N^{p+q-d} \overline{u_i^{p, q}}) + O(|t|^{p+q-d-r-s-1}), & r+s \leq p+q-d \\ 0, & r+s > p+q-d \end{cases} \end{aligned}$$

Then the following lemma holds:

**Lemma 6.3.** *For  $d - q \leq k \leq p$ , let  $B_k^{p, q}(z, \bar{z})$  be the  $(p - k + 1) \times (p - k + 1)$  Hermitian matrix  $(b_{k, r, s}^{p, q})_{0 \leq r, s \leq p-k}$  with*

$$b_{k, r, s}^{p, q} = \begin{cases} (\sqrt{-1})^{d-p+q} (-1)^r \frac{(\bar{z}-z)^{p+q-d-r-s}}{(p+q-d-r-s)!}, & r+s \leq p+q-d \\ 0, & r+s > p+q-d \end{cases}.$$

Then

(1) We have

$$\det(B_k^{p, q}(z, \bar{z})) = \left( (-1)^{\sum_{j=k}^p (d-j)} \right) \frac{C_{q-d+k, p-k+1} (2\text{Im}(z))^{(q-d+k)(p-k+1)}}{((q-d+k)(p-k+1))!},$$

where  $C_{q-d+k,p-k+1}$  is the number of standard Young tableaux of the shape  $(q-d+k) \times (p-k+1)$  squares.

- (2) For  $\text{Im}(z) \neq 0$ , if we perform the block diagonalization to  $B_k^{p,q}(z, \bar{z})$  as in Lemma 2.7, the diagonal matrix  $D_k^{p,q}(z, \bar{z})$  becomes

$$\text{diag}(\dots, (-1)^{d-l} \tilde{C}_l^{p,q} (\text{Im}(z))^{q-d-p+2l}, \dots)_{l=p,p-1,p-2,\dots,k},$$

where

$$\tilde{C}_l^{p,q} = 2^{q-d-p+2l} \left( \frac{C_{q-d+l,p-l+1}}{C_{q-d+l+1,p-l}} \right) \left( \frac{((q-d+l+1)(p-l))!}{((q-d+l)(p-l+1))!} \right).$$

In particular, the signature of  $B_k^{p,q}(z, \bar{z})$  is  $\sum_{j=k}^p (-1)^{d-j}$ .

Assuming Lemma 6.3, then the Hermitian matrix

$$\begin{aligned} & \left( (\sqrt{-1})^d S(e^{zN} N^r u_i^{p,q}, e^{\bar{z}N} N^s \overline{u_i^{p,q}}) \right)_{0 \leq r, s \leq p-k} \\ &= \left( (\sqrt{-1})^{p-q} S(u_i^{p,q}, N^{p+q-d} \overline{u_i^{p,q}}) \right) B_k^{p,q}(z, \bar{z}) + R_{k,i}^{p,q}(z, \bar{z}), \end{aligned}$$

where each entry of  $R_{k,i}^{p,q}(z, \bar{z})$  is either 0 or has degree (in  $z, \bar{z}$ ) strictly less than the corresponding entry of  $B_k^{p,q}(z, \bar{z})$ . As a result, we see that

- (1) The restriction of the filtration  $\hat{F}^\bullet$  on  $V_{k,i}^{p,q}$  satisfies the desired condition in Theorem 2.5 (1) for  $\text{Im}(z)$  sufficiently large.
- (2) If we perform the block diagonalization to the Hermitian matrix

$$\left( (\sqrt{-1})^d S(e^{zN} N^r u_i^{p,q}, e^{\bar{z}N} N^s \overline{u_i^{p,q}}) \right)_{0 \leq r, s \leq p-k}$$

as in Lemma 2.7, the diagonal matrix becomes

$$\text{diag}(\dots, (-1)^{d-p+r} \tilde{C}_{p-r}^{p,q} \lambda_i^{p,q} (\text{Im}(z))^{p+q-d-2r} + O(|t|^{p+q-d-2r-1}), \dots)_{0 \leq r \leq p-k},$$

where

$$\lambda_i^{p,q} = (\sqrt{-1})^{p-q} S(u_i^{p,q}, N^{p+q-d} \overline{u_i^{p,q}}).$$

In particular, the signature of the Hermitian matrix is  $\pm \left( \sum_{j=k}^p (-1)^{d-j} \right)$  where  $\pm$  depends on whether  $i \in J_{\text{prim},+}^{p,q}$  or  $i \in J_{\text{prim},-}^{p,q}$ . This shows that the formula (6.1) holds in this simplest case.

*Proof of Lemma 6.3.* Note that under suitable symmetry (the reflection about the middle row), the matrix  $B_k^{p,q}(z, \bar{z})$  is related to the matrix  $A_{q-d+k,p-k+1}(\bar{z}-z)$  that we define in Corollary 6.2, and then we have

$$\begin{aligned} \det(B_k^{p,q}(z, \bar{z})) &= (\sqrt{-1})^{(d-p+q)(p-k+1)} (-1)^{\frac{(p-k)(p-k+1)}{2}} \\ & \quad (-1)^{\frac{(p-k)(p-k+1)}{2}} \det(A_{q-d+k,p-k+1}(\bar{z}-z)). \end{aligned}$$

The first  $(-1)^{\frac{(p-k)(p-k+1)}{2}}$  corresponds to the  $(-1)^r$  factor in each entry and the second  $(-1)^{\frac{(p-k)(p-k+1)}{2}}$  corresponds the permutation of rows of  $B_k^{p,q}(z, \bar{z})$  to match those of  $A_{q-d+k,p-k+1}(\bar{z}-z)$ . Thus, by Corollary 6.2 we have

$$\det(B_k^{p,q}(z, \bar{z})) = (\sqrt{-1})^{(d-p+q)(p-k+1)} \frac{C_{q-d+k,p-k+1}}{((q-d+k)(p-k+1))!} (\bar{z}-z)^{(q-d+k)(p-k+1)}$$

$$= \left( (-1)^{\sum_{j=k}^p (d-j)} \right) \frac{C_{q-d+k, p-k+1} (2\text{Im}(z))^{(q-d+k)(p-k+1)}}{((q-d+k)(p-k+1))!}.$$

On the other hand, since for  $k \leq l \leq p$ ,  $B_l^{p,q}(z, \bar{z})$  is the leading principal  $(p-l) \times (p-l)$  submatrix of  $B_k^{p,q}(z, \bar{z})$ , one sees in particular that the leading principal  $(p-l) \times (p-l)$  submatrices of  $B_k^{p,q}(z, \bar{z})$  are non-degenerate for  $\text{Im}(z) \neq 0$ . Thus, by Lemma 2.7, the second assertion follows.  $\square$

For all  $p, q, r, p', q', r, s$ , we define the number

$$n_{p,q,p',q',r,s} = \min\{p+q-d-r-s, p'+q'-d-r-s\}.$$

Then for all  $(p, q, r, i)$  and  $(p', q', s, i')$ , we have

$$\begin{aligned} & (\sqrt{-1})^d S(e^{zN} N^r u_i^{p,q}, e^{\bar{z}N} N^s \overline{u_{i'}^{p,q}}) \\ &= (\sqrt{-1})^d (-1)^r S(u_i^{p,q}, e^{(\bar{z}-z)N} N^{r+s} \overline{u_{i'}^{p,q}}) \\ &= \begin{cases} (\sqrt{-1})^d (-1)^r \frac{(\bar{z}-z)^{n_{p,q,p',q',r,s}}}{n_{p,q,p',q',r,s}!} S(u_i^{p,q}, N^{n_{p,q,p',q',r,s}+r+s} \overline{u_{i'}^{p,q}}) + O(|t|^{n_{p,q,p',q',r,s}-1}), & \text{when } n_{p,q,p',q',r,s} \geq 0 \\ 0, & \text{when } n_{p,q,p',q',r,s} < 0. \end{cases} \end{aligned}$$

We can also notice that  $S(u_i^{p,q}, N^{n_{p,q,p',q',r,s}+r+s} \overline{u_{i'}^{p,q}}) = 0$  unless  $(p, q, i) = (p', q', i')$ .

If the mixed Hodge structure is split over  $\mathbb{R}$ , then all of the terms  $O(|t|^{n_{p,q,p',q',r,s}-1})$  and  $R_{k,i}^{p,q}$  above are zero. Moreover, for  $(p, q, i) \neq (p', q', i')$ ,  $V_{k,i}^{p,q}$  and  $V_{k,i'}^{p',q'}$  are mutually orthogonal with respect to the Hermitian form  $(\sqrt{-1})^d S(e^{zN} \bullet, e^{\bar{z}N} \bullet)$ . Hence, according to our computation for the restriction of the Hermitian form  $(\sqrt{-1})^d S(e^{zN} \bullet, e^{\bar{z}N} \bullet)$  on  $V_{k,i}^{p,q}$ , we can conclude that for all  $\text{Im}(z) > 0$ , the filtration  $\hat{F}^\bullet$  satisfies the desired condition in Theorem 2.5 (1) and the signature of the Hermitian form  $(\sqrt{-1})^d S(\bullet, \bar{\bullet})$  on  $\exp(zN)F^k$  is as in (6.1), and the theorem is proved.

In general, consider the Hermitian matrix

$$M = (M_{p,q,p',q',r,s,i,i'})_{p,q,p',q',r,s,i,i'} := \left( (\sqrt{-1})^d S(e^{zN} N^r u_i^{p,q}, e^{\bar{z}N} N^s \overline{u_{i'}^{p,q}}) \right)_{p,q,p',q',r,s,i,i'}$$

where  $p \geq k$ ,  $i \in J_{\text{prim}}^{p,q}$ ,  $0 \leq r \leq \max\{p-k, p+q-d\}$  and  $p' \geq k$ ,  $i' \in J_{\text{prim}}^{p',q'}$ ,  $0 \leq s \leq \max\{p'-k, p'+q'-d\}$ . We will perform the block diagonalization to  $M$  as in Lemma 2.7 with respect to the well-ordering on  $(p, q, i, r)$  (from large to small). We want to show that the terms  $O(|t|^{n_{p,q,p',q',r,s}-1})$  above don't contribute to the leading terms in the block diagonalization process. Theorem 2.5 (1) is proved if we show that the successive diagonal entries are nonzero. The signature formula (6.1) follows if we can show that after the block diagonalization, the diagonal entry  $M_{p,q,p,q,r,r,i,i}$  of  $M$  becomes

$$(-1)^{d-p+r} \tilde{C}_{p-r}^{p,q} \lambda_i^{p,q} (\text{Im}(z))^{p+q-d-2r} + O(|t|^{p+q-d-2r-1})$$

for all  $(p, q, i, r)$ .

We recall that the block diagonalization of a Hermitian matrix  $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$  with  $A$  invertible is given by

$$\begin{pmatrix} A & 0 \\ 0 & C - B^*A^{-1}B \end{pmatrix}.$$

Let's call the terms coming from  $-B^*A^{-1}B$  the supplementary terms to the corresponding entries of  $C$ . We will prove by induction that the following situation holds:

Before we perform the  $(p, q, i, r)$ -diagonalization step, we have

- (1) For  $(p', q', i', r')$  and  $(p'', q'', i'', r'')$  such that at least one of them  $> (p, q, i, r)$  and such that  $(p', q', i', r') \neq (p'', q'', i'', r'')$ ,

$$M_{p',q',p'',q'',r',r'',i',i''} = 0.$$

- (2) For  $(p', q', i', r') \geq (p, q, i, r)$ ,

$$M_{p',q',p',q',r',r',i',i'} = (-1)^{d-p'+r'} \tilde{C}_{p'-r'}^{p',q'} \lambda_i^{p',q'} (\operatorname{Im}(z))^{p'+q'-d-2r'} + O(|t|^{p'+q'-d-2r'-1}).$$

- (3) For  $(p', q', i', r')$  and  $(p'', q'', i'', r'')$ , both  $\leq (p, q, i, r)$  and such that  $(p', q', i') \neq (p'', q'', i'')$ ,

$$M_{p',q',p'',q'',r',r'',i',i''} = O(|t|^{n_{p',q',p'',q'',r',r'',i',i''}-1}).$$

- (4) For  $(p', q', i', r')$  and  $(p', q', i', r'')$ , both  $\leq (p, q, i, r)$ ,

$$M_{p',q',p',q',r',r'',i',i'} = O(|t|^{n_{p',q',p',q',r',r'',i',i'}}).$$

After we perform the  $(p, q, i, r)$ -diagonalization step, we have

- (5) For  $(p', q', i', r')$  and  $(p'', q'', i'', r'')$ , both  $< (p, q, i, r)$ , and such that either  $(p', q', i') \neq (p, q, i)$  or  $(p'', q'', i'') \neq (p, q, i)$ , the supplementary term to the entry  $M_{p',q',p'',q'',r',r'',i',i''}$  has order

$$O(|t|^{n_{p',q',p'',q'',r',r'',i',i''}-1}).$$

- (6) For  $(p, q, i, r')$  and  $(p, q, i, r'')$ , both  $< (p, q, i, r)$ , the supplementary term to  $M_{p,q,p,q,r',r'',i,i}$  has order

$$O(|t|^{n_{p,q,p,q,r',r'',i,i}}).$$

Now, we assume that (1), (2), (3), (4) hold at  $(p, q, i, r)$ -diagonalization step and (5), (6) hold for all  $(p', q', i', r')$ -diagonalization step with  $(p', q', i', r') > (p, q, i, r)$ . Then for  $(p', q', i', r')$  and  $(p'', q'', i'', r'')$ , both  $< (p, q, i, r)$ , and such that either  $(p', q', i') \neq (p, q, i)$  or  $(p'', q'', i'') \neq (p, q, i)$ , the supplementary term to the entry  $M_{p',q',p'',q'',r',r'',i',i''}$  is

$$-M_{p',q',p,q,r',r,i',i} M_{p,q,p,q,r,r,i,i}^{-1} M_{p,q,p'',q'',r,r'',i',i''},$$

which has order less than or equal to

$$\begin{aligned} & n_{p',q',p,q,r',r} - n_{p,q,p,q,r,r} + n_{p,q,p'',q'',r,r''} - 1 \\ & \leq n_{p',q',p'',q'',r',r''} - 1. \end{aligned}$$

The negative one arises since either  $(p', q', i') \neq (p, q, i)$  or  $(p'', q'', i'') \neq (p, q, i)$ . Also, for  $(p, q, i, r')$  and  $(p, q, i, r'')$ , both  $< (p, q, i, r)$ , the supplementary term to the entry  $M_{p,q,p,q,r',r'',i,i}$  is

$$-M_{p,q,p,q,r',r,i,i} M_{p,q,p,q,r,r,i,i}^{-1} M_{p,q,p,q,r,r'',i,i},$$

which has order less than or equal to

$$\begin{aligned} & n_{p,q,p,q,r',r} - n_{p,q,p,q,r,r} + n_{p,q,p,q,r,r''} \\ & = n_{p,q,p,q,r',r''}. \end{aligned}$$

Thus if we let  $(\hat{p}, \hat{q}, \hat{i}, \hat{r})$  be the predecessor of  $(p, q, i, r)$ , then we have seen (5), (6) hold for  $(p, q, i, r)$ -diagonalization step and (1), (3), (4) hold for  $(\hat{p}, \hat{q}, \hat{i}, \hat{r})$ -diagonalization step. Finally, to see (2) for  $(\hat{p}, \hat{q}, \hat{i}, \hat{r})$ , we notice that by (5) and (6) for all the steps before  $(\hat{p}, \hat{q}, \hat{i}, \hat{r})$ , the only contribution of order  $n_{\hat{p},\hat{p},\hat{q},\hat{q},\hat{r},\hat{r}}$  to  $M_{\hat{p},\hat{p},\hat{q},\hat{q},\hat{r},\hat{r},\hat{i},\hat{i}}$  comes from those entries  $M_{\hat{p},\hat{p},\hat{q},\hat{q},r',r'',\hat{i},\hat{i}}$  with  $r', r'' \leq \hat{r}$ . As a consequence, by Lemma 6.3, we see that

$$M_{\hat{p},\hat{p},\hat{q},\hat{q},\hat{r},\hat{r},\hat{i},\hat{i}} = (-1)^{d-\hat{p}+\hat{r}} \tilde{C}_{\hat{p}-\hat{r}}^{\hat{p},\hat{q}} \lambda_{\hat{i}}^{\hat{p},\hat{q}} (\text{Im}(z))^{\hat{p}+\hat{q}-d-2\hat{r}} + O(|t|^{\hat{p}+\hat{q}-d-2\hat{r}-1}).$$

This proves the theorem. □

**Remark 6.4.** In the proof, we also see that there are real analytic orthogonal basis  $v_i(z)$  of  $\exp(zN)F^k \cap \overline{\exp(zN)F^{d-k}}$  with respect to the Hermitian form  $(\sqrt{-1})^d S(\bullet, \bar{\bullet})$  such that the growth of order of  $(\sqrt{-1})^d S(v_i(z), \overline{v_i(z)})$  are  $\text{Im}(z)^{l_i}$  with  $k - d \leq l_i \leq k$ . The asymptotic order  $l_i$  is the smallest integer among those integers  $l$  such that

$$\lim_{\text{Im}(z) \rightarrow \infty} e^{-zN} v_i(z) \in W_{l+d}.$$

This is compatible with Deligne-Schmid's theorem [41, Theorem 6.6'] in the case of the degeneration of polarized Hodge structures, which states that a (multi-valued) flat section  $v(q)$  belongs to  $W_l$  if and only if, along a radical ray,

$$S(Cv(q), \overline{v(q)}) = O((-\log |q|)^{l-d})$$

as  $|q|$  tends to zero.

## References

- [1] Daniele Angella and Hisashi Kasuya. Bott-Chern cohomology of solvmanifolds. *Ann. Global Anal. Geom.*, 52(4):363–411, 2017.
- [2] Daniele Angella and Hisashi Kasuya. Cohomologies of deformations of solvmanifolds and closedness of some properties. *North-West. Eur. J. Math.*, 3:75–105, i, 2017.
- [3] Daniele Angella, Tatsuo Suwa, Nicoletta Tardini, and Adriano Tomassini. Note on Dolbeault cohomology and Hodge structures up to bimeromorphisms. *Complex Manifolds*, 7(1):194–214, 2020.
- [4] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 2004.
- [5] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.
- [6] Constantin Bănică and Octavian Stănăşilă. *Algebraic methods in the global theory of complex spaces*. Editura Academiei, Bucharest; John Wiley & Sons, London-New York-Sydney, 1976. Translated from the Romanian.
- [7] F. Campana. The class  $\mathcal{C}$  is not stable by small deformations. *Math. Ann.*, 290(1):19–30, 1991.
- [8] Eduardo Cattani and Aroldo Kaplan. Polarized mixed Hodge structures and the local monodromy of a variation of Hodge structure. *Invent. Math.*, 67(1):101–115, 1982.
- [9] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid. Degeneration of Hodge structures. *Ann. of Math. (2)*, 123(3):457–535, 1986.

- [10] Kwokwai Chan, Naichung Conan Leung, and Ziming Nikolas Ma. Geometry of the Maurer-Cartan equation near degenerate Calabi-Yau varieties. *J. Differential Geom.*, 125(1):1–84, 2023.
- [11] Brian Conrad. More examples. *an addendum to*, 2001.
- [12] Pierre Deligne. *Équations différentielles à points singuliers réguliers*, volume Vol. 163 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1970.
- [13] Pierre Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.
- [14] Pierre Deligne, Phillip Griffiths, John Morgan, and Dennis Sullivan. Real homotopy theory of Kähler manifolds. *Invent. Math.*, 29(3):245–274, 1975.
- [15] Robert Friedman. Simultaneous resolution of threefold double points. *Math. Ann.*, 274(4):671–689, 1986.
- [16] Robert Friedman. The  $\partial\bar{\partial}$ -lemma for general Clemens manifolds. *Pure Appl. Math. Q.*, 15(4):1001–1028, 2019.
- [17] Robert Friedman and Radu Laza. Deformations of singular Fano and Calabi-Yau varieties. *arXiv preprint arXiv:2203.04823*, 2022.
- [18] Robert Friedman and Radu Laza. Deformations of some local Calabi-Yau manifolds. *arXiv preprint arXiv:2203.11738*, 2022.
- [19] Taro Fujisawa. Polarizations on limiting mixed Hodge structures. *J. Singul.*, 8:146–193, 2014.
- [20] Taro Fujisawa. Limiting mixed Hodge structures on the relative log de Rham cohomology groups of a projective semistable log smooth degeneration. *Kyoto J. Math.*, 63(3):517–577, 2023.
- [21] Taro Fujisawa and Chikara Nakayama. Geometric polarized log Hodge structures with a base of log rank one. *Kodai Math. J.*, 43(1):57–83, 2020.
- [22] F. Guillén and V. Navarro Aznar. Sur le théorème local des cycles invariants. *Duke Math. J.*, 61(1):133–155, 1990.
- [23] Kenji Hashimoto and Taro Sano. Examples of non-Kähler Calabi-Yau 3-folds with arbitrarily large  $b_2$ . *Geom. Topol.*, 27(1):131–152, 2023.
- [24] Hisashi Kasuya. Hodge symmetry and decomposition on non-Kähler solvmanifolds. *J. Geom. Phys.*, 76:61–65, 2014.
- [25] Hisashi Kasuya and Jonas Stelzig. Complex non-Kähler manifolds that are cohomologically close to, or far from, being Kähler, 2023.
- [26] Yujiro Kawamata. Unobstructed deformations. A remark on a paper of Z. Ran: “Deformations of manifolds with torsion or negative canonical bundle” [*J. Algebraic Geom.* **1** (1992), no. 2, 279–291; MR1144440 (93e:14015)]. *J. Algebraic Geom.*, 1(2):183–190, 1992.
- [27] Yujiro Kawamata and Yoshinori Namikawa. Logarithmic deformations of normal crossing varieties and smoothing of degenerate Calabi-Yau varieties. *Invent. Math.*, 118(3):395–409, 1994.
- [28] K. Kodaira. On the structure of compact complex analytic surfaces. III. *Amer. J. Math.*, 90:55–83, 1968.
- [29] Claude LeBrun and Yat Sun Poon. Twistors, Kähler manifolds, and bimeromorphic geometry. II. *J. Amer. Math. Soc.*, 5(2):317–325, 1992.
- [30] Nam-Hoon Lee. An example of non-Kähler Calabi-Yau fourfold. *arXiv preprint arXiv:2102.12656*, 2021.
- [31] Chi Li. Polarized Hodge structures for Clemens manifolds. *arXiv preprint arXiv:2202.10353*, 2022.
- [32] Yoshinori Namikawa and J. H. M. Steenbrink. Global smoothing of Calabi-Yau threefolds. *Invent. Math.*, 122(2):403–419, 1995.
- [33] Chris A. M. Peters and Joseph H. M. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2008.
- [34] J. P. Ramis, G. Ruget, and J. L. Verdier. Dualité relative en géométrie analytique complexe. *Invent. Math.*, 13:261–283, 1971.
- [35] Ziv Ran. Deformations of Calabi-Yau Kleinfolds. In *Essays on mirror manifolds*, pages 451–457. Int. Press, Hong Kong, 1992.
- [36] Sheng Rao, Song Yang, and Xiangdong Yang. Dolbeault cohomologies of blowing up complex manifolds. *J. Math. Pures Appl. (9)*, 130:68–92, 2019.

- [37] Sönke Rollenske and Richard Thomas. Smoothing nodal Calabi-Yau  $n$ -folds. *J. Topol.*, 2(2):405–421, 2009.
- [38] Morihiko Saito. Modules de Hodge polarisables. *Publ. Res. Inst. Math. Sci.*, 24(6):849–995, 1988.
- [39] Taro Sano. Examples of non-Kähler Calabi-Yau manifolds with arbitrarily large  $b_2$ . *J. Topol.*, 14(4):1448–1460, 2021.
- [40] Taro Sano. Construction of non-Kähler Calabi-Yau manifolds by log deformations. *arXiv preprint arXiv:2208.02455*, 2022.
- [41] Wilfried Schmid. Variation of Hodge structure: the singularities of the period mapping. *Inventiones mathematicae*, 22(3-4):211–319, 1973.
- [42] Joseph Steenbrink. Limits of Hodge structures. *Invent. Math.*, 31(3):229–257, 1975/76.
- [43] Jonas Stelzig. The double complex of a blow-up. *Int. Math. Res. Not. IMRN*, (14):10731–10744, 2021.
- [44] Gang Tian. Smoothing 3-folds with trivial canonical bundle and ordinary double points. In *Essays on mirror manifolds*, pages 458–479. Int. Press, Hong Kong, 1992.
- [45] Lee Tsung-Ju. Finite distance problem on the moduli of non-Kähler Calabi-Yau  $\partial\bar{\partial}$ -threefolds. in preparation.
- [46] Sampei Usui. Mixed Torelli problem for Todorov surfaces. *Osaka J. Math.*, 28(3):697–735, 1991.
- [47] Claire Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. Translated from the French original by Leila Schneps.
- [48] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.
- [49] Song Yang and Xiangdong Yang. Bott-Chern blow-up formulae and the bimeromorphic invariance of the  $\partial\bar{\partial}$ -lemma for threefolds. *Trans. Amer. Math. Soc.*, 373(12):8885–8909, 2020.

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NY 10027

*Email address:* kc3396@columbia.edu