

The Essence of Useful Evaluation Through Quantitative Types (Extended Version)

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ABSTRACT

Several evaluation notions for λ -calculus qualify as *reasonable cost models* according to Slot and van Emde Boas' *Invariance Thesis*. A notable result achieved by Accattoli and Dal Lago is that leftmost-outermost reduction is reasonable, where the term representation uses *sharing* and the steps are *useful*. These results, initially studied in call-by-name, have also been extended to call-by-value. However, the existing formulations of usefulness lack inductive structure, making it challenging in particular to define and reason about type systems on top of the untyped syntax. Additionally, no type-based quantitative interpretations exist for useful evaluation.

In this work, we establish the first *inductive* definition of useful evaluation for open weak call-by-value. This new useful strategy connects to a previous implementation of usefulness through a low-level abstract machine, incurring only in linear time overhead, thus providing a *reasonable* cost model for open call-by-value implementation. We also propose a *semantic interpretation* of useful call-by-value using a non-idempotent intersection type system equipped with a notion of tightness. The resulting interpretation is *quantitative*, *i.e.* provides exact step-count information for program evaluation. This turns out to be the first semantical interpretation in the literature for a notion of useful evaluation.

1 INTRODUCTION

The formal model behind functional programming languages and some interactive proof assistants is known as the λ -calculus, which expresses all computable functions. However, not only computability matters, but also *resource efficiency*. In this sense, it is not clear whether different models of computation are interchangeable. The *Invariance Thesis* [49] states that there is a standard class of machine models, including Turing machines and RAMs, that are able to simulate each other with polynomial overhead in time and constant factor overhead in space. They are called *reasonable* models of computation.

One fundamental question is whether the λ -calculus is a reasonable model of computation. To answer that, a time *cost model* needs to be fixed in order to measure the run time cost of a λ -term. An initial idea would be to measure the number of β -steps (function application steps) to normal form. However, since β -evaluation may produce arbitrarily many copies of arbitrarily large terms, implementing β -evaluation in a Turing machine using a simple representation for λ -terms cannot be done with polynomial time overhead, as it does not even suffice to write down the result of the computation on the tape.

Formulating a Cost Model. Building cost models for λ -calculus has received considerable interest in the last decades [20, 43, 44]. To achieve such a formulation it is necessary to choose a particular *evaluation strategy* and a particular *term representation*. Concerning the evaluation strategies for λ -calculus, there are many of them. For example, in *call-by-name* (CBN) a function is applied without evaluating its argument, whereas in *call-by-value* (CBV) a function may be applied only when its argument has become a *value*. Concerning term representation, it is well-known that usual *trees* are not sufficient to obtain a reasonable cost model, as illustrated by the famous *size explosion problem* (see *e.g.* [10]), exhibiting a reduction sequence from a term t_n of linear size n , which evaluates in n β -steps to a term s_n with exponential size in n .

However, one may find more *succinct* representations for λ -terms, such as those based on *sharing*, that can be represented using *explicit substitutions* (ES). The remarkable reasonable cost models achieved by Accattoli and Dal Lago [9, 10] for leftmost-outermost reduction rely on two key ingredients: representing λ -terms using ESs for sharing, and restricting the copy of shared subterms as to avoid size explosion, a strategy which is known as *useful evaluation*. We briefly discuss each of them.

Explicit Substitutions. The ES calculus used to implement sharing in [10] is based on the *Linear Substitution Calculus* (LSC) [1, 14], a variant of Accattoli and Kesner’s structural λ -calculus [8], which in turn generalizes Milner’s Calculus [39, 46]. In the LSC, the truly computational steps are called *distant beta* steps (db) and create new ESs; for example $(\lambda x. x y)[y/I] z \rightarrow_{\text{db}} (x y)[x/z][y/I]$. This operation is *distant* in that there may be ESs between the λ -abstraction and its argument, such as $[y/I]$ in the example. Also, a *single* occurrence of a variable x may be substituted by a term t whenever x is bound to t by an ES. More precisely, there are *linear substitution* steps (ls), of the form $C\langle x \rangle[x/t] \rightarrow_{\text{ls}} C\langle t \rangle[x/t]$, where C denotes an arbitrary context. Substitution is called *linear* because it replaces variables one occurrence at a time, for example $(\lambda x. x x) y \rightarrow_{\text{db}} (x x)[x/y] \rightarrow_{\text{ls}} (x y)[x/y] \rightarrow_{\text{ls}} (y y)[x/y]$.

Useful Evaluation. The second central notion behind Accattoli and Dal Lago’s results is *useful evaluation*, which identifies the conditions under which shared subterms can be copied without producing size explosion. The insight that drives the notion of useful evaluation is that ls-steps, which copy shared subterms, should only be performed when contributing to the creation of a db-redex. For example, the step $(x x)[x/I] \rightarrow_{\text{ls}} (I x)[x/I]$ is useful, because substituting x by I creates the db-redex $I x$. A subtler example is $(x x)[x/y][y/I] \rightarrow_{\text{ls}} (y x)[x/y][y/I]$. This step is useful because it *indirectly* contributes to creating a db-redex, in one more step: $(y x)[x/y][y/I] \rightarrow_{\text{ls}} (I x)[x/y][y/I]$. On the other hand, the steps $(x x)[x/y] \rightarrow_{\text{ls}} ((y y) x)[x/y]$ and $(x x)[x/I] \rightarrow_{\text{ls}} (x I)[x/I]$ are not useful, because the substitutions do not contribute to creating a db-redex.

Open Call-by-Value. In functional programming languages, evaluation is defined on *closed* terms, *i.e.*, evaluation is restricted to terms without occurrences of free variables (*e.g.* $\lambda x. x x$). Accordingly, evaluation is *weak*, not proceeding inside the bodies of abstractions. However, in proof assistants, evaluation is *strong*, allowing evaluation inside abstractions, and thus needs to be able to operate on *open* terms, which may include occurrences of free variables (*e.g.* $\lambda x. x y$).

This work is part of a broader, community-driven effort to understand the concept of useful *strong* CBV. Indeed, it has been noted that usefulness is not really required to obtain a reasonable cost model in the *open* and *weak* case [6]. However, in order to achieve a robust notion of useful strong call-by-value, it is essential to develop tools that enhance our comprehension of usefulness within a less complex framework, which already presents numerous technical challenges. Our work aims to achieve this, starting from an *open weak* setting for CBV.

There are well-established notions of evaluation for *closed* terms, such as Plotkin’s CBV [48], but the situation is more subtle in the open case, as naive extensions of CBV to open terms are not *adequate*. The intuitive idea to obtain adequacy is that normal forms should be *meaningful* terms (technically defined by the notion of *solvable* terms¹). To illustrate non-adequacy, consider the term $t := (\lambda x. \delta) (y y) \delta$, where $\delta := \lambda z. z z$. Note that $\lambda x. \delta$ is applied to an argument $y y$ which is not a value, so t cannot be further reduced (*i.e.* it is a normal form); however, t is not solvable. As evidenced by this example, a naive extension of Plotkin’s CBV to the open framework does not give an adequate calculus, since the term t is a normal form and *meaningless* at the same time.

Our starting point is a formulation of open weak CBV which recovers adequacy, called the *fireball calculus* [2]. *Useful* evaluation for the fireball calculus, also in [2], is specified by imposing *global* restrictions on reduction steps at the meta-level, which is contrary to the inductive way in which one usually reasons about the syntax and semantics of programming languages and proof-assistants. Indeed, inductive methods offer a more structured and rigorous approach to understand, specify, and implement evaluation strategies in programming language theory. They provide clarity and precision, making it easier to achieve formal analysis and proofs. As part of this work, we reformulate useful open weak CBV *inductively*.

Quantitative Interpretations. The denotational semantics of CBV is comparatively less explored and not as well understood as that of CBN. This discrepancy primarily stems from the complexities inherent to open terms and non-erasable terms. One of our goals is to enhance the semantic understanding of useful CBV evaluation in an open setting. This is achieved through the use of a quantitative interpretation specified by intersection types. These types extend simple types with a new type constructor \cap such that a program t becomes typable with $\alpha \cap \beta$ if t is typable with both types α and β independently. They were originally introduced as (*qualitative*) *models* capturing computational properties of functional programs [31]. For example, termination of a particular evaluation strategy can be characterized by typability in an appropriate intersection type system, so that a program t is terminating for the evaluation strategy if and only if t is typable in the associated type system (which means that typability becomes undecidable in these systems). Initially, the intersection type constructor was defined in particular as an idempotent type constructor (*i.e.* $\sigma \cap \sigma = \sigma$). By instead adopting a *non-idempotent* notion of intersection [29, 36], types can be naturally understood as multisets. Just like their idempotent precursors, non-idempotent types still allow for a characterization of operational properties of programs by means of typability [29, 36], but they also grant

¹A general discussion on CBV-solvability can be found in [7, 35, 47].

a substantial improvement: they provide *quantitative* measures of these properties. For example, it is still possible to prove that a program is terminating if and only if it is typable, but now an *upper bound* or even an *exact measure* for the number of steps to normal form can be obtained from the typing derivation. *Quantitative types* based on non-idempotent intersection types have been used to provide upper and exact measures for evaluation strategies in the λ -calculus [15, 24, 26, 32], classical calculi [40, 41], call-by-value [5, 7, 34, 38, 42], call-by-need [11, 21, 23], call-by-push-value [28], languages with effects [19], etc.

One crucial insight is that *exact measures*, instead of upper bounds, can be obtained by considering minimal type derivations, called *tight* [17]. Using appropriate refined tight systems, it is also possible to obtain *independent* measures for different kinds of evaluation steps. More precisely, quantitative typing systems are equipped with constants and counters, together with a condition called tightness, ensuring that a typing derivation is minimal. *Soundness* of the resulting intersection type system means that for any tight type derivation Φ of a program t with a counter m , the term t evaluates to a normal form in exactly m steps (generalized for steps of many possible kinds with counters m_1, \dots, m_n).

On the other hand, *completeness* means that each evaluation sequence of a given size has a corresponding (tight) typing derivation with appropriate counters. Exact measures based on tight systems have been extended to encompass different notions of evaluation such as call-by-name [17], call-by-value [5, 42], call-by-need [16, 45], call-by-push-value [27, 42], and classical calculi [27].

Contributions. As an **initial contribution**, we define a *useful* CBV evaluation strategy for *open* terms. This is the first *inductive* specification of usefulness in the literature. To formulate our strategy, we refine in two stages the *value substitution calculus* (vsc) [12]: first, we introduce the *linear* open CBV calculus (LOCBV°), which refines the vsc with *linear substitution* but it is still not useful (**Section 3**). Second, we introduce the *useful* open CBV evaluation strategy (UOCBV^\bullet) by restricting evaluation to substitute abstractions only *for progress* (**Section 4**). Our inductive approach to usefulness has been inspired by Balabonski *et al.*'s remarkable definition of strong call-by-need evaluation [22]. We then show that our notion of usefulness enjoys the diamond property, and thus confluence. Further, we relate LOCBV° and UOCBV^\bullet (**Section 5**).

As a **second contribution**, we show that UOCBV^\bullet is a *reasonable* implementation of open CBV (**Section 6**). To do this, we follow the methodology of [2], by showing a *high-level* and a *low-level* implementation theorem. Composing these implementation theorems entails that UOCBV^\bullet indeed implements open CBV reasonably.

One of the remarkable benefits of our inductive approach is that it allows us to provide a *semantic* interpretation for usefulness, the first one in the literature. This complements our *syntactic* presentation. Indeed, as a **third contribution**, we propose a *quantitative* interpretation for UOCBV^\bullet , based on non-idempotent intersection types (**Section 7**). This interpretation provides independent and exact measures. More precisely, we define a type system based on non-idempotent intersection types and equip it with a notion of tightness, and we show that useful evaluation according to UOCBV^\bullet is *sound* and *complete*, meaning in particular that for any *tight* type derivation of a program t with counters m and e , the term t evaluates to a normal form in exactly m function application steps and e substitution steps. This is a novel result in the literature, as existing useful evaluation notions lack semantic interpretations, and existing quantitative interpretations do not consider usefulness.

2 PRELIMINARY NOTIONS

We define here the shared notions for our calculi and strategies.

Given a denumerable set of **variables** (x, y, z, \dots) , the sets of **terms** (t, s, u, \dots) , **substitution contexts** (L, L', \dots) and **values** (v, w, \dots) are given by the following grammars:

$$t ::= x \mid \lambda x. t \mid t t \mid t[x/t] \quad L ::= \diamond \mid L[x/t] \quad v ::= x \mid \lambda x. t$$

The set of terms includes **variables**, **abstractions**, **applications**, as well as **closures** $t[x/s]$ representing an **explicit substitution** (ES) $[x/s]$ on a term t . **Free** and **bound occurrences** of variables are defined as expected, where free occurrences of x in t are bound in $t[x/s]$. Terms are considered up to α -renaming of bound variables. We write tL for the *variable-capturing replacement* of the hole \diamond in L by t (keeping the standard notation $C(t)$ for other kinds of contexts). We write $t \in \text{Abs}$ if t is of the form $(\lambda x. t)L$ and $t\{x := s\}$ stands for the capture-avoiding **substitution** of the free occurrences of x by s in t . The sets of **free variables** of a **term** $(\text{fv}(t))$ and a **context** $(\text{fv}(L))$ are defined as expected. The set of **reachable variables** of a term t is written $\text{rv}(t)$ and defined as:

$$\begin{aligned} \text{rv}(x) &::= \{x\} & \text{rv}(t s) &::= \text{rv}(t) \cup \text{rv}(s) \\ \text{rv}(\lambda x. t) &::= \emptyset & \text{rv}(t[x/s]) &::= (\text{rv}(t) \setminus \{x\}) \cup \text{rv}(s) \end{aligned}$$

We now introduce some general notions of reduction that will be used all along the paper. Given a **reduction system** \mathcal{R} , we denote by $\rightarrow_{\mathcal{R}}$ the (one-step) reduction relation associated to system \mathcal{R} . We write $\rightarrow_{\mathcal{R}}^{\bar{\cdot}}$ and $\rightarrow_{\mathcal{R}}^*$ for the reflexive and reflexive-transitive closure of $\rightarrow_{\mathcal{R}}$, and $\rightarrow_{\mathcal{R}}^n$ for the composition of n -steps of $\rightarrow_{\mathcal{R}}$. A term t is said to be **\mathcal{R} -reducible** if there is s such that $t \rightarrow_{\mathcal{R}} s$, and t is said to be **\mathcal{R} -irreducible**, or in **\mathcal{R} -normal form**, written $t \not\rightarrow_{\mathcal{R}}$, if there is no s such that $t \rightarrow_{\mathcal{R}} s$. A term t is said to be **\mathcal{R} -terminating** if there is no infinite \mathcal{R} -sequence starting at t . A term t is **\mathcal{R} -diamond** (or enjoys the \mathcal{R} -diamond property) if $t \rightarrow_{\mathcal{R}} t_0$ and $t \rightarrow_{\mathcal{R}} t_1$ with $t_0 \neq t_1$ imply there is t' such that $t_0 \rightarrow_{\mathcal{R}} t'$ and $t_1 \rightarrow_{\mathcal{R}} t'$. A term t is **\mathcal{R} -locally confluent** (resp. **\mathcal{R} -confluent**) if $t \rightarrow_{\mathcal{R}} t_0$ and $t \rightarrow_{\mathcal{R}} t_1$ (resp. $t \rightarrow_{\mathcal{R}}^* t_0$ and $t \rightarrow_{\mathcal{R}}^* t_1$) imply there is t' such that $t_0 \rightarrow_{\mathcal{R}}^* t'$ and $t_1 \rightarrow_{\mathcal{R}}^* t'$. A relation \mathcal{R} is **terminating** (resp. **diamond**, **locally confluent**, **confluent**) if and only if every term is \mathcal{R} -terminating (resp. \mathcal{R} -diamond, \mathcal{R} -locally confluent, \mathcal{R} -confluent). Any relation verifying the diamond property is in particular confluent, and any relation verifying termination and local confluence is confluent [50]. Moreover, if t is confluent, then its \mathcal{R} -normal form, if it exists, is unique [50].

3 LINEAR OPEN CALL-BY-VALUE

Most notions of open CBV evaluation that have been studied in the literature are not useful, in the sense explained in the introduction. The only exception is the *useful fireball calculus* [2, 4], whose formulation is non-inductive: the evaluation of a complex expression is not given in terms of the evaluation of its immediate subexpressions, but rather by means of side conditions of a global nature. This makes it hard to reason inductively, as typically done when one wants to build type systems on top of an untyped syntax. In this paper, a new inductive notion of usefulness for CBV is developed. This is done in two steps. Indeed, useful evaluation is based on two key components:

Sharing structures. A term is a **structure**² if its *unfolding* (performing all the remaining substitutions) is an application headed by a variable, *i.e.* of the form $x t_1 \dots t_n$ ($n \geq 0$). For example, $(x x)[x/y I][y/z z]$ is a structure because it unfolds to $z z I (z z I)$, which is headed by z . Substituting a variable by a structure never creates a db-redex, thus, in useful evaluation, structures inside ESs must always remain *shared*. This means, for example, that $(x x)[x/y I][y/z z]$ is a normal form in useful evaluation.

Substituting abstractions for progress. In useful evaluation, abstractions not only have to be substituted *on demand*, as in call-by-need, but they also must contribute to creating a db-redex, in order for the computation to make progress, for example, $x[x/I] \rightarrow I[x/I]$ and $(t x)[x/I] \rightarrow (t I)[x/I]$ are *not* useful steps, while $x[x/I] y \rightarrow I[x/I] y$ is. Some of these ideas can already be found in the literature about optimal reduction, *e.g.* [51]. Hence, in this section, we present a CBV calculus, called linear open CBV (LOCBV°), which only implements sharing of structures, so that value substitution remains unrestricted, even if this does not contribute to the progress of the computation. To allow sharing structures, LOCBV° extends the fireball calculus with ESs. This calculus is *linear* in the sense that variables are substituted one occurrence at a time. In the following Section 4, we shall further refine LOCBV° to obtain a notion of *useful open CBV* evaluation, based on the second key component described above to achieve usefulness. In Section 5 we relate these two notions.

The Fireball Calculus. In Plotkin's closed CBV, values are defined to be just abstractions. In the fireball calculus, values are generalized to *fireballs* ($f ::= \lambda x. t \mid i$) defined mutually recursively with the notion of *inert* term ($i ::= x f_1 \dots f_n$ with $n \geq 0$). Since inert terms necessarily contain free variables, fireballs collapse to λ -abstractions in a closed setting, and thus we can see the fireball calculus as a natural extension of Plotkin's closed CBV.

The reduction relation of the fireball calculus is given by the reduction rule $(\lambda x. t) f \rightarrow_{\beta_f} t\{x := f\}$, which is closed by (*pure*) *surface* contexts $S ::= \diamond \mid tS \mid St$. Here $t\{x := f\}$ performs *full* substitution: all the free occurrences of x in t are replaced by f simultaneously, avoiding capture. The fireball calculus as presented above is *not* reasonable as an implementation technique. Indeed, the families of terms of Example 3.1 show that the fireball calculus without sharing still suffers from the size explosion problem:

Example 3.1 (Size explosion). Consider the families of terms $(t_n)_{n \in \mathbb{N}}$ and $(s_n)_{n \in \mathbb{N}}$ given by:

$$t_0 := z \quad t_{n+1} := (\lambda x. x x) t_n \quad s_0 := z \quad s_{n+1} := s_n s_n$$

Then t_n has size linear in n , evaluates in n β -steps to s_n , but the size of s_n is exponential in n .

Sharing Structures. As extensively discussed in [4], it is sufficient in general to avoid the substitution of structures to obtain a reasonable implementation of open CBV and, for that, one can rely on ESs. One calculus implementing this mechanism for open terms is the *value substitution calculus* (vsc) [12]. The vsc is based on two kinds of steps.

²This name is borrowed from [21], but the terminology *inert term* is also used [2].

Distant beta steps: they perform function applications by creating a delayed ES; for example, reducing $(\lambda x. I) (y y) z$ produces $I[x/y y] z$. However, the ES $[x/y y]$ could be seen in principle as blocking the interaction between I and its value argument z . The solution is to adopt a generalized rule for function application which allows to directly reduce $I[x/y y] z$ to $x'[x'/z][x/y y]$, i.e. a list of ESs $[x_1/u_1] \dots [x_n/u_n]$ is allowed now to lie between an abstraction and its argument, as in $(\lambda x. t)[x_1/u_1] \dots [x_n/u_n] s \rightarrow t[x/s][x_1/u_1] \dots [x_n/u_n]$. This rule is written more succinctly as $(\lambda x. t)_L s \rightarrow_{\text{db}} t[x/s]L$, where L is a substitution context. The rule is called *distant beta* (db), originally introduced by [8].

Substitution steps: they perform full substitution, but only of values, by pushing outside the eventual substitution context affecting the substituted value. This rule is written as $t[x/v]L \rightarrow t\{x := v\}L$, where L is a substitution context. For example $(x x)[x/\delta[y/I]] \rightarrow (\delta \delta)[y/I]$, where we recall that $\delta := \lambda z. z z$.

Given that substitution of structures may produce size explosion, and that it does not contribute to the **progress of the computation** (i.e. it does not create function applications), such substitutions are going to be disallowed. Indeed, the *vsc shares structures*, i.e. substitution steps are allowed only to substitute variables by *values*, but not to substitute variables by structures. Thus, an evaluation step like $(x x)[x/z z] \rightarrow z z (z z)$ is disallowed.

Note that substituting variables by variables may be required to contribute to the progress of the computation; for example, substituting the underlined x by y in $(\underline{x} z)[x/y][y/I] \rightarrow (y z)[y/I] \rightarrow \underline{I} z$ contributes to creating later the underlined function application $I z$. Recall that the two key ingredients for useful evaluation are *sharing structures* and *substituting abstractions for progress*. The *vsc* shares structures, as it *never* substitutes a variable by a structure. This is enough to recover adequacy. In particular the term $(\lambda x. \delta) (y y) \delta$ mentioned in the introduction reduces to $\delta[x/y y]\delta$, and this in turn to $(\delta \delta)[x/y y]$, which is now meaningless (i.e. unsolvable). However, this calculus is not useful, since it *always* substitutes a variable by an abstraction, even if this does not contribute to creating a function application.

Linear Substitution. The obvious idea is to restrict *vsc* to substitute variables by abstractions only when contributing to creating a function application. The delicate point is that a variable can occur multiple times: substituting some of them by an abstraction may lead to creating function applications, but others might not. For example, in the step $(z (\underline{x} y) \bar{x})[x/I] \rightarrow z (I y) I$, substituting the first (underlined) occurrence of x by I creates a function application $I y$, while substituting the second (overlined) occurrence of x by I does not contribute to creating a function application.

To formulate a useful notion of open CBV evaluation, we depart from the *vsc* by refining the operation of substitution to be *linear*, i.e. to substitute one occurrence of a variable at a time. One possible way to present this *linear* variant of *vsc* is simply to equip it with the distant beta rule (whose left-hand side is called a **db-redex**), together with a rule to substitute a *single occurrence of a variable* x by a value v . The resulting calculus is called here the *linear vsc*, and written *lvsc*:

$$(\lambda x. t)_L s \rightarrow_{\text{db}} t[x/s]L \quad S\langle x \rangle[x/v]L \rightarrow_{\text{ls}} S\langle v \rangle[x/v]L$$

In these rules, S stands for a *surface context*, i.e. a context that does not go inside abstractions: $S ::= \diamond \mid S t \mid t S \mid S[x/t] \mid t[x/S]$. For example:

$$\begin{aligned} (\lambda x. \lambda y. x y y) (z z) (I z) &\rightarrow (\lambda y. x y y)[x/z z] (I z) \\ \rightarrow (x y y)[y/I z][x/z z] &\rightarrow (x y y)[y/w][w/z][x/z z] \\ \rightarrow (x w y)[y/w][w/z][x/z z] &\rightarrow (x w w)[y/w][w/z][x/z z] \end{aligned}$$

The calculus above implements *linear* substitution, which proceeds by *micro-steps*, replacing one variable at a time. Note that in the last term, the substitution $[y/w]$ cannot be used anymore, because y does not occur free in $x w w$. Some calculi with linear substitution incorporate a reduction rule to erase unused substitutions, called *garbage collection* [8]. In this work, we are not interested in space complexity, so we disregard the issue of garbage collection. Linear substitution models more closely how abstract machines usually work [13].

We now give an *alternative* presentation of *lvsc*, using a style closer to the one we finally used for the uocbv^\bullet in Section 4.

The Linear Open CBV Calculus. The linear open CBV calculus, abbreviated locbv° , is just an alternative presentation of the *lvsc* of above which follows Balabonski et al.'s style, —in fact, they define the same reduction relation—. By *linear* we mean that the *lvsc* and the locbv° use a *linear* substitution operation (in contrast with *vsc* which uses a *full* substitution operation).

Formally, we define a family of binary relations $\xrightarrow{\rho}$, where ρ distinguishes the **step kind**, which is an element of the set $\{\text{db}, \text{lsv}, \text{sub}_{(x,v)}\}$, x being a variable and v a value such that $x \notin \text{fv}(v)$. The set of free variables of a step kind ρ is given by $\text{fv}(\text{db}) = \emptyset$, $\text{fv}(\text{lsv}) = \emptyset$, and $\text{fv}(\text{sub}_{(x,v)}) = \{x\} \cup \text{fv}(v)$. The **linear** relation $\xrightarrow{\rho}$ of locbv° is defined inductively as follows:

$$\frac{}{(\lambda x. t)_L s \xrightarrow{\text{db}} t[x/s]L} \text{DB}^\circ$$

$$\begin{array}{c}
\frac{}{x \xrightarrow{\text{sub}(x,v)} v} \text{SUB}^\circ \quad \frac{t \xrightarrow{\text{sub}(x,v)} t'}{t[x/v] \xrightarrow{\text{lsv}} t'[x/v]} \text{LSV}^\circ \\
\frac{t \xrightarrow{\rho} t'}{t s \xrightarrow{\rho} t' s} \text{APPL}^\circ \quad \frac{s \xrightarrow{\rho} s'}{t s \xrightarrow{\rho} t s'} \text{APPR}^\circ \\
\frac{t \xrightarrow{\rho} t' \quad x \notin \text{fv}(\rho)}{t[x/s] \xrightarrow{\rho} t'[x/s]} \text{ESL}^\circ \quad \frac{s \xrightarrow{\rho} s'}{t[x/s] \xrightarrow{\rho} t[x/s']} \text{ESR}^\circ
\end{array}$$

Rules DB° , SUB° , and LSV° introduce the three kinds of evaluation steps, whereas APPL° , APPR° , ESL° , and ESR° are *the only* congruence rules, because reduction is *weak* (i.e. evaluation does not proceed inside the bodies of λ -abstractions).

A step of the form $t \xrightarrow{\text{db}} s$ represents a distant beta step. The evaluation steps $t \xrightarrow{\text{sub}(x,v)} s$ and $t \xrightarrow{\text{lsv}} s$ correspond to two kinds of *substitution* steps. The first kind of step, $t \xrightarrow{\text{sub}(x,v)} s$, substitutes a *free* occurrence of x in t by the value v . The second kind of step, $t \xrightarrow{\text{lsv}} s$, substitutes a *bound* occurrence of a variable x by a value v , as long as x is bound to a term of the form vL by an ES. The two kinds of substitution steps focus on a *single occurrence* of a variable, i.e. they are *linear* substitution steps. The only rule that allows to create a lsv-step is LSV° , while the contextual rules APPL° , APPR° , ESL° and ESR° are used to propagate any kind of steps, including lsv. Indeed, each step substituting a bound variable ($\xrightarrow{\text{lsv}}$) depends internally on a step that substitutes a free variable ($\xrightarrow{\text{sub}(x,v)}$). In addition to the rule DB° mentioned above, an application may be evaluated using rules APPL° and APPR° , which evaluate within the left and right subterms, respectively. Note that rules DB° , APPL° and APPR° overlap, so reduction is not deterministic. For example:

$$x[x/I] (I I) \xleftarrow{\text{db}} I I (I I) \xrightarrow{\text{db}} I I x[x/I]$$

Congruence rules for ESs allow evaluating the left-hand side of the term (ESL°), as well as the argument of the substitution (ESR°). Again, there is an overlap between these rules. A technical point in rule ESL° is that the variable x bound by the ES $[x/s]$ may *not* occur free in the step kind ρ . This is to avoid variable capture; e.g. it is **not** possible to derive a “pathological” step like $y[x/z] \xrightarrow{\text{sub}(y,x)} x[x/z]$ by means of rule ESL° from the valid step $y \xrightarrow{\text{sub}(y,x)} x$.

Example 3.2. For example, the following is a sequence of evaluation steps to normal form according to LOCBV° :

$$\begin{array}{ccc}
(\lambda x. z x (x y)) I & \xrightarrow{\text{db}} & (z x (x y)) [x/I] \\
\xrightarrow{\text{lsv}} (z I (x y)) [x/I] & \xrightarrow{\text{lsv}} & (z I (I y)) [x/I] \\
\xrightarrow{\text{db}} (z I (w [w/y])) [x/I] & \xrightarrow{\text{lsv}} & (z I (y [w/y])) [x/I]
\end{array}$$

Confluence. Despite the overlaps in the rules mentioned above, it is not difficult to show that toplevel $\xrightarrow{\text{db}}$ reduction is confluent. This claim may be somewhat puzzling, given that the SUB° rule allows to substitute a variable for *any* value, thus $x \xrightarrow{\text{sub}(x,v_1)} v_1$ and $x \xrightarrow{\text{sub}(x,v_2)} v_2$. Indeed, confluence of $\xrightarrow{\text{sub}(_,v)}$ makes no sense: these steps are only an auxiliary mechanism to be able to define lsv reduction steps and in particular to define the notion of *linear* substitution of a single occurrence of a variable by a value. What one actually want to show is that confluence holds for the *toplevel* step kinds (db and lsv) and not for $\text{sub}(_,v)$. More precisely, if $\xrightarrow{\text{top}} := \xrightarrow{\text{db}} \cup \xrightarrow{\text{lsv}}$ stands for toplevel LOCBV° reduction, then:

PROPOSITION 3.3. $\xrightarrow{\text{top}}$ is confluent.

4 USEFUL OPEN CALL-BY-VALUE

The notion of useful evaluation first appeared in [10], in which useful leftmost-outermost CBN evaluation is proposed. Their definition of useful step [10, Definition 6.1] relies on the key notion of the *relative unfolding* of a term t with respect to a surrounding context C , written $t^{\downarrow C}$. The relative unfolding performs all the ESs found in C , so for example $(x x)^{\downarrow (\diamond y)[x/z z]} = z z (z z)$. While function application steps are always deemed to be useful, the situation is more subtle for substitutions. Indeed, consider a step $C\langle x \rangle \rightarrow C\langle t \rangle$, where the variable x is substituted by the term t ; this step occurs when x is bound to t through an ES, appearing in the context C . This step is deemed to be useful depending on the relative unfolding $t^{\downarrow C}$. Specifically, the substitution step is useful if it contributes to the progress of the computation. In CBN, this can be due to two reasons: first, it may be that $t^{\downarrow C}$ itself contains a β -redex; second, it may be that the substitution creates a β -redex, when $t^{\downarrow C}$ is an abstraction and the hole of the context C is applied to an argument. For example, the substitution step $x[x/I z] \rightarrow (I z)[x/I z]$ is useful for the first reason, because $(I z)^{\downarrow \diamond} = I z$ is reducible. The substitution step $x[x/y z][y/I] \rightarrow (y z)[x/y z][y/I]$ is also useful for the first reason, because $(y z)^{\downarrow [y/I]} = I z$ is reducible. On the other hand, $(x z)[x/y][y/I] \rightarrow (y z)[x/y][y/I]$ is useful by the second reason, because $y^{\downarrow [y/I]} = I$ is an abstraction, and the hole of $(\diamond z)[x/y][y/I]$ is applied to an argument z .

Useful evaluation has been extended to CBV [2, 4]. This definition again relies on the notion of relative unfolding: useful CBV evaluation steps are not characterized by *local* predicates, but rather by means of side conditions of a *global* nature. The main difficulty is that their notion is not *inductive*, in the sense that applying congruence rules below term constructors may turn a non-useful step into a useful one. For example, the substitution step $x[x/I] \rightarrow I[x/I]$ is not useful, because x is not applied, whereas $x[x/I] t \rightarrow I[x/I] t$ is a useful step. This makes it hard to reason about the properties of useful evaluation using inductive arguments.

In this section, we refine the notion of LOCBV° reduction introduced in Section 3, in order to encompass usefulness. Indeed, we keep *sharing of structures* as in LOCBV° , but we also restrict evaluation to *substitute abstractions only for progress*. This restriction is non-trivial to impose: in order to achieve an inductive specification of usefulness we define a family of evaluation relations that are indexed by certain *parameters* representing the essential information coming from the surrounding evaluation contexts: this is the minimal data that cannot be ignored to decide what is useful and what is not.

The remainder of this section is organized as follows: We first define the new notion of *useful open* CBV, written UOCBV^\bullet . We then give an inductive characterization of its normal forms. We finally show that UOCBV^\bullet satisfies the Diamond Property (Theorem 4.5), which ensures that the length n of a reduction sequence to normal form does not depend on the particular order chosen to evaluate the terms, thus allowing this natural number n to be taken as a time complexity measure.

The Useful Open CBV Strategy. We now introduce the notion of useful open CBV evaluation, which implements both sharing of structures and substitution of abstractions only when progress is guaranteed. The second feature is subtle. Intuitively, progress is in principle related to the substitution of an applied variable by an abstraction, so that a db-step is *immediately* created. But progress can also be related to the substitution of an applied variable by another variable, which in turn can be substituted by an abstraction, *indirectly* leading to a db-step. We thus need to identify those terms that are already abstractions and those that are currently variables but will be substituted in turn by abstractions; both are necessary to achieve progress. Thus, given a set of variables \mathcal{A} called an **abstraction frame**, the set of terms considered as **hereditary abstractions** under \mathcal{A} , written $\text{HA}_{\mathcal{A}}$, is defined inductively as follows:

$$\frac{}{\lambda x. t \in \text{HA}_{\mathcal{A}}} \text{H-LAM} \quad \frac{t \in \text{HA}_{\mathcal{A}} \quad x \notin \mathcal{A}}{t[x/s] \in \text{HA}_{\mathcal{A}}} \text{H-SUB}_1$$

$$\frac{x \in \mathcal{A}}{x \in \text{HA}_{\mathcal{A}}} \text{H-VAR} \quad \frac{t \in \text{HA}_{\mathcal{A} \cup \{x\}} \quad x \notin \mathcal{A} \quad s \in \text{HA}_{\mathcal{A}}}{t[x/s] \in \text{HA}_{\mathcal{A}}} \text{H-SUB}_2$$

Indeed, every abstraction belongs to the set of hereditary abstractions, for any \mathcal{A} . In the case of variables, only those in the abstraction frame are hereditary abstractions: they will eventually be substituted by abstractions to create —directly or indirectly— a db-step, thus guaranteeing progress in the computation. The set of hereditary abstractions may also contain terms with ES, depending on whether the argument by which the variable will be substituted is a hereditary abstraction or not. For example, performing a substitution step in the term $x[x/y_1 y_2] z$ will never generate a db-step: thus the term $x[x/y_1 y_2]$ is not a hereditary abstraction. Another example is given by $x[x/I] z$: progress is obtained by performing a substitution step on the left-hand side of the application because a db-redex of the form $I[x/I] z$ can be created. This means that $x[x/I]$ is a hereditary abstraction, according to rule H-SUB_2 . Terms that are applications (whether they are db-redexes or not) do not belong to the set of hereditary abstractions under any abstraction frame, as they do not comply with the principle of being abstractions or variables to be substituted by abstractions during the evaluation.

It is also necessary to distinguish the irreducible terms that are not hereditary abstractions. For that, given a set of variables \mathcal{S} called a **structure frame**, the set of **structures** under \mathcal{S} , written $\text{St}_{\mathcal{S}}$, is inductively defined as follows:

$$\frac{x \in \mathcal{S}}{x \in \text{St}_{\mathcal{S}}} \text{S-VAR} \quad \frac{t \in \text{St}_{\mathcal{S}} \quad x \notin \mathcal{S}}{t[x/s] \in \text{St}_{\mathcal{S}}} \text{S-SUB}_1$$

$$\frac{t \in \text{St}_{\mathcal{S}}}{t s \in \text{St}_{\mathcal{S}}} \text{S-APP} \quad \frac{t \in \text{St}_{\mathcal{S} \cup \{x\}} \quad x \notin \mathcal{S} \quad s \in \text{St}_{\mathcal{S}}}{t[x/s] \in \text{St}_{\mathcal{S}}} \text{S-SUB}_2$$

Indeed, no abstraction belongs to the set of structures, for any structure frame. An application is a structure if its left-hand side is itself a structure; this excludes db-redexes from belonging to any structures set. In the case of variables, those that belong to the structure frame are structures. As for the case of terms with ESs, the intuition for rules S-SUB_1 and S-SUB_2 is analogous to the ones for rules H-SUB_1 and H-SUB_2 , respectively. For example, $x[x/y_1 y_2] z$ cannot generate a db-step, hence

$x[x/y_1 y_2]$ is considered to be a structure. The same happens with the term $(x y)[y/I] z$, so $(x y)[y/I]$ is considered to be a structure, even though y is bound to a function.

We summarize below some key (but easy) properties of hereditary abstractions and structures:

Remark 4.1. (1) If $\mathcal{A} \subseteq \mathcal{A}'$ then $\text{HA}_{\mathcal{A}} \subseteq \text{HA}_{\mathcal{A}'}$; (2) A term of the form $t s$ is never in $\text{HA}_{\mathcal{A}}$; (3) If $t \in \text{HA}_{\mathcal{A}}$ then t is of the form $t = vL$; (4) If $\mathcal{S} \subseteq \mathcal{S}'$ then $\text{St}_{\mathcal{S}} \subseteq \text{St}_{\mathcal{S}'}$; (5) A term of the form $(\lambda x. t)L$ is never in $\text{St}_{\mathcal{S}}$; (6) For any L , one has that $(\lambda x. t)L \in \text{HA}_{\mathcal{A}}$.

Given all the previous ingredients, we can move on to the reduction rules for UOCBV^\bullet . We define a family of binary relations $\xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu}$, where ρ is a step kind, \mathcal{A} is an abstraction frame, \mathcal{S} a structure frame and $\mu \in \{ @, \emptyset \}$ a positional flag. The **useful** relation $\xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu}$ is defined inductively as follows:

$$\begin{array}{c}
\frac{}{(\lambda x. t)L s \xrightarrow{\text{db}, \mathcal{A}, \mathcal{S}, \mu} t[x/s]L} \text{DB}^\bullet \quad \frac{}{x \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, @} v} \text{SUB}^\bullet \\
\frac{t \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} t' \quad x \notin \mathcal{A} \cup \mathcal{S} \quad vL \in \text{HA}_{\mathcal{A}}}{t[x/vL] \xrightarrow{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} t'[x/v]L} \text{LSV}^\bullet \\
\frac{t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, @} t'}{t s \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t' s} \text{APPL}^\bullet \quad \frac{t \in \text{St}_{\mathcal{S}} \quad s \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, @} s'}{t s \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t s'} \text{APPR}^\bullet \\
\frac{s \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, @} s'}{t[x/s] \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t[x/s']} \text{ESR}^\bullet \\
\frac{t \xrightarrow{\rho, \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} t' \quad s \in \text{HA}_{\mathcal{A}} \quad x \notin \mathcal{A} \cup \mathcal{S} \quad x \notin \text{fv}(\rho)}{t[x/s] \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'[x/s]} \text{ESLA}^\bullet \\
\frac{t \xrightarrow{\rho, \mathcal{A}, \mathcal{S} \cup \{x\}, \mu} t' \quad s \in \text{St}_{\mathcal{S}} \quad x \notin \mathcal{A} \cup \mathcal{S} \quad x \notin \text{fv}(\rho)}{t[x/s] \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'[x/s]} \text{ESLS}^\bullet
\end{array}$$

Note that each UOCBV^\bullet step is in particular a LOCBV° step, i.e., $\xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} \subseteq \xrightarrow{\rho}$. Note also that the reduction relation defined above is *non-erasing*: $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'$ implies $\text{fv}(t) = \text{fv}(t')$ whenever $\rho = \{\text{db}, \text{lsv}\}$. As in the previous section, rules DB^\bullet , SUB^\bullet and LSV^\bullet introduce the three possible kinds of evaluation steps, whereas all the other cases are congruence rules for steps of an arbitrary step kind $\rho \in \{\text{db}, \text{lsv}, \text{sub}(x,v)\}$. Note also that there are no congruence rules to evaluate under abstractions, so that reduction is again *weak*.

Rule DB^\bullet performs a function application step, the same one performed by rule DB° in LOCBV° . Note that evaluation steps of the form $t \xrightarrow{\text{sub}(x,v), \mathcal{A}, \mathcal{S}, \mu} s$ and $t \xrightarrow{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} s$ correspond to two different kinds of *substitution steps*, as in LOCBV° : the former substitutes a free occurrence of a variable, while the latter substitutes a bound occurrence. The only rule that allows to create an lsv-step is LSV^\bullet , while the congruence rules APPL^\bullet , APPR^\bullet , ESLA^\bullet , ESLS^\bullet and ESR^\bullet are used to propagate any kind of steps, including lsv. Indeed, each application of the LSV^\bullet rule depends internally on a $\text{sub}(x,v)$ -step, where v is necessarily required to be a hereditary abstraction, a restriction which is not required in the LSV° rule of LOCBV° . Notice also that substituting a free variable in a term t using rule SUB^\bullet is only possible when t is in an applied position, meaning that progress of the computation is possible. The following example is an instance of the LSV^\bullet rule:

$$\frac{\frac{}{x \xrightarrow{\text{sub}(x,I), \{x\}, \{z\}, @} I} \text{SUB}^\bullet \quad I[y/z] \in \text{HA}_\emptyset}{x[x/I[y/z]] \xrightarrow{\text{lsv}, \emptyset, \{z\}, @} I[x/I][y/z]} \text{LSV}^\bullet$$

APPL^\bullet and APPR^\bullet are congruence rules for the application constructor. The rule APPR^\bullet also requires the left subterm of the application to be a structure, thus avoiding a possible overlap with rule DB^\bullet . Still, rules APPL^\bullet and APPR^\bullet overlap, so reduction is not deterministic, like in the linear CBV strategy. For example:

$$x y[y/I] (I I) \xleftarrow{\text{db}, \emptyset, \{x\}, @} x (I I) (I I) \xrightarrow{\text{db}, \emptyset, \{x\}, @} x (I I) y[y/I]$$

ESR^\bullet , ESLA^\bullet and ESLS^\bullet are congruence rules for the ES constructor. More precisely, rule ESR^\bullet allows the evaluation of the argument of any ESs, while the two other rules allow the evaluation of the left-hand side of the substitution under some conditions: ESLA^\bullet (resp. ESLS^\bullet) can only be applied if the argument of the substitution is a hereditary abstraction (resp. a structure). Note that rules ESLA^\bullet and ESLS^\bullet force to evaluate the argument s of an ES $t[x/s]$ until it becomes “rigid”, and only

then one is allowed to proceed to evaluate the body t . This is consistent with a CBV strategy. As long as evaluation terminates, these rules cover all possible cases, because a normalizing term is guaranteed to always become either a hereditary abstraction or a structure (cf. Lemma B.4 in Appendix B). These last two rules are subject to the condition that the variable x bound by the ES $[x/s]$ must not occur free in the step kind ρ . This is the same kind of restriction used in the rule esL° presented in Section 3, and it intends to prevent variable capture and pathological steps such as $y[x/I] \xrightarrow{\text{sub}(y,x),\{y\},\emptyset,@} x[x/I]$. Note that there is a possible overlap between rule esR^\bullet and rules esLA^\bullet and esLS^\bullet .

A term t is $(\rho, \mathcal{A}, \mathcal{S}, \mu)$ -**reducible** if there exists a term t' such that $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'$. A term t belongs to the set $\text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$ if t is $(\rho, \mathcal{A}, \mathcal{S}, \mu)$ -reducible for some step kind ρ ; and t belongs to $\text{Irred}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$ if $t \notin \text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$.

Up to this point, hereditary abstractions and structures exhibit distinct and disjoint behaviors within the framework of uocbv^\bullet . This divergence is evident through the use of the abstraction frame and structure frame, each capturing the unique characteristics of these constructs. For that, an abstraction frame \mathcal{A} and a structure frame \mathcal{S} are said to verify the **correctness invariant** for t , written $\text{inv}(\mathcal{A}, \mathcal{S}, t)$, if $\mathcal{A} \cap \mathcal{S} = \emptyset$ and $\text{fv}(t) \subseteq \mathcal{A} \cup \mathcal{S}$. We will usually need to assume this invariant $\text{inv}(\mathcal{A}, \mathcal{S}, t)$ when stating theorems. Remark that $\text{inv}(\emptyset, \text{fv}(t), t)$ always holds, so for a toplevel term t , one takes $\mathcal{A} := \emptyset$, and $\mathcal{S} := \text{fv}(t)$.

Example 4.2. The following is a sequence of evaluation steps in uocbv^\bullet to normal form:

$$\begin{aligned} & (\lambda x. z x (x y)) I \xrightarrow{\text{db}, \emptyset, \{z, y\}, @} (z x (x y)) [x/I] \\ & \xrightarrow{\text{lsv}, \emptyset, \{z, y\}, @} (z x (I y)) [x/I] \xrightarrow{\text{db}, \emptyset, \{z, y\}, @} (z x (w [w/y])) [x/I] \end{aligned}$$

Compare this evaluation with the corresponding evaluation of the same term in locbv° (Example 3.2). Note that in uocbv^\bullet the leftmost occurrence of x and the occurrence of w are not substituted, because they do not contribute to creating a function application.

Example 4.3. The following is a sequence of evaluation steps in uocbv^\bullet to normal form:

$$\begin{aligned} & (x z) [x/y [y/I]] \xrightarrow{\text{lsv}, \emptyset, \{z\}, @} (y z) [x/y] [y/I] \\ & \xrightarrow{\text{lsv}, \emptyset, \{z\}, @} (I z) [x/y] [y/I] \xrightarrow{\text{db}, \emptyset, \{z\}, @} x_1 [x_1/z] [x/y] [y/I] \end{aligned}$$

Operational Properties. We state two theorems about the operational properties of uocbv^\bullet . Theorem 4.4 provides an inductive characterization of the set of normal forms, while Theorem 4.5 shows that it enjoys a very strong form of confluence, namely the diamond property. See the appendix (Appendix B) for more details and proofs.

THEOREM 4.4 (CHARACTERIZATION OF NORMAL FORMS). *The set of irreducible terms $\text{Irred}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$ is exactly the set $\text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$ defined inductively as below:*

$$\begin{aligned} & \frac{x \in \mathcal{A} \Rightarrow \mu = @}{x \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet} \text{NF-VAR}^\bullet \quad \frac{}{\lambda x. t \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^\bullet} \text{NF-LAM}^\bullet \\ & \frac{t \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^\bullet \quad s \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^\bullet}{t s \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet} \text{NF-APP}^\bullet \\ & \frac{t \in \text{NF}_{\mathcal{A} \cup \{x\}, \mathcal{S}, \mu}^\bullet \quad s \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^\bullet \quad s \in \text{HA}_{\mathcal{A}}}{t [x/s] \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet} \text{NF-ESA}^\bullet \\ & \frac{t \in \text{NF}_{\mathcal{A}, \mathcal{S} \cup \{x\}, \mu}^\bullet \quad s \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^\bullet \quad s \in \text{St}_{\mathcal{S}}}{t [x/s] \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet} \text{NF-ESS}^\bullet \end{aligned}$$

THEOREM 4.5 (DIAMOND PROPERTY). *Let $t \xrightarrow{\rho_1, \emptyset, \mathcal{S}, @} t_1$ and $t \xrightarrow{\rho_2, \emptyset, \mathcal{S}, @} t_2$, where $t_1 \neq t_2$ and $\rho_1, \rho_2 \in \{\text{db}, \text{lsv}\}$ and $\mathcal{S} = \text{fv}(t)$. Then there exists t' such that $t_1 \xrightarrow{\rho_2, \emptyset, \mathcal{S}, @} t'$ and $t_2 \xrightarrow{\rho_1, \emptyset, \mathcal{S}, @} t'$.*

The two announced consequences of the previous result follow:

COROLLARY 4.6. *Any two reduction sequences to normal form in uocbv^\bullet have the same number of db and lsv steps.*

Similarly as for locbv° , confluence of uocbv^\bullet holds only for the *toplevel* step kinds (db and lsv). Indeed, if $\xrightarrow{\text{top}, \mathcal{S} := @} \xrightarrow{\text{db}, \emptyset, \mathcal{S}, @} \cup \xrightarrow{\text{lsv}, \emptyset, \mathcal{S}, @}$ stands for toplevel uocbv^\bullet reduction, then:

COROLLARY 4.7. *$\xrightarrow{\text{top}, \mathcal{S}}$ is confluent.*

5 RELATING LINEAR AND USEFUL OPEN CBV

This section aims to establish a connection between LOCBV° and UOCBV^\bullet , defined in Sections 3 and 4 respectively, in particular showing that our concept of usefulness is a (complete) restriction of LOCBV° , in the sense that UOCBV^\bullet achieves evaluation to *equivalent* normal forms by omitting certain substitution steps, specifically those that do not contribute to the creation of function applications.

Going from UOCBV^\bullet to LOCBV° is easy: indeed, each UOCBV^\bullet step is, in particular, a LOCBV° step, because the former disallows some substitution steps that the latter allows. However, relating them in the reverse direction is much more delicate, because not every LOCBV° step is useful. For example, $(x\ y)[y/I] \xrightarrow{\circ} (x\ I)[y/I]$ is not useful. In fact, $(x\ y)[y/I]$ is a normal form for UOCBV^\bullet (details in Proposition 5.1), whereas its normal form is $(x\ I)[y/I]$ in LOCBV° .

Normal forms in both formalisms are different, but structurally equivalent because LOCBV° evaluates terms further than UOCBV^\bullet . We then need a precise way to relate them, which is done by means of an *unfolding* operation. Recall that, in calculi with ESs, unfolding is used to perform *all* of the pending substitutions. For example, unfolding the term $x[x/y\ y][y/z\ z]$ yields $z\ z\ (z\ z)[x/y\ y][y/z\ z]$. In this work, unfolding is much subtler, and it has to be defined in a controlled way. For example, unfolding the term $x[x/I]$, which is a normal form in UOCBV^\bullet , yields its corresponding normal form in LOCBV° , which is $I[y/I]$. But the new unfolding operation does not need to unfold *all* ESs: in particular, $(\lambda x. y)[y/I]$ is *not* unfolded to $(\lambda x. I)[y/I]$ (because evaluation is weak), and $x[x/y\ y]$ is *not* unfolded to $(y\ y)[x/y\ y]$ (because structures are shared and never substituted). Intuitively, this notion of unfolding only performs the substitution of those reachable variables that are bound to values.

The remainder of this section is organized as follows. First, we define the notion of **unfolding** of a term t with respect to a value assignment σ . We then relate LOCBV° and UOCBV^\bullet by means of two technical results (Proposition 5.1): we show that the unfolding of normal forms in UOCBV^\bullet always yields a normal form in LOCBV° and that terms reducible in UOCBV^\bullet remain reducible in LOCBV° .

Recursive Definition of Unfolding. According to the preceding discussion, we start by defining the notion of **value assignment**, written σ , which is a *partial* function mapping each variable to a value. The domain and image of a value assignment σ are denoted by $\text{dom}(\sigma)$ and $\text{im}(\sigma)$ respectively. To ensure idempotence, we require the domain $\text{dom}(\sigma)$ and the free variables $\text{fv}(\text{im}(\sigma))$ of the image to be disjoint (*i.e.* $\text{dom}(\sigma) \cap \text{fv}(\text{im}(\sigma)) = \emptyset$). Additionally, we write \cdot^\downarrow to denote the value assignment with an empty domain.

The **unfolding of a term t under the value assignment σ** , written $t^{\downarrow\sigma}$, is defined recursively as follows.

$$\begin{aligned} (\lambda x. t)^{\downarrow\sigma} &:= \lambda x. t \\ (t\ s)^{\downarrow\sigma} &:= t^{\downarrow\sigma}\ s^{\downarrow\sigma} \\ x^{\downarrow\sigma} &:= \begin{cases} \sigma(x) & \text{if } x \in \text{dom}(\sigma) \\ x & \text{otherwise} \end{cases} \\ t[x/s]^{\downarrow\sigma} &:= \begin{cases} t^{\downarrow\sigma \cup (x \mapsto v)}[x/v]^\downarrow & \text{if } s^{\downarrow\sigma} = v^\downarrow, x \in \text{rv}(t) \\ t^{\downarrow\sigma}[x/s^{\downarrow\sigma}] & \text{otherwise} \end{cases} \end{aligned}$$

Notice that there are two base cases. The unfolding operation causes no effect on abstractions, justified by the fact that evaluation is weak. The other base case is when the term is a variable: if $x \in \text{dom}(\sigma)$, the definition mirrors the behavior of performing a step of the form $x \xrightarrow{\circ}_{\text{sub}(x, \sigma(x))} v$; otherwise, the variable is unchanged, reflecting the fact that it will not be substituted by a value. Note that the expansion of σ with $x \mapsto v$ is still a value assignment since we may always assume by α -conversion that $x \notin \text{dom}(\sigma)$, $x \notin \text{fv}(v)$ and for any $v' \in \text{im}(\sigma)$, $x \notin \text{fv}(v')$. The unfolding of toplevel terms is done under the empty value assignment. Some examples of such unfoldings follow: $(\lambda z. x)[x/y]^\downarrow = (\lambda z. x)[x/y]$, $(x\ y)[y/I]^\downarrow = (x\ I)[y/I]$ and $x[x/y[z/I]]^\downarrow = y[x/y][z/I]$. Sometimes, the value assignment used for the left subterm of an ES is the original one, as in the first example, given that $x \notin \text{rv}(\lambda z. x)$. However, sometimes it must be extended, as in the last example: when we apply the unfolding over the subterm x , the value assignment is extended to $(x \mapsto y)$.

Relating Reduction Steps and Normal Forms. We now relate LOCBV° to UOCBV^\bullet by using as a midpoint the unfolding of a term with respect to a value assignment, as previously explained. Our goal is to prove that UOCBV^\bullet simulates LOCBV° while still obtaining the same normal forms, up to unfolding. To prove this, we state the following result, which has two parts: first, the unfolding of a useful normal form is a linear normal form; second, the unfolding of a reducible term in the useful strategy is either db-reducible in the linear strategy or an abstraction in an applied position. This second point is precisely the one that justifies the name “useful”, as it intuitively means that any substitution step in the useful strategy must contribute to the creation of a db-step (*cf.* Section 1).

Recall that $\overset{\circ}{\rightarrow}_{\text{top}}$ and $\overset{\bullet}{\rightarrow}_{\text{top},\mathcal{S}}$ stand for *toplevel* LOCBV° and UOCBV^\bullet reduction respectively, i.e.:

$$\overset{\circ}{\rightarrow}_{\text{top}} := (\overset{\circ}{\rightarrow}_{\text{db}} \cup \overset{\circ}{\rightarrow}_{\text{lsv}}) \quad \overset{\bullet}{\rightarrow}_{\text{top},\mathcal{S}} := (\overset{\bullet}{\rightarrow}_{\text{db},\mathcal{Q},\mathcal{S}_{\text{@}}} \cup \overset{\bullet}{\rightarrow}_{\text{lsv},\mathcal{Q},\mathcal{S}_{\text{@}}})$$

The following proposition relates LOCBV° and UOCBV^\bullet in the *toplevel* case. In the appendix (Proposition C.24) the statement is generalized for arbitrary parameters ($\mathcal{A}, \mathcal{S}, \mu$, etc.), which has some subtleties.

PROPOSITION 5.1. *Let t be a term and $\mathcal{S} = \text{fv}(t)$. Then:*

1. *If t is $\overset{\bullet}{\rightarrow}_{\text{top},\mathcal{S}}$ -irreducible then t^\downarrow is $\overset{\circ}{\rightarrow}_{\text{top}}$ -irreducible.*
2. *If $t \overset{\bullet}{\rightarrow}_{\text{top},\mathcal{S}} t'$ then there exists t'' such that $t^\downarrow \overset{\circ}{\rightarrow}_{\text{db}} t''$.*

As an example of the first property, $(x y)[y/\mathbb{I}]$ is $\overset{\bullet}{\rightarrow}_{\text{top},\{x\}}$ -irreducible, and its unfolding under the empty value assignment is $(x \mathbb{I})[y/\mathbb{I}]$, which is $\overset{\circ}{\rightarrow}_{\text{top}}$ -irreducible. For an example of the second property, consider the lsv step $t = (x y)[x/\mathbb{I}] \overset{\bullet}{\rightarrow}_{\text{top},\{y\}} (\mathbb{I} y)[x/\mathbb{I}] = t' = t^\downarrow$, and note that $t' \overset{\circ}{\rightarrow}_{\text{db}} x_1[x_1/x][y/\mathbb{I}]$.

The preceding results can be easily combined to conclude that t is $\overset{\bullet}{\rightarrow}_{\text{top},\text{fv}(t)}$ -irreducible if and only if t^\downarrow is $\overset{\circ}{\rightarrow}_{\text{top}}$ -irreducible.

6 USEFUL OPEN CBV IS REASONABLE

In this section, we relate the UOCBV^\bullet strategy with prior work in the literature, showing that UOCBV^\bullet is a *reasonable* implementation of open CBV. We proceed in two stages, following [2]. On one side, we prove a **high-level implementation** theorem, stating that reduction in the fireball calculus [2] can be simulated by reduction in UOCBV^\bullet , with *quadratic* overhead in time. On the other side, we prove a **low-level implementation** theorem, stating that reduction in UOCBV^\bullet can be implemented by the GLAMoUr abstract machine [2], with *linear* overhead in time. By composing these results, we obtain that UOCBV^\bullet implements open CBV reasonably. The key property in this section is a simulation result, which embeds the GLAMoUr abstract machine into UOCBV^\bullet .

The definitions and proofs in this section follow well-known methodologies (e.g. [2, 4, 13]). Due to space limitations, we state the main theorems and leave out most of the technical details in Appendix D.

Stability Notions. We say that a term is **rigid** under $(\mathcal{A}, \mathcal{S})$ if it is a hereditary abstraction ($s \in \text{HA}_{\mathcal{A}}$) or a structure ($s \in \text{St}_{\mathcal{S}}$). When evaluating an ES like $t[x/s]$, UOCBV^\bullet reduction only evaluates the body t when the argument s is rigid. To establish a closer correspondence with a low-level abstract machine, we identify a subset of terms, called **stable terms** under $(\mathcal{A}, \mathcal{S})$, in which the arguments of *all* ESs are rigid. Furthermore, we consider a subset of UOCBV^\bullet reduction, called **stable reduction**, and written $\overset{\blacktriangle}{\rightarrow}_{\rho,\mathcal{A},\mathcal{S},\mu} \subseteq \overset{\blacktriangleright}{\rightarrow}_{\rho,\mathcal{A},\mathcal{S},\mu}$, which forces arguments of applications to be rigid before going on. Sometimes we omit some of the parameters, e.g. writing just $\overset{\blacktriangle}{\rightarrow}$, if they are clear from the context. In the stable reduction, the congruence rule APPL^\bullet allows to evaluate the head t of an application $t s$ only if s is rigid, and the db rule allows to contract a redex $(\lambda x. t)\mathbb{L} s \overset{\blacktriangle}{\rightarrow} t[x/s]\mathbb{L}$ only if s is rigid. Stable reduction preserves stable terms. It can also be shown to enjoy the diamond property.

Embedding the GLAMoUr into useful Open CBV . The abstract machine GLAMoUr [2], operates on *states* s , which contain in particular a term t without ESs —the current **focus** of evaluation—. There are seven kinds of transitions in the GLAMoUr: *multiplicative* transitions ($s \rightsquigarrow_{\text{um}} s'$) corresponding to function application steps, *exponential* transitions ($s \rightsquigarrow_{\text{ue}} s'$) corresponding to linear substitution steps, and *administrative* transitions $s \rightsquigarrow_{c_i} s'$ for $i \in 1..5$ that change the focus of evaluation without performing computation. We refer the reader to [2, 4] (or the appendix) for full details.

Each state s of the GLAMoUr can be **decoded** to a term $\{\!\{s}\!\}$ with ESs. As is typical when relating abstract machines and calculi with ESs [13], the GLAMoUr abstract machine can be simulated into UOCBV^\bullet up to a standard notion of **structural equivalence** (\equiv) between terms [3]. In a call-by-value framework, this notion of structural equivalence allows to commute ESs with applications and other ESs, so for example $t u[x/s] \equiv (t u)[x/s]$ and $t[x/s][y/u] \equiv t[x/s][y/u]$ hold as long as there is no variable capture. The main technical lemma is:

LEMMA 6.1 (GLAMoUr SIMULATION). *Let s be a state reachable from an initial state whose focus is t_0 , and let $\mathcal{S}_0 := \text{fv}(t_0)$. Then:*

1. *If $s \rightsquigarrow_{\text{um}} s'$, then $\{\!\{s}\!\} \overset{\blacktriangle}{\rightarrow}_{\text{db}} \equiv \{\!\{s'}\!\}$.*
2. *If $s \rightsquigarrow_{\text{ue}} s'$, then $\{\!\{s}\!\} \overset{\blacktriangle}{\rightarrow}_{\text{lsv}} \equiv \{\!\{s'}\!\}$.*
3. *If $s \rightsquigarrow_{c_i} s'$, then $\{\!\{s}\!\} = \{\!\{s'}\!\}$, for all $i \in \{1..5\}$.*
4. *Progress: if s is \rightsquigarrow -irreducible then $\{\!\{s}\!\}$ is $\overset{\blacktriangle}{\rightarrow}$ -irreducible.*

As a consequence, a *sequence* of GLAMoUr transitions can be decoded as a *sequence* of (stable) UOCBV^\bullet steps interleaved with equivalences. To be able to *postpone* all the intermediate structural equivalence steps to obtain $\{\{s_1\}\} \xrightarrow{\Delta} \xrightarrow{\Delta} \dots \xrightarrow{\Delta} \equiv \{\{s_n\}\}$, we need the following lemma, stating that \equiv is a strong bisimulation with respect to $\xrightarrow{\Delta}$, thus obtaining in particular a postponement property:

LEMMA 6.2 (STRONG BISIMULATION). *Let t_0, s_0 be stable terms such that $s_0 \equiv t_0 \xrightarrow{\Delta}_{\rho, \mathcal{A}, S, \mu} t_1$ then there exists s_1 such that $s_0 \xrightarrow{\Delta}_{\rho, \mathcal{A}, S, \mu} s_1 \equiv t_1$.*

The preceding result relies crucially on the fact that we work with *stable* terms. Indeed, suppose u is not rigid. Then the equivalence $t[x/s][y/u] \equiv t[y/u][x/s]$ cannot be postponed after the (non-stable) step $t[y/u][x/s] \xrightarrow{\Delta} t[y/u][x/s']$ because the rules ESLA^\bullet and ESLS^\bullet cannot be applied to derive a step $t[x/s][y/u] \xrightarrow{\Delta} t[x/s'][y/u]$, given that u is not rigid.

High and Low-Level Implementation. Relying on the previous results, it can now be shown that reduction in the fireball calculus (\rightarrow_{β_f}) can be implemented through UOCBV^\bullet ($\xrightarrow{\Delta}$) with quadratic overhead. More precisely, if we write t^\downarrow for the *full* unfolding of a term t , performing *all* the pending ESs in t :

THEOREM 6.3 (HIGH-LEVEL IMPLEMENTATION). *Let t be a pure term (without ESs) and $S = \text{fv}(t)$. If $t \rightarrow_{\beta_f}^n t'$ then there exists s such that $t \xrightarrow{\Delta}^k s$ where $s^\downarrow = t'$ and $k \in O(|t| \cdot (n^2 + 1))$.*

To complete the picture, it can be shown that UOCBV^\bullet ($\xrightarrow{\Delta}$) can be implemented in the GLAMoUr (\rightsquigarrow) with linear overhead. More precisely:

THEOREM 6.4 (LOW-LEVEL IMPLEMENTATION). *Let t be a pure term (without ESs). If $t \xrightarrow{\Delta}^n t'$ with t' in normal form and s is an initial state such that $\{\{s\}\} = t$ then $s \rightsquigarrow^k s'$ where $\{\{s'\}\}$ is structurally equivalent to t' and $k \in O(|t| \cdot (n + 1))$.*

We can then conclude,

COROLLARY 6.5. *The UOCBV^\bullet strategy is a reasonable implementation of open CBV.*

7 A QUANTITATIVE INTERPRETATION

In this section, we propose a type system for the new strategy UOCBV^\bullet introduced in Section 4. This system, based on non-idempotent intersection types, can be seen as a *semantical interpretation*, akin to *relational models* in the usual sense of linear logic [25, 37].

The existing formulations of useful evaluation in the literature lack the inductive characteristics required to identify the invariant predicates essential for establishing soundness and completeness of the interpretation. On the other hand, there are no semantical interpretations for CBV designed specifically for useful evaluation, as the existing ones *e.g.* [7, 34, 42] do not implement usefulness.

Our work addresses this gap by introducing the first notion of tightness which provides exact quantitative information about evaluation length for useful evaluation. The resulting interpretation is surprisingly natural and simple. It is given by means of a type system, called \mathcal{U} (for *U*seful), which is *sound* and *complete* with respect to UOCBV^\bullet normalization, *i.e.* a term is typable if and only if it is normalizing. This is why we say that the type system *characterizes normalization*. Furthermore, \mathcal{U} provides quantitative information about the use of resources during evaluation, so it can be considered as a **quantitative interpretation/typing system**. More precisely, types in \mathcal{U} represent quantitative specifications of the behavior of a program when evaluated in UOCBV^\bullet . In particular, we want the interpretation to be *perfectly faithful* to the quantitative behavior of the calculus, *i.e.* we not only want the type system to capture upper bounds for the evaluation length but instead to capture *exactly* the number of times that each function is applied during the evaluation of the program (*i.e.* the number of db steps) as well as the number of substitutions that are performed (*i.e.* the number of lsv steps). In particular, our inductive formulation of useful evaluation turns out to be well-suited for demonstrating the typical properties associated with such tight type systems, including subject reduction (Proposition 7.3), subject expansion (Proposition 7.6), and tight typability of normal forms (Proposition 7.5). These proofs would be difficult using a global definition of usefulness.

The remainder of this section is organized as follows: Section 7.1 introduces the formal definition of the quantitative type system \mathcal{U} , while Section 7.2 shows that \mathcal{U} is **sound** and **complete** with respect to UOCBV^\bullet .

7.1 The Quantitative Type System \mathcal{U}

In quantitative type systems, one important point is that the same expression may play different roles in different contexts. For example, in some programs an occurrence of the identity function \mathbb{I} may be applied twice, while in other programs it may be applied only once. In each case, the type of the subexpression \mathbb{I} should change to reflect this quantitative difference. Hence *terms do not have a unique type*. In fact, as usual in intersection type systems, there is no notion of *principal type* in \mathcal{U} .

To be able to capture quantitative information about evaluation, the type of each λ -abstraction is a *multiset* (rather than a *set*) whose cardinality corresponds exactly to the number of times that the abstraction is applied to some argument during the whole evaluation process. In general, the type of an abstraction is a multiset of the form $\mathcal{T} = [\alpha_1, \dots, \alpha_n]$, where each of the α_i is an *arrow type*. For example, the underlined identity function in the expression $(\lambda f. x (f y) (f z)) \underline{\mathbb{I}}$ takes part in *two* function applications. Hence, one possible type assignment for that subexpression is $\vdash \mathbb{I} : [(\mathcal{T} \rightarrow \mathcal{T}), (\mathcal{S} \rightarrow \mathcal{S})]$, meaning that the identity function is applied *twice* during evaluation, once to an argument of type \mathcal{T} and once to an argument of type \mathcal{S} . However, some abstractions may never be applied. For example, the identity function in the program $x[x/\mathbb{I}]$ does not take part in any function application. These abstractions are typed with the *empty* multiset $[\]$.

As studied in Section 4, all (terminating) terms evaluate to either a *variable*, an *abstraction*, or a *structure*. Abstractions, as well as variables bound to abstractions, are assigned finite multisets of arrow types, as we just said, called *arrow multi-types*. Structures, and variables bound to structures, on the other hand, are always given a distinguished type, just written s .

Formally, types of \mathcal{U} are given by the following grammar:

$$\begin{aligned} \text{(Arrow Types)} \quad \alpha, \beta, \dots & ::= \mathcal{T}^? \rightarrow \mathcal{T} \\ \text{(Arrow Multi-Types)} \quad \mathcal{M}, \mathcal{N}, \dots & ::= [\alpha_k]_{k \in K} \\ \text{(Types)} \quad \mathcal{T}, \mathcal{S}, \dots & ::= s \mid \mathcal{M} \\ \text{(Optional Types)} \quad \mathcal{T}^?, \mathcal{S}^?, \dots & ::= \perp \mid \mathcal{T} \end{aligned}$$

We distinguish a set of **tight constants** given by $\mathbb{t} ::= s \mid [\]$, where s is assigned to terms evaluating to structures, while $[\]$ is assigned to terms evaluating to abstractions that are not going to be applied. Unlike in previous quantitative interpretations of CBV, we distinguish here the type \perp from the type $[\]$, the former meaning that *no typing information* is available, and the latter is used as explained above.

The **counting** function $\text{ta}(_)$ returns the number of **toplevel arrows** in a type (or optional type), and it is defined by $\text{ta}(s) := 0$, $\text{ta}(\perp) := 0$, and $\text{ta}([\alpha_k]_{k \in K}) := |K|$. **Typing environments** Γ, Δ, \dots are functions from variables to optional types, assigning \perp to all but finitely many variables. The **domain** of an environment Γ is defined as $\text{dom}(\Gamma) := \{x \mid \Gamma(x) \neq \perp\}$, and \emptyset denotes the empty typing environment, mapping every variable to \perp . The **union of arrow multi-types**, written $\mathcal{M}_1 \uplus \mathcal{M}_2$, are multisets of types defined as expected, where $[\]$ is the neutral element. The **union of types**, written $\mathcal{T}_1 + \mathcal{T}_2$, is the (associative) *partial* operation on types given by $s + s := s$ and $\mathcal{M}_1 + \mathcal{M}_2 := \mathcal{M}_1 \uplus \mathcal{M}_2$, where all other cases are undefined. Note that $\text{ta}(\mathcal{T}_1 + \mathcal{T}_2) = \text{ta}(\mathcal{T}_1) + \text{ta}(\mathcal{T}_2)$. The **union of optional types** is given by $\perp + \perp := \perp$, $\perp + \mathcal{T} := \mathcal{T}$, and $\mathcal{T} + \perp := \mathcal{T}$, so that \perp is the neutral element of $+$. Given typing environments $(\Gamma_i)_{i \in I}$, we write $+_{i \in I} \Gamma_i$ for the environment mapping each variable x to $+_{i \in I} \Gamma_i(x)$, where $\Gamma + \Delta$ and $\Gamma +_{k \in K} \Delta_k$ are particular instances of the general notation. When $\text{dom}(\Gamma) \cap \text{dom}(\Delta) = \emptyset$ we may write $\Gamma; \Delta$ instead of $\Gamma + \Delta$ to emphasize that the domains are disjoint. As a consequence, $\Gamma; x : \perp$ is identical to Γ . We write $x : \mathcal{T}^?$ for the environment assigning $\mathcal{T}^?$ to x and \perp to any other variable. A binary relation of **subsumption** between optional types and types is defined by two cases, declaring that $\perp \triangleleft \mathbb{t}$ and $\mathcal{T} \triangleleft \mathcal{T}$ hold. This subsumption relation is used to introduce a controlled form of weakening in the system.

Typing judgments in \mathcal{U} are not just triples of the form $\Gamma \vdash t : \mathcal{T}$, as usual, but they are rather annotated with natural numbers m and e called *counters*, *i.e.* judgments are of the form $\Gamma \vdash^{(m,e)} t : \mathcal{T}$. Under appropriate conditions, these counters correspond *exactly* to the number of function application steps (m) and substitution steps (e) required to evaluate terms by means of the useful CBV strategy. Typing rules of **system \mathcal{U}** are:

$$\begin{aligned} & \frac{n = \text{ta}(\mathcal{T})}{x : \mathcal{T} \vdash^{(0,n)} x : \mathcal{T}} \text{VAR} \quad \frac{\Gamma \vdash^{(m,e)} t : s \quad \Delta \vdash^{(m',e')} u : \mathbb{t}}{\Gamma + \Delta \vdash^{(m+m',e+e')} t u : s} \text{APP} \\ & \frac{(\Gamma_i; x : \mathcal{T}_i^? \vdash^{(m_i, e_i)} t : \mathcal{S}_i)_{i \in I}}{+_{i \in I} \Gamma_i \vdash^{(+_{i \in I} m_i, +_{i \in I} e_i)} \lambda x. t : [\mathcal{T}_i^? \rightarrow \mathcal{S}_i]_{i \in I}} \text{ABS} \\ & \frac{\Gamma \vdash^{(m,e)} t : [\mathcal{T}^? \rightarrow \mathcal{S}] \quad \mathcal{T}^? \triangleleft \mathcal{T} \quad \Delta \vdash^{(m',e')} u : \mathcal{T}}{\Gamma + \Delta \vdash^{(1+m+m',e+e')} t u : \mathcal{S}} \text{APPC} \end{aligned}$$

$$\frac{\Gamma; x : \mathcal{T}^? \vdash^{(m,e)} t : \mathcal{S} \quad \mathcal{T}^? \triangleleft \mathcal{T} \quad \Delta \vdash^{(m',e')} u : \mathcal{T}}{\Gamma + \Delta \vdash^{(m+m',e+e')} t[x/u] : \mathcal{S}} \text{ ES}$$

The counters are only *extra* information propagated along with typing rules, but they do not impose any condition on typing derivations. It is in fact easy to show that they can be recovered in a single (linear time) recursive traversal from any derivation without counters. For example, it is sufficient to increment one for every rule APPC in the derivation to compute the first counter. Occasionally, we omit them if they are not relevant, writing just $\Gamma \vdash t : \mathcal{T}$.

System \mathcal{U} is *linear*, i.e. each type assumption must be used once and only once. In proof-theoretical jargon, there are no explicit weakening nor contraction rules. This explains in particular the form of the rule VAR, which only has minimal information on the left. In this same rule, the second counter is the number of arrows in the type of the variable x , which represents the number of (useful) substitutions of the variable by a function. Rule ABS is a standard rule in quantitative CBV, and reflects the fact that uocbv^* evaluation is *weak*, i.e. the strategy does not evaluate terms below λ -abstractions. For example, the judgment $\vdash \lambda x. \Omega : []$ is valid even if Ω is non-terminating.

Rule ES can be easily derived from rules ABS and APPC. Rules for applications deserve some discussion. Indeed, the typing system \mathcal{U} keeps track of the different natures of the application constructors involved during the evaluation process: a term constructor is **consuming** if it is destroyed during evaluation, while a **persistent** constructor remains preserved until the normal form. For example in $x((\lambda y. y)z) \rightarrow x(y[y/z])$, the leftmost application constructor is persistent, while the rightmost one is consuming. Therefore, we split the typing rules for applications in two different cases: APPP and APPC, where P stands for *persistent* and C for *consuming*. Rule APPP types a persistent application constructor, and requires the left-hand side of the application to have type \mathfrak{s} , so that there is a guarantee that no new redex will be created, and thus the (typed) application constructor remains persistent. Rule APPC types a consuming application constructor and requires the left-hand side of the application to have a functional type, so that there is a guarantee that a db-redex will be created, and thus the (typed) application constructor is consuming and the first counter is incremented.

We now discuss the use of the subsumption relation in the consuming rules APPC and ES. This is needed because some abstractions do not depend on their arguments. For example, in the program $(\lambda x. I) (z z)$ the subexpression $z z$ is a structure of type \mathfrak{s} , so one would like to type the body of the abstraction as $x : \mathfrak{s} \vdash I : \mathcal{T}$. But this judgment is not valid because the type assumption $x : \mathfrak{s}$ must be used exactly once, and here it is *not* used because x does not occur free in I . To deal with these kinds of situations, the rules for typing an application $t s$ and for typing an ESs $t[x/s]$ allow to “discard” the type of s as long as it is not used by t . This subsumption relation can then be seen as a controlled form of *weakening*. Assuming that $\perp \triangleleft \mathfrak{s}$, one typing derivation that illustrates the use of subsumption is the following:

$$\frac{\frac{\frac{}{y : \mathfrak{s}; x : \perp \vdash^{(0,0)} y : \mathfrak{s}}{\text{VAR}} \quad \frac{}{z : \mathfrak{s} \vdash^{(0,0)} z : \mathfrak{s}}{\text{VAR}}}{\frac{}{z : \mathfrak{s} \vdash^{(0,0)} z : \mathfrak{s}}{\text{ABS}}} \quad \frac{}{y : \mathfrak{s} \vdash^{(0,0)} \lambda x. y : [\perp \rightarrow \mathfrak{s}]}{\text{VAR}}}{\frac{}{y : \mathfrak{s}; z : \mathfrak{s} \vdash^{(1,0)} (\lambda x. y) z : \mathfrak{s}}{\text{APPC}}}$$

We end this section by mentioning some basic properties and notions of the typing system \mathcal{U} .

Relevance. Given that the typing system is *linear*, all of the type assumptions in the typing environment must be actually used. This is formally expressed by the following property:

LEMMA 7.1 (RELEVANCE). *If $\Gamma \vdash^{(m,e)} t : \mathcal{T}$ then $\text{rv}(t) \subseteq \text{dom}(\Gamma) \subseteq \text{fv}(t)$.*

The inclusion $\text{fv}(t) \subseteq \text{dom}(\Gamma)$ does not always hold; for instance, $\vdash \lambda x. y : []$ but $\{y\} \not\subseteq \emptyset$.

Appropriateness. To be able to reason inductively about typing derivations, sometimes we need to guarantee invariants for the types of the free variables occurring in a term. For example, when we are reasoning about a term $t[x/I]$, we may need to keep track of the fact that x is bound to an abstraction when resorting to the inductive hypothesis for the subterm t . Specifically, we say that a typing environment Γ is **appropriate** with respect to an abstraction frame \mathcal{A} , written $\text{appropriate}_{\mathcal{A}}(\Gamma)$, if for each $x \in \mathcal{A}$ one has that $\Gamma(x) \neq \mathfrak{s}$. i.e. $\Gamma(x) = \perp$ or $\Gamma(x) = \mathcal{M}$ for some \mathcal{M} .

Tightness. Typing derivations in \mathcal{U} contain counters that provide *upper bounds* for the length of evaluations to normal form. To be able to provide *exact bounds*, we identify a subset of typing derivations, called *tight derivations*. Recall that a type is **tight** if it is of the form \mathfrak{s} or of the form $[\]$. An optional type $\mathcal{T}^?$ is **tight** if it is either \perp or a tight type. A typing environment Γ is **tight** if $\Gamma(x)$ is tight for every variable x . A typing judgment $\Gamma \vdash^{(m,e)} t : \mathcal{T}$ is **tight** if both Γ and \mathcal{T} are tight. A derivation

of a typing judgment is **tight** if the judgment is tight. For example, the derivable judgment $x : [] \vdash^{(0,0)} x : []$ is tight, and the counters $(0, 0)$ tell us that the evaluation of x requires no function application and no substitution steps, *i.e.* x is already in normal form. On the other hand, the derivable judgment $x : [\mathcal{T} \rightarrow \mathcal{S}] \vdash^{(0,1)} x : [\mathcal{T} \rightarrow \mathcal{S}]$ is *not* tight, and the counters $(0, 1)$ are an upper bound for the number of computation steps to evaluate x . As another example, the derivation typing the term $(\lambda x. y) z$ given above to illustrate subsumption is tight, while its subderivation typing $\lambda x. y$ is not.

Example 7.2. Let $t = (x z)[x/y[y/I]]$. Taking $\mathcal{T} := [\mathfrak{s} \rightarrow \mathfrak{s}]$, the following typing derivation \mathcal{D} turns out to be tight:

$$\frac{\frac{\frac{}{x : \mathcal{T} \vdash^{(0,1)} x : \mathcal{T}}{\text{VAR}} \quad \frac{}{z : \mathfrak{s} \vdash^{(0,0)} z : \mathfrak{s}}{\text{VAR}}}{\frac{}{z : \mathfrak{s}, x : \mathcal{T} \vdash^{(1,1)} x z : \mathfrak{s}}{\text{APPC}}} \mathcal{D}'}{\frac{}{z : \mathfrak{s} \vdash^{(1,2)} (x z)[x/y[y/I]] : \mathfrak{s}}{\text{ES}}}$$

where \mathcal{D}' is the following derivation:

$$\frac{\frac{\frac{}{y : \mathcal{T} \vdash^{(0,1)} y : \mathcal{T}}{\text{VAR}} \quad \frac{\vdots}{\emptyset \vdash^{(0,0)} I : \mathcal{T}}{\text{ES}}}{\frac{}{\emptyset \vdash^{(0,1)} y[y/I] : \mathcal{T}}{\text{ES}}}$$

and $\mathfrak{s} \triangleleft \mathfrak{s}$, as well as $\mathcal{T} = [\mathfrak{s} \rightarrow \mathfrak{s}] \triangleleft [\mathfrak{s} \rightarrow \mathfrak{s}] = \mathcal{T}$ both hold. Note that all the other subderivations of \mathcal{D} are not tight, except for the derivation corresponding to the judgment $z : \mathfrak{s} \vdash^{(0,1)} z : \mathfrak{s}$.

Tightness is a key ingredient to show that the typing system \mathcal{U} is sound and complete with respect to the quantitative interpretation in Theorems 7.4 and 7.7.

7.2 Soundness and Completeness of System \mathcal{U}

In this subsection, we prove that tight derivations do not only guarantee termination of useful CBV (uocbv^\bullet), but also provide exact quantitative information about this evaluation strategy. More precisely, we show that when a term t is tightly typable with counters (m, e) , *i.e.* $\Gamma \vdash^{(m,e)} t : \mathcal{T}$, then t evaluates in uocbv^\bullet to a normal form s in exactly m function application steps (*i.e.* db-steps) and exactly e substitution steps (lsv-steps). In this sense, \mathcal{U} is a quantitative interpretation.

To prove **soundness** one can follow well-understood techniques: it requires a subject reduction property, based in turn on a substitution lemma (omitted here but presented in the appendix).

However, given the *contextual* specification of evaluation in uocbv^\bullet , these lemmas are not straightforward, since they require formulating complex invariants on the contextual parameters \mathcal{A} , \mathcal{S} , and μ used to define the useful evaluation strategy.

More generally, we have to show that any uocbv^\bullet step preserves typing and decrements the counters correctly.

PROPOSITION 7.3 (SUBJECT REDUCTION). *Let $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'$ where $\rho \in \{\text{db}, \text{lsv}\}$ and $\Gamma \vdash^{(m,e)} t : \mathcal{T}$ and appropriate $\mathcal{A}(\Gamma)$. Suppose moreover that, if $\mu = @$ then either $\mathcal{T} = \mathfrak{s}$ or \mathcal{T} is a singleton, *i.e.* of the form $[\alpha]$. Then $\Gamma \vdash^{(m',e')} t' : \mathcal{T}$, where, if $\rho = \text{db}$ we have that $m > 0$ and $(m', e') = (m - 1, e)$, and if $\rho = \text{lsv}$ we have that $e > 0$ and $(m', e') = (m, e - 1)$.*

We can now prove soundness. That is, a tight derivation of a term t with counters (m, e) necessarily gives a terminating uocbv^\bullet evaluation sequence, containing exactly m function application steps (*i.e.* db-steps) and exactly e substitution steps (lsv-steps).

THEOREM 7.4 (SOUNDNESS OF \mathcal{U}). *Let $\mathcal{S} = \text{fv}(t)$ and let $\Gamma \vdash^{(m,e)} t : \mathcal{T}$ be a tight derivation. Then there exists a $\xrightarrow{\bullet}_{\text{top}, \mathcal{S}}$ -irreducible term s such that $t \xrightarrow{\bullet}_{\text{top}, \mathcal{S}}^{m+e} s$ where m and e are respectively the number of db and lsv steps in the reduction.*

To illustrate this property, take the tight derivation for the term $t = (x z)[x/y[y/I]]$ in Example 7.2, where the final counter is $(1, 2)$. On the other hand, there is a reduction sequence from t that ends in $x_1[x_1/z][x/y][y/I] \in \text{NF}_{\emptyset, \{z\}, @}^\bullet$ in Example 4.3. Note that the first counter coincides with the number of db-steps in the evaluation sequence of Example 4.3 (that is, 1), while the second counter coincides with the number of lsv-steps in the evaluation sequence of Example 4.3 (that is, 2). Notice also that the judgments that are not tight do not necessarily give exact information about the length of evaluation sequences. For example, the judgment $z : \mathfrak{s} \vdash^{(1,1)} x : \mathfrak{s}$ in Example 7.2 is not tight —since \mathcal{T} is not tight—, and the counters $(1, 1)$ do not correspond to the number of steps needed to get a normal form, as $x z$ is already in normal form.

In order to show **completeness** of the typing system \mathcal{U} with respect to uocbv^\bullet , we first guarantee that normal forms are tightly typable in \mathcal{U} . For that, tight environments are constructed for each normal form $t \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$, by typing the *reachable*

variables in \mathcal{A} with $[\]$ and those in \mathcal{S} with \mathfrak{s} . More precisely, given t , \mathcal{A} and \mathcal{S} such that $\text{inv}(\mathcal{A}, \mathcal{S}, t)$, the **tight environment** for t under \mathcal{A} and \mathcal{S} is written $\text{TEnv}(\mathcal{A}, \mathcal{S}, t)$ and defined as the environment Γ such that $\Gamma(x) = [\]$ when $x \in \mathcal{A} \cap \text{rv}(t)$, $\Gamma(x) = \mathfrak{s}$ when $x \in \mathcal{S} \cap \text{rv}(t)$, and $\Gamma(x) = \perp$ otherwise.

Tight environments are used to *tightly* type normal forms:

PROPOSITION 7.5 (NORMAL FORMS ARE TIGHT TYPABLE). *Let t be a term such that $t \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$ and $\text{inv}(\mathcal{A}, \mathcal{S}, t)$. Then there exists a tight type \mathfrak{t} such that $\text{TEnv}(\mathcal{A}, \mathcal{S}, t) \vdash^{(0,0)} t : \mathfrak{t}$. Moreover, if $t \in \text{HA}_{\mathcal{A}}$ then $\mathfrak{t} = [\]$, and if $t \in \text{St}_{\mathcal{S}}$ then $\mathfrak{t} = \mathfrak{s}$.*

Completeness of quantitative type systems can also be proved by following well-understood techniques: it requires a subject expansion property, based in turn on an anti-substitution lemma (omitted here but presented in the appendix). The statements are quite technical because the properties have to be appropriately generalized to be able to reason inductively.

Subject expansion is similar to subject reduction, but going *backward*: given a typed term t' and a reduction step $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'$, the term t is typed as well.

PROPOSITION 7.6 (SUBJECT EXPANSION). *Let $\text{inv}(\mathcal{A}, \mathcal{S}, t)$, and let $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'$ where $\rho \in \{\text{db}, \text{lsv}\}$ and $\Gamma \vdash^{(m', e')} t' : \mathcal{T}$ and $\text{appropriate}_{\mathcal{A}}(\Gamma)$. Suppose moreover that if $\mu = @$ then either $\mathcal{T} = \mathfrak{s}$ or \mathcal{T} is a singleton, i.e. of the form $[\alpha]$. Then $\Gamma \vdash^{(m, e)} t : \mathcal{T}$, where, if $\rho = \text{db}$ we have that $(m, e) = (m' + 1, e')$, and if $\rho = \text{lsv}$ we have that $(m, e) = (m', e' + 1)$.*

We now prove completeness. That is, given a terminating UOCBV^\bullet evaluation sequence from t containing exactly m function application steps (i.e. db-steps) and exactly e substitution steps (lsv-steps), there is necessarily a tight derivation of t with counters (m, e) .

THEOREM 7.7 (COMPLETENESS OF \mathcal{U}). *Let $\mathcal{S} = \text{fv}(t)$ and consider a reduction sequence $t \xrightarrow{\text{top}, \mathcal{S}}^n s$ where s is $\xrightarrow{\text{top}, \mathcal{S}}$ -irreducible. Let $n = m + e$ where m and e are respectively the number of db and lsv steps in the sequence. Then there exists a tight environment Γ and a tight type \mathfrak{t} such that $\Gamma \vdash^{(m, e)} t : \mathfrak{t}$.*

To illustrate this property, take the reduction sequence from $t = (xz)[x/y][y/I]$ that ends in $x_1[x_1/z][x/y][y/I] \in \text{NF}_{\emptyset, \{z\}, @}^\bullet$ in Example 4.3. This reduction sequence is of length $3 = m + e$, m corresponding to the db-steps and e to the lsv-steps. Then Example 7.2 gives a tight environment $z : \mathfrak{s}$ and a tight type \mathfrak{s} such that $z : \mathfrak{s} \vdash^{(1,2)} t : \mathfrak{s}$, where $m = 1$ and $e = 2$.

We can then conclude,

COROLLARY 7.8. *The type system \mathcal{U} is sound and complete for the UOCBV^\bullet strategy.*

8 CONCLUSIONS

This paper contributes to the study of reasonable cost models for functional programming languages in two different ways. At a *syntactic* level, we propose an inductive specification of usefulness for open CBV evaluation, in contrast to previous notions of usefulness (both for CBN and CBV) that are not inductive. The kind of technique that we use to achieve such an inductive definition is inspired by [21], which focuses on strong *call-by-need* evaluation, another evaluation strategy being dependent on essential information coming from the surrounding evaluation context. We think that this technique scales to other languages; in particular, it could be applied to provide an inductive formulation of useful call-by-name evaluation [10], which is formulated non-inductively. Moreover, we show that our new formulation of usefulness provides a reasonable implementation of open CBV. This is done by connecting our formalism to previous work in the literature.

At a *semantical* level, we propose the first model for usefulness in the literature. Our interpretation is based on intersection types, thus allowing to characterize normalization of useful open CBV (UOCBV^\bullet) by means of typing. Moreover, intersection types are non-idempotent, so that the model is quantitative, and provides independent and exact measures for evaluation lengths. Our semantic interpretation achieves high precision while staying surprisingly simple. This highlights the ability of our semantical approach based on intersection types to capture intricate operational details by streamlined and intuitive means.

Several complementary properties are worth studying. Firstly, it would be interesting to understand if the linear time algorithm in [30], comparing unshared λ -terms represented by sharing, still applies. Secondly, we would like to understand how flexible is the quantitative model (i.e. type system) so that some optimizations in [18] can be captured by both the syntactical specification and the semantical one.

Another interesting question is related to the extension of our formalization to *strong* CBV so that evaluation is also allowed inside abstractions. This is relevant for the implementation of proof assistants based on dependent type theory, in which

type checking requires deciding the definitional equality of type expressions up to full β -conversion, thus requiring strong evaluation. Even more challenging would be to adapt all this technology to call-by-need, and to call-by-push-value.

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SUPPLEMENTARY MATERIAL

In this appendix we use the following complementary notations:

If X, Y are sets, we write $X \# Y$ to mean that X and Y are **disjoint**, i.e. $X \cap Y = \emptyset$.

We write $\text{dom}(L)$ for the **domain** of L , i.e. $\text{dom}([x_1/t_1] \dots [x_n/t_n]) := \{x_1, \dots, x_n\}$.

A PROOFS OF SECTION 3 “LINEAR OPEN CALL-BY-VALUE”

In this section of the appendix we discuss technical details and show results regarding the LOCBV° calculus.

LEMMA A.1 (WEAKENING LEMMA). *Let $t \in \text{NF}_{\mathcal{V}, \mu}^\circ$ and let \mathcal{V}' be such that $\mathcal{V}' \# \text{rv}(t)$. Then $t \in \text{NF}_{\mathcal{V} \cup \mathcal{V}', \mu}^\circ$.*

PROOF. The proof is straightforward by induction on the derivation of $t \in \text{NF}_{\mathcal{V}, \mu}^\circ$. □

Now we move to provide a syntactic characterization of normal forms for LOCBV° (Corollary A.7).

We take inspiration from Balabonski *et al.* [22] to obtain an *inductive* syntactic characterization of normal forms. Two parameters are used to obtain an *inductive* definition: a **value frame** \mathcal{V} , which is a set of variables, and a **positional flag** $\mu \in \{\emptyset, @\}$. These two parameters give information about the evaluation context in which the term is considered to be a normal form. More precisely, the set \mathcal{V} keeps track of the variables that are bound to a value in the context, while the constant μ keeps track of applied positions of subterms w.r.t. the context. For example, x appears in an applied position in the term $x y$, while y appears in a non-applied position. For toplevel terms, the positional element is always $@$. The set of **normal forms** under \mathcal{V} and μ , written $\text{NF}_{\mathcal{V}, \mu}^\circ$, is defined inductively as follows:

$$\begin{array}{c} \frac{x \notin \mathcal{V}}{x \in \text{NF}_{\mathcal{V}, \mu}^\circ} \text{NF-VAR}^\circ \quad \frac{t \in \text{NF}_{\mathcal{V} \cup \{x\}, \mu}^\circ \quad s \in \text{NF}_{\mathcal{V}, @}^\circ \quad s \in \text{Val}}{t[x/s] \in \text{NF}_{\mathcal{V}, \mu}^\circ} \text{NF-ESVAL}^\circ \\ \\ \frac{}{\lambda x. t \in \text{NF}_{\mathcal{V}, @}^\circ} \text{NF-LAM}^\circ \quad \frac{t \in \text{NF}_{\mathcal{V}, @}^\circ \quad s \in \text{NF}_{\mathcal{V}, @}^\circ}{t s \in \text{NF}_{\mathcal{V}, \mu}^\circ} \text{NF-APP}^\circ \\ \\ \frac{t \in \text{NF}_{\mathcal{V}, \mu}^\circ \quad s \in \text{NF}_{\mathcal{V}, @}^\circ \quad \neg(s \in \text{Val})}{t[x/s] \in \text{NF}_{\mathcal{V}, \mu}^\circ} \text{NF-ESNONVAL}^\circ \end{array}$$

Value frames are sets of variables bound to values, so in rule NF-ESVAL° , we extend the value frame of the left premise with the variable bound by the ES. Accordingly, in rule NF-VAR° , the variable x must not be in the value frame; otherwise it would mean that it must be substituted by some value. As an example, $x[x/y] \xrightarrow{\text{lsv}} y[x/y]$ can be derived using the inductive rule LSV° because $x \xrightarrow{\text{sub}_{(x,y)}} y$, so intuitively $x[x/y] \notin \text{NF}_{\emptyset, @}^\circ$. However, $y[x/y] \in \text{NF}_{\emptyset, @}^\circ$ as stated below:

$$\frac{\frac{}{y \in \text{NF}_{\{x\}, @}^\circ} \text{NF-VAR}^\circ \quad \frac{}{y \in \text{NF}_{\emptyset, @}^\circ} \text{NF-VAR}^\circ}{y[x/y] \in \text{NF}_{\emptyset, @}^\circ} \text{NF-ESVAL}^\circ$$

If we wanted to derive a judgment $x[x/y] \in \text{NF}_{\emptyset, @}^\circ$, we would end up with a premise stating $x \notin \{x\}$, which is false. The predicate $y \in \text{NF}_{\{x\}, @}^\circ$ in the previous derivation tree reflects the fact that y is in normal form with respect to a value frame containing x , which comes from the reduction step $x \xrightarrow{\text{sub}_{(x,y)}} y$. Just as normal forms are parameterized by a value frame \mathcal{V} , which represent the set of variables that are bound to values, we also need to generalize evaluation, parameterizing it with respect to a value frame, in order to establish a precise relation between reduction and normal forms. Consequently, we define the set of **reduction rules related to a value frame** \mathcal{V} as:

$$\mathcal{R}_{\mathcal{V}} := \{\text{db}, \text{lsv}\} \cup \{\text{sub}_{(x,v)} \mid x \in \mathcal{V}\}$$

A term t belongs to the set $\text{Red}_{\mathcal{V}}^\circ$ of **reducible terms under a value frame** \mathcal{V} if there is a step kind $\rho \in \mathcal{R}_{\mathcal{V}}$ and a term t' such that $t \xrightarrow{\rho} t'$; and t belongs to the set $\text{Irred}_{\mathcal{V}}^\circ$ of **irreducible terms under** \mathcal{V} if $t \notin \text{Red}_{\mathcal{V}}^\circ$.

We can now show the two main results. First, we show soundness of the syntactic characterization of normal forms w.r.t. the parametrized reduction rules i.e. we show that given any value frame \mathcal{V} and any positional flag μ , a term in normal form under \mathcal{V} and μ is in $\text{Irred}_{\mathcal{V}}^\circ$.

LEMMA A.2. $(\lambda x. t)L \notin \text{NF}_{\mathcal{V}, @}^\circ$ for any \mathcal{V} .

PROOF. The proof is straightforward by induction on L . □

PROPOSITION A.3 (SOUNDNESS OF LINEAR NORMAL FORMS). *If $t \in \text{NF}_{\mathcal{V},\mu}^\circ$ then $t \in \text{Irred}_{\mathcal{V}}^\circ$.*

PROOF. By induction on the derivation of the judgment $t \in \text{NF}_{\mathcal{V},\mu}^\circ$.

1. NF-VAR $^\circ$. Then $t = y$ and

$$\frac{y \notin \mathcal{V}}{y \in \text{NF}_{\mathcal{V},\mu}^\circ} \text{NF-VAR}^\circ$$

Suppose $y \in \text{Red}_{\mathcal{V}}^\circ$. Then, the only rule that would allow us to reduce y is SUB $^\circ$, as follows: $y \xrightarrow{\text{sub}_{(y,v)}} v$ with $y \in \mathcal{V}$. But the premise of rule NF-VAR $^\circ$ states that $y \notin \mathcal{V}$, which gives a contradiction. We conclude then $y \in \text{Irred}_{\mathcal{V}}^\circ$.

2. NF-LAM $^\circ$. Then $t = \lambda y. s$, $s \in \text{NF}_{\mathcal{V},\emptyset}^\circ$, where $\mu = \emptyset$. There are no rules to reduce an abstraction, so it is immediate to conclude $\lambda y. s \in \text{Irred}_{\mathcal{V}}^\circ$.

3. NF-APP $^\circ$. Then $t = s u$ and

$$\frac{s \in \text{NF}_{\mathcal{V},\emptyset}^\circ \quad u \in \text{NF}_{\mathcal{V},\emptyset}^\circ}{s u \in \text{NF}_{\mathcal{V},\mu}^\circ} \text{NF-APP}^\circ$$

There are three rules that would allow us to reduce $s u$. We argue that no reduction rule applies:

- 3.1 $s \in \text{Irred}_{\mathcal{V}}^\circ$ by *i.h.* on s , so the term does not reduce via rule APP $^\circ$,
- 3.2 $u \in \text{Irred}_{\mathcal{V}}^\circ$ by *i.h.* on u , so the term does not reduce via rule APP $^\circ$,
- 3.3 $s u$ does not reduce via rule DB $^\circ$, otherwise $s = (\lambda y. s')L \in \text{NF}_{\mathcal{V},\emptyset}^\circ$, which is impossible by Lemma A.2.

4. NF-ESVAL $^\circ$. Then $t = s[y/u]$ and

$$\frac{s \in \text{NF}_{\mathcal{V} \cup \{y\},\mu}^\circ \quad u \in \text{NF}_{\mathcal{V},\emptyset}^\circ \quad u \in \text{Val}}{s[y/u] \in \text{NF}_{\mathcal{V},\mu}^\circ} \text{NF-ESVAL}^\circ$$

There are three rules that would allow us to reduce $s[y/u]$. We argue that no reduction rule applies:

- 4.1 $u \in \text{Irred}_{\mathcal{V}}^\circ$ by *i.h.* on u , so the term does not reduce via rule ES $^\circ$,
 - 4.2 $s \in \text{Irred}_{\mathcal{V} \cup \{y\}}^\circ$ by *i.h.* on s , so the term does not reduce via rule ES $^\circ$, whenever $y \notin \text{fv}(\rho)$,
 - 4.3 $s \in \text{Irred}_{\mathcal{V} \cup \{y\}}^\circ$ by *i.h.* on s , so the term does not reduce via rule LSV $^\circ$.
5. NF-ESNONVAL $^\circ$. We have that $t = s[y/u]$ and

$$\frac{s \in \text{NF}_{\mathcal{V},\mu}^\circ \quad u \in \text{NF}_{\mathcal{V},\emptyset}^\circ \quad \neg(u \in \text{Val})}{s[y/u] \in \text{NF}_{\mathcal{V},\mu}^\circ} \text{NF-ESNONVAL}^\circ$$

There are three rules that would allow us to reduce $s[y/u]$. We argue that no reduction rule applies:

- 5.1 $u \in \text{Irred}_{\mathcal{V}}^\circ$ by *i.h.* on u , so the term does not reduce via rule ES $^\circ$,
- 5.2 $s \in \text{Irred}_{\mathcal{V}}^\circ$ by *i.h.* on s , so the term does not reduce via rule ES $^\circ$, whenever $y \notin \text{fv}(\rho)$,
- 5.3 given that $\neg(u \in \text{Val})$, then the term does not reduce via rule LSV $^\circ$.

□

Completeness intuitively states that any term in $\text{Irred}_{\mathcal{V}}^\circ$ is in normal form under the same \mathcal{V} and any positional flag μ . However, we need to consider the set $\text{Irred}_{\mathcal{V}}^\circ$ with caution, since given an irreducible abstraction in an applied position, the normal form predicate would fail, as it means that the computation of the whole term can continue with a db-step. Formally,

LEMMA A.4. *If $t \xrightarrow{\text{sub}_{(x,v)}} t'$ then for every value w there exists t'' such that $t \xrightarrow{\text{sub}_{(x,w)}} t''$.*

PROOF. The proof is straightforward by induction on the derivation of $t \xrightarrow{\text{sub}_{(x,v)}} t'$. □

LEMMA A.5. *Let $x \notin \mathcal{V}$. If $t \in \text{Red}_{\mathcal{V}'}^\circ$ then $t[x/s] \in \text{Red}_{\mathcal{V}}^\circ$ with $\mathcal{V}' = \mathcal{V} \cup \{x\}$ if s is of the form vL , and $\mathcal{V}' = \mathcal{V}$ otherwise.*

PROOF. By definition there exist a rule $\rho \in \mathcal{R}_{\mathcal{V}'}$ and a term t' such that $t \xrightarrow{\rho_{\mathcal{V}'}} t'$. We have two cases:

1. $x \in \text{fv}(\rho)$. Then $\rho = \text{sub}_{(y,w)}$ with $y \in \mathcal{V}'$. We have two subcases:
 - 1.1 $x = y$. Then $t \xrightarrow{\text{sub}_{(x,w)}} t'$, and we have to analyze the form of s :
 - 1.1.1 $s = vL$. We can apply rule LSV $^\circ$ and derive $t[x/vL] \xrightarrow{\text{lsv}} t'[x/v]L$ that is, $t[x/s] \xrightarrow{\rho_{\mathcal{V}'}} t'[x/v]L$.
 - 1.1.2 $s \neq vL$: Then $\mathcal{V}' = \mathcal{V}$. Since $x \notin \mathcal{V}$ by hypothesis and $x = y \in \mathcal{V}$, then this case is impossible.
 - 1.2 $x \neq y$. Then $x \in \text{fv}(w)$. Let w' be a value such that $x \notin \text{fv}(w')$. By Lemma A.4 there exists t'' such that $t \xrightarrow{\text{sub}_{(y,w')}} t''$. Applying rule ES $^\circ$ we derive $t[x/s] \xrightarrow{\text{sub}_{(y,w')}} t'[x/s]$, with $y \in \mathcal{V}$, that is, $t[x/s] \xrightarrow{\rho_{\mathcal{V}'}} t'[x/s]$.

2. $x \notin \text{fv}(\rho)$. Then ρ is db or lsv or $\text{sub}_{(y,v)}$ with $x \notin \{y\} \cup \text{fv}(v)$. In all cases, applying rule esL° we derive $t[x/s] \xrightarrow{\rho} t'[x/s]$ that is, $t[x/s] \xrightarrow{\rho} t'[x/s]$.

In either case $t[x/s] \in \text{Red}_{\mathcal{V}}^\circ$. □

PROPOSITION A.6 (COMPLETENESS OF LINEAR NORMAL FORMS). *If $t \in \text{Irred}_{\mathcal{V}}^\circ$ and $(t \in \text{Abs} \Rightarrow \mu = \textcircled{\emptyset})$, then $t \in \text{NF}_{\mathcal{V},\mu}^\circ$.*

PROOF. By induction on t .

1. $t = x$. The only rule that would reduce x is SUB° . By hypothesis $x \in \text{Irred}_{\mathcal{V}}^\circ$ so that $x \notin \mathcal{V}$. Then applying rule NF-VAR° we derive $x \in \text{NF}_{\mathcal{V},\mu}^\circ$.
2. $t = \lambda y. s$. Then $\mu = \textcircled{\emptyset}$ by hypothesis, so applying rule NF-LAM° we derive $\lambda y. s \in \text{NF}_{\mathcal{V},\textcircled{\emptyset}}^\circ$.
3. $t = s u$. We necessarily have that $s \in \text{Irred}_{\mathcal{V}}^\circ$, because otherwise $s u$ would reduce by rule APPL° ; also $s \neq (\lambda y. s')$ because otherwise $s u$ would reduce by rule DB° , and both cases contradict the hypothesis. Then $s \in \text{NF}_{\mathcal{V},\textcircled{\emptyset}}^\circ$ by *i.h.* on s . Likewise, we necessarily have that $u \in \text{Irred}_{\mathcal{V}}^\circ$, because otherwise $s u$ would reduce by rule APPR° , contradicting the hypothesis. Then $u \in \text{NF}_{\mathcal{V},\textcircled{\emptyset}}^\circ$ by *i.h.* on u . Applying rule NF-APP° we derive $s u \in \text{NF}_{\mathcal{V},\mu}^\circ$.
4. $t = s[y/u]$. By α -conversion we can assume $y \notin \mathcal{V}$. We necessarily have $u \in \text{Irred}_{\mathcal{V}}^\circ$, because otherwise $s[y/u]$ would reduce by rule esR° , contradicting the hypothesis. Then $u \in \text{NF}_{\mathcal{V},\textcircled{\emptyset}}^\circ$ by *i.h.* on u . By α -conversion, we may assume $y \notin \mathcal{V}$. We have to analyze two cases:
 - 4.1 $u \in \text{Val}$. Since $s[y/u] \in \text{Irred}_{\mathcal{V}}^\circ$, then $s \in \text{Irred}_{\mathcal{V} \cup \{y\}}^\circ$ by contraposition of Lemma A.5. Moreover, the implication of the statement also trivially holds. Therefore $s \in \text{NF}_{\mathcal{V} \cup \{y\},\mu}^\circ$ by *i.h.* on s . Applying rule NF-ESVAL° we derive $s[y/u] \in \text{NF}_{\mathcal{V},\mu}^\circ$.
 - 4.2 $\neg u \in \text{Val}$. Since $s[y/u] \in \text{Irred}_{\mathcal{V}}^\circ$, then $s \in \text{Irred}_{\mathcal{V}}^\circ$ by contraposition of Lemma A.5. Moreover, the implication of the statement also trivially holds. Therefore $s \in \text{NF}_{\mathcal{V},\mu}^\circ$ by *i.h.* on s . Applying rule NF-ESNONVAL° we derive $s[y/u] \in \text{NF}_{\mathcal{V},\mu}^\circ$.

□

The following corollary combines the previous soundness and completeness results, so that the set of terms that are in normal form according to the inductive predicate $\text{NF}_{\mathcal{V},\mu}^\circ$ is exactly the set $\text{Irred}_{\mathcal{V}}^\circ$. Recall that the parameters \mathcal{V} and μ are used in the definition of LOCBV° in order to define evaluation *inductively*. However, we are actually interested in the *toplevel* situation, that is, in the evaluation of an *isolated* term. In the case of an isolated term, the value frame \mathcal{V} is empty, because there is no surrounding context binding any variable, and the positional flag μ is taken to be non-applied ($\textcircled{\emptyset}$), because an isolated term is never applied.

COROLLARY A.7 (CHARACTERIZATION OF LINEAR NORMAL FORMS). *$t \in \text{NF}_{\emptyset,\textcircled{\emptyset}}^\circ$ iff $t \in \text{Irred}_{\emptyset}^\circ$.*

An example of this result is the term $y[x/y]$: it is in $\text{NF}_{\emptyset,\textcircled{\emptyset}}^\circ$, as shown above, and it is in $\text{Irred}_{\emptyset}^\circ$, since none of the rules in \mathcal{R}_{\emptyset} are applicable to it.

B PROOFS OF SECTION 4 “USEFUL OPEN CALL-BY-VALUE”

In this section of the appendix we show results regarding the UOCBV^\bullet strategy. First, we start in Appendix B.1 by characterizing normal forms. Second, in Appendix B.2 we show that the UOCBV^\bullet strategy enjoys the diamond property.

B.1 Normal Forms for Useful Open Call-by-Value

This section provides a syntactic characterization of normal forms of UOCBV^\bullet , together with their corresponding soundness and completeness results (Corollary B.11).

Characterizing normal forms for useful evaluation is not simple [11]. The main idea of our inductive characterization bears some resemblance to the one given for the LOCBV° calculus, in the sense that a *inductive definition* is obtained by means of some parameters, as done *e.g.* in [21]. Here, we use an abstraction frame \mathcal{A} , a structure frame \mathcal{S} , and a positional flag $\mu \in \{\textcircled{\emptyset}, \textcircled{\emptyset}\}$. These three parameters give information about the evaluation context in which the (sub)term is considered to be a normal form. As explained before, the purpose of the frames \mathcal{A} and \mathcal{S} is to keep track of the variables that are hereditary abstractions or structures respectively, whereas the positional flag keeps track of applied positions of subterms w.r.t. the context. For the *toplevel* terms, the positional element is always $\textcircled{\emptyset}$, as in the characterization of normal forms of LOCBV° . The set of **normal**

forms of UOCBV^\bullet under \mathcal{A}, \mathcal{S} and μ , also called $(\mathcal{A}, \mathcal{S}, \mu)$ -**normal forms**, written $\text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$, is inductively defined as follows:

$$\begin{array}{c}
\frac{x \in \mathcal{A} \Rightarrow \mu = @}{x \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet} \text{NF-VAR}^\bullet \quad \frac{}{\lambda x. t \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^\bullet} \text{NF-LAM}^\bullet \\
\\
\frac{t \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^\bullet \quad s \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^\bullet}{t s \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet} \text{NF-APP}^\bullet \\
\\
\frac{t \in \text{NF}_{\mathcal{A} \cup \{x\}, \mathcal{S}, \mu}^\bullet \quad s \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^\bullet \quad s \in \text{HA}_{\mathcal{A}}}{t[x/s] \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet} \text{NF-ESA}^\bullet \\
\\
\frac{t \in \text{NF}_{\mathcal{A}, \mathcal{S} \cup \{x\}, \mu}^\bullet \quad s \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^\bullet \quad s \in \text{St}_{\mathcal{S}}}{t[x/s] \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet} \text{NF-ESS}^\bullet
\end{array}$$

If a variable is an hereditary abstraction ($x \in \mathcal{A}$) and it is applied ($\mu = @$), substituting the variable contributes to creating a db-redex, so the variable is not in normal form. Otherwise, it is in normal form, according to NF-VAR^\bullet . For example, the term $x[x/I]$ is in normal form if non-applied ($\mu = @$), whereas it reduces to $I[x/I]$ if applied ($\mu = @$). To derive the judgment $x[x/I] \in \text{NF}_{@, @, @}^\bullet$, we would end up with a premise stating $x \in \{x\} \Rightarrow @ = @$, which does not hold.

Abstractions are in normal form if they are not applied. In rule NF-ESA^\bullet (resp. NF-ESS^\bullet) the abstraction (resp. structure) frame of the left premise is extended with the bound variable in the ES, given that it is bound to an hereditary abstraction (resp. structure), exactly as in the left premises of the reduction rule ESLA^\bullet (resp. ESLS^\bullet).

We can now present the two main results of this section. The first one is soundness of the syntactic characterization of normal forms w.r.t. the parametrized reduction rules *i.e.* we state that given any abstraction and structure frames \mathcal{A} and \mathcal{S} and a positional flag μ , a term in normal form under these parameters is in the set $\text{Irred}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$.

LEMMA B.1. $(\lambda x. t) \notin \text{NF}_{\mathcal{A}, \mathcal{S}, @}^\bullet$ for any \mathcal{A}, \mathcal{S} .

PROOF. The proof is straightforward by induction on L. □

LEMMA B.2. If $\mathcal{A} \# \mathcal{S}$ then either $t \notin \text{HA}_{\mathcal{A}}$ or $t \notin \text{St}_{\mathcal{S}}$.

PROOF. By induction on t .

1. $t = x$: Note that x is not simultaneously in \mathcal{A} and \mathcal{S} since $\mathcal{A} \# \mathcal{S}$, so either $x \notin \mathcal{A}$ and $x \notin \text{HA}_{\mathcal{A}}$, or $x \notin \mathcal{S}$ and $x \notin \text{St}_{\mathcal{S}}$.
2. $t = \lambda x. s$: We conclude that $\lambda x. s \in \text{St}_{\mathcal{S}}$ does not hold since it cannot be derived by any rule.
3. $t = s u$: We conclude that $s u \in \text{HA}_{\mathcal{S}}$ does not hold since it cannot be derived by any rule.
4. $t = s[x/u]$: To show that $t \notin \text{HA}_{\mathcal{A}}$ or $t \notin \text{St}_{\mathcal{S}}$, we assume $t \in \text{HA}_{\mathcal{A}}$, and argue that $t \notin \text{St}_{\mathcal{S}}$. We have the following subcases, depending on the rule used to derive $t \in \text{HA}_{\mathcal{A}}$:
 - 4.1 H-SUB₁: then $s \in \text{HA}_{\mathcal{A}}$, where we may assume $x \notin \mathcal{A} \cup \mathcal{S}$ by α -conversion. Note that $s \notin \text{St}_{\mathcal{S}}$ by *i.h.* on s , so $s[x/u] \in \text{St}_{\mathcal{S}}$ cannot be derived using s-SUB₁, and, similarly, $s \notin \text{St}_{\mathcal{S} \cup \{x\}}$ by *i.h.* on s , so $s[x/u] \in \text{St}_{\mathcal{S}}$ cannot be derived using s-SUB₂.
 - 4.2 H-SUB₂: then $s \in \text{HA}_{\mathcal{A} \cup \{x\}}$ and $u \in \text{HA}_{\mathcal{A}}$ holds, where we may assume $x \notin \mathcal{A} \cup \mathcal{S}$ by α -conversion. Note that by *s* $\notin \text{St}_{\mathcal{S}}$ *i.h.* on s , so $s[x/u] \in \text{St}_{\mathcal{S}}$ cannot be derived using s-SUB₁, and $u \notin \text{St}_{\mathcal{S}}$ by *i.h.* on u , so $s[x/u] \in \text{St}_{\mathcal{S}}$ cannot be derived using s-SUB₂. □

PROPOSITION B.3 (SOUNDNESS OF USEFUL NORMAL FORMS). If $\text{inv}(\mathcal{A}, \mathcal{S}, t)$ and $t \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$ then $t \in \text{Irred}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$.

PROOF. By induction on the derivation of $t \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$.

1. NF-VAR^\bullet . Then $(x \in \mathcal{A} \Rightarrow \mu = @)$ and $x \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$ holds. The only rule that would allow us to reduce x is SUB^\bullet but this rule requires $x \in \mathcal{A}$, so μ must be $@$, contrary to the fact that the SUB^\bullet requests it to be $@$. Hence no reduction rule applies.
2. NF-LAM^\bullet . Immediate, since there are no rules to reduce an abstraction.

3. NF-APP[•] Then

$$\frac{s \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^{\bullet} \quad u \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^{\bullet}}{s u \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}} \text{NF-APP}^{\bullet}$$

Moreover, $\text{inv}(\mathcal{A}, \mathcal{S}, s u)$ implies $\text{inv}(\mathcal{A}, \mathcal{S}, s)$ and $\text{inv}(\mathcal{A}, \mathcal{S}, u)$. There are three rules that would allow us to reduce $t = s u$. We argue that no reduction rule applies:

3.1 By *i.h.* on s , we have $s \in \text{Irred}_{\mathcal{A}, \mathcal{S}, @}^{\bullet}$, so the rule APPL^{\bullet} does not allow us to conclude that $t \in \text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$.

3.2 By *i.h.* on s , we have $u \in \text{Irred}_{\mathcal{A}, \mathcal{S}, @}^{\bullet}$, so the rule APPR^{\bullet} does not allow us to conclude that $t \in \text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$.

3.3 Furthermore, rule DB^{\bullet} does not allow us to conclude that $t \in \text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$. Indeed, when $s = (\lambda x. t_1)\text{L}$ then $(\lambda x. t_1)\text{L} \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^{\bullet}$ by hypothesis, which is impossible by Lemma B.1.

 4. NF-ESA[•]. Then

$$\frac{s \in \text{NF}_{\mathcal{A} \cup \{x\}, \mathcal{S}, \mu}^{\bullet} \quad u \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^{\bullet} \quad u \in \text{HA}_{\mathcal{A}}}{s[x/u] \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}} \text{NF-ESA}^{\bullet}$$

We may assume $x \notin \mathcal{A} \cup \mathcal{S}$ by α -conversion. Moreover, $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ implies $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)$ and $\text{inv}(\mathcal{A}, \mathcal{S}, u)$.

There are four rules that would allow us to reduce $t = s[x/u]$. We argue that no reduction rule applies:

4.1 By *i.h.* on u , we have $u \in \text{Irred}_{\mathcal{A}, \mathcal{S}, @}^{\bullet}$, so the rule ESR^{\bullet} does not allow us to conclude that $t \in \text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$.

4.2 Note that $\mathcal{A} \# \mathcal{S}$ holds by the invariant and $u \in \text{HA}_{\mathcal{A}}$ holds by the hypothesis, so $u \notin \text{St}_{\mathcal{S}}$ by Lemma B.2. Hence the rule ESLS^{\bullet} does not allow us to conclude that $s[x/u] \in \text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$.

4.3 By *i.h.* on s , we have $s \in \text{Irred}_{\mathcal{A} \cup \{x\}, \mathcal{S}, \mu}^{\bullet}$. Hence the rule ESLA^{\bullet} does not allow us to conclude that $s[x/u] \in \text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$.

4.4 By *i.h.* on s , we have $s \in \text{Irred}_{\mathcal{A} \cup \{x\}, \mathcal{S}, \mu}^{\bullet}$. Hence the rule LSV^{\bullet} does not allow us to conclude that $s[x/u] \in \text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$.

 5. NF-ES[•]. Then

$$\frac{s \in \text{NF}_{\mathcal{A}, \mathcal{S} \cup \{x\}, \mu}^{\bullet} \quad u \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^{\bullet} \quad u \in \text{St}_{\mathcal{S}}}{s[x/u] \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}} \text{NF-ES}^{\bullet}$$

We may assume $x \notin (\mathcal{A} \cup \mathcal{S})$ by α -conversion. Moreover, $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ implies $\text{inv}(\mathcal{A}, \mathcal{S} \cup \{x\}, s)$ and $\text{inv}(\mathcal{A}, \mathcal{S} \cup \{x\}, u)$. There are four rules that would allow us to reduce $t = s[x/u]$. We argue that no reduction rule applies:

5.1 By *i.h.* on u , we have $u \in \text{Irred}_{\mathcal{A}, \mathcal{S} \cup \{x\}, @}^{\bullet}$, so the rule ESR^{\bullet} does not allow us to conclude that $t \in \text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$.

5.2 Note that $\mathcal{A} \# \mathcal{S}$ holds by the invariant and $u \in \text{St}_{\mathcal{S}}$ holds by the hypothesis, so $u \notin \text{HA}_{\mathcal{A}}$ by Lemma B.2. Hence the rule ESLA^{\bullet} does not allow us to conclude that $t \in \text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$.

5.3 By *i.h.* on s , we have $s \in \text{Irred}_{\mathcal{A}, \mathcal{S} \cup \{x\}, \mu}^{\bullet}$. Hence the rule ESLS^{\bullet} does not allow us to conclude that $s[x/u] \in \text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$.

5.4 Note that $s[x/u]$ does not reduce using the rule LSV^{\bullet} , since $u = v\text{L} \in \text{St}_{\mathcal{S}}$, but the rule LSV^{\bullet} requires that $u \in \text{HA}_{\mathcal{A}}$, which contradicts Lemma B.2. \square

Completeness states that a term $t \in \text{Irred}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$ is in normal form with respect to the same parameters \mathcal{A} , \mathcal{S} and μ . Actually, there is an exception to this rule, since the context surrounding t has to be taken into account. In particular, an irreducible abstraction is *not* considered to be a normal form if it occurs in applied position, because the evaluation of the whole term (including the surrounding context) can proceed by means of a db-step. An hereditary abstraction t must be either a term of the form $(\lambda x. t')\text{L}$, or a term which is reducible in an applied position, such as $x[x/I]$. This means that an applied hereditary abstraction is always reducible. The first part of the following proposition covers the case in which an irreducible term is a normal form, while the second and third parts cover the case of applied hereditary abstractions, which are not in normal form even if they are irreducible.

LEMMA B.4. *Let $\text{inv}(\mathcal{A}, \mathcal{S}, t)$. If $t \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$ then $t \in \text{HA}_{\mathcal{A}} \cup \text{St}_{\mathcal{S}}$. Furthermore, if $\mu = @$ then $t \in \text{St}_{\mathcal{S}}$.*

PROOF. By induction on the derivation of $t \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$.

 1. NF-VAR[•]: Then

$$\frac{x \in \mathcal{A} \Rightarrow \mu = @}{x \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}} \text{NF-VAR}^{\bullet}$$

Since $\text{inv}(\mathcal{A}, \mathcal{S}, x)$ holds, there are two cases depending on whether $x \in \mathcal{A}$ or $x \in \mathcal{S}$. If $x \in \mathcal{A}$, by rule H-VAR we derive $x \in \text{HA}_{\mathcal{A}}$ and $\mu = @$. If $x \in \mathcal{S}$, by rule S-VAR we derive $x \in \text{St}_{\mathcal{S}}$. Then we get $x \in \text{HA}_{\mathcal{A}} \cup \text{St}_{\mathcal{S}}$.

 2. NF-LAM[•]: Then $\lambda x. s \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^{\bullet}$. By rule H-LAM we derive $\lambda x. s \in \text{HA}_{\mathcal{A}}$, so we can conclude that $\lambda x. s \in \text{HA}_{\mathcal{A}} \cup \text{St}_{\mathcal{S}}$.

3. NF-APP[•]: Then

$$\frac{s \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^{\bullet} \quad u \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^{\bullet}}{s u \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}} \text{NF-APP}^{\bullet}$$

Note that $\text{inv}(\mathcal{A}, \mathcal{S}, s u)$ implies in particular $\text{inv}(\mathcal{A}, \mathcal{S}, s)$, so $s \in \text{St}_{\mathcal{S}}$ by *i.h.* on s , since the positional argument is $@$. Then, by applying rule $s\text{-APP}$, we derive $s u \in \text{St}_{\mathcal{S}}$, so in particular $s u \in \text{HA}_{\mathcal{A}} \cup \text{St}_{\mathcal{S}}$.

4. NF-ESA[•]: Then

$$\frac{s \in \text{NF}_{\mathcal{A} \cup \{x\}, \mathcal{S}, \mu}^{\bullet} \quad u \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^{\bullet} \quad u \in \text{HA}_{\mathcal{A}}}{s[x/u] \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}} \text{NF-ESA}^{\bullet}$$

We can assume $x \notin \mathcal{A} \cup \mathcal{S}$ by α -conversion, so that $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ implies in particular $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)$. Then, by applying the *i.h.* on s , we have two possible cases depending on whether $s \in \text{HA}_{\mathcal{A} \cup \{x\}}$ or $s \in \text{St}_{\mathcal{S}}$. If $s \in \text{HA}_{\mathcal{A} \cup \{x\}}$, applying rule H-SUB_2 we derive $s[x/u] \in \text{HA}_{\mathcal{A}}$. If $s \in \text{St}_{\mathcal{S}}$, we can apply rule $s\text{-SUB}_1$ and derive $s[x/u] \in \text{St}_{\mathcal{S}}$. Note that if $\mu = @$, then the last case is the only possible case by *i.h.* on s .

5. NF-ESS[•]: Then

$$\frac{s \in \text{NF}_{\mathcal{A}, \mathcal{S} \cup \{x\}, \mu}^{\bullet} \quad u \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^{\bullet} \quad u \in \text{St}_{\mathcal{S}}}{s[x/u] \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}} \text{NF-ESS}^{\bullet}$$

We can assume $x \notin \mathcal{A} \cup \mathcal{S}$ by α -conversion, so that $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ implies in particular $\text{inv}(\mathcal{A}, \mathcal{S} \cup \{x\}, s)$. Then, by applying the *i.h.* on s , we have two possible cases depending on whether $s \in \text{HA}_{\mathcal{A}}$ or $s \in \text{St}_{\mathcal{S} \cup \{x\}}$. If $s \in \text{HA}_{\mathcal{A}}$, applying rule H-SUB_1 we derive $s[x/u] \in \text{HA}_{\mathcal{A}}$. If $s \in \text{St}_{\mathcal{S} \cup \{x\}}$, we can apply rule $s\text{-SUB}_2$ and derive $s[x/u] \in \text{St}_{\mathcal{S}}$. Note that if $\mu = @$, then the last case is the only possible case by *i.h.* on s .

□

Remark B.5. If $t \xrightarrow{\text{sub}_{(x,v), \mathcal{A}, \mathcal{S}, \mu}} t'$ then $x \in \text{fv}(t)$.

LEMMA B.6. Let $\text{inv}(\mathcal{A}, \mathcal{S}, t)$. If $t \xrightarrow{\text{sub}_{(x,v), \mathcal{A}, \mathcal{S}, \mu}} t'$ then $x \in \mathcal{A}$.

PROOF. By induction on the derivation of $t \xrightarrow{\text{sub}_{(x,v), \mathcal{A}, \mathcal{S}, \mu}} t'$. The interesting case is the SUB^{\bullet} rule. Then $x \xrightarrow{\text{sub}_{(x,v), \mathcal{A}' \cup \{x\}, \mathcal{S}, \mu}} v$ with $\mathcal{A} = \mathcal{A}' \cup \{x\}$, so $x \in \mathcal{A}$ trivially. The remaining cases are straightforward by resorting to the *i.h.*. For example, in the ESLA^{\bullet} case, we have that:

$$\frac{s \xrightarrow{\text{sub}_{(x,v), \mathcal{A} \cup \{y\}, \mathcal{S}, \mu}} s' \quad u \in \text{HA}_{\mathcal{A}} \quad y \notin \mathcal{A} \cup \mathcal{S} \quad y \notin \text{fv}(\text{sub}_{(x,v)})}{s[y/u] \xrightarrow{\text{sub}_{(x,v), \mathcal{A}, \mathcal{S}, \mu}} s'[y/u]} \text{ESLA}^{\bullet}$$

Note that $\text{inv}(\mathcal{A}, \mathcal{S}, s[y/u])$ implies in particular $\text{inv}(\mathcal{A} \cup \{y\}, \mathcal{S}, s)$, so by *i.h.* $x \in \mathcal{A} \cup \{y\}$. Furthermore we may assume $y \neq x$ by α -conversion, so $x \in \mathcal{A}$ as required. □

LEMMA B.7. Let $\text{inv}(\mathcal{A}, \mathcal{S}, t)$. If $t \xrightarrow{\text{sub}_{(x,v), \mathcal{A}, \mathcal{S}, \mu}} t'$ then for every value w there exists t'' such that $t \xrightarrow{\text{sub}_{(x,w), \mathcal{A}, \mathcal{S}, \mu}} t''$.

PROOF. By induction on the derivation of $t \xrightarrow{\text{sub}_{(x,v), \mathcal{A}, \mathcal{S}, \mu}} t'$. The interesting case is the SUB^{\bullet} rule. In that case, we have that $x \xrightarrow{\text{sub}_{(x,v), \mathcal{A}, \mathcal{S}, \mu}} v$, so $x \in \mathcal{A}$. Hence $x \xrightarrow{\text{sub}_{(x,w), \mathcal{A}, \mathcal{S}, \mu}} w$ by rule SUB^{\bullet} . The remaining cases are straightforward by resorting to the *i.h.*. For example, in the case of the ESLA^{\bullet} rule, we have that:

$$\frac{s \xrightarrow{\text{sub}_{(x,v), \mathcal{A} \cup \{y\}, \mathcal{S}, \mu}} s' \quad u \in \text{HA}_{\mathcal{A}} \quad y \notin \mathcal{A} \cup \mathcal{S} \quad y \notin \text{fv}(\text{sub}_{(x,v)})}{s[y/u] \xrightarrow{\text{sub}_{(x,v), \mathcal{A}, \mathcal{S}, \mu}} s'[y/u]} \text{ESLA}^{\bullet}$$

Note that $\text{inv}(\mathcal{A} \cup \{y\}, \mathcal{S}, s)$, so by *i.h.* on s there exists s'' such that $s \xrightarrow{\text{sub}_{(x,w), \mathcal{A} \cup \{y\}, \mathcal{S}, \mu}} s''$. Applying ESLA^{\bullet} we derive $s[y/u] \xrightarrow{\text{sub}_{(x,w), \mathcal{A}, \mathcal{S}, \mu}} s''[y/u]$, as required. □

LEMMA B.8. Let $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, t)$ and $s \in \text{HA}_{\mathcal{A}}$, with $x \notin \mathcal{A}$. If $t \in \text{Red}_{\mathcal{A} \cup \{x\}, \mathcal{S}, \mu}^{\bullet}$ then $t[x/s] \in \text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$.

PROOF. By definition there exist ρ, t' such that $t \xrightarrow{\rho, \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} t'$. Notice that $x \notin (\mathcal{A} \cup \mathcal{S})$ holds by the hypothesis. There are two cases depending on whether $x \in \text{fv}(\rho)$ or not:

1. $x \in \text{fv}(\rho)$. Then $\rho = \text{sub}_{(y,v)}$. We have two subcases, depending on whether $x = y$ or not:

1.1 $x = y$. Notice that s is of the form wL by Remark 4.1. Taking the value w and applying Lemma B.7, we get that there exists t'' such that $t \xrightarrow{\text{sub}_{(x,w), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu}} t''$. We can apply rule LSV^{\bullet} , yielding $t[x/wL] \xrightarrow{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} t'[x/w]L$.

- 1.2 $x \neq y$. Let w be a value such that $x \notin \text{fv}(w)$. There exists t'' such that $t \xrightarrow{\text{sub}_{(y,w), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu}} t''$ by Lemma B.7. We can apply rule esLA^\bullet , yielding $t[x/s] \xrightarrow{\text{sub}_{(y,w), \mathcal{A}, \mathcal{S}, \mu}} t''[x/s]$.
2. $x \notin \text{fv}(\rho)$. We can apply rule esLA^\bullet , yielding $t[x/s] \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'[x/s]$.

In either case $t[x/s] \in \text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$. □

LEMMA B.9. *Let $\text{inv}(\mathcal{A}, \mathcal{S} \cup \{x\}, t)$ and $s \in \text{St}_{\mathcal{S}}$, with $x \notin \mathcal{S}$. If $t \in \text{Red}_{\mathcal{A}, \mathcal{S} \cup \{x\}, \mu}^\bullet$, then $t[x/s] \in \text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$.*

PROOF. By definition there exist ρ, t' such that $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S} \cup \{x\}, \mu} t'$. Notice that $x \notin (\mathcal{A} \cup \mathcal{S})$ holds by the hypothesis. There are two cases depending on whether $x \in \text{fv}(\rho)$ or not:

1. $x \in \text{fv}(\rho)$. Then $\rho = \text{sub}_{(y,v)}$. We have two subcases, depending on whether $x = y$ or not:
 - 1.1 $x = y$. This case is not possible, since $x = y \in \mathcal{A}$ by Lemma B.6, and at the same time $x \notin \mathcal{A}$, because $\mathcal{A} \# (\mathcal{S} \cup \{x\})$ by hypothesis.
 - 1.2 $x \neq y$. Let w be a value such that $x \notin \text{fv}(w)$. There exists t'' such that $t \xrightarrow{\text{sub}_{(y,w), \mathcal{A}, \mathcal{S} \cup \{x\}, \mu}} t''$ by Lemma B.7. We can apply rule esLS^\bullet , yielding $t[x/s] \xrightarrow{\text{sub}_{(y,w), \mathcal{A}, \mathcal{S}, \mu}} t''[x/s]$.
2. $x \notin \text{fv}(\rho)$. We can apply rule esLS^\bullet , yielding $t[x/s] \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'[x/s]$.

In either case $t[x/s] \in \text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$. □

PROPOSITION B.10 (COMPLETENESS OF USEFUL NORMAL FORMS). *Let $\text{inv}(\mathcal{A}, \mathcal{S}, t)$.*

1. *If $t \in \text{Irred}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$ and $(t \in \text{HA}_{\mathcal{A}} \Rightarrow \mu = \emptyset)$, then $t \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$.*
2. *If $t \in \text{HA}_{\mathcal{A}}$, then either $t \in \text{Abs}$ or $t \in \text{Red}_{\mathcal{A}, \mathcal{S}, \emptyset}^\bullet$.*
3. *If $t \in \text{HA}_{\mathcal{A}}$, then $t s \in \text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$, for any term s .*

PROOF. Part 3 is an immediate consequence of part 2, since if $t \in \text{Abs}$ then $t s \xrightarrow{\text{db}_{\mathcal{A}, \mathcal{S}, \mu}} t s$ by rule DB^\bullet , while if $t \in \text{Red}_{\mathcal{A}, \mathcal{S}, \emptyset}^\bullet$, then $t s \in \text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$ by rule APPL^\bullet . Parts 1 and 2 are shown by simultaneous induction on t .

1.

- 1.1 $t = x$. Since $x \in \text{Irred}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$, we cannot apply the reduction rule SUB^\bullet . If $x \in \mathcal{A}$, then also $x \in \text{HA}_{\mathcal{A}}$, so $\mu = \emptyset$ by hypothesis. Hence we may apply NF-VAR^\bullet , yielding $x \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$.
- 1.2 $t = \lambda x. s$. Since $t \in \text{HA}_{\mathcal{A}}$, then $\mu = \emptyset$ by hypothesis, and by applying NF-LAM^\bullet we derive $\lambda x. s \in \text{NF}_{\mathcal{A}, \mathcal{S}, \emptyset}^\bullet$.
- 1.3 $t = s u$. We necessarily have $s \in \text{Irred}_{\mathcal{A}, \mathcal{S}, \emptyset}^\bullet$, because otherwise $s u$ would be in $\text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$ by rule APPL^\bullet . Moreover, $s u \notin \text{HA}_{\mathcal{A}}$. Note that $\text{inv}(\mathcal{A}, \mathcal{S}, s u)$ implies, in particular, that $\text{inv}(\mathcal{A}, \mathcal{S}, s)$. We have $s \in \text{St}_{\mathcal{S}}$ by Lemma B.4, so applying Lemma B.2 we obtain $s \notin \text{HA}_{\mathcal{A}}$. By *i.h.* (1) on s we have $s \in \text{NF}_{\mathcal{A}, \mathcal{S}, \emptyset}^\bullet$. Lastly, notice that $u \in \text{Irred}_{\mathcal{A}, \mathcal{S}, \emptyset}^\bullet$, because otherwise, $s u$ would reduce by rule APPR^\bullet . Moreover, $\text{inv}(\mathcal{A}, \mathcal{S}, s u)$ implies, in particular, that $\text{inv}(\mathcal{A}, \mathcal{S}, u)$. Then $u \in \text{NF}_{\mathcal{A}, \mathcal{S}, \emptyset}^\bullet$ by *i.h.* (1) on u . Applying rule NF-APP^\bullet we derive $s u \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$.
- 1.4 $t = s[x/u]$. We necessarily have $u \in \text{Irred}_{\mathcal{A}, \mathcal{S}, \emptyset}^\bullet$, because otherwise $s[x/u]$ would be in $\text{Red}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$ by rule ESR^\bullet . Moreover, $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ implies in particular $\text{inv}(\mathcal{A}, \mathcal{S}, u)$. Since the positional element associated to u is \emptyset , then $u \in \text{NF}_{\mathcal{A}, \mathcal{S}, \emptyset}^\bullet$ by *i.h.* (1) on u . Note that we may assume $x \notin (\mathcal{A} \cup \mathcal{S})$ by α -conversion, and $\text{inv}(\mathcal{A}, \mathcal{S}, t)$ implies $\mathcal{A} \# \mathcal{S}$. Then u is either in $\text{HA}_{\mathcal{A}}$ or in $\text{St}_{\mathcal{S}}$ by Lemma B.4. We analyze both cases:
 - 1.4.1 $u \in \text{HA}_{\mathcal{A}}$. Since $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ holds, it implies in particular $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)$. Then $s \in \text{Irred}_{\mathcal{A} \cup \{x\}, \mathcal{S}, \mu}^\bullet$ by the contraposition of Lemma B.8. Moreover, if $s[x/u] \in \text{HA}_{\mathcal{A}}$ holds, it can be derived either by (1) H-SUB_1 , meaning that $s \in \text{HA}_{\mathcal{A}}$, and thus $s \in \text{HA}_{\mathcal{A} \cup \{x\}}$ by Remark 4.1, or by (2) H-SUB_2 , meaning that $s \in \text{HA}_{\mathcal{A} \cup \{x\}}$. By the hypothesis in 1 we have $\mu = \emptyset$. Then we can apply *i.h.* (1) on s , yielding $s \in \text{NF}_{\mathcal{A} \cup \{x\}, \mathcal{S}, \mu}^\bullet$. By applying rule NF-ESA^\bullet we derive $s[x/u] \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$.
 - 1.4.2 $u \in \text{St}_{\mathcal{S}}$. Since $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ holds, it implies in particular $\text{inv}(\mathcal{A}, \mathcal{S} \cup \{x\}, s)$. Then $s \in \text{Irred}_{\mathcal{A}, \mathcal{S} \cup \{x\}, \mu}^\bullet$ by the contraposition of Lemma B.9. Since $u \in \text{St}_{\mathcal{S}}$, and $\mathcal{A} \# \mathcal{S}$ holds by hypothesis, then $u \notin \text{HA}_{\mathcal{A}}$ by Lemma B.2. As a consequence, if $s[x/u] \in \text{HA}_{\mathcal{A}}$, we necessarily have $s \in \text{HA}_{\mathcal{A}}$ by rule H-SUB_1 , and not by rule H-SUB_2 . By hypothesis 1 we have $\mu = \emptyset$. Then we can apply *i.h.* (1) on s , yielding $s \in \text{NF}_{\mathcal{A}, \mathcal{S} \cup \{x\}, \mu}^\bullet$. By applying rule NF-ESS^\bullet we derive $s[x/u] \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$.

2.

- 2.1 $t = x$. Then $x \in \text{HA}_{\mathcal{A}}$ is derived from rule H-VAR , so $x \in \mathcal{A}$. We can apply rule SUB^\bullet with $\rho = \text{sub}_{(x,v)}$ and $t' = v$, yielding $x \xrightarrow{\text{sub}_{(x,v), \mathcal{A}, \mathcal{S}, \emptyset}} v$.
- 2.2 $t = \lambda z. t'$. Immediate.

2.3 $t = t_1 t_2$. This case is not possible since $t_1 t_2 \in \text{HA}_{\mathcal{A}}$ never holds.

2.4 $t = t_1[x/t_2]$. There are two cases depending on which rule we use to derive $t_1[x/t_2] \in \text{HA}_{\mathcal{A}}$:

2.4.1 H-SUB₁: Then

$$\frac{t_1 \in \text{HA}_{\mathcal{A}} \quad x \notin \mathcal{A}}{t_1[x/t_2] \in \text{HA}_{\mathcal{A}}} \text{H-SUB}_1$$

If there exist ρ and t'_2 such that $t_2 \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, @} t'_2$, then applying rule ESR^\bullet we derive $t_1[x/t_2] \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t_1[x/t'_2]$ for any μ , in particular $\mu = @$.

If there is no ρ and t'_2 such that $t_2 \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, @} t'_2$, then $t_2 \in \text{Irred}^\bullet_{\mathcal{A}, \mathcal{S}, @}$. By *i.h.* (1) on t_2 we have that $t_2 \in \text{NF}^\bullet_{\mathcal{A}, \mathcal{S}, @}$. Since $\text{inv}(\mathcal{A}, \mathcal{S}, t_1[x/t_2])$ implies in particular $\text{inv}(\mathcal{A}, \mathcal{S}, t_2)$, then $t_2 \in \text{HA}_{\mathcal{A}} \cup \text{St}_{\mathcal{S}}$ by Lemma B.4. We reason by cases:

2.4.1.1 $t_2 \in \text{HA}_{\mathcal{A}}$. We have that $\text{inv}(\mathcal{A}, \mathcal{S}, t_1[x/t_2])$ implies in particular $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, t_1)$, so by *i.h.* (2) on t_1 , we have that either $t_1 \in \text{Abs}$ or $t_1 \in \text{Red}^\bullet_{\mathcal{A} \cup \{x\}, \mathcal{S}, @}$:

- If $t_1 \in \text{Abs}$, then $t_1[x/t_2] \in \text{Abs}$ and we are done.
- If $t_1 \in \text{Red}^\bullet_{\mathcal{A} \cup \{x\}, \mathcal{S}, @}$, then $t_1[x/t_2] \in \text{Red}^\bullet_{\mathcal{A}, \mathcal{S}, @}$ by Lemma B.8, and we are done.

2.4.1.2 $t_2 \in \text{St}_{\mathcal{S}}$. We have that $\text{inv}(\mathcal{A}, \mathcal{S}, t_1[x/t_2])$ implies in particular $\text{inv}(\mathcal{A}, \mathcal{S} \cup \{x\}, t_1)$, so by *i.h.* (2) on t_1 , we have that either $t_1 \in \text{Abs}$ or $t_1 \in \text{Red}^\bullet_{\mathcal{A}, \mathcal{S} \cup \{x\}, @}$:

- If $t_1 \in \text{Abs}$, then $t_1[x/t_2] \in \text{Abs}$ and we are done.
- If $t_1 \in \text{Red}^\bullet_{\mathcal{A}, \mathcal{S} \cup \{x\}, @}$, then $t_1[x/t_2] \in \text{Red}^\bullet_{\mathcal{A}, \mathcal{S}, @}$ by Lemma B.9, and we are done.

2.4.2 H-SUB₂: Then

$$\frac{t_1 \in \text{HA}_{\mathcal{A} \cup \{x\}} \quad x \notin \mathcal{A} \quad t_2 \in \text{HA}_{\mathcal{A}}}{t_1[x/t_2] \in \text{HA}_{\mathcal{A}}}$$

By *i.h.* (2) on t_1 we have that either $t_1 \in \text{Abs}$ or $t_1 \in \text{Red}^\bullet_{\mathcal{A} \cup \{x\}, \mathcal{S}, @}$:

- If $t_1 \in \text{Abs}$, then $t_1[x/t_2] \in \text{Abs}$, and we are done.
- If $t_1 \in \text{Red}^\bullet_{\mathcal{A} \cup \{x\}, \mathcal{S}, @}$, then $t_1[x/t_2] \in \text{Red}^\bullet_{\mathcal{A}, \mathcal{S}, @}$ by Lemma B.8, and we are done. □

The following corollary combines soundness (Proposition B.3) and completeness (Proposition B.10) for a term in toplevel position, *i.e.* when the abstraction frame \mathcal{A} is empty, the structure frame \mathcal{S} is the set of all free variables, and the positional element is taken to be $@$.

COROLLARY B.11 (CHARACTERIZATION OF USEFUL NORMAL FORMS). $t \in \text{NF}^\bullet_{\emptyset, \text{fv}(t), @}$ iff $t \in \text{Irred}^\bullet_{\emptyset, \text{fv}(t), @}$.

An example of this result is the term $(xy)[y/I]$: on one hand the term is in $\text{Red}^\bullet_{\emptyset, \{x\}, @}$, since it cannot reduce using any reduction rule, on the other hand it is normal (names of starting rules in the derivation appear on the top for lack of space).

$$\frac{\frac{\text{NF-VAR}^\bullet}{x \in \text{NF}^\bullet_{\{y\}, \{x\}, @}} \quad \frac{\text{NF-VAR}^\bullet}{y \in \text{NF}^\bullet_{\{y\}, \{x\}, @}} \quad \text{NF-APP}^\bullet \quad \frac{\text{NF-LAM}^\bullet}{I \in \text{NF}^\bullet_{\emptyset, \{x\}, @}} \quad \frac{\text{H-LAM}}{I \in \text{HA}_{\emptyset}}}{(xy)[y/I] \in \text{NF}^\bullet_{\emptyset, \{x\}, @}} \text{NF-ESA}^\bullet$$

B.2 Diamond Property

Definition B.12 (Expansion of abstraction and structure frames). Let \mathcal{A} be an abstraction frame. We inductively define the **expansion** of \mathcal{A} under L, written \mathcal{A}^\diamond , as follows:

$$\begin{aligned} \mathcal{A}^\diamond &:= \mathcal{A} \\ \mathcal{A}^{L'[x/t]} &:= \begin{cases} \mathcal{A}^{L'} \cup \{x\} & \text{if } t \in \text{HA}_{\mathcal{A}} \\ \mathcal{A}^{L'} & \text{otherwise} \end{cases} \end{aligned}$$

Analogously, let \mathcal{S} be a structure frame. We inductively define the **expansion** of \mathcal{S} under L, written \mathcal{S}^\diamond , as follows:

$$\begin{aligned} \mathcal{S}^\diamond &:= \mathcal{S} \\ \mathcal{S}^{L'[x/t]} &:= \begin{cases} \mathcal{S}^{L'} \cup \{x\} & \text{if } t \in \text{St}_{\mathcal{S}} \\ \mathcal{S}^{L'} & \text{otherwise} \end{cases} \end{aligned}$$

LEMMA B.13. $tL \in \text{St}_{\mathcal{S}}$ iff $t \in \text{St}_{\mathcal{S}^\diamond}$.

PROOF. We prove both implications by induction on the length of L .

We first show $tL \in \text{St}_{\mathcal{S}}$ implies $t \in \text{St}_{\mathcal{S}^L}$.

- $L = \diamond$. Immediate.
- $L = L'[x/s]$. The judgment $tL'[x/s] \in \text{St}_{\mathcal{S}}$ can be derived either by rule s-SUB_1 or by rule s-SUB_2 :
 1. s-SUB_1 . Then $tL' \in \text{St}_{\mathcal{S}}$ and $x \notin \mathcal{S}$. We apply *i.h.* on L' , yielding $t \in \text{St}_{\mathcal{S}^{L'}}$. Since $\mathcal{S}^{L'} \subseteq \mathcal{S}^{L'[x/s]}$ by definition of $\mathcal{S}^{L'[x/s]}$, then $\text{St}_{\mathcal{S}^{L'}} \subseteq \text{St}_{\mathcal{S}^{L'[x/s]}}$ by Remark 4.1. Therefore $t \in \text{St}_{\mathcal{S}^L}$.
 2. s-SUB_2 . Then $tL' \in \text{St}_{\mathcal{S} \cup \{x\}}$, $x \notin \mathcal{S}$ and $s \in \text{St}_{\mathcal{S}}$. We apply *i.h.* on L' , yielding $t \in \text{St}_{\mathcal{S}^{L' \cup \{x\}}}$. Since $s \in \text{St}_{\mathcal{S}}$, then $\mathcal{S}^L = \mathcal{S}^{L'} \cup \{x\}$. We then conclude $t \in \text{St}_{\mathcal{S}^L}$.

We now show $t \in \text{St}_{\mathcal{S}^L}$ implies $tL \in \text{St}_{\mathcal{S}}$.

- $L = \diamond$. Immediate.
- $L = L'[x/s]$. Then $t \in \text{St}_{\mathcal{S}^{L'[x/s]}}$. We have two cases.
 1. $s \in \text{St}_{\mathcal{S}}$. Then $\mathcal{S}^{L'[x/s]} = \mathcal{S}^{L'} \cup \{x\}$, so $t \in \text{St}_{\mathcal{S}^{L' \cup \{x\}}}$, and $tL' \in \text{St}_{\mathcal{S} \cup \{x\}}$ by *i.h.* on L' . By α -conversion we may assume $x \notin \mathcal{S}^{L'}$, and in particular $x \notin \mathcal{S}$, since $\mathcal{S} \subseteq \mathcal{S}^{L'}$. Applying rule s-SUB_2 we conclude $tL'[x/s] \in \text{St}_{\mathcal{S}}$.
 2. $s \notin \text{St}_{\mathcal{S}}$. Then $\mathcal{S}^{L'[x/s]} = \mathcal{S}^{L'}$, so $t \in \text{St}_{\mathcal{S}^{L'}}$. By α -conversion we may assume $x \notin \mathcal{S}^{L'}$, and in particular $x \notin \mathcal{S}$. We apply *i.h.* on L' , yielding $tL' \in \text{St}_{\mathcal{S}}$. By rule s-SUB_1 we conclude $tL'[x/s] \in \text{St}_{\mathcal{S}}$.

□

LEMMA B.14. *Let $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, t)$, and let \mathcal{B} be a set of variables disjoint from \mathcal{A} . If $t \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} t'$ with $v \in \text{HA}_{\mathcal{A} \cup \mathcal{B}}$ and $t \in \text{HA}_{\mathcal{A} \cup \{x\}}$, then $t' \in \text{HA}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}}$.*

PROOF. By induction on the derivation of $t \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} t'$. Note that cases APPL^\bullet and APPR^\bullet are impossible since $t = s u$, which cannot be an element of $\text{HA}_{\mathcal{A} \cup \{x\}}$ by Remark 4.1. We analyze the remaining cases.

1. SUB^\bullet . Then $t = x \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, @} v = t'$, with $\mu = @$. Since $v \in \text{HA}_{\mathcal{A} \cup \mathcal{B}}$ by hypothesis, then $v \in \text{HA}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}}$ by Remark 4.1, as $\mathcal{A} \cup \mathcal{B} \subseteq \mathcal{A} \cup \{x\} \cup \mathcal{B}$.
2. ESR^\bullet . Then

$$\frac{u \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, @} u'}{t = s[y/u] \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s[y/u'] = t'} \text{ESR}^\bullet$$

By α -conversion we may assume $y \notin \mathcal{B}$. We consider two cases depending on whether $s[y/u] \in \text{HA}_{\mathcal{A} \cup \{x\}}$ is derived either by rule H-SUB_1 or by rule H-SUB_2 :

- 2.1 H-SUB_1 . Then $s \in \text{HA}_{\mathcal{A} \cup \{x\}}$ and $y \notin \mathcal{A} \cup \{x\}$. We have $s \in \text{HA}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}}$ by Remark 4.1. We apply rule H-SUB_1 , yielding $s[y/u'] \in \text{HA}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}}$.
- 2.2 H-SUB_2 . Then $s \in \text{HA}_{\mathcal{A} \cup \{x\} \cup \{y\}}$, $y \notin \mathcal{A} \cup \{x\}$ and $u \in \text{HA}_{\mathcal{A} \cup \{x\}}$. We have $s \in \text{HA}_{\mathcal{A} \cup \{x\} \cup \{y\} \cup \mathcal{B}}$ and $u \in \text{HA}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}}$ by Remark 4.1, so $u' \in \text{HA}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}}$ by *i.h.* on u . We apply rule H-SUB_2 , yielding $s[y/u'] \in \text{HA}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}}$.
3. ESLA^\bullet . Then

$$\frac{s \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\} \cup \{y\}, \mathcal{S}, \mu} s' \quad u \in \text{HA}_{\mathcal{A} \cup \{x\}} \quad y \notin (\mathcal{A} \cup \{x\}) \cup \mathcal{S} \quad y \notin \text{fv}(\text{sub}(x,v)})}{t = s[y/u] \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s'[y/u] = t'} \text{ESLA}^\bullet$$

The judgment $s[x/u] \in \text{HA}_{\mathcal{A} \cup \{x\}}$ can be derived either by rule H-SUB_1 or by rule H-SUB_2 . The former has $s \in \text{HA}_{\mathcal{A} \cup \{x\}}$ as premise, and since $\mathcal{A} \cup \{x\} \subseteq \mathcal{A} \cup \{x\} \cup \{y\}$ then $s \in \text{HA}_{\mathcal{A} \cup \{x\} \cup \{y\}}$ by Remark 4.1. And it is also the case that $u \in \text{HA}_{\mathcal{A} \cup \{x\}}$ by premise of the rule ESLA^\bullet , so in both cases we proceed as follows. Since $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s[x/u])$ then in particular $\text{inv}(\mathcal{A} \cup \{x\} \cup \{y\}, \mathcal{S}, s)$. Moreover, note that $v \in \text{HA}_{\mathcal{A} \cup \{y\} \cup \mathcal{B}}$ by Remark 4.1, so we apply *i.h.* on s , yielding $s' \in \text{HA}_{(\mathcal{A} \cup \{y\}) \cup \{x\} \cup \mathcal{B}}$. Moreover, $u \in \text{HA}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}}$ by Remark 4.1, as $\mathcal{A} \cup \{x\} \subseteq \mathcal{A} \cup \{x\} \cup \mathcal{B}$. And we may assume $y \notin \mathcal{B}$ by α -conversion, so we apply rule H-SUB_2 , yielding $s'[x/u] \in \text{HA}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}}$.

4. ESLS^\bullet . Then

$$\frac{s \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S} \cup \{y\}, \mu} s' \quad u \in \text{St}_{\mathcal{S}} \quad y \notin (\mathcal{A} \cup \{x\}) \cup \mathcal{S} \quad y \notin \text{fv}(\text{sub}(x,v)})}{t = s[y/u] \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s'[y/u] = t'} \text{ESLS}^\bullet$$

We consider two cases depending on whether $s[x/u] \in \text{HA}_{\mathcal{A} \cup \{x\}}$ is derived either by rule H-SUB_1 or by rule H-SUB_2 :

- 4.1 H-SUB_1 . Then $s \in \text{HA}_{\mathcal{A} \cup \{x\}}$ and $y \notin \mathcal{A} \cup \{x\}$. Since $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s[x/u])$ then in particular $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S} \cup \{y\}, s)$. We apply *i.h.* on s , yielding $s' \in \text{HA}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}}$. Applying rule H-SUB_1 we obtain $s'[x/u] \in \text{HA}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}}$.

4.2 H-SUB₂. Then $s \in \text{HA}_{\mathcal{A} \cup \{x\} \cup \{y\}}$, $y \notin \mathcal{A} \cup \{x\}$ and $u \in \text{HA}_{\mathcal{A} \cup \{x\}}$. Moreover, $u \in \text{St}_{\mathcal{S}}$ by premise of the rule esLS^\bullet . Then $(\mathcal{A} \cup \{x\}) \cap \mathcal{S} \neq \emptyset$ by contraposition of Lemma B.2, which contradicts $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s[x/u])$. Hence this case is not possible. \square

Definition B.15 (Hereditary variables). The set of *hereditary variables* under a variable x , written HVar_x , is defined inductively as follows:

$$\frac{}{x \in \text{HVar}_x} \text{HVAR-VAR} \quad \frac{t \in \text{HVar}_x \quad x \neq y}{t[y/s] \in \text{HVar}_x} \text{HVAR-SUB}_1 \quad \frac{t \in \text{HVar}_y \quad s \in \text{HVar}_x}{t[y/s] \in \text{HVar}_x} \text{HVAR-SUB}_2$$

Remark B.16. If $t \in \text{HVar}_x$ and $x \notin \text{dom}(L)$ then $tL \in \text{HVar}_x$.

LEMMA B.17. *Let \mathcal{A} be an abstraction frame and x any variable. If $t \in \text{HVar}_x$ then $t \in \text{HA}_{\mathcal{A} \cup \{x\}}$.*

PROOF. The proof is straightforward by induction on the derivation of the judgment $t \in \text{HVar}_x$. \square

LEMMA B.18. *Let \mathcal{A} be an abstraction frame and consider an arbitrary partition $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. If $t \in \text{HA}_{\mathcal{A}}$ then $t \in \text{HA}_{\mathcal{A}_1}$ or there exists $x \in \mathcal{A}_2$ such that $t \in \text{HVar}_x$.*

PROOF. By induction on the judgment $t \in \text{HA}_{\mathcal{A}_1 \cup \mathcal{A}_2}$.

1. H-VAR. Then $t = y$, and $y \in \mathcal{A}_1 \cup \mathcal{A}_2$ by premise of the rule H-VAR. There are two cases, depending on whether $y \in \mathcal{A}_1$ or $y \in \mathcal{A}_2$. If $y \in \mathcal{A}_1$, then $y \in \text{HA}_{\mathcal{A}_1}$ by rule H-VAR. If $y \in \mathcal{A}_2$, then $y \in \text{HVar}_y$ by rule HVAR-VAR.
2. H-LAM. Then $t = \lambda y. s$. We apply rule H-LAM, yielding $\lambda y. s \in \text{HA}_{\mathcal{A}_1}$.
3. H-SUB₁. Then $t = s[y/u]$. The judgment $s[y/u] \in \text{HA}_{\mathcal{A}_1 \cup \mathcal{A}_2}$ is derived from $s \in \text{HA}_{\mathcal{A}_1 \cup \mathcal{A}_2}$ and $y \notin \mathcal{A}_1 \cup \mathcal{A}_2$. We take $\mathcal{A}' = \mathcal{A}_1 \cup (\mathcal{A}_2 \cup \{y\})$ and apply *i.h.* on s , yielding two possible cases:
 - 3.1 $s \in \text{HA}_{\mathcal{A}_1}$. Since $y \notin \mathcal{A}_1$ we can apply rule H-SUB₁, yielding $s[y/u] \in \text{HA}_{\mathcal{A}_1}$.
 - 3.2 $s \in \text{HVar}_x$, for some $x \in \mathcal{A}_2$. Since $y \notin \mathcal{A}_2$, then $x \neq y$ and we can apply rule HVAR-SUB₁, yielding $s[y/u] \in \text{HVar}_y$.
4. H-SUB₂. Then $t = s[y/u]$. The judgment $s[y/u] \in \text{HA}_{\mathcal{A}_1 \cup \mathcal{A}_2}$ is derived from $s \in \text{HA}_{(\mathcal{A}_1 \cup \mathcal{A}_2) \cup \{y\}}$, $y \notin \mathcal{A}_1 \cup \mathcal{A}_2$ and $u \in \text{HA}_{\mathcal{A}_1 \cup \mathcal{A}_2}$. By *i.h.* on s we have two possible cases:
 - 4.1 $s \in \text{HA}_{\mathcal{A}_1}$. Since $y \notin \mathcal{A}_1$, we can apply rule H-SUB₁, yielding $s[y/u] \in \text{HA}_{\mathcal{A}_1}$.
 - 4.2 $s \in \text{HVar}_x$ for some $x \in (\mathcal{A}_2 \cup \{y\})$. We have two cases depending on whether $x = y$ or not:
 - 4.2.1 $x = y$. By *i.h.* on u we have two possible subcases:
 - 4.2.1.1 $u \in \text{HA}_{\mathcal{A}_1}$. Since $y \notin \mathcal{A}_1$, and $s \in \text{HA}_{\mathcal{A}_1 \cup \{y\}}$ by Lemma B.17, applying rule H-SUB₂ we conclude $s[y/u] \in \text{HA}_{\mathcal{A}_1}$.
 - 4.2.1.2 $u \in \text{HVar}_z$ for some $z \in \mathcal{A}_2$. Then $s[y/u] \in \text{HVar}_z$ by rule HVAR-SUB₂.
 - 4.2.2 $x \neq y$. Then $s[y/u] \in \text{HVar}_x$ by rule HVAR-SUB₁.

\square

LEMMA B.19. *$tL \in \text{HA}_{\mathcal{A}}$ if and only if $t \in \text{HA}_{\mathcal{A}^\perp}$.*

PROOF. We prove both implications by induction on the length of L .

We first show $tL \in \text{HA}_{\mathcal{A}}$ implies $t \in \text{HA}_{\mathcal{A}^\perp}$.

- $L = \diamond$. Immediate.
- $L = L'[x/s]$. The judgment $tL'[x/s] \in \text{HA}_{\mathcal{A}}$ can be derived either by rule H-SUB₁ or by rule H-SUB₂:
 1. H-SUB₁. Then $tL' \in \text{HA}_{\mathcal{A}}$ and $x \notin \mathcal{A}$. We apply *i.h.* on L' , yielding $t \in \text{HA}_{\mathcal{A}^\perp}$. Since $\mathcal{A}^{L'} \subseteq \mathcal{A}^{L'[x/s]}$ by definition of $\mathcal{A}^{L'[x/s]}$, then $\text{HA}_{\mathcal{A}^\perp} \subseteq \text{HA}_{\mathcal{A}^{L'[x/s]}}$ by Remark 4.1. Therefore $t \in \text{HA}_{\mathcal{A}^\perp}$.
 2. H-SUB₂. Then $tL' \in \text{HA}_{\mathcal{A} \cup \{x\}}$, $x \notin \mathcal{A}$ and $s \in \text{HA}_{\mathcal{A}}$. We apply *i.h.* on L' , yielding $t \in \text{HA}_{\mathcal{A}^\perp \cup \{x\}}$. Since $s \in \text{HA}_{\mathcal{A}}$, then $\mathcal{A}^\perp = \mathcal{A}^\perp \cup \{x\}$.

We now show $t \in \text{HA}_{\mathcal{A}^\perp}$ implies $tL \in \text{HA}_{\mathcal{A}}$.

- $L = \diamond$. Immediate.
- $L = L'[x/s]$. Then $t \in \text{HA}_{\mathcal{A}^\perp[x/s]}$. We have to analyze two cases.
 1. $s \in \text{HA}_{\mathcal{A}}$. Then $\mathcal{A}^{L'[x/s]} = \mathcal{A}^\perp \cup \{x\}$, so $t \in \text{HA}_{\mathcal{A}^\perp \cup \{x\}}$. By α -conversion we may assume $x \notin \mathcal{A}^\perp$. There are two possible subcases by Lemma B.18:
 - 1.1 $t \in \text{HA}_{\mathcal{A}^\perp}$. Then $tL' \in \text{HA}_{\mathcal{A}}$ by *i.h.* on L' , and $tL'[x/u] \in \text{HA}_{\mathcal{A}}$ by rule H-SUB₁.

1.2 $t \in \text{HVar}_x$. Then $tL' \in \text{HVar}_x$ by Remark B.16, and $\text{HVar}_x \subseteq \text{HA}_{\mathcal{A} \cup \{x\}}$ by Lemma B.17, so we have $tL' \in \text{HA}_{\mathcal{A} \cup \{x\}}$.

By rule H-SUB_2 we conclude $tL'[x/s] \in \text{HA}_{\mathcal{A}}$.

2. $s \notin \text{HA}_{\mathcal{A}}$. Then $\mathcal{A}^{L'[x/s]} = \mathcal{A}^{L'}$, so $t \in \text{HA}_{\mathcal{A}^{L'}}$. We apply *i.h.* on L' , yielding $tL' \in \text{HA}_{\mathcal{A}}$. By α -conversion we may assume $x \notin \mathcal{A}^{L'}$, and in particular $x \notin \mathcal{A}$. Applying rule H-SUB_1 we conclude $tL'[x/u] \in \text{HA}_{\mathcal{A}}$. \square

LEMMA B.20 (HEREDITARY ABSTRACTIONS AND STRUCTURES ARE CLOSED BY REDUCTION). *Let $\text{inv}(\mathcal{A}, \mathcal{S}, t)$.*

1. *Let $t \in \text{HA}_{\mathcal{A}}$ and $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'$ where $\rho \in \{\text{db}, \text{lsv}\}$. Then $t' \in \text{HA}_{\mathcal{A}}$.*
2. *If $t \in \text{St}_{\mathcal{S}}$ and $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'$ then $t' \in \text{St}_{\mathcal{S}}$.*

PROOF. By induction on the derivation of $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'$. \square

Given a substitution context L , we can use the reduction rules defined in Section 4 to define the notion of **useful reduction for substitution contexts**, by just understanding these elements as terms, where \diamond is taken as a free variable.

Remark B.21.

1. If $(\lambda x. t)L \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, @} s$ and $\diamond \notin \mathcal{A} \cup \mathcal{S}$, then s is of the form $(\lambda x. t)L'$, and $L \xrightarrow{\rho, \mathcal{A}, \mathcal{S} \cup \{\diamond\}, \mu} L'$.
2. If $vL \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, @} t$ and $\diamond \notin \mathcal{A} \cup \mathcal{S}$, then there exists L' such that $t = vL'$, and $L \xrightarrow{\rho, \mathcal{A}, \mathcal{S} \cup \{\diamond\}, \mu} L'$.
3. If $L \xrightarrow{\rho, \mathcal{A}, \mathcal{S} \cup \{\diamond\}, \mu} L'$ then $tL \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} tL'$.

LEMMA B.22. *If $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'$, then for all sets \mathcal{A}' and \mathcal{S}' such that $\mathcal{A} \subseteq \mathcal{A}'$ and $\mathcal{S} \subseteq \mathcal{S}'$ it holds that $t \xrightarrow{\rho, \mathcal{A}', \mathcal{S}', \mu} t'$.*

PROOF. By induction on the derivation of the judgment $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'$.

1. DB^\bullet . Then $t = (\lambda x. s)L u \xrightarrow{\text{db}, \mathcal{A}, \mathcal{S}, \mu} s[x/u]L = t'$, where $\rho = \text{db}$. We conclude $(\lambda x. s)L u \xrightarrow{\text{db}, \mathcal{A}', \mathcal{S}', \mu} s[x/u]L$ by rule DB^\bullet .
2. SUB^\bullet . Then $t = x \xrightarrow{\text{sub}_{(x,v)}, \mathcal{A} \cup \{x\}, \mathcal{S}, @} v = t'$, where $\rho = \text{sub}_{(x,v)}$ and $\mu = @$. We conclude $x \xrightarrow{\text{sub}_{(x,v)}, \mathcal{A}' \cup \{x\}, \mathcal{S}', @} v$ by rule SUB^\bullet .
3. LSV^\bullet . Then

$$\frac{s \xrightarrow{\text{sub}_{(x,v)}, \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s' \quad x \notin \mathcal{A} \cup \mathcal{S} \quad vL \in \text{HA}_{\mathcal{A}}}{t = s[x/vL] \xrightarrow{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} s'[x/v]L = t'} \text{LSV}^\bullet$$

where $\rho = \text{lsv}$. If $\mathcal{A} \subseteq \mathcal{A}'$ and $\mathcal{S} \subseteq \mathcal{S}'$, then in particular $\mathcal{A} \cup \{x\} \subseteq \mathcal{A}' \cup \{x\}$. Therefore $s \xrightarrow{\text{sub}_{(x,v)}, \mathcal{A}' \cup \{x\}, \mathcal{S}', \mu} s'$ by *i.h.* on s . Moreover $vL \in \text{HA}_{\mathcal{A}'}$ by Remark 4.1, and we can always assume $x \notin \mathcal{A}' \cup \mathcal{S}'$. We can then apply rule LSV^\bullet and conclude $s[x/vL] \xrightarrow{\text{lsv}, \mathcal{A}', \mathcal{S}', \mu} s'[x/v]L$.

4. APPL^\bullet . Then

$$\frac{s \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, @} s'}{t = s u \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} s' u = t'} \text{APPL}^\bullet$$

Then $s \xrightarrow{\rho, \mathcal{A}', \mathcal{S}', @} s'$ by *i.h.* on s . We can then apply rule APPL^\bullet , yielding $s u \xrightarrow{\rho, \mathcal{A}', \mathcal{S}', \mu} s' u$.

5. APPR^\bullet . Then

$$\frac{s \in \text{St}_{\mathcal{S}} \quad u \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, @} u'}{t = s u \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} s u' = t'} \text{APPR}^\bullet$$

Then $u \xrightarrow{\rho, \mathcal{A}', \mathcal{S}', @} u'$ by *i.h.* on u . Moreover $s \in \text{St}_{\mathcal{S}'}$ by Remark 4.1, so we can apply rule APPR^\bullet and conclude $s u \xrightarrow{\rho, \mathcal{A}', \mathcal{S}', \mu} s u'$.

6. ESR^\bullet . Then

$$\frac{u \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, @} u'}{t = s[x/u] \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} s[x/u'] = t'} \text{ESR}^\bullet$$

Then $u \xrightarrow{\rho, \mathcal{A}', \mathcal{S}', @} u'$ by *i.h.* on u . We can then apply rule ESR^\bullet , yielding $s[x/u] \xrightarrow{\rho, \mathcal{A}', \mathcal{S}', \mu} s[x/u']$.

7. ESLA^\bullet . Then

$$\frac{s \xrightarrow{\rho, \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s' \quad u \in \text{HA}_{\mathcal{A}} \quad x \notin \mathcal{A} \cup \mathcal{S} \quad x \notin \text{fv}(\rho)}{t = s[x/u] \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} s'[x/u] = t'} \text{ESLA}^\bullet$$

Then $\mathcal{A} \cup \{x\} \subseteq \mathcal{A}' \cup \{x\}$ and thus $s \xrightarrow{\rho, \mathcal{A}' \cup \{x\}, \mathcal{S}', \mu} s'$ by *i.h.* on s . Moreover $u \in \text{HA}_{\mathcal{A}'}$ by Remark 4.1. We can also assume $x \notin \mathcal{A}' \cup \mathcal{S}'$ by α -conversion. We can then apply rule ESLA^\bullet , yielding $s[x/u] \xrightarrow{\rho, \mathcal{A}', \mathcal{S}', \mu} s'[x/u]$.

8. esLS^\bullet . Then

$$\frac{s \xrightarrow{\rho, \mathcal{A}, \mathcal{S} \cup \{x\}, \mu} s' \quad u \in \text{St}_{\mathcal{S}} \quad x \notin \mathcal{A} \cup \mathcal{S} \quad x \notin \text{fv}(\rho)}{t = s[x/u] \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} s'[x/u] = t'} \text{esLS}^\bullet$$

Then $\mathcal{S} \cup \{x\} \subseteq \mathcal{S}' \cup \{x\}$ and thus $s \xrightarrow{\rho, \mathcal{A}', \mathcal{S}' \cup \{x\}, \mu} s'$ by *i.h.* on s . Moreover $u \in \text{St}_{\mathcal{S}'}$ by Remark 4.1, and we can always assume $x \notin \mathcal{A}' \cup \mathcal{S}'$. We can then apply rule esLS^\bullet , yielding $s[x/u] \xrightarrow{\rho, \mathcal{A}', \mathcal{S}', \mu} s'[x/u]$. \square

PROPOSITION B.23 (TOWARDS THE DIAMOND PROPERTY). *Let $\text{inv}(\mathcal{A}, \mathcal{S}, t)$ and let $t \xrightarrow{\rho_1, \mathcal{A}, \mathcal{S}, \mu} t_1$ and $t \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} t_2$, with $t_1 \neq t_2$. Moreover, assume that if $\rho_1 = \text{sub}_{(x, v_1)}$ and $\rho_2 = \text{sub}_{(x, v_2)}$ then $v_1 = v_2$. Moreover, let $\mathcal{B}^1, \mathcal{B}^2$ be sets of variables disjoint from \mathcal{A} , such that if $\rho_i = \text{sub}_{(x, v_i)}$ then $v_i \in \text{HA}_{\mathcal{A} \cup \mathcal{B}^i}$, for all $i \in \{1, 2\}$. Then there exists t' such that $t_1 \xrightarrow{\rho_2, \mathcal{A} \cup \mathcal{B}^1, \mathcal{S}, \mu} t'$ and $t_2 \xrightarrow{\rho_1, \mathcal{A} \cup \mathcal{B}^2, \mathcal{S}, \mu} t'$.*

PROOF. By induction on t . Case $t = \lambda x. s$ is impossible, as there are no rules to reduce abstractions. We analyze the remaining cases.

1. $t = x$. This case does not apply since we would have $x \xrightarrow{\text{sub}_{(x, v_1)}, \mathcal{A} \cup \{x\}, \mathcal{S}, @} v_1 = t_1$ and $x \xrightarrow{\text{sub}_{(x, v_2)}, \mathcal{A} \cup \{x\}, \mathcal{S}, @} v_2 = t_2$, with $v_1 = v_2 = v$ by the hypothesis, which contradicts the hypothesis that states $t_1 \neq t_2$.
2. $t = s u$. Since $\text{inv}(\mathcal{A}, \mathcal{S}, s u)$, then both $\text{inv}(\mathcal{A}, \mathcal{S}, s)$ and $\text{inv}(\mathcal{A}, \mathcal{S}, u)$ holds. We can reduce an application via rules DB^\bullet , APPL^\bullet and APPR^\bullet , so we have the following subcases:
 - 2.1 $\text{DB}^\bullet\text{-DB}^\bullet$. This case does not apply since it ends up being that $t_1 = t_2$, which contradicts the hypothesis.
 - 2.2 $\text{DB}^\bullet\text{-APPL}^\bullet$. We have $t = (\lambda x. s')L u \xrightarrow{\text{db}, \mathcal{A}, \mathcal{S}, \mu} s'[x/u]L = t_1$, where $s = (\lambda x. s')L$ and $\rho_1 = \text{db}$. On the other hand $t = (\lambda x. s')L u \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} s_2 u = t_2$ is derived from $(\lambda x. s')L \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, @} s_2$, where s_2 is of the form $(\lambda x. s')L'$ by Remark B.21. Then $t_2 = (\lambda x. s')L' u \xrightarrow{\text{db}, \mathcal{A}, \mathcal{S}, \mu} s'[x/u]L' = t'$ by rule DB^\bullet , and $t_1 = s'[x/u]L \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} s'[x/u]L' = t'$ by Remark B.21. And we conclude by applying Lemma B.22 on both reductions to extend \mathcal{A} to $\mathcal{A} \cup \mathcal{B}^1$ and $\mathcal{A} \cup \mathcal{B}^2$ respectively. The following diagram summarizes the proof:

$$\begin{array}{ccc} t = (\lambda x. s')L u & \xrightarrow{\text{db}, \mathcal{A}, \mathcal{S}, \mu} & s'[x/u]L = t_1 \\ \rho_2, \mathcal{A}, \mathcal{S}, \mu \downarrow \bullet & & \rho_2, \mathcal{A} \cup \mathcal{B}^1, \mathcal{S}, \mu \downarrow \bullet \\ t_2 = (\lambda x. s')L' u & \xrightarrow{\text{db}, \mathcal{A} \cup \mathcal{B}^2, \mathcal{S}, \mu} & s'[x/u]L' = t' \end{array}$$

- 2.3 $\text{DB}^\bullet\text{-APPR}^\bullet$. We have $t = (\lambda x. s')L u \xrightarrow{\text{db}, \mathcal{A}, \mathcal{S}, \mu} s'[x/u]L = t_1$, where $s = (\lambda x. s')L$ and $\rho_1 = \text{db}$. On the other hand $t = (\lambda x. s')L u \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} (\lambda x. s')L u_2 = t_2$ is derived from $(\lambda x. s')L \in \text{St}_{\mathcal{S}}$ and $u \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, @} u_2$. This case is not possible since $(\lambda x. s')L$ cannot be an element of $\text{St}_{\mathcal{S}}$ by Remark 4.1.
- 2.4 $\text{APPL}^\bullet\text{-APPR}^\bullet$. We have $t = s u \xrightarrow{\rho_1, \mathcal{A}, \mathcal{S}, \mu} s_1 u = t_1$ derived from (1) $s \xrightarrow{\rho_1, \mathcal{A}, \mathcal{S}, @} s_1$, and we have $t = s u \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} s u_2 = t_2$ is derived from (2) $s \in \text{St}_{\mathcal{S}}$ and (3) $u \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, @} u_2$. We can apply rule APPL^\bullet with (1) as premise, yielding $t_2 = s u_2 \xrightarrow{\rho_1, \mathcal{A}, \mathcal{S}, \mu} s_1 u_2 = t'$. Having (1) and (2), then $s_1 \in \text{St}_{\mathcal{S}}$ by Lemma B.20. With this result and (3) we can apply rule APPR^\bullet , yielding $t_1 = s_1 u \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} s_1 u_2 = t'$. And we conclude by applying Lemma B.22 on both reductions to extend \mathcal{A} to $\mathcal{A} \cup \mathcal{B}^1$ and $\mathcal{A} \cup \mathcal{B}^2$ respectively. The following diagram summarizes the proof:

$$\begin{array}{ccc} t = s u & \xrightarrow{\rho_1, \mathcal{A}, \mathcal{S}, \mu} & s_1 u = t_1 \\ \rho_2, \mathcal{A}, \mathcal{S}, \mu \downarrow \bullet & & \rho_2, \mathcal{A} \cup \mathcal{B}^1, \mathcal{S}, \mu \downarrow \bullet \\ t_2 = s u_2 & \xrightarrow{\rho_1, \mathcal{A} \cup \mathcal{B}^2, \mathcal{S}, \mu} & s_1 u_2 = t' \end{array}$$

- 2.5 $\text{APPL}^\bullet\text{-APPL}^\bullet$. We have $t = s u \xrightarrow{\rho_1, \mathcal{A}, \mathcal{S}, \mu} s_1 u = t_1$ derived from $s \xrightarrow{\rho_1, \mathcal{A}, \mathcal{S}, @} s_1$, and we have $t = s u \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} s_2 u = t_2$ derived from $s \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, @} s_2$, where $s_1 \neq s_2$ since $s_1 u \neq s_2 u$ by hypothesis. We apply *i.h.* on s , yielding s' such that $s_1 \xrightarrow{\rho_2, \mathcal{A} \cup \mathcal{B}^1, \mathcal{S}, @} s'$ and $s_2 \xrightarrow{\rho_1, \mathcal{A} \cup \mathcal{B}^2, \mathcal{S}, @} s'$. Applying rule APPL^\bullet to reduce both $s_1 u$ and $s_2 u$, we obtain $t_1 = s_1 u \xrightarrow{\rho_2, \mathcal{A} \cup \mathcal{B}^1, \mathcal{S}, \mu} s' u = t'$ and $t_2 = s_2 u \xrightarrow{\rho_1, \mathcal{A} \cup \mathcal{B}^2, \mathcal{S}, \mu} s' u = t'$ respectively. The following diagram summarizes

the proof:

$$\begin{array}{ccc}
 t = s u & \xrightarrow{\rho_1, \mathcal{A}, \mathcal{S}, \mu} & s_1 u = t_1 \\
 \rho_2, \mathcal{A}, \mathcal{S}, \mu \downarrow \bullet & & \rho_2, \mathcal{A} \cup \mathcal{B}^1, \mathcal{S}, \mu \downarrow \bullet \\
 t_2 = s_2 u & \xrightarrow{\rho_1, \mathcal{A} \cup \mathcal{B}^2, \mathcal{S}, \mu} & s' u = t'
 \end{array}$$

2.6 APPR[•]-APPR[•]. Analogous to the previous case.

3. $t = s[x/u]$. Since $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ then $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)$, $\text{inv}(\mathcal{A}, \mathcal{S} \cup \{x\}, s)$ and $\text{inv}(\mathcal{A}, \mathcal{S}, u)$. We can reduce t via rules LSV[•], esR[•], esLA[•] and esLS[•], so we have the following cases:

3.1 LSV[•]-LSV[•]. We have

$$\frac{s \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s_1 \quad x \notin \mathcal{A} \cup \mathcal{S} \quad vL \in \text{HA}_{\mathcal{A}}}{t = s[x/vL] \xrightarrow{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} s_1[x/v]L = t_1} \text{LSV}^{\bullet}$$

where $u = vL$ and $\rho_1 = \text{lsv}$, and we have

$$\frac{s \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s_2 \quad x \notin \mathcal{A} \cup \mathcal{S} \quad vL \in \text{HA}_{\mathcal{A}}}{t = s[x/vL] \xrightarrow{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} s_2[x/v]L = t_2} \text{LSV}^{\bullet}$$

where $\rho_2 = \text{LSV}^{\bullet}$ and $s_1 \neq s_2$ since $s_1[x/vL] \neq s_2[x/vL]$. We apply *i.h.* on s , yielding s' such that $s_1 \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\} \cup \mathcal{B}^1, \mathcal{S}, \mu} s'$ and $s_2 \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\} \cup \mathcal{B}^2, \mathcal{S}, \mu} s'$. Moreover, $s_1 \xrightarrow{\text{sub}(x,v), \mathcal{A}^{\perp} \cup \{x\} \cup \mathcal{B}^1, \mathcal{S}^{\perp}, \mu} s'$ and $s_2 \xrightarrow{\text{sub}(x,v), \mathcal{A}^{\perp} \cup \{x\} \cup \mathcal{B}^2, \mathcal{S}^{\perp}, \mu} s'$ by Lemma B.22. Furthermore, the hypothesis $vL \in \text{HA}_{\mathcal{A}}$ implies $v \in \text{HA}_{\mathcal{A}^{\perp}}$ by Lemma B.19 and $v \in \text{HA}_{\mathcal{A}^{\perp} \cup \mathcal{B}^i}$ by Lemma B.22. Since $x \notin \mathcal{A} \cup \mathcal{S}$ by hypothesis, and $x \notin \text{dom}(L) \cup \mathcal{B}^1 \cup \mathcal{B}^2$ by α -conversion, then $x \notin \mathcal{A}^{\perp} \cup \mathcal{B}^1 \cup \mathcal{S}^{\perp}$ and $x \notin \mathcal{A}^{\perp} \cup \mathcal{B}^2 \cup \mathcal{S}^{\perp}$. We can then apply rule LSV[•] on both $s_1[x/v]$ and $s_2[x/v]$, yielding $s_1[x/v] \xrightarrow{\text{lsv}, \mathcal{A}^{\perp} \cup \mathcal{B}^1, \mathcal{S}^{\perp}, \mu} s'[x/v]$ and $s_2[x/v] \xrightarrow{\text{lsv}, \mathcal{A}^{\perp} \cup \mathcal{B}^2, \mathcal{S}^{\perp}, \mu} s'[x/v]$ resp. To conclude, $t_1 = s_1[x/v]L \xrightarrow{\text{lsv}, \mathcal{A} \cup \mathcal{B}^1, \mathcal{S}, \mu} s'[x/v]L = t'$ and $t_2 = s_2[x/v]L \xrightarrow{\text{lsv}, \mathcal{A} \cup \mathcal{B}^2, \mathcal{S}, \mu} s'[x/v]L = t'$ by successively applying rule esLA[•] or esLS[•] accordingly. The following diagram summarizes the proof:

$$\begin{array}{ccc}
 t = s[x/vL] & \xrightarrow{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} & s_1[x/v]L = t_1 \\
 \text{lsv}, \mathcal{A}, \mathcal{S}, \mu \downarrow \bullet & & \text{lsv}, \mathcal{A} \cup \mathcal{B}^2, \mathcal{S}, \mu \downarrow \bullet \\
 t_2 = s_2[x/v]L & \xrightarrow{\text{lsv}, \mathcal{A} \cup \mathcal{B}^1, \mathcal{S}, \mu} & s'[x/v]L = t'
 \end{array}$$

3.2 LSV[•]-esR[•]. We have

$$\frac{s \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s_1 \quad (1) \quad x \notin \mathcal{A} \cup \mathcal{S} \quad (2) \quad vL \in \text{HA}_{\mathcal{A}} \quad (3)}{t = s[x/vL] \xrightarrow{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} s_1[x/v]L = t_1} \text{LSV}^{\bullet}$$

where $u = vL$ and $\rho_1 = \text{lsv}$, and we have $t = s[x/vL] \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} s[x/u_2] = t_2$ which is derived from (4) $vL \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} u_2$. Moreover, $u_2 = vL'$ by Remark B.21. We can then apply rule esR[•], esLA[•] or esLS[•] to $(\rho_2, \mathcal{A}, \mathcal{S}, \mu)$ -reduce $t_1 = s_1[x/v]L$ to the term $s_1[x/v]L' = t'$. And we conclude by applying Lemma B.22 to extend \mathcal{A} to $\mathcal{A} \cup \mathcal{B}^1$. On the other hand, we analyze two possible cases, depending on the form ρ_2 can have:

3.2.1 $\rho_2 \in \{\text{db}, \text{lsv}\}$. Since (3) and (4) then $vL' \in \text{HA}_{\mathcal{A}}$ by Lemma B.20. With this result and having that (1) and (2) holds, then we can apply rule LSV[•], yielding $t_2 = s[x/vL'] \xrightarrow{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} s_1[x/v]L' = t'$. To conclude, we extend \mathcal{A} to $\mathcal{A} \cup \mathcal{B}^2$ by Lemma B.22.

3.2.2 $\rho_2 = \text{sub}(x_2, v_2)$. By α -conversion, we may assume $x_2 \neq x$, and $x_2 \in \mathcal{A}$ by Lemma B.6. Moreover, $v_2 \in \text{HA}_{\mathcal{A} \cup \mathcal{B}^2}$ by hypothesis. Since (4) and (3) then $vL' \in \text{HA}_{\mathcal{A} \cup \mathcal{B}^2}$ by Lemma B.14. Moreover, since (1) holds then $s \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\} \cup \mathcal{B}^1, \mathcal{S}, \mu} s_1$ by Lemma B.22. We apply rule LSV[•], yielding $t_2 = s[x/vL'] \xrightarrow{\text{lsv}, \mathcal{A} \cup \mathcal{B}^2, \mathcal{S}, \mu} s_1[x/v]L' = t'$.

The following diagram summarizes the proof:

$$\begin{array}{ccc}
 t = s[x/vL] & \xrightarrow{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} & s_1[x/v]L = t_1 \\
 \rho_2, \mathcal{A}, \mathcal{S}, \mu \downarrow \bullet & & \rho_2, \mathcal{A} \cup \mathcal{B}^1, \mathcal{S}, \mu \downarrow \bullet \\
 t_2 = s[x/vL'] & \xrightarrow{\text{lsv}, \mathcal{A} \cup \mathcal{B}^2, \mathcal{S}, \mu} & s_1[x/v]L' = t'
 \end{array}$$

3.3 $\text{LSV}^\bullet\text{-ESLA}^\bullet$. We have

$$\frac{s \xrightarrow{\rho_2, \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s_1 \quad x \notin \mathcal{A} \cup \mathcal{S} \quad vL \in \text{HA}_{\mathcal{A}}}{t = s[x/vL] \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} s_1[x/v]L = t_1} \text{LSV}^\bullet$$

where $u = vL$ and $\rho_1 = \text{lsv}$, and we have

$$\frac{s \xrightarrow{\rho_2, \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s_2 \quad vL \in \text{HA}_{\mathcal{A}} \quad x \notin \mathcal{A} \cup \mathcal{S} \quad x \notin \text{fv}(\rho_2) \quad (1)}{t = s[x/vL] \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} s_2[x/vL] = t_2} \text{ESLA}^\bullet$$

We apply *i.h.* on s , yielding s' such that $s_1 \xrightarrow{\rho_2, \mathcal{A} \cup \{x\} \cup \mathcal{B}^1, \mathcal{S}, \mu} s'$ and (2) $s_2 \xrightarrow{\rho_2, \mathcal{A} \cup \{x\} \cup \mathcal{B}^2, \mathcal{S}, \mu} s'$. Moreover (3) $s_1 \xrightarrow{\rho_2, \mathcal{A} \cup \{x\} \cup \mathcal{B}^1, \mathcal{S}^\perp, \mu} s'$ by Lemma B.22. The hypothesis $vL \in \text{HA}_{\mathcal{A}}$ implies $v \in \text{HA}_{\mathcal{A}^\perp}$ by Lemma B.19 and $v \in \text{HA}_{\mathcal{A}^\perp \cup \mathcal{B}^i}$ by Lemma B.22, with $i \in \{1, 2\}$. Hence, we have (4) $v \in \text{HA}_{\mathcal{A}^\perp \cup \mathcal{B}^1}$ and (5) $vL \in \text{HA}_{\mathcal{A} \cup \mathcal{B}^2}$ by Lemma B.19. Since $x \notin \mathcal{A} \cup \mathcal{S}$ by hypothesis, and $x \notin \mathcal{B}^1 \cup \mathcal{B}^2 \cup \text{dom}(L)$ by α -conversion, then (6) $x \notin (\mathcal{A} \cup \mathcal{B}^2) \cup \mathcal{S}$ and (7) $x \notin (\mathcal{A}^\perp \cup \mathcal{B}^1) \cup \mathcal{S}^\perp$. Therefore, $s_1[x/v] \xrightarrow{\rho_2, \mathcal{A} \cup \mathcal{B}^1, \mathcal{S}, \mu} s'[x/v]$ by rule ESLA^\bullet , with (3), (4), (7) and (1) as premises. And we conclude $t_1 = s_1[x/v]L \xrightarrow{\rho_2, \mathcal{A} \cup \mathcal{B}^1, \mathcal{S}, \mu} s'[x/v]L = t'$ by successively applying rule ESLA^\bullet or rule ESL^\bullet accordingly. On the other hand, applying rule LSV^\bullet with (2), (6) and (5) as premises we obtain $t_2 = s_2[x/vL] \xrightarrow{\rho_2, \mathcal{A} \cup \mathcal{B}^2, \mathcal{S}, \mu} s'[x/v]L = t'$. The following diagram summarizes the proof:

$$\begin{array}{ccc} t = s[x/vL] & \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} & s_1[x/v]L = t_1 \\ \rho_2, \mathcal{A}, \mathcal{S}, \mu \downarrow \bullet & & \rho_2, \mathcal{A} \cup \mathcal{B}^1, \mathcal{S}, \mu \downarrow \bullet \\ t_2 = s_2[x/vL] & \xrightarrow{\rho_2, \mathcal{A} \cup \mathcal{B}^2, \mathcal{S}, \mu} & s'[x/v]L = t' \end{array}$$

3.4 $\text{LSV}^\bullet\text{-ESL}^\bullet$. We have

$$\frac{s \xrightarrow{\rho_2, \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s_1 \quad x \notin \mathcal{A} \cup \mathcal{S} \quad vL \in \text{HA}_{\mathcal{A}}}{t = s[x/vL] \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} s_1[x/v]L = t_1} \text{LSV}^\bullet$$

where $u = vL$ and $\rho_1 = \text{lsv}$; and we have $t = s[x/vL] \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} s_2[x/vL] = t_2$ derived from $s \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S} \cup \{x\}, \mu} s_2$, $vL \in \text{St}_{\mathcal{S}}$, $x \notin \mathcal{A} \cup \mathcal{S}$ and $x \notin \text{fv}(\rho_2)$. Since $\text{inv}(\mathcal{A}, \mathcal{S}, vL)$, then in particular $\mathcal{A} \# \mathcal{S}$. Hence we have that $vL \in \text{HA}_{\mathcal{A}}$ and $vL \in \text{St}_{\mathcal{S}}$, and at the same time we have $vL \notin \text{HA}_{\mathcal{A}}$ or $vL \notin \text{St}_{\mathcal{S}}$ by Lemma B.2. Therefore we reach a contradiction, so this case is not possible.

3.5 $\text{ESR}^\bullet\text{-ESR}^\bullet$. We have $t = s[x/u] \xrightarrow{\rho_1, \mathcal{A}, \mathcal{S}, \mu} s[x/u_1] = t_1$ derived from $u \xrightarrow{\rho_1, \mathcal{A}, \mathcal{S} \oplus} u_1$; and we have $t = s[x/u] \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} s[x/u_2] = t_2$ derived from $u \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S} \oplus} u_2$; where $u_1 \neq u_2$ since $s[x/u_1] \neq s[x/u_2]$ by hypothesis. We apply *i.h.* on u , yielding u' such that $u_1 \xrightarrow{\rho_2, \mathcal{A} \cup \mathcal{B}^1, \mathcal{S} \oplus} u'$ and $u_2 \xrightarrow{\rho_1, \mathcal{A} \cup \mathcal{B}^2, \mathcal{S} \oplus} u'$. Applying rule ESR^\bullet on both $s[x/u_1]$ and $s[x/u_2]$, we obtain $t_1 = s[x/u_1] \xrightarrow{\rho_2, \mathcal{A} \cup \mathcal{B}^1, \mathcal{S}, \mu} s[x/u'] = t'$ and $t_2 = s[x/u_2] \xrightarrow{\rho_1, \mathcal{A} \cup \mathcal{B}^2, \mathcal{S}, \mu} s[x/u'] = t'$ respectively. The following diagram summarizes the proof:

$$\begin{array}{ccc} t = s[x/u] & \xrightarrow{\rho_1, \mathcal{A}, \mathcal{S}, \mu} & s[x/u_1] = t_1 \\ \rho_2, \mathcal{A}, \mathcal{S}, \mu \downarrow \bullet & & \rho_2, \mathcal{A} \cup \mathcal{B}^1, \mathcal{S}, \mu \downarrow \bullet \\ t_2 = s[x/u_2] & \xrightarrow{\rho_1, \mathcal{A} \cup \mathcal{B}^2, \mathcal{S}, \mu} & s[x/u'] = t' \end{array}$$

3.6 $\text{ESR}^\bullet\text{-ESLA}^\bullet$. We have $t = s[x/u] \xrightarrow{\rho_1, \mathcal{A}, \mathcal{S}, \mu} s[x/u_1] = t_1$ derived from (1) $u \xrightarrow{\rho_1, \mathcal{A}, \mathcal{S} \oplus} u_1$, and we have $t = s[x/u] \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} s_2[x/u] = t_2$ derived from (2) $s \xrightarrow{\rho_2, \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s_2$, (3) $u \in \text{HA}_{\mathcal{A}}$, (4) $x \notin \mathcal{A} \cup \mathcal{S}$ and (5) $x \notin \text{fv}(\rho_2)$. We can apply rule ESLA^\bullet with (2), (3), (4) and (5) as premises, yielding $t_1 = s[x/u_1] \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} s_2[x/u_1] = t'$; and we can apply rule ESR^\bullet with (1) as premise, yielding $t_2 = s_2[x/u] \xrightarrow{\rho_1, \mathcal{A}, \mathcal{S}, \mu} s_2[x/u_1] = t'$. We conclude by applying Lemma B.22 on both reductions to extend \mathcal{A} to $\mathcal{A} \cup \mathcal{B}^1$ and $\mathcal{A} \cup \mathcal{B}^2$ respectively. The following diagram summarizes the proof:

$$\begin{array}{ccc} t = s[x/u] & \xrightarrow{\rho_1, \mathcal{A}, \mathcal{S}, \mu} & s[x/u_1] = t_1 \\ \rho_2, \mathcal{A}, \mathcal{S}, \mu \downarrow \bullet & & \rho_2, \mathcal{A} \cup \mathcal{B}^1, \mathcal{S}, \mu \downarrow \bullet \\ t_2 = s_2[x/u] & \xrightarrow{\rho_1, \mathcal{A} \cup \mathcal{B}^2, \mathcal{S}, \mu} & s_2[x/u_1] = t' \end{array}$$

3.7 $\text{ESR}^\bullet\text{-ESL}^\bullet$. Analogous to the previous case.

- 3.8 esLA^\bullet - esLS^\bullet . We have $t = s[x/u] \xrightarrow{\rho_1, \mathcal{A}, \mathcal{S}, \mu} s_1[x/u] = t_1$ which is derived by $s \xrightarrow{\rho_1, \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s_1$, $u \in \text{HA}_{\mathcal{A}}$, $x \notin \mathcal{A} \cup \mathcal{S}$ and $x \notin \text{fv}(\rho_1)$; and we have $t = s[x/u] \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} s_2[x/u] = t_2$ which is derived by $s \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S} \cup \{x\}, \mu} s_2$, $u \in \text{St}_{\mathcal{S}}$, $x \notin \mathcal{A} \cup \mathcal{S}$ and $x \notin \text{fv}(\rho_2)$. Since $\text{inv}(\mathcal{A}, \mathcal{S}, u)$ then in particular $\mathcal{A} \# \mathcal{S}$. At the same time that $u \in \text{HA}_{\mathcal{A}}$ and $u \in \text{St}_{\mathcal{S}}$, we have $u \notin \text{HA}_{\mathcal{A}}$ or $u \notin \text{St}_{\mathcal{S}}$ by Lemma B.2. Therefore we reach a contradiction, so this case is not possible.
- 3.9 esLA^\bullet - esLA^\bullet . We have $t = s[x/u] \xrightarrow{\rho_1, \mathcal{A}, \mathcal{S}, \mu} s_1[x/u] = t_1$ which is derived from $s \xrightarrow{\rho_1, \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s_1$, $u \in \text{HA}_{\mathcal{A}}$, $x \notin \mathcal{A} \cup \mathcal{S}$ and $x \notin \text{fv}(\rho_1)$, and we have $t = s[x/u] \xrightarrow{\rho_2, \mathcal{A}, \mathcal{S}, \mu} s_2[x/u] = t_2$ which is derived from $s \xrightarrow{\rho_2, \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s_2$, $u \in \text{HA}_{\mathcal{A}}$, $x \notin \mathcal{A} \cup \mathcal{S}$ and $x \notin \text{fv}(\rho_2)$; where $s_1 \neq s_2$ since $s_1[x/u] \neq s_2[x/u]$ by hypothesis. We apply *i.h.* on s , yielding s' such that $s_1 \xrightarrow{\rho_2, \mathcal{A} \cup \{x\} \cup \mathcal{B}^1, \mathcal{S}, \mu} s'$ and $s_2 \xrightarrow{\rho_1, \mathcal{A} \cup \{x\} \cup \mathcal{B}^2, \mathcal{S}, \mu} s'$. Applying rule esLA^\bullet to reduce both $s_1[x/u]$ and $s_2[x/u]$, we obtain $t_1 = s_1[x/u] \xrightarrow{\rho_2, \mathcal{A} \cup \mathcal{B}^1, \mathcal{S}, \mu} s'[x/u] = t'$ and $t_2 = s_2[x/u] \xrightarrow{\rho_1, \mathcal{A} \cup \mathcal{B}^2, \mathcal{S}, \mu} s'[x/u] = t'$ respectively. The following diagram summarizes the proof:

$$\begin{array}{ccc}
 t = s[x/u] & \xrightarrow{\rho_1, \mathcal{A}, \mathcal{S}, \mu} & s_1[x/u] = t_1 \\
 \rho_2, \mathcal{A}, \mathcal{S}, \mu \downarrow \bullet & & \rho_2, \mathcal{A} \cup \mathcal{B}^1, \mathcal{S}, \mu \downarrow \bullet \\
 t_2 = s_2[x/u] & \xrightarrow{\rho_1, \mathcal{A} \cup \mathcal{B}^2, \mathcal{S}, \mu} & s'[x/u] = t'
 \end{array}$$

- 3.10 esLS^\bullet - esLS^\bullet . Analogous to the previous case. □

THEOREM 4.5 (DIAMOND PROPERTY). *Let $t \xrightarrow{\rho_1, \varnothing, \mathcal{S} \otimes} t_1$ and $t \xrightarrow{\rho_2, \varnothing, \mathcal{S} \otimes} t_2$, where $t_1 \neq t_2$ and $\rho_1, \rho_2 \in \{\text{db}, \text{lsv}\}$ and $\mathcal{S} = \text{fv}(t)$. Then there exists t' such that $t_1 \xrightarrow{\rho_2, \varnothing, \mathcal{S} \otimes} t'$ and $t_2 \xrightarrow{\rho_1, \varnothing, \mathcal{S} \otimes} t'$.*

PROOF. This is a particular case of Proposition B.23. □

C PROOFS OF SECTION 5 “RELATING LINEAR AND USEFUL OPEN CBV”

This section presents the technical details regarding the relation between the LOCBV° calculus and the uOCBV^\bullet strategy. We start first by discussing in Appendix C.1 the unfolding operation we introduced in Section 5: we give in particular a characterization of the unfolding operation via rewriting. Our main goal here is to show that the relation \rightarrow_σ is terminating (Theorem C.14). To achieve this result, we prove that there is a decreasing measure, which we define in this same subsection. Then we move on to Appendix C.2, where we show the technical results that are necessary to relate LOCBV° and uOCBV^\bullet (Corollary C.25).

C.1 Characterizing the Unfolding Operation via Rewriting

The relation of **evaluation under a value assignment** σ , written \rightarrow_σ , is defined as follows:

$$\rightarrow_\sigma := \overset{\circ}{\rightarrow}_{\text{lsv}} \cup_{x \in \text{dom}(\sigma)} (\overset{\circ}{\rightarrow}_{\text{sub}(x, \sigma(x))})$$

A term t is said to be σ -**reducible** if there exists s such that $t \rightarrow_\sigma s$. Notice that $\text{dom}(\sigma)$ plays the role of a value frame, so this notion is closely related to the set Red_V° (defined in Appendix A).

We start by proving that the reduction relation \rightarrow_σ is *terminating* (cf. Section 2). To prove this, we define a *measure* $\#^\sigma(_)$ on terms and we show that it is strictly decreasing w.r.t. \rightarrow_σ (Lemma C.7). The measure $\#^\sigma(_)$ is inspired by de Vrijer’s direct proof of the finite developments theorem [33], and defined by means of intermediate functions $\#_x(_)$ and $\#(_)$.

So, given a term t and a variable x , the **potential number of occurrences of x in t** , written $\#_x(t)$, is defined as 0 if $x \notin \text{fv}(t)$, and otherwise is defined recursively as follows:

$$\begin{array}{ll}
 \#_x(x) & := 1 & \#_x(ts) & := \#_x(t) + \#_x(s) \\
 \#_x(\lambda y. t) & := 0 & \#_x(t[y/s]) & := \#_x(t) + \#_x(s) \cdot (1 + \#_y(t))
 \end{array}$$

This gives an *overapproximation* of the number of free reachable occurrences of x in the unfolding of t . By this we mean that $\#_x(t)$ counts the *potential* number of free occurrences of x in the unfolding of t , even if not all the substitutions are performed during useful reduction for some reason. For example, $\#_y(x[x/y]y) = 4$, although the substitution of x by y is never executed because it is not useful. Another example is $t_0 := x[x/y[y/z]]$. We have $\#_z(t_0) = 4$, despite the fact that z only occurs three times in the corresponding unfolding $z[x/z][y/z]$.

We now define the **measure of** t , written $\#(t)$, as follows:

$$\begin{aligned} \#(x) &:= 0 & \#(ts) &:= \#(t) + \#(s) \\ \#(\lambda x. t) &:= 0 & \#(t[x/s]) &:= \#(t) + \#_x(t) + \#(s) \cdot (1 + \#_x(t)) \end{aligned}$$

This measure can be seen as the number of steps that the evaluation takes to perform all the substitutions in the longest reduction sequence starting at t . Notice that $\#(v) = 0$ for any value v .

Furthermore, given σ a value assignment, we define the **measure of** t **under** σ , written $\#^\sigma(t)$, as $\#^\sigma(t) := \#(t) + \sum_{x \in \text{dom}(\sigma)} \#_x(t)$.

For example, take again the term t_0 from above. Since \rightarrow_σ -reduction is non-deterministic, we consider two different reduction sequences from t_0 , of lengths 2 and 3 respectively, where the second sequence is the longest possible. In this example, σ is the empty value assignment ($\sigma = \cdot$):

$$\begin{aligned} t_0 = x[x/y[y/z]] &\rightarrow s = x[x/z[y/z]] &\rightarrow t_3 = z[x/z][y/z] \\ t_0 = x[x/y[y/z]] &\rightarrow t_1 = y[x/y][y/z] &\rightarrow t_2 = y[x/z][y/z] \\ &&&\rightarrow t_3 = z[x/z][y/z] \end{aligned}$$

Then $\#_z(t_0) = 4$, $\#_z(t_1) = 3$, $\#_z(t_2) = 3$, $\#_z(t_3) = 3$, and $\#_z(s) = 4$. Note for example that $\#_z(t_0) = 4$ is greater than the actual number of free reachable occurrences of z in its unfolding (which is 3).

Note also that $\#(t_0) = 3$, $\#(t_1) = 2$, $\#(t_2) = 1$, $\#(t_3) = 0$, and $\#(s) = 1$. These are upper bounds for the number of substitution steps required to completely unfold these terms. For instance, $\#(t_0) = 3$, even though the first reduction sequence reaches the unfolding in only two steps. The important point is that the measure strictly decreases at each step of any reduction sequence, as we prove at the end of this subsection.

Given a value frame \mathcal{V} , the set of **substitution contexts in normal form under** \mathcal{V} is written $\text{CtxNF}_\mathcal{V}^\circ$ and is defined inductively as follows:

$$\begin{aligned} \frac{}{\diamond \in \text{CtxNF}_\mathcal{V}^\circ} \text{CTX-NF-EMPTY} & \quad \frac{L \in \text{CtxNF}_{\mathcal{V} \cup \{x\}}^\circ \quad t \in \text{NF}_{\mathcal{V} \oplus}^\circ \quad t \in \text{Val}}{L[x/t] \in \text{CtxNF}_\mathcal{V}^\circ} \text{CTX-NF-ADDVAL} \\ \frac{L \in \text{CtxNF}_\mathcal{V}^\circ \quad t \in \text{NF}_{\mathcal{V} \oplus}^\circ \quad \neg(t \in \text{Val})}{L[x/t] \in \text{CtxNF}_\mathcal{V}^\circ} \text{CTX-NF-ADDNONVAL} \end{aligned}$$

Definition C.1 (Expansion of value frames). Let \mathcal{V} be a value frame. We inductively define the *expansion of* \mathcal{V} *under* L , written \mathcal{V}^\diamond , as follows:

$$\begin{aligned} \mathcal{V}^\diamond &:= \mathcal{V} \\ \mathcal{V}^{L[x/t]} &:= \begin{cases} \mathcal{V}^{L'} \cup \{x\} & \text{if } t \in \text{Val} \\ \mathcal{V}^{L'} & \text{otherwise} \end{cases} \end{aligned}$$

LEMMA C.2. *The following are equivalent:*

1. $tL \in \text{NF}_{\mathcal{V}, \mu}^\circ$
2. $t \in \text{NF}_{\mathcal{V}^\diamond, \mu}^\circ$ and $L \in \text{CtxNF}_\mathcal{V}^\circ$.

PROOF. Simultaneously by induction on L . □

Definition C.3. Let $\varphi : \text{Var} \rightarrow \mathbb{N}$. Given a substitution context L , the **potential number of occurrences of** x **in** L **under** φ , written $\#_x^\varphi(L)$, is recursively defined as follows:

$$\#_x^\varphi(\diamond) := \varphi(x) \quad \#_x^\varphi(L'[y/t]) := \#_x^\varphi(L') + \#_x(t) \cdot (1 + \#_y^\varphi(L'))$$

We define the **measure of** L **under** φ , written $\#^\varphi(L)$, as follows:

$$\#^\varphi(\diamond) := 0 \quad \#^\varphi(L'[x/t]) := \#^\varphi(L') + \#_x^\varphi(L') + \#(t) \cdot (1 + \#_x^\varphi(L'))$$

For any term t , we define $\varphi_t(x) := \#_x(t)$.

LEMMA C.4 (SPLITTING OF MEASURES). *Let* t *be a term and* L *be a substitution context. Then*

1. $\#_x(tL) = \#_x^{\varphi_t}(L)$
2. $\#(tL) = \#(t) + \#^{\varphi_t}(L)$

PROOF. Simultaneously by induction on L . □

LEMMA C.5. *Let t and s be terms, and L be a substitution context. If $\text{fv}(t) \# \text{dom}(L)$ then*

1. $\#_x^{\rho_{\{y/s\}}}(L) \leq \#_x(t) + \#_x^{\rho_s}(L) \cdot (1 + \#_y(t))$ for any variable x
2. $\#_x^{\rho_{\{x/s\}}}(L) \leq \#_x^{\rho_s}(L) \cdot (1 + \#_x(t))$

PROOF. Simultaneously by induction on L . □

LEMMA C.6. *Let t be a term such that $t \rightarrow_\sigma t'$ for some term t' , and let x be a variable such that if t ($\text{sub}_{(y,v)}$)-reduces to t' then $x \notin \text{fv}(v)$. Then $\#_x(t) \geq \#_x(t')$. Moreover, if $x = y$ then $\#_x(t) > \#_x(t')$.*

PROOF. By induction on the derivation of the judgment $t \rightarrow_\sigma t'$. □

Now we can state a lemma that involves the general notion of decreasing measure:

LEMMA C.7. *Let t be a term.*

- If $t \xrightarrow{\rho} t'$ then $\#(t) \geq \#(t')$ when $\rho = \text{sub}_{(x,v)}$ or $\#(t) > \#(t')$ when $\rho = \text{lsv}$.
- If $t \rightarrow_\sigma t'$ then $\#^\sigma(t) > \#^\sigma(t')$.

PROOF. The proof uses Lemmas C.8, C.9 and C.10. □

LEMMA C.8. *Let $t \xrightarrow{\text{sub}_{(x,v)}} t'$. Then $\#(t) \geq \#(t')$.*

PROOF. By induction on the derivation of the judgment $t \xrightarrow{\text{sub}_{(x,v)}} t'$. □

LEMMA C.9. *Let $t \xrightarrow{\text{lsv}} t'$. Then $\#(t) > \#(t')$.*

PROOF. By induction on the derivation of the judgment $t \xrightarrow{\text{lsv}} t'$. □

LEMMA C.10. *Let $t \rightarrow_\sigma t'$. Then, $\#^\sigma(t) > \#^\sigma(t')$.*

PROOF. By case analysis on the rule used to derive $t \rightarrow_\sigma t'$. □

LEMMA C.11. *Let v be a value and L a substitution context.*

1. *If there exists t such that $vL \xrightarrow{\rho} t$ then there exist v' and L' such that $t = v'L'$.*
2. *If $vL \xrightarrow{\rho} v'L'$ then $t[x/v]L \xrightarrow{\rho} t[x/v']L'$, for any term t .*

PROOF. We prove each item independently.

1. By induction on the derivation of $vL \xrightarrow{\rho} t$. Note that it is not possible to apply rules DB° , APPL° nor APPR° , given that vL is not an application. Then we are left to analyze the following cases:
 - SUB° . Then $x \xrightarrow{\text{sub}_{(x,w)}} w$, where $v = x$, $L = \diamond$ and $t = w$, so we are done, with $v' = w$ and $L' = \diamond$.
 - LSV° . Then $L = L_1[x/wL_2]$, thus deriving

$$\frac{vL_1 \xrightarrow{\text{sub}_{(x,w)}} t_1}{vL_1[x/wL_2] \xrightarrow{\text{lsv}} t_1[x/w]L_2 = t} \text{LSV}^\circ$$

We apply *i.h.* on vL_1 , yielding v'_1 and L'_1 such that $t_1 = v'_1L'_1$. And we are done, with $v' = v'_1$ and $L' = L'_1[x/w]L_2$.

- ESL° . Then $L = L_1[x/s]$, thus deriving

$$\frac{vL_1 \xrightarrow{\rho} t_1 \quad x \notin \text{fv}(\rho)}{vL_1[x/s] \xrightarrow{\rho} t_1[x/s]} \text{ESL}^\circ$$

We apply *i.h.* on vL_1 , yielding v'_1 and L'_1 such that $t_1 = v'_1L'_1$. And we are done, with $v' = v'_1$ and $L' = L'_1[x/s]$.

- ESR° . Then $L = L_1[x/s]$, thus deriving $vL_1[x/s] \xrightarrow{\rho} vL_1[x/s']$ from $s \xrightarrow{\rho} s'$. We are done with $v' = v$ and $L' = L_1[x/s']$.
2. By induction on the length of L .
 - $L = \diamond$. Then $v \xrightarrow{\rho} v'$ which can only be derived by rule SUB° . We obtain $t[x/v] \xrightarrow{\rho} t[x/v']$ by applying rule ESR° , so we are done.
 - $L = L_1[y/s]$. We analyze different cases, depending on the rule which is used to derive $vL_1[y/s] \xrightarrow{\rho} v'L'$. Since it is not possible to apply rules SUB° , DB° , APPL° and APPR° , given that $vL_1[y/s]$ is neither a variable nor an application, we are left to analyze the following three cases:

– LSV° . Then $s = wL_2$ and $\rho = \text{lsv}$, deriving

$$\frac{vL_1 \xrightarrow{\text{sub}(y,w)} u}{vL_1[y/wL_2] \xrightarrow{\text{lsv}} u[y/w]L_2} \text{LSV}^\circ$$

Moreover, $u = v'L_3$ by point (1), hence we necessarily have $L' = L_3[y/w]L_2$. Thus in particular $vL_1 \xrightarrow{\rho} v'L_3$. We apply *i.h.* on L_1 , yielding $t[x/v]L_1 \xrightarrow{\text{sub}(y,w)} t[x/v']L_3$. Applying rule LSV° , we obtain $t[x/v]L = t[x/v]L_1[y/wL_2] \xrightarrow{\text{sub}(y,w)} t[x/v']L_3[y/w]L_2 = t[x/v']L'$.

– ESL° . Then

$$\frac{vL_1 \xrightarrow{\rho} u \quad y \notin \text{fv}(\rho)}{vL_1[y/s] \xrightarrow{\rho} u[y/s]} \text{ESL}^\circ$$

Moreover, $u = v'L_3$ by point (1), hence we necessarily have $L' = L_3[y/s]$. Thus in particular $vL_1 \xrightarrow{\rho} v'L_3$. We apply *i.h.* on L_1 , yielding $t[x/v]L_1 \xrightarrow{\rho} t[x/v']L_3$. Given that $y \notin \text{fv}(\rho)$, we can then apply rule ESL° , yielding $t[x/v]L = t[x/v]L_1[y/s] \xrightarrow{\rho} t[x/v']L_3[y/s] = t[x/v']L'$.

– ESR° . Then $vL_1[y/s] \xrightarrow{\rho} vL_1[y/s']$ is derived from $s \xrightarrow{\rho} s'$. Hence $v' = v$ and $L' = L_1[y/s']$. By applying ESR° , we obtain $t[x/v]L = t[x/v]L_1[y/s] \xrightarrow{\rho} t[x/v]L_1[y/s'] = t[x/v']L'$. \square

LEMMA C.12. *If $t \xrightarrow{\text{sub}(x,v)} t_1$ and $vL \xrightarrow{\rho} v'L'$, then there exists t'_1 such that $t_1 \xrightarrow{\rho} t'_1$.*

PROOF. By induction on the derivation of $t \xrightarrow{\text{sub}(x,v)} t_1$. Cases SUB° and ESL° are the most interesting, while the rest are analogous to the ESL° case.

- SUB° . Then $t = x \xrightarrow{\text{sub}(x,v)} v = t_1$, so we take $t'_1 = v$, as v reduces to itself in zero steps.
- ESL° . Then

$$\frac{s \xrightarrow{\text{sub}(x,v)} s_1 \quad y \notin \text{fv}(\text{sub}(x,v))}{t = s[y/u] \xrightarrow{\text{sub}(x,v)} s_1[y/u] = t_1} \text{ESL}^\circ$$

We apply *i.h.* on s , yielding s'_1 such that $s_1 \xrightarrow{\rho} s'_1$. We may assume $y \notin \text{fv}(\rho)$ by α -conversion, so we can apply rule ESL° , yielding $t_1 = s_1[y/u] \xrightarrow{\rho} s'_1[y/u] = t'_1$. \square

PROPOSITION C.13 (LOCAL CONFLUENCE / WCR OF \rightarrow_σ). *Let t be a σ -reducible term such that $t \xrightarrow{\rho_1} t_1$ and $t \xrightarrow{\rho_2} t_2$, with $\rho_1, \rho_2 \in \rightarrow_\sigma$. Then there exists t' such that $t_1 \xrightarrow{\rho_2^*} t'$ and $t_2 \xrightarrow{\rho_1^*} t'$.*

PROOF. By induction on t . Case $t = \lambda x. s$ is impossible, as there are no rules to reduce abstractions. We analyze the remaining cases.

- $t = x$. The only rule to reduce x is SUB° , and thus $t = x \xrightarrow{\text{sub}(x, \text{dom}(\sigma))} \sigma(x) = t_1 = t_2$. Hence we are done, as t_1 reduces in zero steps to itself.
- $t = s u$. We can σ -reduce t with the rules APPL° and APPR° , so we have the following cases:
 1. APPL° - APPR° . We have $t = s u \xrightarrow{\rho_1} s_1 u = t_1$, which is derived from (1) $s \xrightarrow{\rho_1} s_1$, and we have $t = s u \xrightarrow{\rho_2} s u_2 = t_2$, which is derived from (2) $u \xrightarrow{\rho_2} u_2$. We apply rule APPL° with (1) as premise, yielding $t_2 = s u_2 \xrightarrow{\rho_1} s_1 u_2 = t'$. And we apply rule APPR° with (2) as premise, yielding $t_1 = s_1 u \xrightarrow{\rho_2} s_1 u_2 = t'$. The following diagram summarizes the proof:

$$\begin{array}{ccc} t = s u & \xrightarrow{\rho_1} & s_1 u = t_1 \\ \rho_2 \downarrow \circ & & \rho_2 \downarrow \circ \\ t_2 = s u_2 & \xrightarrow{\rho_1} & s_1 u_2 = t' \end{array}$$

2. APPL° - APPL° . We have $t = s u \xrightarrow{\rho_1} s_1 u = t_1$, which is derived from $s \xrightarrow{\rho_1} s_1$, and we have $t = s u \xrightarrow{\rho_2} s_2 u = t_2$, which is derived from $s \xrightarrow{\rho_2} s_2$. We apply *i.h.* on s , yielding s' such that $s_1 \xrightarrow{\rho_2^*} s'$ and $s_2 \xrightarrow{\rho_1^*} s'$. Applying rule APPL° to reduce both $s_1 u$ and $s_2 u$, we obtain $t_1 = s_1 u \xrightarrow{\rho_2^*} s' u = t'$ and $t_2 = s_2 u \xrightarrow{\rho_1^*} s' u = t'$ respectively. The following

diagram summarizes the proof:

$$\begin{array}{ccc}
 t = s u & \xrightarrow[\rho_1]{\circ} & s_1 u = t_1 \\
 \rho_2 \downarrow \circ & & \rho_2 \downarrow \circ \\
 t_2 = s_2 u & \xrightarrow[\rho_1]{\circ} & s' u = t'
 \end{array}$$

3. APPR[◦]-APPR[◦]. Analogous to the previous case.

• $t = s[x/u]$. We can σ -reduce t with the rules LSV[◦], esL[◦] and esR[◦], so we have the following cases:

1. LSV[◦]-LSV[◦]. We have

$$\frac{s \xrightarrow{\circ}_{\text{sub}(x,v)} s_1}{t = s[x/vL] \xrightarrow{\circ}_{\text{lsv}} s_1[x/v]L = t_1} \text{LSV}^\circ$$

where $u = vL$ and $\rho_1 = \text{lsv}$, and we have

$$\frac{s \xrightarrow{\circ}_{\text{sub}(x,v)} s_2}{t = s[x/vL] \xrightarrow{\circ}_{\text{lsv}} s_2[x/v]L = t_2} \text{LSV}^\circ$$

where $\rho_2 = \text{lsv}^\circ$. We apply *i.h.* on s , yielding s' such that $s_1 \xrightarrow{\circ}_{\text{sub}(x,v)}^* s'$ and $s_2 \xrightarrow{\circ}_{\text{sub}(x,v)}^* s'$. We can then apply rule LSV[◦] on both $s_1[x/v]$ and $s_2[x/v]$, yielding $t_1 = s_1[x/v] \xrightarrow{\circ}_{\text{lsv}}^* s'[x/v]$ and $t_2 = s_2[x/v] \xrightarrow{\circ}_{\text{lsv}}^* s'[x/v]$ resp. To conclude, $s_1[x/v]L \xrightarrow{\circ}_{\text{lsv}}^* s'[x/v]L$ and $s_2[x/v]L \xrightarrow{\circ}_{\text{lsv}}^* s'[x/v]L$ is derived by successively applying rule esL[◦], as $\text{fv}(\text{lsv}) = \emptyset$. The following diagram summarizes the proof:

$$\begin{array}{ccc}
 t = s[x/vL] & \xrightarrow[\text{lsv}]{\circ} & s_1[x/v]L = t_1 \\
 \text{lsv} \downarrow \circ & & \text{lsv} \downarrow \circ \\
 t_2 = s_2[x/vL] & \xrightarrow[\text{lsv}]{\circ} & s'[x/v]L = t'
 \end{array}$$

2. LSV[◦]-esL[◦]. We have

$$\frac{s \xrightarrow{\circ}_{\text{sub}(x,v)} s_1}{t = s[x/vL] \xrightarrow{\circ}_{\text{lsv}} s_1[x/v]L = t_1} \text{LSV}^\circ$$

where $u = vL$ and $\rho_1 = \text{lsv}$, and we have

$$\frac{s \xrightarrow{\circ}_{\rho_2} s_2 \quad x \notin \text{fv}(\rho_2) \quad (1)}{t = s[x/vL] \xrightarrow{\circ}_{\rho_2} s_2[x/vL] = t_2} \text{esL}^\circ$$

We apply *i.h.* on s , yielding s' such that (2) $s_1 \xrightarrow{\circ}_{\rho_2}^* s'$ and (3) $s_2 \xrightarrow{\circ}_{\text{sub}(x,v)}^* s'$. Then $s_1[x/v] \xrightarrow{\circ}_{\rho_2}^* s'[x/v]$ by rule esL[◦], with (2) and (1) as premises. And we conclude that $t_1 = s_1[x/v]L \xrightarrow{\circ}_{\rho_2}^* s'[x/v]L$ is derived by successively applying rule esL[◦]. On the other hand, $t_2 = s_2[x/vL] \xrightarrow{\circ}_{\text{lsv}}^* s'[x/v]L$ by rule LSV[◦] with (3) as premise. The following diagram summarizes the proof:

$$\begin{array}{ccc}
 t = s[x/vL] & \xrightarrow[\text{lsv}]{\circ} & s_1[x/v]L = t_1 \\
 \rho_2 \downarrow \circ & & \rho_2 \downarrow \circ \\
 t_2 = s_2[x/vL] & \xrightarrow[\text{lsv}]{\circ} & s'[x/v]L = t'
 \end{array}$$

3. LSV[◦]-esR[◦]. We have

$$\frac{s \xrightarrow{\circ}_{\text{sub}(x,v)} s_1 \quad (1)}{t = s[x/vL] \xrightarrow{\circ}_{\text{lsv}} s_1[x/v]L = t_1} \text{LSV}^\circ$$

where $u = vL$ and $\rho_1 = \text{lsv}$, and we have $t = s[x/vL] \xrightarrow{\circ}_{\rho_2} s[x/u_2] = t_2$, which is derived from (2) $vL \xrightarrow{\circ}_{\rho_2} u_2$. Moreover, $u_2 = v'L'$ by Lemma C.11 (1). By Lemma A.4, there exists s'_1 such that $s \xrightarrow{\circ}_{\text{sub}(x,v')} s'_1$, hence we can apply rule LSV[◦], yielding $t_2 = s[x/v'L'] \xrightarrow{\circ}_{\text{lsv}} s'_1[x/v']L' = t'$. On the other hand, we have $s_1[x/v]L \xrightarrow{\circ}_{\rho_2} s_1[x/v']L'$ by Lemma C.11 (2). And since $s_1 \xrightarrow{\circ}_{\rho_2} s'_1$ by Lemma C.12, we can derive $s_1[x/v'] \xrightarrow{\circ}_{\rho_2} s'_1[x/v']$ by rule esL[◦], given that $x \notin \text{fv}(\rho_2)$

by α -conversion. Then $s_1[x/v']L' \xrightarrow{\rho_2} s'_1[x/v']L'$ by applying rule ESL° (length of L') times. The following diagram summarizes the proof:

$$\begin{array}{ccc}
 t = s[x/vL] & \xrightarrow{\text{lsv}^\circ} & s_1[x/v]L = t_1 \\
 \downarrow \rho_2^\circ & & \downarrow \rho_2^\circ \\
 & & s_1[x/v']L' \\
 & & \downarrow \rho_2^\circ \\
 t_2 = s[x/v'L'] & \xrightarrow{\text{lsv}^\circ} & s'_1[x/v']L' = t'
 \end{array}$$

4. ESL° - ESL° . We have $t = s[x/u] \xrightarrow{\rho_1} s_1[x/u] = t_1$ which is derived from $s \xrightarrow{\rho_1} s_1$ and $x \notin \text{fv}(\rho_1)$, and we have $t = s[x/u] \xrightarrow{\rho_2} s_2[x/u] = t_2$ which is derived from $s \xrightarrow{\rho_2} s_2$ and $x \notin \text{fv}(\rho_2)$. We apply *i.h.* on s , yielding s' such that $s_1 \xrightarrow{\rho_2^*} s'$ and $s_2 \xrightarrow{\rho_1^*} s'$. Applying rule ESL° we derive $t_1 = s_1[x/u] \xrightarrow{\rho_2^*} s'[x/u] = t'$ and $t_2 = s_2[x/u] \xrightarrow{\rho_1^*} s'[x/u] = t'$ respectively. The following diagram summarizes the proof:

$$\begin{array}{ccc}
 t = s[x/u] & \xrightarrow{\rho_1^\circ} & s_1[x/u] = t_1 \\
 \downarrow \rho_2^\circ & & \downarrow \rho_2^\circ \\
 t_2 = s_2[x/u] & \xrightarrow{\rho_1^*} & s'[x/u] = t'
 \end{array}$$

5. ESL° - ESR° . We have $t = s[x/u] \xrightarrow{\rho_1} s_1[x/u] = t_1$, which is derived from (1) $s \xrightarrow{\rho_1} s_1$ and (2) $x \notin \text{fv}(\rho_1)$, and we have $t = s[x/u] \xrightarrow{\rho_2} s[x/u_2] = t_2$, which is derived from (3) $u \xrightarrow{\rho_2} u_2$. We can apply rule ESR° with (3) as premise, yielding $t_1 = s_1[x/u] \xrightarrow{\rho_2} s_1[x/u_2]$. On the other hand we can apply rule ESL° with (1) and (2) as premises, yielding $t_2 = s[x/u_2] \xrightarrow{\rho_1} s_1[x/u_2]$. The following diagram summarizes the proof:

$$\begin{array}{ccc}
 s[x/u] & \xrightarrow{\rho_1^\circ} & s_1[x/u] = t_1 \\
 \downarrow \rho_2^\circ & & \downarrow \rho_2^\circ \\
 t_2 = s[x/u_2] & \xrightarrow{\rho_1^\circ} & s_1[x/u_2] = t'
 \end{array}$$

6. ESR° - ESR° . We have $t = s[x/u] \xrightarrow{\rho_1} s[x/u_1] = t_1$, which is derived from $u \xrightarrow{\rho_1} u_1$; and we have $t = s[x/u] \xrightarrow{\rho_2} s[x/u_2] = t_2$ which is derived from $u \xrightarrow{\rho_2} u_2$. We apply *i.h.* on u , yielding u' such that $u_1 \xrightarrow{\rho_2^*} u'$ and $u_2 \xrightarrow{\rho_1^*} u'$. Applying rule ESR° we derive $t_1 = s[x/u_1] \xrightarrow{\rho_2^*} s[x/u'] = t'$ and $t_2 = s[x/u_2] \xrightarrow{\rho_1^*} s[x/u'] = t'$ respectively. The following diagram summarizes the proof:

$$\begin{array}{ccc}
 s[x/u] & \xrightarrow{\rho_1^\circ} & s[x/u_1] = t_1 \\
 \downarrow \rho_2^\circ & & \downarrow \rho_2^\circ \\
 t_2 = s[x/u_2] & \xrightarrow{\rho_1^*} & s[x/u'] = t'
 \end{array}$$

□

Observe that the second point of Lemma C.7 provides a *decreasing measure* for \rightarrow_σ reduction, and the previous lemma states that \rightarrow_σ is locally confluent. Hence:

THEOREM C.14. *The reduction relation \rightarrow_σ is terminating and confluent. In particular, a term t always has a unique \rightarrow_σ -normal form.*

PROOF. Termination is a straightforward consequence of Lemma C.7. Confluence is a consequence of the fact that \rightarrow_σ is also locally confluent (Proposition C.13 in Appendix C), and Newman's Lemma (*cf.* Section 2). Confluence trivially entails the uniqueness of normal forms. □

The following corollary relates the normal form of the reduction relation \rightarrow_σ and the unfolding of a term under a value assignment σ .

COROLLARY C.15. *For any term t and any value assignment σ , the (unique) σ -normal form of t is $t^{\downarrow\sigma}$.*

PROOF. By Theorem C.14, \rightarrow_σ is terminating so there exists a term s such that $t \rightarrow_\sigma^* s$ and s is in σ -normal form. By Lemma C.17(2) (see Appendix C) we have $t \rightarrow_\sigma^* t^{\downarrow\sigma}$. Moreover by Lemma C.17(1) $t^{\downarrow\sigma}$ is in σ -normal form. Therefore by confluence of \rightarrow_σ (Theorem C.14) we conclude $t^{\downarrow\sigma} = s$. \square

In particular, in the toplevel case, t^{\downarrow} is the (unique) lsv-normal form of t .

C.2 Relating LOCBV° and UOCBV^\bullet

LEMMA C.16. *Let vL be a term in σ -normal form and t be a term in $(\sigma \cup (x \mapsto v))$ -normal form. Then $t[x/v]L$ is in σ -normal form.*

PROOF. By induction on L .

- $L = \diamond$. There are three rules to evaluate $t[x/v]$. We argue that none of them apply.

- (1) If the term reduces by rule LSV° , then

$$\frac{t \xrightarrow{\text{sub}_{(x,v)}} t'}{t[x/v] \xrightarrow{\text{lsv}} t'[x/v]} \text{LSV}^\circ$$

but we reach a contradiction since t is in $(\sigma \cup (x \mapsto v))$ -normal form.

- (2) If the term reduces by rule ESL° , then, given a rule name ρ such that $\xrightarrow{\rho} \in \rightarrow_\sigma$,

$$\frac{t \xrightarrow{\rho} t' \quad x \notin \text{fv}(\rho)}{t[x/v] \xrightarrow{\rho} t'[x/v]} \text{ESL}^\circ$$

but we reach a contradiction since t is in $(\sigma \cup (x \mapsto v))$ -normal form.

- (3) If the term reduces by rule ESR° , then, given a rule name ρ such that $\xrightarrow{\rho} \in \rightarrow_\sigma$,

$$\frac{v \xrightarrow{\rho} v'}{t[x/v] \xrightarrow{\rho} t[x/v']} \text{ESR}^\circ$$

but we reach a contradiction since v is a value, and so it is in σ -normal form.

- $L = L'[y/s]$. There are three rules to evaluate $t[x/v]L'[y/s]$. We argue that none of them apply.

- (1) If the term reduces by rule LSV° , then $s = wL_1$ and

$$\frac{t[x/v]L' \xrightarrow{\text{sub}_{(y,w)}} r}{t[x/v]L'[y/wL_1] \xrightarrow{\text{lsv}} r[y/w]L_1} \text{LSV}^\circ$$

Since $vL'[y/wL_1]$ is in σ -normal form, then vL' is in $(\sigma \cup (y \mapsto w))$ -normal form. By α -conversion, $y \notin \text{fv}(t)$, so we also have that t is in $(\sigma \cup (x \mapsto v) \cup (y \mapsto w))$ -normal form. Then $t[x/v]L'$ is in $(\sigma \cup (y \mapsto w))$ -normal form by *i.h.* on L' , so we reach a contradiction. Hence $t[x/v]L$ is in σ -normal form.

- (2) If the term reduces by rule ESL° , then, given a rule name ρ such that $\xrightarrow{\rho} \in \rightarrow_\sigma$,

$$\frac{t[x/v]L' \xrightarrow{\rho} r \quad y \notin \text{fv}(\rho)}{t[x/v]L'[x/s] \xrightarrow{\rho} r[x/s]} \text{ESL}^\circ$$

since vL is in σ -normal form, then vL' is in $(\sigma \cup (y \mapsto w))$ -normal form. By α -conversion, $y \notin \text{fv}(t)$, so we also have that t is in $(\sigma \cup (x \mapsto v) \cup (y \mapsto w))$ -normal form. Then $t[x/v]L'$ is in $(\sigma \cup (y \mapsto w))$ -normal form by *i.h.* on L' , so we reach a contradiction. Hence $t[x/v]L$ is in σ -normal form.

- (3) If the term reduces by rule ESR° , then, given a rule name ρ such that $\xrightarrow{\rho} \in \rightarrow_\sigma$,

$$\frac{s \xrightarrow{\rho} s'}{t[x/v]L'[y/s] \xrightarrow{\rho} t[x/v]L'[y/s']} \text{ESR}^\circ$$

but we reach a contradiction since vL is in σ -normal form by hypothesis. \square

LEMMA C.17. *Let t be a term and σ a value assignment. Then*

1. $t^{\downarrow\sigma}$ is in σ -normal form.

$$2. t \rightarrow_{\sigma}^* t^{\downarrow\sigma}$$

PROOF.

1. By induction on t .

- $t = x$. There are two cases depending on the form of $x^{\downarrow\sigma}$:
 - If $x \in \text{dom}(\sigma)$ then $x^{\downarrow\sigma} = \sigma(x)$. If $\sigma(x)$ is a variable, then $\sigma(x)$ is in σ -normal form as there are no reduction rules in \rightarrow_{σ} to reduce $\sigma(x)$: $\sigma(x) \notin \text{dom}(\sigma)$ because σ is idempotent. If $\sigma(x)$ is an abstraction, then $\sigma(x)$ is in σ -normal form as there are no reduction rules to reduce abstractions.
 - Otherwise, $x \notin \text{dom}(\sigma)$ and $x^{\downarrow\sigma} = x$. We conclude x is in σ -normal form since there are no reduction rules in \rightarrow_{σ} to reduce x .
- $t = \lambda x. s$. Then $(\lambda x. s)^{\downarrow\sigma} = \lambda x. s$, which is in σ -normal form since there are no reduction rules in \rightarrow_{σ} to reduce an abstraction.
- $t = s u$. Then $(s u)^{\downarrow\sigma} = s^{\downarrow\sigma} u^{\downarrow\sigma}$. There are two rules that would allow us to reduce $s^{\downarrow\sigma} u^{\downarrow\sigma}$. We argue that neither applies:
 - We can apply *i.h.* on s , yielding $s^{\downarrow\sigma}$ is in σ -normal form, so $s^{\downarrow\sigma} u^{\downarrow\sigma}$ does not reduce via rule APPL° .
 - We can apply *i.h.* on u , yielding $u^{\downarrow\sigma}$ is in σ -normal form, so $s^{\downarrow\sigma} u^{\downarrow\sigma}$ does not reduce via rule APPR° .
- $t = s[x/u]$. There are two cases depending on the form of $s[x/u]^{\downarrow\sigma}$:
 - If $u^{\downarrow\sigma} = vL$ and $x \in \text{rv}(s)$, then $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma \cup (x \mapsto v)}[x/v]L$, which is in σ -normal form by the *i.h.* and by Lemma C.16.
 - Otherwise $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma}[x/u^{\downarrow\sigma}]$. Let us reason by contradiction, assuming that the term is σ -reducible. Firstly, if $s^{\downarrow\sigma}[x/u^{\downarrow\sigma}]$ reduces by rule ESR° , then it is also the case that $u^{\downarrow\sigma}$ is σ -reducible, therefore we reach a contradiction since $u^{\downarrow\sigma}$ is in σ -normal form by *i.h.* on u . Then the only way to reduce the whole term is by first reducing $s^{\downarrow\sigma}$, but we reach again a contradiction, since $s^{\downarrow\sigma}$ is in σ -normal form by *i.h.* on s . Then $s^{\downarrow\sigma}[x/u^{\downarrow\sigma}]$ is in σ -normal form.

2. By induction on t .

- $t = x$. There are two cases depending on the form of $x^{\downarrow\sigma}$:
 - If $x \in \text{dom}(\sigma)$ then $x^{\downarrow\sigma} = \sigma(x)$. We conclude $x \xrightarrow{\text{sub}(x, \sigma(x))}_{\sigma} \sigma(x)$ by rule SUB° .
 - Otherwise $x \rightarrow_{\sigma}^* x = x^{\downarrow\sigma}$, so we are done.
- $t = \lambda x. s$. Then $\lambda x. s \rightarrow_{\sigma}^* \lambda x. s = (\lambda x. s)^{\downarrow\sigma}$, so we are done.
- $t = s u$. Then $(s u)^{\downarrow\sigma} = s^{\downarrow\sigma} u^{\downarrow\sigma}$. We can apply *i.h.* on s , yielding $s \rightarrow_{\sigma}^* s^{\downarrow\sigma}$. Hence we have the reduction sequence $s u \rightarrow_{\sigma}^* s^{\downarrow\sigma} u$ in which each step is obtained by applying rule APPL° . Analogously, we can apply *i.h.* on u , yielding $u \rightarrow_{\sigma}^* u^{\downarrow\sigma}$. Hence we have the reduction sequence $s^{\downarrow\sigma} u \rightarrow_{\sigma}^* s^{\downarrow\sigma} u^{\downarrow\sigma}$ in which each step is obtained by applying rule APPR° . Then we can conclude $s u \rightarrow_{\sigma}^* s^{\downarrow\sigma} u^{\downarrow\sigma}$.
- $t = s[x/u]$. There are two cases depending on the form of $s[x/u]^{\downarrow\sigma}$:
 - If $u^{\downarrow\sigma} = vL$ and $x \in \text{rv}(s)$ then $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma \cup (x \mapsto v)}[x/v]L$. We can apply *i.h.* on u , yielding $u \rightarrow_{\sigma}^* u^{\downarrow\sigma}$. Hence we have the reduction sequence $s[x/u] \rightarrow_{\sigma}^* s[x/vL]$ in which each step is obtained by applying rule ESR° . On the other hand, we can apply *i.h.* on s , yielding $s \xrightarrow{\sigma \cup (x \mapsto v)}^* s^{\downarrow\sigma \cup (x \mapsto v)}$; we can then write this reduction sequence as $s \xrightarrow{\sigma}^* s' \rightarrow_{\sigma \cup (x \mapsto v)} s^{\downarrow\sigma \cup (x \mapsto v)}$, for some s' . Then $s[x/vL] \rightarrow_{\sigma}^* s'[x/vL]$ by rule ESL° , and $s'[x/vL] \rightarrow_{\sigma} s^{\downarrow\sigma \cup (x \mapsto v)}[x/v]L$ by rule LSV° . Then we can conclude $s[x/u] \rightarrow_{\sigma}^* s^{\downarrow\sigma \cup (x \mapsto v)}[x/v]L$.
 - Otherwise, $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma}[x/u^{\downarrow\sigma}]$. We can apply *i.h.* on s , yielding $s \rightarrow_{\sigma}^* s^{\downarrow\sigma}$. Hence we have the reduction sequence $s[x/u] \rightarrow_{\sigma}^* s^{\downarrow\sigma}[x/u]$ in which each step is obtained by applying rule ESL° , since x does not occur free in the rule name by α -conversion. On the other hand, we can apply *i.h.* on u , yielding $u \rightarrow_{\sigma}^* u^{\downarrow\sigma}$. Hence we have the reduction sequence $s^{\downarrow\sigma}[x/u] \rightarrow_{\sigma}^* s^{\downarrow\sigma}[x/u^{\downarrow\sigma}]$ in which each step is obtained by applying rule ESR° . Therefore we can conclude $s[x/u] \rightarrow_{\sigma}^* s^{\downarrow\sigma}[x/u^{\downarrow\sigma}]$.

□

Remark C.18. If t is of the form vL , then $t^{\downarrow\sigma}$ is of the form $v'L'$.

Remark C.19. If $t^{\downarrow\sigma} \in \text{Abs}$ then $t^{\downarrow\sigma'} \in \text{Abs}$, with $\text{dom}(\sigma) \subseteq \text{dom}(\sigma')$.

LEMMA C.20. Let σ_1, σ_2 be value assignments. Let t be a term.

1. $t^{\downarrow\sigma_1} \in \text{Val}$ if and only if $t^{\downarrow\sigma_2} \in \text{Val}$.

2. Let v be a value. If $t \in \text{St}_{\mathcal{S}}$ and $(x \downarrow^{\sigma_1} = x \downarrow^{\sigma_2} \text{ for all } x \in \mathcal{S})$, then there exists L_1 such that $t \downarrow^{\sigma_1} = v_{L_1}$ if and only if there exists L_2 such that $t \downarrow^{\sigma_2} = v_{L_2}$.

PROOF. We prove the two statements simultaneously.

1. We reason by induction on t , showing only the left-to-right implication since the other is similar.
 - $t = x$. There are two possible cases, depending on the form of $x \downarrow^{\sigma_2}$. If $x \downarrow^{\sigma_2} = \sigma_2(x)$, then we conclude $\sigma_2(x) \in \text{Val}$ by definition. Otherwise, $x \downarrow^{\sigma_2} = x$, and we conclude since $x \in \text{Val}$.
 - $t = \lambda x. s$. Immediate, since $(\lambda x. s) \downarrow^{\sigma_2} = \lambda x. s$ holds by definition.
 - $t = s u$. This case is not possible since there are no v_1, L_1 such that $(t s) \downarrow^{\sigma_1} = s \downarrow^{\sigma_1} u \downarrow^{\sigma_1} = v_1 L_1$.
 - $t = s[x/u]$. We analyze two cases, depending on the form of $s[x/u] \downarrow^{\sigma_1}$.
 - If $u \downarrow^{\sigma_1} = v_a L_a$ and $x \in \text{rv}(s)$, then $s[x/u] \downarrow^{\sigma_1} = s \downarrow^{\sigma_1 \cup (x \mapsto v_a)} [x/v_a] L_a$. Thus by hypothesis $s \downarrow^{\sigma_1 \cup (x \mapsto v_a)} \in \text{Val}$. By *i.h.* (1) on u , there exist v_b and L_b such that $u \downarrow^{\sigma_2} = v_b L_b$. We are then in the case where $s[x/u] \downarrow^{\sigma_2} = s \downarrow^{\sigma_2 \cup (x \mapsto v_b)} [x/v_b] L_b$. We can apply *i.h.* (1) on s , yielding $s \downarrow^{\sigma_2 \cup (x \mapsto v_b)} \in \text{Val}$. Therefore $s[x/u] \downarrow^{\sigma_2} \in \text{Val}$.
 - Otherwise, $s[x/u] \downarrow^{\sigma_1} = s \downarrow^{\sigma_1} [x/u \downarrow^{\sigma_1}]$. Thus by hypothesis $s \downarrow^{\sigma_1} \in \text{Val}$. If $x \notin \text{rv}(s)$, then we are in the case $s[x/u] \downarrow^{\sigma_2} = s \downarrow^{\sigma_2} [x/u \downarrow^{\sigma_2}]$. If $\neg u \downarrow^{\sigma_1} \in \text{Val}$, then the *i.h.* (1) on u states $\neg u \downarrow^{\sigma_2} \in \text{Val}$, and so we are also in the case $s[x/u] \downarrow^{\sigma_2} = s \downarrow^{\sigma_2} [x/u \downarrow^{\sigma_2}]$. Hence, we can apply *i.h.* (1) on s , yielding $s \downarrow^{\sigma_2} \in \text{Val}$. Therefore $s[x/u] \downarrow^{\sigma_2} \in \text{Val}$.
2. By induction on the derivation of the judgment $t \in \text{St}_{\mathcal{S}}$, showing only the left-to-right implication since the other is similar.
 - S-VAR. Then $t = y \in \text{St}_{\mathcal{S}}$, with $y \in \mathcal{S}$. By hypothesis $y \downarrow^{\sigma_1} = v_{L_1}$ and $y \downarrow^{\sigma_2} = y \downarrow^{\sigma_2}$. Hence we conclude $y \downarrow^{\sigma_2} = v_{L_2}$ with $L_2 = L_1$.
 - S-APP. Then $t = s u \in \text{St}_{\mathcal{S}}$, which is derived from $s \in \text{St}_{\mathcal{S}}$. This case is not possible since there are no v, L_1 such that $(s u) \downarrow^{\sigma_1} = s \downarrow^{\sigma_1} u \downarrow^{\sigma_1} = v_{L_1}$.
 - S-SUB₁. Then

$$\frac{s \in \text{St}_{\mathcal{S}} \quad y \notin \mathcal{S}}{t = s[y/u] \in \text{St}_{\mathcal{S}}} \text{S-SUB}_1$$

By hypothesis $s[y/u] \downarrow^{\sigma_1} = v_{L_1}$. We analyze two cases depending on the form of the unfolding:

- If $u \downarrow^{\sigma_1} = v_a L_a$ and $y \in \text{rv}(s)$, then $s[y/u] \downarrow^{\sigma_1} = s \downarrow^{\sigma_1 \cup (y \mapsto v_a)} [y/v_a] L_a$. Thus by hypothesis there exists L_c such that $s \downarrow^{\sigma_1 \cup (y \mapsto v_a)} = v_{L_c}$, where $L_1 = L_c[y/v_a] L_a$. By *i.h.* (1) on u , there exist v_b, L_b such that $u \downarrow^{\sigma_2} = v_b L_b$. Hence we are in the case $s[y/u] \downarrow^{\sigma_2} = s \downarrow^{\sigma_2 \cup (y \mapsto v_b)} [y/v_b] L_b$. We are still in the case $x \downarrow^{\sigma_1 \cup (y \mapsto v_a)} = x \downarrow^{\sigma_2 \cup (y \mapsto v_b)}$ for all $x \in \mathcal{S}$, since $y \notin \mathcal{S}$. We can then apply *i.h.* (2) on s , yielding L_d such that $s \downarrow^{\sigma_2 \cup (y \mapsto v_b)} = v_{L_d}$. Therefore $s[y/u] \downarrow^{\sigma_2} = v_{L_2}$, where $L_2 = L_d[y/v_b] L_b$.
- Otherwise, $s[y/u] \downarrow^{\sigma_1} = s \downarrow^{\sigma_1} [y/u \downarrow^{\sigma_1}]$. By hypothesis there exists L_c such that $s \downarrow^{\sigma_1} = v_{L_c}$, where $L_1 = L_c[y/u \downarrow^{\sigma_1}]$. If $x \notin \text{rv}(s)$, then we are in the case where $s[x/u] \downarrow^{\sigma_2} = s \downarrow^{\sigma_2} [x/u \downarrow^{\sigma_2}]$. If $\neg u \downarrow^{\sigma_1} \in \text{Val}$, then the *i.h.* (1) on u states $\neg u \downarrow^{\sigma_2} \in \text{Val}$ and so we are also in the case $s[x/u] \downarrow^{\sigma_2} = s \downarrow^{\sigma_2} [x/u \downarrow^{\sigma_2}]$. Hence, we can apply *i.h.* (2) on s , yielding L_d such that $s \downarrow^{\sigma_2} = v_{L_d}$. Therefore $s[y/u] \downarrow^{\sigma_2} = v_{L_2}$, where $L_2 = L_d[y/u \downarrow^{\sigma_2}]$.
- S-SUB₂. Then

$$\frac{s \in \text{St}_{\mathcal{S} \cup \{y\}} \quad y \notin \mathcal{S} \quad u \in \text{St}_{\mathcal{S}}}{t = s[y/u] \in \text{St}_{\mathcal{S}}} \text{S-SUB}_2$$

By hypothesis, there exists L_1 such that $s[y/u] \downarrow^{\sigma_1} = v_{L_1}$. We analyze two cases depending on the form of the unfolding under σ_1 :

- If $u \downarrow^{\sigma_1} = v_a L_a$ and $x \in \text{rv}(s)$, then $v_{L_1} = s[y/u] \downarrow^{\sigma_1} = s \downarrow^{\sigma_1 \cup (y \mapsto v_a)} [y/v_a] L_a$. Thus by hypothesis it must be the case that $s \downarrow^{\sigma_1 \cup (y \mapsto v_a)} = v_{L_c}$, with $L_1 = L_c[y/v_a] L_a$, for some L_c . By *i.h.* (2) on u , there exists L_b such that $u \downarrow^{\sigma_2} = v_a L_b$. Moreover, $x \in \text{rv}(s)$, so $s[y/u] \downarrow^{\sigma_2} = s \downarrow^{\sigma_2 \cup (y \mapsto v_a)} [y/v_a] L_b$. On the other hand, $x \downarrow^{\sigma_1 \cup (y \mapsto v_a)} = x \downarrow^{\sigma_2 \cup (y \mapsto v_a)}$ for all $x \in \mathcal{S} \cup \{y\}$. We then apply *i.h.* (2) on s , yielding $L_{c'}$ such that $s \downarrow^{\sigma_2 \cup (y \mapsto v_a)} = v_{L_{c'}}$. Hence we conclude $t \downarrow^{\sigma_2} = s \downarrow^{\sigma_2 \cup (y \mapsto v_a)} L_{c'} [y/v_a] L_b$, where $L_2 = L_{c'} [y/v_a] L_b$.
- Otherwise $v_{L_1} = s[y/u] \downarrow^{\sigma_1} = s \downarrow^{\sigma_1} [y/u \downarrow^{\sigma_1}]$. Thus by hypothesis it must be the case that $s \downarrow^{\sigma_1} = v_{L_c}$, with $L_1 = L_c[y/u \downarrow^{\sigma_1}]$, for some L_c . If $x \notin \text{rv}(s)$, then we are in the case where $s[x/u] \downarrow^{\sigma_2} = s \downarrow^{\sigma_2} [x/u \downarrow^{\sigma_2}]$. If $\neg u \downarrow^{\sigma_1} \in \text{Val}$, then the *i.h.* (1) on u states $\neg u \downarrow^{\sigma_2} \in \text{Val}$, and so we are also in the case $s[x/u] \downarrow^{\sigma_2} = s \downarrow^{\sigma_2} [x/u \downarrow^{\sigma_2}]$. We then apply *i.h.* (2) on s , yielding $L_{c'}$ such that $s \downarrow^{\sigma_2} = v_{L_{c'}}$. Hence we conclude $t \downarrow^{\sigma_2} = s \downarrow^{\sigma_2} L_{c'} [y/u \downarrow^{\sigma_2}]$, where $L_2 = L_{c'} [y/u \downarrow^{\sigma_2}]$.

□

Definition C.21 (Compatibility). Given a value assignment σ , we say σ is **compatible with** the sets of variables \mathcal{A} and \mathcal{S} , written $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S})$ if the following three conditions hold:

1. The value assignment must affect all variables in \mathcal{A} and some of the variables in \mathcal{S} (i.e. $\mathcal{A} \subseteq \text{dom}(\sigma) \subseteq \mathcal{A} \cup \mathcal{S}$).
2. Variables in \mathcal{A} must be mapped to abstractions (i.e. $\sigma(x) \in \text{Abs}$ must hold for every $x \in \mathcal{A}$).
3. Variables in \mathcal{S} affected by σ must be mapped to variables (i.e. $\sigma(x)$ must be a variable for every $x \in \mathcal{S} \cap \text{dom}(\sigma)$).

LEMMA C.22. Let $t[x/s]$, with $\text{inv}(\mathcal{A}, \mathcal{S}, t[x/s])$ and $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S})$.

1. If $s \in \text{HA}_{\mathcal{A}}$ and $s^{\downarrow\sigma} = v\text{L}$, then $\text{compatible}(\sigma \cup (x \mapsto v), \mathcal{A} \cup \{x\}, \mathcal{S})$.
2. If $s \in \text{St}_{\mathcal{S}}$, then $\text{compatible}(\sigma', \mathcal{A}, \mathcal{S} \cup \{x\})$, where $\sigma' = \sigma \cup (x \mapsto v)$ if $x \in \text{rv}(t)$ and $s^{\downarrow\sigma}$ is of the form $v\text{L}$ for some v, L , or $\sigma' = \sigma$ otherwise.

PROOF. Since $\text{inv}(\mathcal{A}, \mathcal{S}, t[x/s])$ then $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, t)$, $\text{inv}(\mathcal{A}, \mathcal{S} \cup \{x\}, t)$ and $\text{inv}(\mathcal{A}, \mathcal{S}, s)$. We prove each item independently.

1. We check that conditions of the definition of $\text{compatible}(\sigma \cup (x \mapsto v), \mathcal{A} \cup \{x\}, \mathcal{S})$ holds:
 - (a) Since $\mathcal{A} \subseteq \text{dom}(\sigma) \subseteq \mathcal{A} \cup \mathcal{S}$ by the hypothesis $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S})$, then $\mathcal{A} \cup \{x\} \subseteq \text{dom}(\sigma) \cup \{x\} \subseteq (\mathcal{A} \cup \{x\}) \cup \mathcal{S}$.
 - (b) Let $y \in \mathcal{A} \cup \{x\}$. If $y \neq x$, then $(\sigma \cup (x \mapsto v))(y) = \sigma(y)$, and $\sigma(y) \in \text{Abs}$ holds by the hypothesis $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S})$. Otherwise $y = x$, and $v\text{L} \in \text{Abs}$ by Lemma C.23 and thus $v \in \text{Abs}$ holds.
 - (c) Let $y \in \mathcal{S} \cap \text{dom}(\sigma \cup (x \mapsto v))$. Since $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, t)$ then $x \notin \mathcal{S}$, therefore $x \notin \mathcal{S} \cap \text{dom}(\sigma \cup (x \mapsto v))$. Then $x \neq y$ and thus the property holds by the hypothesis $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S})$.
2. We check that conditions of the definition of $\text{compatible}(\sigma', \mathcal{A}, \mathcal{S} \cup \{x\})$ holds:
 - (a) Since $\mathcal{A} \subseteq \text{dom}(\sigma) \subseteq \mathcal{A} \cup \mathcal{S}$ by the hypothesis $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S})$, then $\mathcal{A} \subseteq \text{dom}(\sigma') \subseteq \mathcal{A} \cup (\mathcal{S} \cup \{x\})$.
 - (b) Let $y \in \mathcal{A}$. Since $\text{inv}(\mathcal{A}, \mathcal{S} \cup \{x\}, t)$ implies $x \notin \mathcal{A}$, then $x \neq y$ so $\sigma'(y) = \sigma(y)$ and thus $\sigma'(y) \in \text{Abs}$ by the hypothesis $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S})$.
 - (c) Let $y \in (\mathcal{S} \cup \{x\}) \cap \text{dom}(\sigma')$. If $y \neq x$, then $\sigma'(y) = \sigma(y)$ and thus $\sigma'(y)$ is a variable by the hypothesis $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S})$. If $y = x$, there are two subcases, depending on whether $x \in \text{rv}(s)$. If $x \in \text{rv}(s)$, then $\sigma' = \sigma \cup (x \mapsto v)$, so we have that $\sigma'(y) = v$, and we are in the case $s^{\downarrow\sigma} = v\text{L}$, where v is a variable by Lemma C.23 (2). If $x \notin \text{rv}(s)$, then $\sigma' = \sigma$, so $\sigma'(y) = \sigma(x)$. Note that, by α -conversion, we may assume that $x \notin \text{dom}(\sigma)$, so $\sigma(x) = x$, which is a variable.

□

LEMMA C.23. Let $\text{inv}(\mathcal{A}, \mathcal{S}, t)$. Let σ be a value assignment, and \mathcal{A} and \mathcal{S} sets of variables such that $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S})$ holds. Then:

1. If $t \in \text{HA}_{\mathcal{A}}$ then $t^{\downarrow\sigma} \in \text{Abs}$.
2. If $t \in \text{St}_{\mathcal{S}}$ and $t^{\downarrow\sigma} = v\text{L}$, then v is a variable.

PROOF. We prove each item independently.

1. By induction on the derivation of $t \in \text{HA}_{\mathcal{A}}$.
 - 1.1 H-VAR. Then $t = x \in \text{HA}_{\mathcal{A}}$, with $x \in \mathcal{A}$. Since $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S})$ then $x^{\downarrow\sigma} = \sigma(x)$, and $\sigma(x) \in \text{Abs}$.
 - 1.2 H-LAM. Then $t = \lambda x. s$, and $(\lambda x. s)^{\downarrow\sigma} = \lambda x. s$, which satisfies the predicate abs .
 - 1.3 H-SUB₁. Then $t = s[x/u] \in \text{HA}_{\mathcal{A}}$ which is derived from $s \in \text{HA}_{\mathcal{A}}$ and $x \notin \mathcal{A}$. Since $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ then $\text{inv}(\mathcal{A}, \mathcal{S} \cup \{x\}, s)$ holds. We proceed by showing that $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S} \cup \{x\})$ holds:
 - (a) Since $\mathcal{A} \subseteq \text{dom}(\sigma) \subseteq \mathcal{A} \cup \mathcal{S}$ by the hypothesis $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S})$, then $\mathcal{A} \subseteq \text{dom}(\sigma) \subseteq \mathcal{A} \cup (\mathcal{S} \cup \{x\})$.
 - (b) Let $y \in \mathcal{A}$. Since $\text{inv}(\mathcal{A}, \mathcal{S} \cup \{x\}, s)$ implies $x \notin \mathcal{A}$, then $x \neq y$ and thus $\sigma(y) \in \text{Abs}$ by the hypothesis $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S})$.
 - (c) Let $y \in (\mathcal{S} \cup \{x\}) \cap \text{dom}(\sigma)$. Since $x \notin \text{dom}(\sigma)$ by α -conversion then $y \neq x$. Thus $\sigma(y)$ is a variable by the hypothesis $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S})$.
We can now apply *i.h.* (1) on s , yielding $s^{\downarrow\sigma} \in \text{Abs}$. We analyze two cases:
 - If $u^{\downarrow\sigma} = v\text{L}$ and $x \in \text{rv}(s)$, then $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma \cup (x \mapsto v)}[x/v]\text{L}$. Since $s^{\downarrow\sigma} \in \text{Abs}$ by *i.h.* (1) on s , then $s^{\downarrow\sigma \cup (x \mapsto v)} \in \text{Abs}$ by Remark C.19. Hence we conclude that $s^{\downarrow\sigma \cup (x \mapsto v)}[x/v]\text{L} \in \text{Abs}$ holds.
 - Otherwise, $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma}[x/u^{\downarrow\sigma}]$. Since $s^{\downarrow\sigma} \in \text{Abs}$ by *i.h.* (1) on s , then $s^{\downarrow\sigma}[x/u^{\downarrow\sigma}] \in \text{Abs}$.
 - 1.4 H-SUB₂. Then $t = s[x/u] \in \text{HA}_{\mathcal{A}}$ which is derived from $s \in \text{HA}_{\mathcal{A} \cup \{x\}}$, $x \notin \mathcal{A}$, and $u \in \text{HA}_{\mathcal{A}}$. Since $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ then $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)$ and $\text{inv}(\mathcal{A}, \mathcal{S}, u)$. Moreover, $u \in \text{Val}$ holds by Remark 4.1, so $u^{\downarrow\sigma} = v\text{L}$ by Remark C.18. Thus we analyze two cases, depending on whether $x \in \text{rv}(s)$ or not:

- If $x \in \text{rv}(s)$, then $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma \cup (x \mapsto v)}[x/v]\text{L}$. Moreover, $\text{compatible}(\sigma \cup (x \mapsto v), \mathcal{A} \cup \{x\}, \mathcal{S})$ holds by Lemma C.22 (1). Then $s^{\downarrow\sigma \cup (x \mapsto v)} \in \text{Abs}$ by *i.h.* (1) on s . Therefore we conclude $s^{\downarrow\sigma \cup (x \mapsto v)}[x/v]\text{L} \in \text{Abs}$.
 - If $x \notin \text{rv}(s)$, then $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma}[x/u^{\downarrow\sigma}]$. Moreover, $x \notin \text{rv}(s)$ implies $s^{\downarrow\sigma} = s^{\downarrow\sigma \cup (x \mapsto v)}$. Similarly to the previous case, we obtain $s^{\downarrow\sigma \cup (x \mapsto v)} \in \text{Abs}$ by *i.h.* (1) on s , thus having $s^{\downarrow\sigma} \in \text{Abs}$, so that we can conclude $s^{\downarrow\sigma}[x/u^{\downarrow\sigma}] \in \text{Abs}$ as well.
2. By induction on the derivation of $t \in \text{St}_{\mathcal{S}}$.
- 2.1 s-VAR. Then $t = x \in \text{St}_{\mathcal{S}}$, with $x \in \mathcal{S}$. There are two possible cases, depending on whether $x \in \text{dom}(\sigma)$ or not:
- 2.1.1 If $x \in \text{dom}(\sigma)$, then $x^{\downarrow\sigma} = \sigma(x)$. Since $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S})$, then $\sigma(x)$ is a variable.
- 2.1.2 Otherwise, $x^{\downarrow\sigma} = x$, which has the form $v\text{L}$ with $\text{L} = \diamond$ and v a variable.
- 2.2 s-APP. Then $t = s u \in \text{St}_{\mathcal{S}}$. This case is not possible, given that $(s u)^{\downarrow\sigma} = s^{\downarrow\sigma} u^{\downarrow\sigma}$ is not of the form $v\text{L}$.
- 2.3 s-SUB₁. Then $t = s[x/u] \in \text{St}_{\mathcal{S}}$ which is derived from $s \in \text{St}_{\mathcal{S}}$ and $x \notin \mathcal{S}$. We analyze two cases:
- 2.3.1 If $u^{\downarrow\sigma} = v_a\text{L}_a$ and $x \in \text{rv}(s)$, then $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma \cup (x \mapsto v_a)}[x/v_a]\text{L}_a$. Since $t^{\downarrow\sigma}$ is of the form $v\text{L}$ then so is $s^{\downarrow\sigma \cup (x \mapsto v_a)}$. Then there exist L_1 such that $s^{\downarrow\sigma} = v\text{L}_1$ by Lemma C.20 (2), given that $x \notin \mathcal{S}$ by hypothesis so that $y^{\downarrow\sigma \cup (x \mapsto v_a)} = y^{\downarrow\sigma}$ for all $y \in \mathcal{S}$. We can then apply *i.h.* (2) on s , and conclude that v is a variable.
- 2.3.2 Otherwise, $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma}[x/u^{\downarrow\sigma}]$. Since $t^{\downarrow\sigma}$ is of the form $v\text{L}$ then so is $s^{\downarrow\sigma}$. Moreover, $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S} \cup \{x\})$ holds, as shown in (1), case H-SUB₁, and $s \in \text{St}_{\mathcal{S} \cup \{x\}}$ by Remark 4.1. Given that $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ implies $\text{inv}(\mathcal{A}, \mathcal{S} \cup \{x\}, s)$, we can then apply *i.h.* (2) on s , yielding that v is a variable.
- 2.4 s-SUB₂. Then $t = s[x/u] \in \text{St}_{\mathcal{S}}$, which is derived from $s \in \text{St}_{\mathcal{S} \cup \{x\}}$, $x \notin \mathcal{S}$ and $u \in \text{St}_{\mathcal{S}}$. Since $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ then in particular $\text{inv}(\mathcal{A}, \mathcal{S} \cup \{x\}, s)$. Moreover, $\text{compatible}(\sigma', \mathcal{A}, \mathcal{S} \cup \{x\})$ holds by Lemma C.22 (2), where $\sigma' = \sigma \cup (x \mapsto v_a)$ if $x \in \text{rv}(s)$ and $u^{\downarrow\sigma} = v_a\text{L}_a$, for some v_a, L_a , or $\sigma' = \sigma$ otherwise. Since $t^{\downarrow\sigma}$ is of the form $v\text{L}$ then so is $s^{\downarrow\sigma'}$. We can then apply *i.h.* (2) on s , yielding that v is a variable.

□

PROPOSITION C.24. *Let t be a term, σ a value assignment, and \mathcal{A}, \mathcal{S} sets of variables. Suppose $\text{inv}(\mathcal{A}, \mathcal{S}, t)$ and $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S})$ hold. Then:*

1. If $t \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$ then $t^{\downarrow\sigma} \in \text{NF}_{\text{dom}(\sigma), \mu}^{\circ}$.
2. If $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'$ then either there exists a term t'' such that $t^{\downarrow\sigma} \xrightarrow{\text{db}} t''$, or $t^{\downarrow\sigma} \in \text{Abs}$ and $\mu = @$.

PROOF. We prove each item independently.

1. By induction on the derivation of the judgment $t \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$. The interesting cases are NF-VAR[•], NF-ESA[•], and NF-ES[•].
 - NF-VAR[•]. Then $t = x$, and $x \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$ is derived from $x \in \mathcal{A} \Rightarrow \mu = @$. Remark that $\text{inv}(\mathcal{A}, \mathcal{S}, t)$ implies $\text{fv}(x) = \{x\} \subseteq \mathcal{A} \cup \mathcal{S}$ and $\mathcal{A} \# \mathcal{S}$. We then analyze two possible cases:
 - 1.1 $x \in \mathcal{A}$. Then $\mu = @$. Since $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S})$, then $x \in \text{dom}(\sigma)$, and $x^{\downarrow\sigma}$ is an abstraction. Applying rule NF-LAM[◦] we conclude $x^{\downarrow\sigma} \in \text{NF}_{\text{dom}(\sigma), @}^{\circ}$.
 - 1.2 $x \in \mathcal{S}$. Then $x \in \text{dom}(\sigma)$ or not. If $x \in \text{dom}(\sigma)$ then $x \in \text{dom}(\sigma) \cap \mathcal{S}$, and $x^{\downarrow\sigma} = y$, $x \neq y$. By idempotency of value assignments we have $y \notin \text{dom}(\sigma)$, so $y \in \text{NF}_{\text{dom}(\sigma), \mu}^{\circ}$ by rule NF-VAR[◦]. If $x \notin \text{dom}(\sigma)$, then we conclude $x^{\downarrow\sigma} = x \in \text{NF}_{\text{dom}(\sigma), \mu}^{\circ}$ by rule NF-VAR[◦].
 - NF-ESA[•]. Then

$$\frac{s \in \text{NF}_{\mathcal{A} \cup \{x\}, \mathcal{S}, \mu}^{\bullet} (1) \quad u \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^{\bullet} (2) \quad u \in \text{HA}_{\mathcal{A}} (3)}{t = s[x/u] \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}} \text{NF-ESA}^{\bullet}$$

Since (3), then $u \in \text{Val}$ by Remark 4.1, and thus (4) $u^{\downarrow\sigma} = v\text{L}$ by Remark C.18, for some v and L . Moreover, $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ implies $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)$ and $\text{inv}(\mathcal{A}, \mathcal{S}, u)$; and given (3) and (4), we obtain $\text{compatible}(\sigma \cup (x \mapsto v), \mathcal{A} \cup \{x\}, \mathcal{S})$ by Lemma C.22 (1). Given (2), we apply *i.h.* on u , yielding $u^{\downarrow\sigma} = v\text{L} \in \text{NF}_{\text{dom}(\sigma), @}^{\circ}$. Given (1), we apply *i.h.* on s , yielding $s^{\downarrow\sigma \cup (x \mapsto v)} \in \text{NF}_{\text{dom}(\sigma) \cup \{x\}, \mu}^{\circ}$. There are two possible cases, depending on whether $x \in \text{rv}(s)$ or not:

- 1.1 If $x \in \text{rv}(s)$, then $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma \cup (x \mapsto v)}[x/v]\text{L}$. We obtain (5) $v \in \text{NF}_{\text{dom}(\sigma)^{\perp}, @}^{\circ}$ and $\text{L} \in \text{CtxNF}_{\text{dom}(\sigma)}^{\circ}$ by Lemma C.2. We may assume $\text{dom}(\text{L}) \# \text{fv}(s^{\downarrow\sigma \cup (x \mapsto v)})$ by α -conversion, so $\text{dom}(\text{L}) \# \text{rv}(s^{\downarrow\sigma \cup (x \mapsto v)})$. Hence we can apply Lemma A.1, yielding (6) $s^{\downarrow\sigma \cup (x \mapsto v)} \in \text{NF}_{\text{dom}(\sigma)^{\perp} \cup \{x\}, \mu}^{\circ}$. Applying rule NF-ESVAL[◦] with (6) and (5) as premises, we obtain that $s^{\downarrow\sigma \cup (x \mapsto v)}[x/v] \in \text{NF}_{\text{dom}(\sigma)^{\perp}, \mu}^{\circ}$. We conclude $t^{\downarrow\sigma} = s^{\downarrow\sigma \cup (x \mapsto v)}[x/v]\text{L} \in \text{NF}_{\text{dom}(\sigma), \mu}^{\circ}$ by Lemma C.2.

1.2 If $x \notin \text{rv}(s)$, then $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma}[x/u^{\downarrow\sigma}]$. Since $x \notin \text{rv}(s)$ then $s^{\downarrow\sigma \cup (x \mapsto v)} = s^{\downarrow\sigma}$. We conclude $t^{\downarrow\sigma} = s^{\downarrow\sigma}[x/u^{\downarrow\sigma}] \in \text{NF}_{\text{dom}(\sigma), \mu}^{\circ}$ by rule NF-ESVAL° .

- NF-ES^{\bullet} . Then

$$\frac{s \in \text{NF}_{\mathcal{A}, \mathcal{S} \cup \{x\}, \mu}^{\bullet} \quad (1) \quad u \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^{\bullet} \quad (2) \quad u \in \text{St}_{\mathcal{S}} \quad (3)}{s[x/u] \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}} \text{NF-ES}^{\bullet}$$

There are two possible cases, depending on the form of $s[x/u]^{\downarrow\sigma}$:

– If $u^{\downarrow\sigma} = vL$ and $x \in \text{rv}(s)$, then $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma \cup (x \mapsto v)}[x/v]L$, and the proof is the analogous as case 1.1 in rule NF-ESA^{\bullet} .

– Otherwise, $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma}[x/u^{\downarrow\sigma}]$. There are two subcases, depending on whether $u^{\downarrow\sigma} \in \text{Val}$ or not. If $u^{\downarrow\sigma} \in \text{Val}$ then the proof is analogous, yet simpler, then case 1.2 in rule NF-ESA^{\bullet} . Otherwise, $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ implies $\text{inv}(\mathcal{A}, \mathcal{S} \cup \{x\}, s)$ and $\text{inv}(\mathcal{A}, \mathcal{S}, u)$. Given (2) we can apply *i.h.* on u , yielding (4) $u^{\downarrow\sigma} \in \text{NF}_{\text{dom}(\sigma), @}^{\circ}$. Moreover, given (3), we have that $\text{compatible}(\sigma, \mathcal{A}, \mathcal{S} \cup \{x\})$ holds by Lemma C.22 (2). Given (1) we can apply *i.h.* on s , yielding (5) $s^{\downarrow\sigma} \in \text{NF}_{\text{dom}(\sigma), \mu}^{\circ}$. Applying rule $\text{NF-ESNONVAL}^{\circ}$ with (4) and (5) as premises, we obtain $t^{\downarrow\sigma} = s^{\downarrow\sigma}[x/u^{\downarrow\sigma}] \in \text{NF}_{\text{dom}(\sigma), \mu}^{\circ}$.

2. By induction on the derivation of $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'$.

- SUB^{\bullet} . Then $t = x \xrightarrow{\text{sub}_{(x,v)}, \mathcal{A}' \cup \{x\}, \mathcal{S}, @} v = t'$, where $\rho = \text{sub}_{(x,v)}$, $\mathcal{A} = \mathcal{A}' \cup \{x\}$ and $\mu = @$. Since $x \in \mathcal{A}' \cup \{x\}$ and $\text{compatible}(\sigma, \mathcal{A}' \cup \{x\}, \mathcal{S})$, then $\sigma(x) \in \text{Abs}$ (i.e. $x^{\downarrow\sigma} \in \text{Abs}$).

- DB^{\bullet} . Then $(\lambda x. s)L u \xrightarrow{\text{db}, \mathcal{A}, \mathcal{S}, \mu} s[x/u]L$, where $t = (\lambda x. s)L u$, $t' = s[x/u]L$ and $\rho = \text{db}$. We have $((\lambda x. s)L u)^{\downarrow\sigma} = (\lambda x. s)L^{\downarrow\sigma} u^{\downarrow\sigma}$. Moreover, $(\lambda x. s)L \in \text{HA}_{\mathcal{A}}$ by Remark 4.1, then $(\lambda x. s)L^{\downarrow\sigma} \in \text{Abs}$ by Lemma C.23 (1), so $(\lambda x. s)L^{\downarrow\sigma} = (\lambda x. s)L'$. Then $((\lambda x. s)L') u^{\downarrow\sigma} \xrightarrow{\text{db}} s[x/u^{\downarrow\sigma}]L'$ by rule DB° .

- LSV^{\bullet} . Then

$$\frac{s \xrightarrow{\text{sub}_{(x,v)}, \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s' \quad x \notin \mathcal{A} \cup \mathcal{S} \quad vL \in \text{HA}_{\mathcal{A}}}{t = s[x/v]L \xrightarrow{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} s'[x/v]L = t'} \text{LSV}^{\bullet}$$

where $\rho = \text{lsv}$. Since $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/v]L)$ then in particular $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, vL)$. We then have $(vL)^{\downarrow\sigma} \in \text{Abs}$ by Lemma C.23, and $(vL)^{\downarrow\sigma}$ is of the form $v'L'$ by Remark C.18. Moreover, $\text{compatible}(\sigma \cup (x \mapsto v'), \mathcal{A} \cup \{x\}, \mathcal{S})$ by Lemma C.22 (1). We can apply *i.h.* on s , yielding two possible cases:

2.1 There exists s'' such that $s^{\downarrow\sigma \cup (x \mapsto v')} \xrightarrow{\text{db}} s''$. We analyze two different cases, depending on whether $x \in \text{rv}(s)$ or not:

- If $x \in \text{rv}(s)$, then $s[x/v]L^{\downarrow\sigma} = s^{\downarrow\sigma \cup (x \mapsto v')}[x/v']L' \xrightarrow{\text{db}} s''[x/v']L'$ by applying (length of $L' + 1$) times rule ESL° , so we are done.
- If $x \notin \text{rv}(s)$, then $s[x/v]L^{\downarrow\sigma} = s^{\downarrow\sigma}[x/v]L' = s^{\downarrow\sigma \cup (x \mapsto v')}[x/(vL)]^{\downarrow\sigma} \xrightarrow{\text{db}} s''[x/v]L' = t'$ by rule ESL° .

2.2 $s^{\downarrow\sigma \cup (x \mapsto v')} \in \text{Abs}$ and $\mu = @$. Then $s[x/v]L^{\downarrow\sigma} \in \text{Abs}$, and $\mu = @$, so we are done.

- APPL^{\bullet} . Then

$$\frac{s \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, @} s'}{t = s u \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} s' u = t'} \text{APPL}^{\bullet}$$

Since $\text{inv}(\mathcal{A}, \mathcal{S}, s u)$ then in particular $\text{inv}(\mathcal{A}, \mathcal{S}, s)$. We have two possible cases by *i.h.* on s :

- 2.1 There exists s'' such that $s^{\downarrow\sigma} \xrightarrow{\text{db}} s''$. Then $s^{\downarrow\sigma} u^{\downarrow\sigma} \xrightarrow{\text{db}} s'' u^{\downarrow\sigma} = t'$ by rule APPL° .
- 2.2 $s^{\downarrow\sigma} \in \text{Abs}$ and $\mu = @$. Then $s^{\downarrow\sigma}$ is of the form $(\lambda x. r)L$, and $(\lambda x. r)L u^{\downarrow\sigma} \xrightarrow{\text{db}} r[x/u^{\downarrow\sigma}]L$ by rule DB° .

- APPR^{\bullet} . Then

$$\frac{s \in \text{St}_{\mathcal{S}} \quad u \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, @} u'}{t = s u \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} s u' = t'} \text{APPR}^{\bullet}$$

Since $\text{inv}(\mathcal{A}, \mathcal{S}, s u)$ then in particular $\text{inv}(\mathcal{A}, \mathcal{S}, u)$. We have two possible cases by *i.h.* on u :

- 2.1 There exists u'' such that $u^{\downarrow\sigma} \xrightarrow{\text{db}} u''$. Then $s^{\downarrow\sigma} u^{\downarrow\sigma} \xrightarrow{\text{db}} s^{\downarrow\sigma} u'' = t'$ by rule APPR° .
- 2.2 $u^{\downarrow\sigma} \in \text{Abs}$ and $\mu = @$. This case is not possible since $\mu = @$ by premise of rule APPR^{\bullet} .

- ESR^{\bullet} . Then

$$\frac{u \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, @} u'}{t = s[x/u] \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} s[x/u'] = t'} \text{ESR}^{\bullet}$$

Since $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ then in particular $\text{inv}(\mathcal{A}, \mathcal{S}, u)$. Two cases are possible cases by *i.h.* on u :

- 2.1 There exists u'' such that $u^{\downarrow\sigma} \xrightarrow{\text{db}} u''$. By definition of $s[x/u]^{\downarrow\sigma}$ there are two subcases:
- If $u^{\downarrow\sigma} = vL$ and $x \in \text{rv}(s)$, then $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma \cup (x \mapsto v)}[x/v]L$. We have that vL should have the form $vL_1[y/r]L_2$, with $r \xrightarrow{\text{db}} r'$. Then $s^{\downarrow\sigma \cup (x \mapsto v)}[x/v]L_1[y/r]L_2 \xrightarrow{\text{db}} s^{\downarrow\sigma \cup (x \mapsto v)}[x/v]L_1[y/r']L_2 = t''$ by applying rules esL° and esR° as appropriate.
 - Otherwise, $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma}[x/u^{\downarrow\sigma}]$. Applying rule esR° we obtain $s^{\downarrow\sigma}[x/u^{\downarrow\sigma}] \xrightarrow{\text{db}} s^{\downarrow\sigma}[x/u''] = t''$.
- 2.2 $u^{\downarrow\sigma} \in \text{Abs}$ and $\mu = @$. This case is not possible since $\mu = @$ by premise of rule esR^\bullet .
- esLA^\bullet . Then

$$\frac{s \xrightarrow{\rho, \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s' \quad (1) \quad u \in \text{HA}_{\mathcal{A}}(2) \quad x \notin \mathcal{A} \cup \mathcal{S}(3) \quad x \notin \text{fv}(\rho)(4)}{t = s[x/u] \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} s'[x/u] = t'} \text{esLA}^\bullet$$

Since $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ then in particular $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)$. Given (2), then $u^{\downarrow\sigma} = vL$ by Remark C.18. By Lemma C.22 (1) then $\text{compatible}(\sigma \cup (x \mapsto v), \mathcal{A} \cup \{x\}, \mathcal{S})$, so we can apply *i.h.* on s , yielding two possible cases:

- 2.1 There exists s'' such that $s^{\downarrow\sigma \cup (x \mapsto v)} \xrightarrow{\text{db}} s''$. By definition of $s[x/u]^{\downarrow\sigma}$ there are two possible subcases:
- 2.1.1 If $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma \cup (x \mapsto v)}[x/v]L$, we conclude $s^{\downarrow\sigma \cup (x \mapsto v)}[x/v]L \xrightarrow{\text{db}} s''[x/v]L = t''$ by successively applying rule esL° .
- 2.1.2 If $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma}[x/u^{\downarrow\sigma}]$, then $x \notin \text{rv}(s)$, so $s^{\downarrow\sigma \cup (x \mapsto v)} = s^{\downarrow\sigma}$. Applying rule esL° we obtain $s^{\downarrow\sigma}[x/u^{\downarrow\sigma}] \xrightarrow{\text{db}} s''[x/u^{\downarrow\sigma}] = t''$.
- 2.2 $s^{\downarrow\sigma \cup (x \mapsto v)} \in \text{Abs}$ and $\mu = @$. Then it is immediate to conclude $s[x/u]^{\downarrow\sigma} \in \text{Abs}$ and $\mu = @$.
- esLS^\bullet . Then

$$\frac{s \xrightarrow{\rho, \mathcal{A}, \mathcal{S} \cup \{x\}, \mu} s' \quad u \in \text{St}_{\mathcal{S}} \quad x \notin \mathcal{A} \cup \mathcal{S} \quad x \notin \text{fv}(\rho)}{t = s[x/u] \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} s'[x/u] = t'} \text{esLS}^\bullet$$

Since $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ then in particular $\text{inv}(\mathcal{A}, \mathcal{S} \cup \{x\}, s)$, and $\text{compatible}(\sigma', \mathcal{A}, \mathcal{S} \cup \{x\})$ holds by Lemma C.22 (2). We can apply *i.h.* on s , yielding two possible cases:

- 2.1 There exists s'' such that $s^{\downarrow\sigma'} \xrightarrow{\text{db}} s''$. There are two subcases by definition of $s[x/u]^{\downarrow\sigma}$:
- 2.1.1 If $u^{\downarrow\sigma} = vL$ and $x \in \text{rv}(s)$, then $\sigma' = \sigma \cup (x \mapsto v)$, and $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma \cup (x \mapsto v)}[x/v]L$. Thus $s^{\downarrow\sigma \cup (x \mapsto v)}[x/v]L \xrightarrow{\text{db}} s''[x/v]L = t''$ by successively applying rule esL° .
- 2.1.2 Otherwise, $\sigma' = \sigma$ and $s[x/u]^{\downarrow\sigma} = s^{\downarrow\sigma}[x/u^{\downarrow\sigma}]$. Applying rule esL° we obtain $s^{\downarrow\sigma}[x/u^{\downarrow\sigma}] \xrightarrow{\text{db}} s''[x/u^{\downarrow\sigma}] = t''$.
- 2.2 $s^{\downarrow\sigma} \in \text{Abs}$ and $\mu = @$. Then it is immediate to conclude $s[x/u]^{\downarrow\sigma} \in \text{Abs}$ and $\mu = @$. □

COROLLARY C.25. $t \in \text{NF}_{\emptyset, \text{fv}(t), @}^\bullet$ iff $t^\downarrow \in \text{NF}_{\emptyset, @}^\circ$.

PROOF. The ‘only if’ direction is an immediate consequence of Proposition C.24(1). For the ‘if’ direction, it suffices to show the contrapositive, namely that $t \notin \text{NF}_{\emptyset, \text{fv}(t), @}^\bullet$ implies $t^\downarrow \notin \text{NF}_{\emptyset, @}^\circ$. Using soundness and completeness of the characterization of normal forms (Corollary B.11), this is equivalent to showing that if $t \xrightarrow{\rho, \emptyset, \text{fv}(t), @} t'$ then there exist a rule name ρ' and a term t'' such that $t^\downarrow \xrightarrow{\rho'} t''$. This is a consequence of Proposition C.24(2). □

D PROOFS OF SECTION 6 “USEFUL OPEN CBV IS REASONABLE”

In this section we develop the technical details that are necessary to show that our inductive characterization of useful evaluation is reasonable. Recall that we proceed in two stages, following the technique in [2]: on one hand we prove a high-level implementation result, presented in Theorem 6.3. On the other hand, we show a low-level implementation result, presented in Theorem 6.4. This second stage requires more development than the first one, given that we need to relate our uocbv^\bullet strategy with the GLAMoUr abstract machine in [2].

D.1 Low-level implementation

Definition D.1 (Stable terms). A term t is **pure**, written $t \in \text{Pure}$, if it does not contains explicit substitutions. Let \mathcal{A} be an abstraction frame and \mathcal{S} a structure frame. The set of **stable terms** under \mathcal{A}, \mathcal{S} , written $\text{Stable}_{\mathcal{A}, \mathcal{S}}$, is inductively defined as follows:

$$\frac{}{x \in \text{Stable}_{\mathcal{A}, \mathcal{S}}} \text{STABLE-VAR} \quad \frac{t \in \text{Pure}}{\lambda x. t \in \text{Stable}_{\mathcal{A}, \mathcal{S}}} \text{STABLE-ABS}$$

$$\begin{array}{c}
\frac{t \in \text{Stable}_{\mathcal{A},\mathcal{S}} \quad s \in \text{Stable}_{\mathcal{A},\mathcal{S}}}{ts \in \text{Stable}_{\mathcal{A},\mathcal{S}}} \text{ STABLE-APP} \\
\frac{t \in \text{Stable}_{\mathcal{A} \cup \{x\},\mathcal{S}} \quad s \in \text{Stable}_{\mathcal{A},\mathcal{S}} \quad s \in \text{HA}_{\mathcal{A}} \quad x \notin \mathcal{A} \cup \mathcal{S}}{t[x/s] \in \text{Stable}_{\mathcal{A},\mathcal{S}}} \text{ STABLE-ES-HABS} \\
\frac{t \in \text{Stable}_{\mathcal{A},\mathcal{S} \cup \{x\}} \quad s \in \text{Stable}_{\mathcal{A},\mathcal{S}} \quad s \in \text{St}_{\mathcal{S}} \quad x \notin \mathcal{A} \cup \mathcal{S}}{t[x/s] \in \text{Stable}_{\mathcal{A},\mathcal{S}}} \text{ STABLE-ES-STRUCT}
\end{array}$$

Remark D.2. If $t \in \text{Pure}$, then $t \in \text{Stable}_{\mathcal{A},\mathcal{S}}$, for any frames \mathcal{A}, \mathcal{S} .

Remark D.3. Let t be a term, \mathcal{A} an abstraction frame and \mathcal{S} a structure frame. Then:

1. Let \mathcal{B} be an abstraction frame such that $\mathcal{B} \# \text{fv}(t)$, and \mathcal{T} be a structure frame such that $\mathcal{T} \# \text{fv}(t)$ and $\mathcal{B} \# \mathcal{T}$. If $t \in \text{Stable}_{\mathcal{A},\mathcal{S}}$, then $t \in \text{Stable}_{\mathcal{A} \cup \mathcal{B},\mathcal{S} \cup \mathcal{T}}$.
2. Let \mathcal{B} be an abstraction frame such that $\mathcal{A} \# \mathcal{B}$, and \mathcal{T} be a structure frame such that $\mathcal{S} \# \mathcal{T}$ and $\mathcal{B} \# \mathcal{T}$. If $t \in \text{Stable}_{\mathcal{A},\mathcal{S}}$, then $t \in \text{Stable}_{\mathcal{A} \cup \mathcal{B},\mathcal{S} \cup \mathcal{T}}$.

Definition D.4 (Stable substitution contexts). Let \mathcal{A} be an abstraction frame and \mathcal{S} a structure frame. The set of **stable substitution contexts** under \mathcal{A}, \mathcal{S} , written $\text{StableCtx}_{\mathcal{A},\mathcal{S}}$, is inductively defined as follows:

$$\begin{array}{c}
\frac{}{\diamond \in \text{StableCtx}_{\mathcal{A},\mathcal{S}}} \text{ STABLECTX-EMPTY} \\
\frac{\text{L} \in \text{StableCtx}_{\mathcal{A} \cup \{x\},\mathcal{S}} \quad t \in \text{Stable}_{\mathcal{A},\mathcal{S}} \quad t \in \text{HA}_{\mathcal{A}} \quad x \notin \mathcal{A} \cup \mathcal{S}}{\text{L}[x/t] \in \text{StableCtx}_{\mathcal{A},\mathcal{S}}} \text{ STABLECTX-HABS} \\
\frac{\text{L} \in \text{StableCtx}_{\mathcal{A},\mathcal{S} \cup \{x\}} \quad t \in \text{Stable}_{\mathcal{A},\mathcal{S}} \quad t \in \text{St}_{\mathcal{S}} \quad x \notin \mathcal{A} \cup \mathcal{S}}{\text{L}[x/t] \in \text{StableCtx}_{\mathcal{A},\mathcal{S}}} \text{ STABLECTX-STRUCT}
\end{array}$$

LEMMA D.5. Let \mathcal{A} be an abstraction frame and \mathcal{S} be a structure frame, t be a term and L be a substitution context such that $\text{inv}(\mathcal{A}, \mathcal{S}, t\text{L})$ holds. Then $t\text{L} \in \text{Stable}_{\mathcal{A},\mathcal{S}}$ if and only if $t \in \text{Stable}_{\mathcal{A}^\diamond, \mathcal{S}^\diamond}$ and $\text{L} \in \text{StableCtx}_{\mathcal{A},\mathcal{S}}$.

PROOF. We prove both sides by induction on the length of L .

1 \Rightarrow 2)

- $\text{L} = \diamond$. On the one hand, $\mathcal{A}^\diamond = \mathcal{A}$ and $\mathcal{S}^\diamond = \mathcal{S}$ hold by definition, therefore we conclude $t \in \text{Stable}_{\mathcal{A},\mathcal{S}}$ by hypothesis. Lastly, we conclude $\diamond \in \text{StableCtx}_{\mathcal{A},\mathcal{S}}$ applying rule **STABLECTX-EMPTY**.
- $\text{L} = \text{L}'[x/s]$. We can derive $t\text{L}'[x/s]$ either by rule **STABLE-ES-HABS** or rule **STABLE-ES-STRUCT**. Both cases are analogous, so we will only show case **STABLE-ES-HABS**, where we have $t\text{L}' \in \text{Stable}_{\mathcal{A} \cup \{x\},\mathcal{S}}$ and (1) $s \in \text{Stable}_{\mathcal{A},\mathcal{S}}$, with (2) $s \in \text{HA}_{\mathcal{A}}$ and (3) $x \notin \mathcal{A} \cup \mathcal{S}$. We can apply *i.h.* on L' , yielding $t \in \text{Stable}_{(\mathcal{A} \cup \{x\})^{\text{L}'}, \mathcal{S}'}$ and (4) $\text{L}' \in \text{StableCtx}_{\mathcal{A} \cup \{x\},\mathcal{S}}$. Note that $(\mathcal{A} \cup \{x\})^{\text{L}'} = \mathcal{A}^{\text{L}'} \cup \{x\} = \mathcal{A}^{\text{L}'[x/s]}$, and since $\text{inv}(\mathcal{A}, \mathcal{S}, t\text{L}'[x/s])$ implies $\text{inv}(\mathcal{A}, \mathcal{S}, s)$, then we have $s \notin \text{St}_{\mathcal{S}}$ by Lemma B.2. Hence by definition $\mathcal{S}^{\text{L}'} = \mathcal{S}^{\text{L}'[x/s]}$. Therefore $t \in \text{Stable}_{\mathcal{A}^{\text{L}'[x/s]}, \mathcal{S}^{\text{L}'[x/s]}}$. On the other hand, we can apply rule **STABLECTX-HABS** with (1), (2), (3) and (4) as premises, yielding $\text{L}'[x/s] \in \text{StableCtx}_{\mathcal{A},\mathcal{S}}$.

2 \Rightarrow 1)

- $\text{L} = \diamond$. Then $\mathcal{A}^\diamond = \mathcal{A}$ and $\mathcal{S}^\diamond = \mathcal{S}$ hold by definition, therefore we conclude $t\text{L} = t \in \text{Stable}_{\mathcal{A},\mathcal{S}}$ by hypothesis.
- $\text{L} = \text{L}'[x/s]$. We have two cases, depending on whether $s \in \text{HA}_{\mathcal{A}}$ or $s \in \text{St}_{\mathcal{S}}$. Both cases are analogous, so we will only show case $s \in \text{HA}_{\mathcal{A}}$. Given $\text{inv}(\mathcal{A}, \mathcal{S}, t\text{L}'[x/s])$ then $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, t\text{L}')$ and $\text{inv}(\mathcal{A}, \mathcal{S}, s)$ hold, and the last statement implies $s \notin \text{St}_{\mathcal{S}}$ by Lemma B.2. Hence by definition $\mathcal{A}^{\text{L}'[x/s]} = \mathcal{A}^{\text{L}'} \cup \{x\} = (\mathcal{A} \cup \{x\})^{\text{L}'}$, and $\mathcal{S}^{\text{L}'[x/s]} = \mathcal{S}^{\text{L}'}$. Therefore (1) $t \in \text{Stable}_{(\mathcal{A} \cup \{x\})^{\text{L}'}, \mathcal{S}'}$. On the other hand, $\text{L}'[x/s]$ can only be derived by rule **STABLECTX-HABS**, with (2) $\text{L}' \in \text{StableCtx}_{\mathcal{A} \cup \{x\},\mathcal{S}}$, (3) $s \in \text{Stable}_{\mathcal{A},\mathcal{S}}$, (4) $s \in \text{HA}_{\mathcal{A}}$, and (5) $x \notin \mathcal{A} \cup \mathcal{S}$. We can apply *i.h.* on L' with (1) and (2) as hypothesis, yielding (6) $t\text{L}' \in \text{Stable}_{\mathcal{A} \cup \{x\},\mathcal{S}}$. We conclude $t\text{L}'[x/s] \in \text{Stable}_{\mathcal{A},\mathcal{S}}$ by applying rule **STABLE-ES-HABS**, with (3), (4), (5) and (6) as hypothesis. □

LEMMA D.6. Let t be a term, \mathcal{A}, \mathcal{B} be two abstraction frames and \mathcal{S}, \mathcal{T} be two structure frames such that $\mathcal{A} \# \mathcal{B}$ and $\mathcal{S} \# \mathcal{T}$ and $\mathcal{B} \# \mathcal{T}$, and suppose that $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, t)$ holds. If $t \xrightarrow{\text{sub}_{(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu}} t'$ with $v \in \text{Stable}_{\mathcal{A} \cup \mathcal{B}, \mathcal{S} \cup \mathcal{T}}$ and $t \in \text{Stable}_{\mathcal{A} \cup \{x\}, \mathcal{S}}$, then $t' \in \text{Stable}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}, \mathcal{S} \cup \mathcal{T}}$.

PROOF. By induction on the derivation of $t \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} t'$.

1. SUB[•]. Then $t = x \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, @} v = t'$, with $\mu = @$. Since $v \in \text{Stable}_{\mathcal{A} \cup \mathcal{B}, \mathcal{S} \cup \mathcal{T}}$ by hypothesis, then $v \in \text{Stable}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}, \mathcal{S} \cup \mathcal{T}}$ by Remark D.3.
2. APPL[•]. Then

$$\frac{s \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, @} s'}{t = s u \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s' u = t'} \text{APPL}^{\bullet}$$

and $s u \in \text{Stable}_{\mathcal{A} \cup \{x\}, \mathcal{S}}$ can only be derived from rule STABLE-APP, so that $s \in \text{Stable}_{\mathcal{A} \cup \{x\}, \mathcal{S}}$ and $u \in \text{Stable}_{\mathcal{A} \cup \{x\}, \mathcal{S}}$ hold. Moreover, $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s u)$ implies $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)$, therefore we can apply *i.h.* on s , yielding $s' \in \text{Stable}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}, \mathcal{S} \cup \mathcal{T}}$. By Remark D.3, we have $u \in \text{Stable}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}, \mathcal{S} \cup \mathcal{T}}$. We conclude $s' u \in \text{Stable}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}, \mathcal{S} \cup \mathcal{T}}$ by applying rule STABLE-APP.

3. APPR[•]. Analogous to the previous case.
4. ESR[•]. Then

$$\frac{u \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, @} u'}{t = s[y/u] \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s[y/u'] = t'} \text{ESR}^{\bullet}$$

By α -conversion we may assume $y \notin \mathcal{B} \cup \mathcal{T}$. We can derive $s[y/u] \in \text{Stable}_{\mathcal{A} \cup \{x\}, \mathcal{S}}$ either by rule STABLE-ES-HABS or by rule STABLE-ES-STRUCT. Both cases are analogous, so we only show case STABLE-ES-HABS. Hence $s \in \text{Stable}_{\mathcal{A} \cup \{x\} \cup \{y\}, \mathcal{S}}$, $u \in \text{Stable}_{\mathcal{A} \cup \{x\}, \mathcal{S}}$, $u \in \text{HA}_{\mathcal{A} \cup \{x\}}$, and $y \notin \mathcal{A} \cup \{x\} \cup \mathcal{B} \cup \mathcal{S} \cup \mathcal{T}$. Given that $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s[y/u])$ implies $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, u)$, we can apply *i.h.* on u , yielding (2) $u' \in \text{Stable}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}, \mathcal{S} \cup \mathcal{T}}$. Moreover, (3) $s \in \text{Stable}_{\mathcal{A} \cup \{x\} \cup \{y\} \cup \mathcal{B}, \mathcal{S} \cup \mathcal{T}}$ by Remark D.3. We have (4) $u' \in \text{HA}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}}$ by Lemma B.14. We apply rule STABLE-ES-HABS with (1), (2), (3) and (4) as premises, yielding $s[y/u'] \in \text{Stable}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}, \mathcal{S} \cup \mathcal{T}}$.

5. ESLA[•]. Then

$$\frac{s \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\} \cup \{y\}, \mathcal{S}, \mu} s' \quad u \in \text{HA}_{\mathcal{A} \cup \{x\}} \quad y \notin (\mathcal{A} \cup \{x\}) \cup \mathcal{S} \quad y \notin \text{fv}(\text{sub}(x,v))}{t = s[y/u] \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s'[y/u] = t'} \text{ESLA}^{\bullet}$$

Note that the judgment $s[x/u] \in \text{Stable}_{\mathcal{A} \cup \{x\}, \mathcal{S}}$ can be derived only by rule STABLE-ES-HABS. Then $s \in \text{Stable}_{\mathcal{A} \cup \{x\} \cup \{y\}, \mathcal{S}}$ and $u \in \text{Stable}_{\mathcal{A} \cup \{x\}, \mathcal{S}}$. By α -conversion, we may assume that $y \notin \mathcal{B} \cup \mathcal{T}$. Moreover, $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s[x/u])$ implies $\text{inv}(\mathcal{A} \cup \{x\} \cup \{y\}, \mathcal{S}, s)$, and note that $v \in \text{Stable}_{\mathcal{A} \cup \{y\} \cup \mathcal{B}, \mathcal{S} \cup \mathcal{T}}$ by hypothesis and Remark 4.1. We can apply *i.h.* on s , yielding (1) $s' \in \text{Stable}_{\mathcal{A} \cup \{x\} \cup \{y\} \cup \mathcal{B}, \mathcal{S} \cup \mathcal{T}}$. By Remark D.3 we have $u \in \text{Stable}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}, \mathcal{S} \cup \mathcal{T}}$. And we have $u \in \text{HA}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}}$ by Remark 4.1, and $y \notin \mathcal{A} \cup \{x\} \cup \mathcal{B} \cup \mathcal{S} \cup \mathcal{T}$. We conclude $s'[y/u] \in \text{Stable}_{\mathcal{A} \cup \{x\} \cup \mathcal{B}, \mathcal{S} \cup \mathcal{T}}$ by applying rule STABLE-ES-HABS.

6. ESLS[•]. Analogous to the previous case. □

LEMMA D.7. Let t be a term, \mathcal{A} an abstraction frame and \mathcal{S} a structure frame such that $\text{inv}(\mathcal{A}, \mathcal{S}, t)$ holds. If $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'$ with $\rho \in \{\text{db}, \text{lsv}\}$ and $t \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$, then $t' \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$.

PROOF. We proceed by induction on $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'$.

- DB[•]. Then $t = (\lambda x. s) L u \xrightarrow{\text{db}, \mathcal{A}, \mathcal{S}, \mu} s[x/u] L = t'$, where $\rho = \text{db}$. The hypothesis $(\lambda x. s) L u \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$ is derived from:

$$\frac{(\lambda x. s) L \in \text{Stable}_{\mathcal{A}, \mathcal{S}} \quad u \in \text{Stable}_{\mathcal{A}, \mathcal{S}}}{(\lambda x. s) L u \in \text{Stable}_{\mathcal{A}, \mathcal{S}}} \text{STABLE-APP}$$

Moreover, $\text{inv}(\mathcal{A}, \mathcal{S}, (\lambda x. s) L u)$ implies $\text{inv}(\mathcal{A}, \mathcal{S}, (\lambda x. s) L)$ and $\text{inv}(\mathcal{A}, \mathcal{S}, u)$. We then have $\lambda x. s \in \text{Stable}_{\mathcal{A}^{\perp}, \mathcal{S}^{\perp}}$ and (1) $L \in \text{StableCtx}_{\mathcal{A}, \mathcal{S}}$ by Lemma D.5, and this judgment can only be derived from rule STABLE-ABS, hence $s \in \text{Pure}$. By Remark D.2, $s \in \text{Stable}_{\mathcal{A}^{\perp} \cup \{x\}, \mathcal{S}^{\perp}}$, and we have to analyze whether $u \in \text{HA}_{\mathcal{A}^{\perp}}$ or $u \in \text{St}_{\mathcal{S}^{\perp}}$. Since both cases are analogous, we will proceed with the first one. Since $\text{dom}(L) \# \text{fv}(u)$ by α -conversion, then $u \in \text{Stable}_{\mathcal{A}^{\perp}, \mathcal{S}^{\perp}}$ holds by Remark D.3, and applying rule STABLE-ES-HABS we have that (2) $s[x/u] \in \text{Stable}_{\mathcal{A}^{\perp}, \mathcal{S}^{\perp}}$. Given that $\text{inv}(\mathcal{A}, \mathcal{S}, (\lambda x. s) L u)$ implies $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u] L)$, we can apply Lemma D.5 with (1) and (2), to obtain $s[x/u] L \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$.

- LSV[•]. Then $t = s[x/vL] \xrightarrow{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} s'[x/v] L = t'$, where $\rho = \text{lsv}$ and it is derived from $s \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s'$, $x \notin \mathcal{A} \cup \mathcal{S}$, and (1) $vL \in \text{HA}_{\mathcal{A}}$. The hypothesis $s[x/vL] \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$ is derived from rule STABLE-ES-HABS by (1), so (2) $s \in \text{Stable}_{\mathcal{A} \cup \{x\}, \mathcal{S}}$ and $vL \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$. By Lemma D.5, we have (3) $v \in \text{Stable}_{\mathcal{A}^{\perp}, \mathcal{S}^{\perp}}$ and (4) $L \in \text{StableCtx}_{\mathcal{A}, \mathcal{S}}$. On the other hand, since $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/vL])$ then $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)$ holds and we can apply Lemma D.6 with (2) and (3) as

hypothesis, yielding (5) $s' \in \text{Stable}_{(\mathcal{A} \cup \{x\})^\perp, \mathcal{S}^\perp} = \text{Stable}_{\mathcal{A}^\perp \cup \{x\}, \mathcal{S}^\perp}$. Moreover (6) $v \in \text{HA}_{\mathcal{A}^\perp}$ by Lemma B.19. we may assume $x \notin \text{dom}(L)$ by α -conversion, hence we can apply rule STABLE-ES-HABS with (3), (5), (6) and $x \notin \mathcal{A}^\perp \cup \mathcal{S}^\perp$ as premises, thus having (7) $s'[x/v] \in \text{Stable}_{\mathcal{A}^\perp, \mathcal{S}^\perp}$. We conclude $s'[x/v]L \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$ by applying Lemma D.5 with (4) and (7) as hypothesis.

- **APPL^{*}**. Then $t = s u \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} s' u = t'$, derived from $s \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, @} s'$. The hypothesis $s u \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$ can only be derived from rule STABLE-APP, so (1) $s \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$ and $u \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$. Given that $\text{inv}(\mathcal{A}, \mathcal{S}, s u)$ implies $\text{inv}(\mathcal{A}, \mathcal{S}, s)$, we can apply *i.h.* on s , yielding (2) $s' \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$. By rule STABLE-APP with (1) and (2) as premises, we conclude $s' u \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$.
- **APPR^{*}**. Analogous to the previous case.
- **ESR^{*}**. Then $t = s[x/u] \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} s[x/u'] = t'$, derived from $u \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, @} u'$. The hypothesis $s[x/u] \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$ can be derived either from rule STABLE-ES-HABS or rule STABLE-ES-STRUCT, so $u \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$. Given that $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ implies $\text{inv}(\mathcal{A}, \mathcal{S}, u)$, we can apply *i.h.* on u , yielding $u' \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$. We can apply either rule STABLE-ES-HABS or STABLE-ES-STRUCT to conclude $s[x/u'] \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$.
- **ESLA^{*}**. Then $t = s[x/u] \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} s'[x/u] = t'$, derived from $s \xrightarrow{\rho, \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s'$, (1) $u \in \text{HA}_{\mathcal{A}}$, $x \notin \mathcal{A} \cup \mathcal{S}$, and $x \notin \text{fv}(\rho)$. The hypothesis $s[x/u] \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$ can only be derived from rule STABLE-ES-HABS, so $s \in \text{Stable}_{\mathcal{A} \cup \{x\}, \mathcal{S}}$ and (2) $u \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$. Given that $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ implies $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)$, we can apply *i.h.* on s , yielding (3) $s' \in \text{Stable}_{\mathcal{A} \cup \{x\}, \mathcal{S}}$. By rule STABLE-ES-HABS with (1), (2) and (3) as premises, we conclude $s'[x/u] \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$.
- **ESLS^{*}**. Analogous to the previous case.

□

We restrict the reduction rule $\xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu}$ to stable terms. The relation of **evaluation of stable terms**, written $\xrightarrow{\blacktriangle, \rho, \mathcal{A}, \mathcal{S}, \mu}$, is defined by the same reduction rules as the relation $\xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu}$, but rule DB^{*} is replaced by

$$\frac{s \in \text{HA}_{\mathcal{A}} \cup \text{St}_{\mathcal{S}}}{(\lambda x. t)L s \xrightarrow{\blacktriangle, \text{db}, \mathcal{A}, \mathcal{S}, \mu} t[x/s]L} \text{DB-STABLE}^{\bullet}$$

Definition D.8 (Structural equivalence). We define a relation of **structural equivalence**, written \equiv , recursively as follows:

$$\begin{array}{c} \frac{x \notin \text{fv}(u) \quad y \notin \text{fv}(s)}{t[x/s][y/u] \equiv t[y/u][x/s]} \text{ES-COMM} \quad \frac{y \notin \text{fv}(t)}{t[x/s][y/u] \equiv t[x/s][y/u]} \text{ES-ASSOC} \\ \\ \frac{x \notin \text{fv}(s)}{(ts)[x/u] \equiv t[x/u]s} \text{ES-L-DIST} \quad \frac{x \notin \text{fv}(t)}{(ts)[x/u] \equiv ts[x/u]} \text{ES-R-DIST} \\ \\ \frac{t \equiv t' \quad s \equiv s'}{ts \equiv t's'} \text{CONG-APP} \quad \frac{t \equiv t' \quad s \equiv s'}{t[x/s] \equiv t'[x/s']} \text{CONG-ES} \\ \\ \frac{}{t \equiv t} \text{REFL} \quad \frac{t \equiv s}{s \equiv t} \text{SYM} \quad \frac{t \equiv s \quad s \equiv u}{t \equiv u} \text{TRANS} \end{array}$$

Remark D.9. If $vL \equiv t$ then t is of the form $v'L'$, and the proof of equivalence necessarily uses rule REFL.

Remark D.10 (Strengthening of abstraction and value frames). Let \mathcal{A} be an abstraction frame and \mathcal{S} a structure set. Let t be a term such that $x \notin \text{fv}(t)$. Then the following holds:

1. If $t \in \text{HA}_{\mathcal{A} \cup \{x\}}$ then $t \in \text{HA}_{\mathcal{A}}$.
2. If $t \in \text{St}_{\mathcal{S} \cup \{x\}}$ then $t \in \text{St}_{\mathcal{S}}$.

LEMMA D.11 (HEREDITARY ABSTRACTIONS AND STRUCTURES ARE CLOSED BY STRUCTURAL EQUIVALENCE). *Let \mathcal{A} and \mathcal{S} be sets of variables, and t be a term such that $\text{inv}(\mathcal{A}, \mathcal{S}, t)$. If $t \equiv s$ then the following holds:*

1. $t \in \text{HA}_{\mathcal{A}}$ if and only if $s \in \text{HA}_{\mathcal{A}}$.
2. $t \in \text{St}_{\mathcal{S}}$ if and only if $s \in \text{St}_{\mathcal{S}}$.

PROOF. Each item is proved by induction on the derivation of $t \equiv s$. □

LEMMA 6.2 (STRONG BISIMULATION). *Let t_0, s_0 be stable terms such that $s_0 \equiv t_0 \xrightarrow{\blacktriangle, \rho, \mathcal{A}, \mathcal{S}, \mu} t_1$ then there exists s_1 such that $s_0 \xrightarrow{\blacktriangle, \rho, \mathcal{A}, \mathcal{S}, \mu} s_1 \equiv t_1$.*

PROOF. We start by introducing an auxiliary equivalence relation \equiv_c on terms, recursively defined by:

$$\begin{array}{c}
 \frac{x \notin \text{fv}(u) \quad y \notin \text{fv}(s)}{t[x/s][y/u] \equiv_c t[y/u][x/s]} \text{ES-COMM} \\
 \\
 \frac{y \notin \text{fv}(t)}{t[x/s][y/u] \equiv_c t[x/s][y/u]} \text{ES-ASSOC(1)} \quad \frac{y \notin \text{fv}(t)}{t[x/s][y/u] \equiv_c t[x/s][y/u]} \text{ES-ASSOC(2)} \\
 \\
 \frac{x \notin \text{fv}(s)}{(ts)[x/u] \equiv_c t[x/u]s} \text{ES-L-DIST(1)} \quad \frac{x \notin \text{fv}(s)}{t[x/u]s \equiv_c (ts)[x/u]} \text{ES-L-DIST(2)} \\
 \\
 \frac{x \notin \text{fv}(t)}{(ts)[x/u] \equiv_c ts[x/u]} \text{ES-R-DIST(1)} \quad \frac{x \notin \text{fv}(t)}{ts[x/u] \equiv_c (ts)[x/u]} \text{ES-R-DIST(2)} \\
 \\
 \frac{t \equiv_c t' \quad s \equiv_c s'}{ts \equiv_c t's'} \text{CONG-APP} \quad \frac{t \equiv_c t' \quad s \equiv_c s'}{t[x/s] \equiv_c t'[x/s']} \text{CONG-ES}
 \end{array}$$

Note in particular that \equiv is the reflexive-transitive closure of \equiv_c . We divide the proof into two parts:

1. We show that if $t_0 \xrightarrow{\Delta}_{\rho, \mathcal{A}, \mathcal{S}, \mu} t_1$ and $t_0 \equiv_c s_0$, then there exists s_1 such that $s_0 \xrightarrow{\Delta}_{\rho, \mathcal{A}, \mathcal{S}, \mu} s_1$ and $t_1 \equiv s_1$.
2. Given that \equiv is the reflexive-transitive closure of \equiv_c , we show the same result but for the \equiv relation by resorting to Item 1.

Item 2 is proved by induction on the reflexive and transitive closure of \equiv_c and is immediate. The proof of Item 1 is by induction on the derivation of $t_0 \xrightarrow{\Delta}_{\rho, \mathcal{A}, \mathcal{S}, \mu} t_1$ and case analysis.

- **DB-STABLE[•]**. Then $t_0 = (\lambda x. u)Lr \xrightarrow{\Delta}_{\text{db}, \mathcal{A}, \mathcal{S}, \mu} u[x/r]L = t_1$, with $r \in \text{HA}_{\mathcal{A}} \cup \text{St}_{\mathcal{S}}$. We also have $t_0 = (\lambda x. u)Lr \equiv_c s_0$. We analyze the different cases according to which rule was used to derive the equivalence. Note that cases ES-COMM, ES-ASSOC and CONG-ES are impossible due to the form of t_0 . The relevant cases are ES-L-DIST, ES-R-DIST, and CONG-APP:
 - a. **ES-L-DIST(2)**. Then $t_0 = (\lambda x. u)L' r \equiv_c ((\lambda x. u)L' r)[y/p] = s_0$, with $y \notin \text{fv}(r)$ and $L = L'[y/p]$. There are two cases for reducing $((\lambda x. u)L' r)[y/p]$, depending on whether $p \in \text{HA}_{\mathcal{A}}$ or $p \in \text{St}_{\mathcal{S}}$; both are analogous, so we show only the first case. Then we obtain $s_0 = ((\lambda x. u)L' r)[y/p] \xrightarrow{\Delta}_{\text{db}, \mathcal{A}, \mathcal{S}, \mu} u[x/r]L'[y/p] = s_1$ by applying rules **ESLA[•]** and **DB-STABLE[•]**. Note that $s_1 \equiv t_1$ by rule **REFL**. The following diagram summarizes the proof:

$$\begin{array}{ccc}
 t_0 = (\lambda x. u)L' r & \xrightarrow{\Delta}_{\text{db}, \mathcal{A}, \mathcal{S}, \mu} & u[x/r]L'[y/p] = t_1 \\
 \equiv_c & & \equiv \\
 s_0 = ((\lambda x. u)L' r)[y/p] & \xrightarrow{\Delta}_{\text{db}, \mathcal{A}, \mathcal{S}, \mu} & u[x/r]L'[y/p] = s_1
 \end{array}$$

- b. **ES-R-DIST(2)**. Then $t_0 = (\lambda x. u)L(r'[y/p]) \equiv_c ((\lambda x. u)Lr')[y/p] = s_0$, with $y \notin \text{fv}((\lambda x. u)L)$ and $r = r'[y/p]$ with $r' \in \text{HA}_{\mathcal{A}'} \cup \text{St}_{\mathcal{S}'}$ where $\mathcal{A}' = \mathcal{A}^{[y/p]}$ and $\mathcal{S}' = \mathcal{S}^{[y/p]}$. In particular (1) $y \notin \text{fv}(u) \cup \text{fv}(L)$, since $x \neq y$ by α -conversion. There are two cases for reducing $((\lambda x. u)Lr')[y/p]$, depending on whether $p \in \text{HA}_{\mathcal{A}}$ or $p \in \text{St}_{\mathcal{S}}$; both are analogous, so we show only the first one. Then we obtain $s_0 = ((\lambda x. u)Lr')[y/p] \xrightarrow{\Delta}_{\text{db}, \mathcal{A}, \mathcal{S}, \mu} u[x/r']L[y/p] = s_1$ by applying rules **ESLA[•]** and **DB-STABLE[•]**, since we already know $r = r'[y/p] \in \text{HA}_{\mathcal{A}} \cup \text{St}_{\mathcal{S}}$, so it is easy to note in particular $r' \in \text{HA}_{\mathcal{A} \cup \{y\}} \cup \text{St}_{\mathcal{S}}$. Then $t_1 = u[x/r'[y/p]]L \equiv_c u[x/r']L[y/p] = t'_1$ by applying rules **CONG-ES** and **ES-ASSOC**, and since $y \notin \text{fv}(u)$ by (1). We conclude $t'_1 \equiv s_1$ by rules **CONG-ES** and **ES-COMM**, since $y \notin \text{fv}(L)$ by (1), and $\text{dom}(L) \# \text{fv}(p)$ by α -conversion, thus $t_1 \equiv s_1$. The following diagram summarizes the proof:

$$\begin{array}{ccc}
 t_0 = (\lambda x. u)Lr'[y/p] & \xrightarrow{\Delta}_{\text{db}, \mathcal{A}, \mathcal{S}, \mu} & u[x/r'[y/p]]L = t_1 \\
 \equiv_c & & \equiv \\
 & & u[x/r']L[y/p] = t'_1 \\
 & & \equiv \\
 s_0 = ((\lambda x. u)Lr')[y/p] & \xrightarrow{\Delta}_{\text{db}, \mathcal{A}, \mathcal{S}, \mu} & u[x/r']L[y/p] = s_1
 \end{array}$$

- c. CONG-APP. Then $t_0 = (\lambda x. u)L r \equiv_c p r' = s_0$, where (1) $(\lambda x. u)L \equiv_c p$, and $r \equiv_c r'$. We split into subcases, according to the rule used to derive the equivalence (1). Note that the only relevant cases are the axioms of the relation \equiv_c , since cases ES-L-DIST and ES-R-DIST are impossible due to the form of $(\lambda x. u)L$.
- ES-COMM. Then $(\lambda x. u)L = (\lambda x. u)L'[y_1/q_1][y_2/q_2] \equiv_c (\lambda x. u)L'[y_2/q_2][y_1/q_1] = p$, where $y_1 \notin \text{fv}(q_2)$ and $y_2 \notin \text{fv}(q_1)$. Therefore $t_0 = (\lambda x. u)L'[y_1/q_1][y_2/q_2] r \equiv_c (\lambda x. u)L'[y_2/q_2][y_1/q_1] r' = s_0$, and s_0 (db, \mathcal{A} , \mathcal{S} , μ)-reduces to $s_1 = u[x/r']L'[y_2/q_2][y_1/q_1]$ by rule DB-STABLE $^\bullet$, since $r' \in \text{HA}_{\mathcal{A}} \cup \text{St}_{\mathcal{S}}$ by Lemma D.11 given that $r \in \text{HA}_{\mathcal{A}} \cup \text{St}_{\mathcal{S}}$. From $t_1 = u[x/r']L'[y_1/q_1][y_2/q_2]$, we build the derivation \mathcal{D} of $s_1 \equiv u[x/r']L'[y_1/q_1][y_2/q_2] = t'_1$:

$$\mathcal{D} := \left(\frac{\frac{\frac{\mathcal{D}' \quad \text{REFL}}{q_1 \equiv q_1} \quad \text{CONG-ES} \quad \frac{\text{REFL}}{q_2 \equiv q_2}}{u[x/r']L'[y_1/q_1] \equiv u[x/r']L'[y_1/q_1]} \quad \text{CONG-ES}}{t_1 \equiv t'_1} \right)$$

where

$$\mathcal{D}' := \left(\frac{\frac{\frac{\text{REFL}}{u \equiv u} \quad \frac{\text{By hypothesis}}{r \equiv r'}}{u[x/r] \equiv u[x/r']} \quad \text{CONG-ES} \quad \dots}{\frac{\dots}{u[x/r]L' \equiv u[x/r']L'} \quad \text{CONG-ES}} \quad \text{CONG-ES} \right)$$

and since $y_1 \notin \text{fv}(q_2)$ and $y_2 \notin \text{fv}(q_1)$, we can apply rule ES-COMM, yielding $t'_1 \equiv u[x/r']L'[y_2/q_2][y_1/q_1] = s_1$, thus $t_1 \equiv s_1$. The following diagram summarizes the proof:

$$\begin{aligned} t_0 &= (\lambda x. u)L'[y_1/q_1][y_2/q_2] r \xrightarrow{\text{db, } \mathcal{A}, \mathcal{S}, \mu} u[x/r]L'[y_1/q_1][y_2/q_2] = t_1 \\ &\equiv \\ &\equiv_c u[x/r']L'[y_1/q_1][y_2/q_2] = t'_1 \\ &\equiv \\ s_0 &= (\lambda x. u)L'[y_2/q_2][y_1/q_1] r' \xrightarrow{\text{db, } \mathcal{A}, \mathcal{S}, \mu} u[x/r']L'[y_2/q_2][y_1/q_1] = s_1 \end{aligned}$$

- ES-ASSOC(1). Then $(\lambda x. u)L = (\lambda x. u)L'[y_1/q_1][y_2/q_2] \equiv_c (\lambda x. u)L'[y_1/q_1][y_2/q_2] = p$, where $y_2 \notin \text{fv}((\lambda x. u)L')$. Hence $t_0 = (\lambda x. u)L'[y_1/q_1][y_2/q_2] r \equiv_c (\lambda x. u)L'[y_1/q_1][y_2/q_2] r' = s_0$, and s_0 (db, \mathcal{A} , \mathcal{S} , μ)-reduces to $s_1 = u[x/r']L'[y_1/q_1][y_2/q_2]$ by rule DB-STABLE $^\bullet$, given that $r' \in \text{HA}_{\mathcal{A}} \cup \text{St}_{\mathcal{S}}$ by Lemma D.11 since $r \in \text{HA}_{\mathcal{A}} \cup \text{St}_{\mathcal{S}}$. We have $t_1 \equiv u[x/r']L'[y_1/q_1][y_2/q_2] = t'_1$, obtained from the derivation \mathcal{D} from the previous subcase. To conclude $t'_1 \equiv s_1$ by rule ES-ASSOC since $y_2 \notin \text{fv}(u[x/r']L')$: on one hand $y_2 \notin \text{fv}((\lambda x. u)L')$ by hypothesis and on the other hand $y_2 \notin \text{fv}(r')$ by α -conversion. Thus $t_1 \equiv s_2$. The following diagram summarizes the proof:

$$\begin{aligned} t_0 &= (\lambda x. u)L'[y_1/q_1][y_2/q_2] r \xrightarrow{\text{db, } \mathcal{A}, \mathcal{S}, \mu} u[x/r]L'[y_1/q_1][y_2/q_2] = t_1 \\ &\equiv \\ &\equiv_c u[x/r']L'[y_1/q_1][y_2/q_2] = t'_1 \\ &\equiv \\ s_0 &= (\lambda x. u)L'[y_1/q_1][y_2/q_2] r' \xrightarrow{\text{db, } \mathcal{A}, \mathcal{S}, \mu} u[x/r']L'[y_1/q_1][y_2/q_2] = s_1 \end{aligned}$$

- ES-ASSOC(2). Then $(\lambda x. u)L = (\lambda x. u)L'[y_1/q_1][y_2/q_2] \equiv_c (\lambda x. u)L'[y_1/q_1][y_2/q_2] = p$, where $y_2 \notin \text{fv}((\lambda x. u)L')$. The steps are analogous to the previous case, with following diagram summarizing the proof:

$$\begin{aligned}
 t_0 &= (\lambda x. u)L'[y_1/q_1][y_2/q_2] r \xrightarrow[\text{db}, \mathcal{A}, \mathcal{S}, \mu]{\Delta} u[x/r]L'[y_1/q_1][y_2/q_2] = t_1 \\
 &\equiv \\
 &\equiv_c \qquad \qquad \qquad u[x/r']L'[y_1/q_1][y_2/q_2] = t'_1 \\
 &\equiv \\
 s_0 &= (\lambda x. u)L'[y_1/q_1][y_2/q_2] r' \xrightarrow[\text{db}, \mathcal{A}, \mathcal{S}, \mu]{\Delta} u[x/r']L'[y_1/q_1][y_2/q_2] = s_1
 \end{aligned}$$

- CONG-ES. Then $(\lambda x. u)L = (\lambda x. u)L'[y/p]$, and $(\lambda x. u)L'[y/p] \equiv_c q'[y/p']$, with $(\lambda x. u)L' \equiv_c q'$ and $p \equiv_c p'$. Then this case is not possible since we need to use rule REFL, by Remark D.9.

- LSV^{*}. Then $t_0 = u_0[x/vL] \xrightarrow[\text{lsv}, \mathcal{A}, \mathcal{S}, \mu]{\Delta} u_1[x/v]L = t_1$, where $\rho = \text{lsv}$, and it is derived from $u_0 \xrightarrow[\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu]{\Delta} u_1$ and $x \notin \mathcal{A} \cup \mathcal{S}$, and $vL \in \text{HA}_{\mathcal{A}}$. We also have $t_0 = u[x/vL] \equiv_c s_0$. We analyze the different cases according to which rule was used to derive the equivalence. Note that case CONG-APP is impossible due to the form of t_0 .

- a. ES-COMM. Then $t_0 = u'_0[y/r][x/vL] \equiv_c u'_0[x/vL][y/r] = s_0$, with $y \notin \text{fv}(vL)$ and $x \notin \text{fv}(r)$. The step $u'_0[y/r] \xrightarrow[\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu]{\Delta} u_1$ can be derived either from rule ESR^{*}, ESLA^{*} or ESLS^{*}:

- ESR^{*}. Then $u_1 = u'_0[y/r']$, with $r \xrightarrow[\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu]{\Delta} r'$. By Remark B.5, we have $x \in \text{fv}(r)$, but at the same time $x \notin \text{fv}(r)$ by hypothesis of rule ES-COMM. Therefore this case is not possible.
- ESLA^{*}. Then $u_1 = u'_0[y/r]$, with $u'_0 \xrightarrow[\text{sub}(x,v), \mathcal{A} \cup \{x\} \cup \{y\}, \mathcal{S}, \mu]{\Delta} u''_0$. Then $s_0 = u'_0[x/vL][y/r] \xrightarrow[\text{lsv}, \mathcal{A}, \mathcal{S}, \mu]{\Delta} u''_0[x/v]L[y/r] = s_1$ derived from rules ESLA^{*} and LSV^{*}. Since $y \notin \text{fv}(vL)$, we can apply several times rules CONG-ES and ES-COMM, yielding $s_1 \equiv t_1$. The following diagram summarizes the proof:

$$\begin{aligned}
 t_0 &= u'_0[y/r][x/vL] \xrightarrow[\text{lsv}, \mathcal{A}, \mathcal{S}, \mu]{\Delta} u''_0[y/r][x/v]L = t_1 \\
 &\equiv_c \qquad \qquad \qquad \equiv \\
 s_0 &= u'_0[x/vL][y/r] \xrightarrow[\text{lsv}, \mathcal{A}, \mathcal{S}, \mu]{\Delta} u''_0[x/v]L[y/r] = s_1
 \end{aligned}$$

- ESLS^{*}. Analogous to the previous case.

- b. ES-ASSOC(1). Then $t_0 = u'_0[y/r][x/vL] \equiv_c u'_0[y/r][x/vL] = s_0$, with $x \notin \text{fv}(u'_0)$. The step $u'_0[y/r] \xrightarrow[\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu]{\Delta} u_1$ can be derived either from rule ESR^{*}, ESLA^{*} or ESLS^{*}:

- ESR^{*}. Then $u_1 = u'_0[y/r']$, with $r \xrightarrow[\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu]{\Delta} r'$. An we can perform the reduction step $s_0 = u'_0[y/r][x/vL] \xrightarrow[\text{lsv}, \mathcal{A}, \mathcal{S}, \mu]{\Delta} u'_0[y/r'][x/v]L = s_1$, derived from rules ESR^{*} and LSV^{*}. Then $t_1 = u'_0[y/r'][x/v]L \equiv_c u'_0[y/r'][x/v]L = t'_1$ by rules CONG-ES and ES-ASSOC, since $x \notin \text{fv}(u'_0)$ by hypothesis. We conclude $t'_1 \equiv s_1$ by applying several times rules CONG-ES and ES-ASSOC, given that we may assume $\text{dom}(L) \# \text{fv}(u'_0)$ by α -conversion. Thus $t_1 \equiv s_1$. The following diagram summarizes the proof:

$$\begin{aligned}
 t_0 &= u'_0[y/r][x/vL] \xrightarrow[\text{lsv}, \mathcal{A}, \mathcal{S}, \mu]{\Delta} u'_0[y/r'][x/v]L = t_1 \\
 &\equiv \\
 &\equiv_c \qquad \qquad \qquad u'_0[y/r'][x/v]L = t'_1 \\
 &\equiv \\
 s_0 &= u'_0[y/r][x/vL] \xrightarrow[\text{lsv}, \mathcal{A}, \mathcal{S}, \mu]{\Delta} u'_0[y/r'][x/v]L = s_1
 \end{aligned}$$

- ESLA^{*}. Then $u_1 = u''_0[y/r]$, with $u'_0 \xrightarrow[\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu]{\Delta} u''_0$. By Remark B.5, we have $x \in \text{fv}(u'_0)$, but at the same time $x \notin \text{fv}(u'_0)$ by hypothesis of rule ES-ASSOC. Therefore this case is not possible.

- ESLS^{*}. This case is analogous to the previous one, hence it is impossible.

- c. ES-ASSOC(2). Then $t_0 = u_0[x/vL][y/r] \equiv_c u_0[x/vL][y/r] = s_0$, with $y \notin \text{fv}(u_0)$ and $L = L'[y/r]$. The predicate $vL' \in \text{HA}_{\mathcal{A}}$ from the premise of rule LSV^{*} can be derived either by rule H-SUB₁ or H-SUB₂, so we have two subcases to derive $s_0 \xrightarrow[\text{lsv}, \mathcal{A}, \mathcal{S}, \mu]{\Delta} u_1[x/v]L'[y/r] = s_1$. Since both are analogous, we only focus on the case where $vL'[y/r] \in \text{HA}_{\mathcal{A}}$

derived by rule H-SUB_2 , with $vL' \in \text{HA}_{\mathcal{A} \cup \{y\}}$ and $r \in \text{HA}_{\mathcal{A}}$ and $y \notin \mathcal{A}$. Hence we can build a derivation of $s_0 \xrightarrow{\Delta}_{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} s_1$ by applying rules ESLA^\bullet and LSV^\bullet . To conclude, we have $t_1 \equiv s_1$ by applying rule REFL . The following diagram summarizes the proof:

$$\begin{array}{ccc} t_0 = u_0[x/vL'[y/r]] & \xrightarrow{\Delta}_{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} & u_1[x/v]L'[y/r] = t_1 \\ \equiv_c & & \equiv \\ s_0 = u_0[x/vL'] [y/r] & \xrightarrow{\Delta}_{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} & u_1[x/v]L' [y/r] = s_1 \end{array}$$

d. $\text{ES-L-DIST}(1)$. Then $t_0 = (r_0 r_1)[x/vL] \equiv_c r_0[x/vL] r_1 = s_0$, with $x \notin \text{fv}(r_1)$. The step $r_0 r_1 \xrightarrow{\Delta}_{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} u_1$ can be derived from rules APPL^\bullet and APPR^\bullet :

– APPL^\bullet . Then $u_1 = r'_0 r_1$, where $r_0 \xrightarrow{\Delta}_{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} r'_0$ with $t_1 = (r'_0 r_1)[x/v]L$, and $s_0 = r_0[x/vL] r_1 \xrightarrow{\Delta}_{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} r'_0[x/v]L r_1 = s_1$ by rules APPL^\bullet and LSV^\bullet . Since $x \notin \text{fv}(r_1)$ by hypothesis and $\text{dom}(L) \# \text{fv}(r_1)$ by α -conversion, we can then apply rule ES-L-DIST to conclude $t_1 \equiv s_1$. The following diagram summarizes the proof:

$$\begin{array}{ccc} t_0 = (r_0 r_1)[x/vL] & \xrightarrow{\Delta}_{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} & (r'_0 r_1)[x/v]L = t_1 \\ \equiv_c & & \equiv \\ s_0 = r_0[x/vL] r_1 & \xrightarrow{\Delta}_{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} & r'_0[x/v]L r_1 = s_1 \end{array}$$

– APPR^\bullet . Then $u_1 = r_0 r'_1$, derived from $r_0 \in \text{St}_{\mathcal{S}}$ and $r_1 \xrightarrow{\Delta}_{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, @} r'_1$. By Remark B.5, we have $x \in \text{fv}(r_1)$, but at the same time $x \notin \text{fv}(r_1)$ by hypothesis of rule ES-L-DIST . Therefore this case is not possible.

e. $\text{ES-R-DIST}(1)$. Analogous to the previous case.

f. CONG-ES . Then $t_0 = u_0[x/vL] \equiv_c u'_0[x/r] = s_0$, derived from $u_0 \equiv_c u'_0$ and $vL \equiv_c r$; note that $r = v'L'$ by Remark D.9, hence it must be the case that we need to use rule REFL , so this case is not possible.

- SUB^\bullet . Then $t_0 = x \xrightarrow{\Delta}_{\text{sub}(x,v), \mathcal{A}' \cup \{x\}, \mathcal{S}, @} v = t_1$, where $\rho = \text{sub}(x,v)$, $\mathcal{A} = \mathcal{A}' \cup \{x\}$ and $\mu = @$. We also have $t_0 = x \equiv_c s_0$. This case is not possible since there are no rules to derive $x \equiv_c s_0$.
- The remaining cases are treated in a similar way.

□

D.1.1 Simulation of GLAMOUR steps in Useful Open CBV.

Definition D.12 (Syntax of the GLAMOUR). The set of **states** (s, s') , **dumps** (D, D', \dots) , **stacks** (π, π', \dots) , **stack items** (ϕ, ψ, \dots) , and **global environments** (E, E', \dots) are given by the following grammars:

$$\begin{array}{ll} s & ::= (D, \mathbf{t}, \pi, E) \\ D & ::= \epsilon \mid D : (\mathbf{t}, \pi) \\ \pi & ::= \epsilon \mid \phi^l : \pi \quad \text{where } l \in \{\mathbb{A}, \mathbb{S}\} \\ \phi & ::= \mathbf{t} \mid (\mathbf{t}, \pi) \\ E & ::= \epsilon \mid [x/\phi^l] : E \quad \text{where } l \in \{\mathbb{A}, \mathbb{S}\} \end{array}$$

where **codes** (\mathbf{t}, s, \dots) are terms with no explicit substitutions (*i.e.* pure terms), but *they are not* considered up to α -equivalence. We use to decorate stack items with **labels** $l \in \{\mathbb{A}, \mathbb{S}\}$, writing ϕ^l rather than ϕ . By convention this label always indicates the shape of ϕ , so in particular, $l = \mathbb{A}$ if and only if ϕ is of the form t , and $l = \mathbb{S}$ if and only if ϕ is of the form (t, π) . Intuitively, these labels indicate whether a stack item unfolds to a hereditary abstraction (\mathbb{A}) or a structure (\mathbb{S}).

Let $s = (D, \mathbf{t}, \pi, E)$. A **binding occurrence** for a variable x in s is either the leftmost occurrence of x in an abstraction $\lambda x. \mathbf{t}$ or the leftmost occurrence of x in an element $[x/\phi]$ of the environment. We say that s is **well-named** if the three following conditions hold:

1. Each variable x has at most one binding occurrence. For example, $((\lambda x. \lambda y. y, \epsilon), w, \epsilon, [z/w])$ is well-named, while $(\epsilon, \lambda x. x, \epsilon, [x/w])$ and $(\epsilon, \lambda x. \lambda x. y, \epsilon, \epsilon)$ are not.
2. If there is some binding occurrence for a variable x in an abstraction, then, all occurrences of x only occur inside the body of this abstraction. For example, $(\epsilon, \lambda x. \lambda y. x, x, \epsilon)$ is not well-named.
3. If there is some binding occurrence for a variable x in an environment of the form $E_1 : [x/\phi] : E_2$, then there are no other occurrences of x in E_2 . For example, $((\epsilon, \epsilon), y, y, [x/y][z/x])$ is not well-named.

Definition D.13 (Decoding of components). The decoding of the components of the GLAMoUr abstract machine into the syntax of our calculus is given by the following function:

$$\begin{aligned}
 \{(D, \mathbf{t}, \pi, E)\} &:= \{E\} \langle \{D\} \langle \{\pi\} \langle t \rangle \rangle \rangle \\
 \{\epsilon\} &:= \diamond \\
 \{D : (\mathbf{t}, \pi)\} &:= \{D\} \langle \{\pi\} \langle t \diamond \rangle \rangle \\
 \{\phi^l : \pi\} &:= \{\pi\} \langle \diamond \{ \phi^l \} \rangle \\
 \{[x/\phi^l] : E\} &:= \{E\} \langle \diamond [x/\{ \phi^l \}] \rangle \\
 \{(\mathbf{t}, \pi)^l\} &:= \{\pi\} \langle t \rangle \\
 \{\mathbf{t}^l\} &:= t \\
 \{\mathbf{t}\} &:= t
 \end{aligned}$$

In the two last lines, t denotes a term which is α -equivalent to \mathbf{t} .

Definition D.14 (Transitions of the GLAMoUr abstract machine). The transitions of the GLAMoUr abstract machine are defined as follows:

D	$\mathbf{t} \mathbf{u}$	π	E	\rightsquigarrow_{c_1}	$D : (\mathbf{t}, \pi)$	\mathbf{u}	ϵ	E
D	$\lambda \mathbf{x}. \mathbf{t}$	$\phi^l : \pi$	E	$\rightsquigarrow_{\text{um}}$	D	\mathbf{t}	π	$[x/\phi^l] : E$
$D : (\mathbf{t}, \pi)$	$\lambda \mathbf{x}. \mathbf{u}$	ϵ	E	\rightsquigarrow_{c_2}	D	\mathbf{t}	$(\lambda \mathbf{x}. \mathbf{u})^\Delta : \pi$	E
$D : (\mathbf{t}, \pi)$	\mathbf{x}	π'	E	\rightsquigarrow_{c_3}	D	\mathbf{t}	$(\mathbf{x}, \pi')^\mathbb{S} : \pi$	E $\mathbf{x} \notin \text{dom}(E)$
$D : (\mathbf{t}, \pi)$	\mathbf{x}	π'	$E_1 : [x/\phi^\mathbb{S}] : E_2$	\rightsquigarrow_{c_4}	D	\mathbf{t}	$(\mathbf{x}, \pi')^\mathbb{S} : \pi$	$E_1 : [x/\phi^\mathbb{S}] : E_2$
$D : (\mathbf{t}, \pi)$	\mathbf{x}	ϵ	$E_1 : [x/\mathbf{u}^\Delta] : E_2$	\rightsquigarrow_{c_5}	D	\mathbf{t}	$\mathbf{x}^\Delta : \pi$	$E_1 : [x/\mathbf{u}^\Delta] : E_2$
D	\mathbf{x}	$\phi^l : \pi$	$E_1 : [x/\mathbf{u}^\Delta] : E_2$	$\rightsquigarrow_{\text{ue}}$	D	\mathbf{u}^α	$\phi^l : \pi$	$E_1 : [x/\mathbf{u}^\Delta] : E_2$

where \mathbf{s}^α is any code α -equivalent to \mathbf{s} that preserves well-naming of the machine. Note that in [2] there are two syntactically different sorts of variables, corresponding to free and bound variables respectively. Here we use only one sort of variables, but this is just a matter of presentation.

Definition D.15. A state s_0 is **initial** if and only if it is of the form $s_0 = (\epsilon, \mathbf{t}, \epsilon, \epsilon)$ for some code \mathbf{t} . We say that a state s is **reachable** if and only if there exists an initial state s_0 such that $s_0 \rightsquigarrow^* s$.

Definition D.16. Let $s_0 = (\epsilon, \mathbf{t}_0, \epsilon, \epsilon)$ be an initial state in the abstract machine, and E a global environment. We say that a stack item ϕ is **E -rigid** if and only if:

1. $\phi^l = \mathbf{t}$ implies $\{\phi^l\} = t \in \text{HA}_{\mathcal{A}}$, where $\mathcal{A} = \emptyset^{\{\{E\}\}}$ is an abstraction frame, and $\{\{E\}\}$ is the list of substitution contexts resulting from the decoding of E . This is equivalent to saying that if $l = \mathbb{A}$ then $\{\phi^l\} \in \text{HA}_{\mathcal{A}}$.
2. $\phi^l = (\mathbf{t}, \pi)$ implies $\{\phi^l\} = \{\{\pi\}\} \langle t \rangle \in \text{St}_{\mathcal{S}}$, where $\mathcal{S} = \text{fv}(\mathbf{t}_0)^{\{\{E\}\}}$ is a structure frame, and $\{\{E\}\}$ is the list of substitution contexts resulting from the decoding of E . This is equivalent to saying that if $l = \mathbb{S}$ then $\{\phi^l\} \in \text{St}_{\mathcal{S}}$.

Remark D.17. The following holds:

1. $\{\{D\}\} \langle t \rangle \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$ if and only if $t \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$ and $\{\{D\}\} \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$.
2. $\{\{\pi\}\} \langle t \rangle \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$ if and only if $t \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$ and $\{\{\pi\}\} \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$.
3. $\{\{E\}\} \langle t \rangle \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$ if and only if $t \in \text{Stable}_{\mathcal{A}^{\{\{E\}\}}, \mathcal{S}^{\{\{E\}\}}}$ and $\{\{E\}\} \in \text{Stable}_{\mathcal{A}, \mathcal{S}}$.

LEMMA D.18 ((NEW) INVARIANTS FOR THE GLAMoUr). *Let s_r be a reachable state from an initial state $(\epsilon, \mathbf{t}_0, \epsilon, \epsilon)$. Then s_r verifies the following invariant:*

1. $\{s_r\} \in \text{Stable}_{\emptyset, \text{fv}(\mathbf{t}_0)}$
2. If $s_r = (D, \mathbf{t}, \pi_1 : \phi^l : \pi_2, E)$, then ϕ^l is E -rigid.
3. If $s_r = (D, \mathbf{t}, \pi, E_1 : [x/\phi^l] : E_2)$, then ϕ^l is E_2 -rigid.

PROOF. We prove that each transition step \rightsquigarrow in the GLAMoUr machine preserves the invariant *i.e.* if $s \rightsquigarrow s'$ and s verifies the invariant, then s' verifies the invariant as well. We proceed by induction on the transition step \rightsquigarrow .

- If $s = (D, \mathbf{t}, \mathbf{u}, \pi, E) \rightsquigarrow_{c_1} (D : (\mathbf{t}, \pi), \mathbf{u}, \epsilon, E) = s'$ then

$$\begin{aligned}
\{\!\{D, \mathbf{t}, \mathbf{u}, \pi, E\}\!\} &= (\{\!\{E\}\!\} \langle \{\!\{D\}\!\} \langle \{\!\{\pi\}\!\} \rangle \rangle \langle \mathbf{t}, \mathbf{u} \rangle) \\
&= \{\!\{E\}\!\} \langle \{\!\{D\}\!\} \langle \{\!\{\pi\}\!\} \langle \mathbf{t}, \mathbf{u} \rangle \rangle \rangle \\
&= (\{\!\{E\}\!\} \langle \{\!\{D\}\!\} \langle \{\!\{\pi\}\!\} \langle \mathbf{t}, \diamond \rangle \rangle \rangle \langle \mathbf{u} \rangle) \\
&= (\{\!\{E\}\!\} \langle (\{\!\{D\}\!\} \langle \{\!\{\pi\}\!\} \langle \mathbf{t}, \diamond \rangle \rangle) \langle \diamond \rangle \rangle \langle \mathbf{u} \rangle) \\
&= (\{\!\{E\}\!\} \langle (\{\!\{D\}\!\} \langle \{\!\{\pi\}\!\} \langle \mathbf{t}, \diamond \rangle \rangle) \langle \{\!\{\epsilon\}\!\} \rangle \rangle \langle \mathbf{u} \rangle) \\
&= (\{\!\{E\}\!\} \langle \{\!\{D : (\mathbf{t}, \pi)\}\!\} \langle \{\!\{\epsilon\}\!\} \rangle \rangle \langle \mathbf{u} \rangle) \\
&= \{\!\{D : (\mathbf{t}, \pi), \mathbf{u}, \epsilon, E\}\!\}
\end{aligned}$$

Therefore $\{\!\{s'\}\!\} \in \text{Stable}_{\emptyset, \text{fv}(t_0)}$ holds as the translation of both states is the same. Item 2 of the invariant trivially holds for s' since it has an empty stack, and item 3 holds for s' since it holds for s which has the same environment E .

- If $s = (D, \lambda x. \mathbf{t}, \phi^l : \pi, E) \rightsquigarrow_{\text{um}} (D, \mathbf{t}, \pi, [x/\phi^l] : E) = s'$, then s verifies the invariant:
 - 1'. $\{\!\{D, \lambda x. \mathbf{t}, \phi^l : \pi, E\}\!\} = \{\!\{E\}\!\} \langle \{\!\{D\}\!\} \langle \{\!\{\pi\}\!\} \langle (\lambda x. \mathbf{t}) \{\!\{\phi^l\}\!\} \rangle \rangle \rangle \in \text{Stable}_{\emptyset, \text{fv}(t_0)}$. Then by Remark D.17 $\{\!\{D\}\!\}$, $\{\!\{\pi\}\!\}$ and $\{\!\{\phi^l\}\!\}$ are all in $\text{Stable}_{\emptyset, \text{fv}(t_0) \{\!\{E\}\!\}}$, and $\{\!\{E\}\!\} \in \text{Stable}_{\emptyset, \text{fv}(t_0)}$
 - 2'. ϕ^l is E -rigid, and if π is of the form $\pi_1 : \psi^l : \pi_2$, then ψ^l is E -rigid
 - 3'. If E is of the form $E_1 : [y/\psi^l] : E_2$, then ψ^l is E_2 -rigid.

Let us show that the invariant holds for s' :

1. $\{\!\{E\}\!\} \langle \{\!\{D\}\!\} \langle \{\!\{\pi\}\!\} \langle \mathbf{t} [x/\{\!\{\phi^l\}\!\}] \rangle \rangle \rangle \in \text{Stable}_{\emptyset, \text{fv}(t_0)}$: this holds directly from (1').
 2. If π is of the form $\pi_1 : \psi^l : \pi_2$, then ψ^l is E -rigid: this holds directly from (2').
 3. ϕ^l is E -rigid, by item (2'), and the rest of this condition holds by (3').
- If $s = (D, x, \phi^l : \pi, E) \rightsquigarrow_{\text{ue}} (D, \mathbf{t}^\mathbb{A}, \phi^l : \pi, E) = s'$, where $E = E_1 : [x/\mathbf{t}^\mathbb{A}] : E_2$, then s verifies the invariant:
 - 1'. $\{\!\{D, x, \phi^l : \pi, E_1 : [x/\mathbf{t}^\mathbb{A}] : E_2\}\!\} = \{\!\{E_2\}\!\} \langle \{\!\{D\}\!\} \langle \{\!\{\pi\}\!\} \langle x \{\!\{\phi^l\}\!\} \rangle \rangle \{\!\{E_1 [x/\mathbf{t}^\mathbb{A}]\}\!\} \rangle \in \text{Stable}_{\emptyset, \text{fv}(t_0)}$. Then by Remark D.17 $\{\!\{D\}\!\}$, $\{\!\{\pi\}\!\}$ and $\{\!\{\phi^l\}\!\}$ are all in $\text{Stable}_{\emptyset, \text{fv}(t_0) \{\!\{E\}\!\}}$, and $\{\!\{E\}\!\} = \{\!\{E_1 : [x/\mathbf{t}^\mathbb{A}] : E_2\}\!\} \in \text{Stable}_{\emptyset, \text{fv}(t_0)}$
 - 2'. ϕ^l is E -rigid
 - 3'. $\mathbf{t}^\mathbb{A}$ is E_2 -rigid

Let us show that the invariant holds for s' :

1. $\{\!\{E_2\}\!\} \langle \{\!\{D\}\!\} \langle \{\!\{\pi\}\!\} \langle \mathbf{t} \{\!\{\phi^l\}\!\} \rangle \rangle \{\!\{E_1 [x/\mathbf{t}^\mathbb{A}]\}\!\} \rangle \in \text{Stable}_{\emptyset, \text{fv}(t_0)}$: this holds directly from (1')
 2. ϕ^l is $E_1 : [x/\mathbf{t}^\mathbb{A}] : E_2$ -rigid: this holds directly from (2')
 3. $\mathbf{t}^\mathbb{A}$ is E_2 -rigid: this holds directly from (3').
- If $s = (D : (\mathbf{t}, \pi), \lambda x. \mathbf{u}, \epsilon, E) \rightsquigarrow_{c_2} (D, \mathbf{t}, (\lambda x. \mathbf{u})^\mathbb{A} : \pi, E) = s'$ then

$$\begin{aligned}
\{\!\{D : (\mathbf{t}, \pi), \lambda x. \mathbf{u}, \epsilon, E\}\!\} &= (\{\!\{E\}\!\} \langle \{\!\{D : (\mathbf{t}, \pi)\}\!\} \langle \{\!\{\epsilon\}\!\} \rangle \rangle \langle \lambda x. \mathbf{u} \rangle) \\
&= (\{\!\{E\}\!\} \langle \{\!\{D\}\!\} \langle \{\!\{\pi\}\!\} \langle \mathbf{t}, \diamond \rangle \rangle \langle \{\!\{\epsilon\}\!\} \rangle \rangle \langle \lambda x. \mathbf{u} \rangle) \\
&= (\{\!\{E\}\!\} \langle \{\!\{D\}\!\} \langle \{\!\{\pi\}\!\} \langle \mathbf{t}, \diamond \rangle \rangle \langle \diamond \rangle \rangle \langle \lambda x. \mathbf{u} \rangle) \\
&= (\{\!\{E\}\!\} \langle \{\!\{D\}\!\} \langle \{\!\{\pi\}\!\} \langle \mathbf{t}, \diamond \rangle \rangle \rangle \langle \lambda x. \mathbf{u} \rangle) \\
&= (\{\!\{E\}\!\} \langle \{\!\{D\}\!\} \langle \{\!\{\pi\}\!\} \langle \mathbf{t}, \lambda x. \mathbf{u} \rangle \rangle \rangle) \\
&= (\{\!\{E\}\!\} \langle \{\!\{D\}\!\} \langle \{\!\{\pi\}\!\} \langle \diamond (\lambda x. \mathbf{u}) \rangle \rangle \rangle \langle \mathbf{t} \rangle) \\
&= (\{\!\{E\}\!\} \langle \{\!\{D\}\!\} \langle \{\!\{(\lambda x. \mathbf{u})^\mathbb{A} : \pi\}\!\} \rangle \rangle \langle \mathbf{t} \rangle) \\
&= \{\!\{D, \mathbf{t}, (\lambda x. \mathbf{u})^\mathbb{A} : \pi, E\}\!\}
\end{aligned}$$

Therefore $\{\!\{s'\}\!\} \in \text{Stable}_{\emptyset, \text{fv}(t_0)}$ holds as the translation of both states is the same. Item 2 of the invariant trivially holds for s' since $\lambda x. \mathbf{u} \in \text{HA}_{\mathcal{A}}$ for any abstraction frame \mathcal{A} . Item 3 holds for s' since it holds for s which has the same environment E .

- If $s = (D : (\mathbf{t}, \pi), x, \pi', E) \rightsquigarrow_{c_3} (D, \mathbf{t}, (x, \pi')^{\mathbb{S}} : \pi, E) = s'$, with $x \notin \text{dom}(E)$, then

$$\begin{aligned}
 \llbracket (D : (\mathbf{t}, \pi), x, \pi', E) \rrbracket &= (\llbracket E \rrbracket \langle \llbracket D : (\mathbf{t}, \pi) \rrbracket \langle \llbracket \pi' \rrbracket \rangle \rangle \langle x \rangle) \\
 &= (\llbracket E \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle t \diamond \rangle \langle \llbracket \pi' \rrbracket \rangle \rangle \rangle \langle x \rangle) \\
 &= (\llbracket E \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle t \llbracket \pi' \rrbracket \rangle \rangle \rangle \langle x \rangle) \\
 &= \llbracket E \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle t \llbracket \pi' \rrbracket \langle x \rangle \rangle \rangle \rangle \\
 &= (\llbracket E \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle \diamond \llbracket \pi' \rrbracket \langle x \rangle \rangle \rangle \rangle \langle t \rangle) \\
 &= (\llbracket E \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle \diamond \llbracket (x, \pi')^{\mathbb{S}} \rrbracket \rangle \rangle \rangle \rangle \langle t \rangle) \\
 &= (\llbracket E \rrbracket \langle \llbracket D \rrbracket \langle \llbracket (x, \pi')^{\mathbb{S}} : \pi \rrbracket \rangle \rangle \rangle \langle t \rangle) \\
 &= \llbracket (D, \mathbf{t}, (x, \pi')^{\mathbb{S}} : \pi, E) \rrbracket
 \end{aligned}$$

Therefore $\llbracket s' \rrbracket \in \text{Stable}_{\emptyset, \text{fv}(t_0)}$ holds as the translation of both states is the same. Item 2 of the invariant holds for s' since $\llbracket \pi' \rrbracket \langle x \rangle \in \text{St}_{\text{fv}(t_0), \llbracket E \rrbracket}$, given the fact that $x \in \text{fv}(t_0)$. And item 3 holds for s' since it holds for s which has the same environment E .

- If $s = (D : (\mathbf{t}, \pi), x, \pi', E_1 : [x/\phi^{\mathbb{S}}] : E_2) \rightsquigarrow_{c_4} (D, \mathbf{t}, (x, \pi')^{\mathbb{S}} : \pi, E_1 : [x/\phi^{\mathbb{S}}] : E_2) = s'$ then

$$\begin{aligned}
 \llbracket (D : (\mathbf{t}, \pi), x, \pi', E_1 : [x/\phi^{\mathbb{S}}] : E_2) \rrbracket &= (\llbracket E_1[x/\phi^{\mathbb{S}}]E_2 \rrbracket \langle \llbracket D : (\mathbf{t}, \pi) \rrbracket \langle \llbracket \pi' \rrbracket \rangle \rangle \langle x \rangle) \\
 &= (\llbracket E_2 \rrbracket \langle \diamond \llbracket E_1[x/\phi^{\mathbb{S}}] \rrbracket \rangle \langle \llbracket D : (\mathbf{t}, \pi) \rrbracket \langle \llbracket \pi' \rrbracket \rangle \rangle \langle x \rangle) \\
 &= (\llbracket E_2 \rrbracket \langle \llbracket D : (\mathbf{t}, \pi) \rrbracket \langle \llbracket \pi' \rrbracket \rangle \llbracket E_1[x/\phi^{\mathbb{S}}] \rrbracket \rangle \langle x \rangle) \\
 &= (\llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle t \diamond \rangle \langle \llbracket \pi' \rrbracket \rangle \llbracket E_1[x/\phi^{\mathbb{S}}] \rrbracket \rangle \rangle \langle x \rangle) \\
 &= (\llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle t \llbracket \pi' \rrbracket \rangle \rangle \llbracket E_1[x/\phi^{\mathbb{S}}] \rrbracket \rangle \langle x \rangle) \\
 &= \llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle t \llbracket \pi' \rrbracket \langle x \rangle \rangle \rangle \llbracket E_1[x/\phi^{\mathbb{S}}] \rrbracket \rangle \\
 &= (\llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle \diamond \llbracket \pi' \rrbracket \langle x \rangle \rangle \rangle \llbracket E_1[x/\phi^{\mathbb{S}}] \rrbracket \rangle \rangle \langle t \rangle) \\
 &= (\llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle \diamond \llbracket (x, \pi')^{\mathbb{S}} \rrbracket \rangle \rangle \rangle \llbracket E_1[x/\phi^{\mathbb{S}}] \rrbracket \rangle \rangle \langle t \rangle) \\
 &= (\llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket (x, \pi')^{\mathbb{S}} : \pi \rrbracket \rangle \rangle \llbracket E_1[x/\phi^{\mathbb{S}}] \rrbracket \rangle \rangle \langle t \rangle) \\
 &= (\llbracket E_2 \rrbracket \langle \diamond \llbracket E_1[x/\phi^{\mathbb{S}}] \rrbracket \rangle \langle \llbracket D \rrbracket \langle \llbracket (x, \pi')^{\mathbb{S}} : \pi \rrbracket \rangle \rangle \rangle \langle t \rangle) \\
 &= (\llbracket E_1[x/\phi^{\mathbb{S}}]E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket (x, \pi')^{\mathbb{S}} : \pi \rrbracket \rangle \rangle \rangle \langle t \rangle) \\
 &= \llbracket (D, \mathbf{t}, (x, \pi')^{\mathbb{S}} : \pi, E_1 : [x/\phi^{\mathbb{S}}] : E_2) \rrbracket
 \end{aligned}$$

Therefore $\llbracket s' \rrbracket \in \text{Stable}_{\emptyset, \text{fv}(t_0)}$ holds as the translation of both states is the same. Item 2 of the invariant holds for s' since $\llbracket \pi' \rrbracket \langle x \rangle \in \text{St}_{\text{fv}(t_0), \llbracket E_1 : [x/\phi^{\mathbb{S}}] : E_2 \rrbracket}$, since the stack item is decorated with the label \mathbb{S} . And item 3 holds for s' since it holds for s which has the same environment E .

- If $s = (D : (\mathbf{t}, \pi), x, \epsilon, E_1 : [x/\mathbf{u}^{\mathbb{A}}] : E_2) \rightsquigarrow_{c_5} (D, \mathbf{t}, x^{\mathbb{A}} : \pi, E_1 : [x/\mathbf{u}^{\mathbb{A}}] : E_2) = s'$ then

$$\begin{aligned}
 \llbracket (D : (\mathbf{t}, \pi), x, \epsilon, E_1 : [x/\mathbf{u}^{\mathbb{A}}] : E_2) \rrbracket &= (\llbracket E_1[x/\mathbf{u}^{\mathbb{A}}]E_2 \rrbracket \langle \llbracket D : (\mathbf{t}, \pi) \rrbracket \langle \llbracket \epsilon \rrbracket \rangle \rangle \langle x \rangle) \\
 &= (\llbracket E_2 \rrbracket \langle \diamond \llbracket E_1[x/\mathbf{u}^{\mathbb{A}}] \rrbracket \rangle \langle \llbracket D : (\mathbf{t}, \pi) \rrbracket \langle \llbracket \epsilon \rrbracket \rangle \rangle \langle x \rangle) \\
 &= (\llbracket E_2 \rrbracket \langle \llbracket D : (\mathbf{t}, \pi) \rrbracket \langle \llbracket \epsilon \rrbracket \rangle \llbracket E_1[x/\mathbf{u}^{\mathbb{A}}] \rrbracket \rangle \langle x \rangle) \\
 &= (\llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle t \diamond \rangle \langle \llbracket \epsilon \rrbracket \rangle \llbracket E_1[x/\mathbf{u}^{\mathbb{A}}] \rrbracket \rangle \rangle \langle x \rangle) \\
 &= (\llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle t \diamond \rangle \langle \diamond \llbracket E_1[x/\mathbf{u}^{\mathbb{A}}] \rrbracket \rangle \rangle \rangle \langle x \rangle) \\
 &= (\llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle t \diamond \rangle \rangle \llbracket E_1[x/\mathbf{u}^{\mathbb{A}}] \rrbracket \rangle \rangle \langle x \rangle) \\
 &= \llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle t x \rangle \rangle \llbracket E_1[x/\mathbf{u}^{\mathbb{A}}] \rrbracket \rangle \\
 &= (\llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle \diamond x \rangle \rangle \llbracket E_1[x/\mathbf{u}^{\mathbb{A}}] \rrbracket \rangle \rangle \langle t \rangle) \\
 &= (\llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket x^{\mathbb{A}} : \pi \rrbracket \rangle \llbracket E_1[x/\mathbf{u}^{\mathbb{A}}] \rrbracket \rangle \rangle \langle t \rangle) \\
 &= (\llbracket E_2 \rrbracket \langle \diamond \llbracket E_1[x/\mathbf{u}^{\mathbb{A}}] \rrbracket \rangle \langle \llbracket D \rrbracket \langle \llbracket x^{\mathbb{A}} : \pi \rrbracket \rangle \rangle \rangle \langle t \rangle) \\
 &= (\llbracket E_1[x/\mathbf{u}^{\mathbb{A}}]E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket x^{\mathbb{A}} : \pi \rrbracket \rangle \rangle \rangle \langle t \rangle) \\
 &= \llbracket (D, \mathbf{t}, x^{\mathbb{A}} : \pi, E_1 : [x/\mathbf{u}^{\mathbb{A}}] : E_2) \rrbracket
 \end{aligned}$$

Therefore $\llbracket s' \rrbracket \in \text{Stable}_{\emptyset, \text{fv}(t_0)}$ holds as the translation of both states is the same. Item 2 of the invariant holds since $x \in \text{HA}_{\emptyset, \llbracket E_1 : E_2 \rrbracket \cup \{x\}}$, since the code \mathbf{u} is decorated with the label \mathbb{A} . And item 3 for s' holds since it holds for s which has the same environment E . □

LEMMA D.19. *Let $x \notin \text{fv}(\llbracket \pi \rrbracket)$ and $x \notin \text{fv}(\llbracket D \rrbracket)$. The following holds:*

1. $\llbracket \pi \rrbracket \langle t[x/s] \rangle \equiv \llbracket \pi \rrbracket \langle t \rangle [x/s]$
2. $\llbracket D \rrbracket \langle t[x/s] \rangle \equiv \llbracket D \rrbracket \langle t \rangle [x/s]$

PROOF.

1. We proceed by induction on the structure of π .

- If $\pi = \epsilon$, then $\{\!\{ \epsilon \}\!\} \langle t[x/s] \rangle = \diamond \langle t[x/s] \rangle = t[x/s] \equiv \diamond \langle t \rangle [x/s] = \{\!\{ \epsilon \}\!\} \langle t \rangle [x/s]$.
- If $\pi = \phi^l : \pi'$, then by α -equivalence, we may assume in particular $x \notin \text{fv}(\{\!\{ \phi^l \}\!\})$. Therefore

$$\begin{aligned}
\{\!\{ \phi^l : \pi' \}\!\} \langle t[x/s] \rangle &= \{\!\{ \pi' \}\!\} \langle \diamond \{\!\{ \phi^l \}\!\} \rangle \langle t[x/s] \rangle \\
&= \{\!\{ \pi' \}\!\} \langle t[x/s] \{\!\{ \phi^l \}\!\} \rangle \\
&\equiv \{\!\{ \pi' \}\!\} \langle (t \{\!\{ \phi^l \}\!\}) [x/s] \rangle && \text{(by ES-L-DIST, since } x \notin \text{fv}(\{\!\{ \phi^l \}\!\}) \text{)} \\
&\equiv \{\!\{ \pi' \}\!\} \langle t \{\!\{ \phi^l \}\!\} \rangle [x/s] && \text{(by i.h. on } \pi' \text{)} \\
&= \{\!\{ \phi^l : \pi' \}\!\} \langle t \rangle [x/s]
\end{aligned}$$

2. We proceed by induction on the structure of D .

- If $D = \epsilon$, then it is analogous to the base case in the previous item.
- If $D = D' : (\mathbf{u}, \pi)$, then by α -equivalence, we may assume in particular $x \notin \text{fv}(\mathbf{u})$. Therefore

$$\begin{aligned}
\{\!\{ D' : (\mathbf{u}, \pi) \}\!\} \langle t[x/s] \rangle &= \{\!\{ D' \}\!\} \langle \{\!\{ \pi \}\!\} \langle \mathbf{u} \diamond \rangle \rangle \langle t[x/s] \rangle \\
&= \{\!\{ D' \}\!\} \langle \{\!\{ \pi \}\!\} \langle \mathbf{u} t[x/s] \rangle \rangle \\
&\equiv \{\!\{ D' \}\!\} \langle \{\!\{ \pi \}\!\} \langle (\mathbf{u} t) [x/s] \rangle \rangle && \text{(by ES-R-DIST, since } x \notin \text{fv}(\mathbf{u}) \text{)} \\
&\equiv \{\!\{ D' \}\!\} \langle \{\!\{ \pi \}\!\} \langle \mathbf{u} t \rangle [x/s] \rangle && \text{(by (1))} \\
&\equiv \{\!\{ D' \}\!\} \langle \{\!\{ \pi \}\!\} \langle \mathbf{u} t \rangle \rangle [x/s] && \text{(by i.h. on } D' \text{)} \\
&= \{\!\{ D' : (\mathbf{u}, \pi) \}\!\} \langle t \rangle [x/s]
\end{aligned}$$

□

LEMMA 6.1 (GLAMOUR SIMULATION). *Let s be a state reachable from an initial state whose focus is t_0 , and let $S_0 := \text{fv}(t_0)$. Then:*

1. If $s \rightsquigarrow_{\text{um}} s'$, then $\{\!\{ s \}\!\} \xrightarrow{\text{db}} \{\!\{ s' \}\!\}$.
2. If $s \rightsquigarrow_{\text{ue}} s'$, then $\{\!\{ s \}\!\} \xrightarrow{\text{lsv}} \{\!\{ s' \}\!\}$.
3. If $s \rightsquigarrow_{c_i} s'$, then $\{\!\{ s \}\!\} = \{\!\{ s' \}\!\}$, for all $i \in \{1..5\}$.
4. Progress: if s is \rightsquigarrow -irreducible then $\{\!\{ s \}\!\}$ is $\xrightarrow{\text{db}}$ -irreducible.

PROOF. We proceed by case analysis of \rightsquigarrow . To lighten the proof, we simplify the notation $[x/\phi^l] : E$ by $[x/\phi^l]E$.

1. If $(D, \lambda x. \mathbf{t}, \phi^l : \pi, E) \rightsquigarrow_{\text{um}} (D, \mathbf{t}, \pi, [x/\phi^l] : E)$ then

$$\begin{aligned}
\{\!\{ (D, \lambda x. \mathbf{t}, \phi^l : \pi, E) \}\!\} &= (\{\!\{ E \}\!\} \langle \{\!\{ D \}\!\} \langle \{\!\{ \pi \}\!\} \rangle \rangle \langle \lambda x. \mathbf{t} \rangle \rangle \\
&= (\{\!\{ E \}\!\} \langle \{\!\{ D \}\!\} \langle \{\!\{ \pi \}\!\} \langle \diamond \{\!\{ \phi^l \}\!\} \rangle \rangle \rangle \langle \lambda x. \mathbf{t} \rangle \rangle \\
&= \{\!\{ E \}\!\} \langle \{\!\{ D \}\!\} \langle \{\!\{ \pi \}\!\} \langle (\lambda x. \mathbf{t}) \{\!\{ \phi^l \}\!\} \rangle \rangle \rangle \\
&\xrightarrow{\text{db}, \emptyset, \text{fv}(s_0) \text{ @}} && (\star) \\
&\{\!\{ E \}\!\} \langle \{\!\{ D \}\!\} \langle \{\!\{ \pi \}\!\} \langle t[x/\{\!\{ \phi^l \}\!\}] \rangle \rangle \rangle \\
&\equiv \{\!\{ E \}\!\} \langle \{\!\{ D \}\!\} \langle \{\!\{ \pi \}\!\} \langle t \rangle \rangle [x/\{\!\{ \phi^l \}\!\}] \rangle && \text{By Lemma D.19(*)} \\
&= (\{\!\{ E \}\!\} \langle \{\!\{ D \}\!\} \langle \{\!\{ \pi \}\!\} \rangle [x/\{\!\{ \phi^l \}\!\}] \rangle \rangle \langle t \rangle \rangle \\
&= (\{\!\{ E \}\!\} \langle \diamond [x/\{\!\{ \phi^l \}\!\}] \rangle \langle \{\!\{ D \}\!\} \langle \{\!\{ \pi \}\!\} \rangle \rangle \rangle \langle t \rangle \rangle \\
&= (\{\!\{ [x/\phi^l]E \}\!\} \langle \{\!\{ D \}\!\} \langle \{\!\{ \pi \}\!\} \rangle \rangle \rangle \langle t \rangle \rangle \\
&= \{\!\{ (D, \mathbf{t}, \pi, [x/\phi^l]E) \}\!\}
\end{aligned}$$

The step (\star) holds by the following: $\{\!\{ E \}\!\} \langle \{\!\{ D \}\!\} \langle \{\!\{ \pi \}\!\} \rangle \langle (\lambda x. \mathbf{t}) \{\!\{ \phi^l \}\!\} \rangle \rangle \in \text{Stable}_{\emptyset, \text{fv}(t_0)}$ and ϕ^l is E -rigid by Lemma D.18 (items 1 and 2) respectively. Then $\{\!\{ \phi^l \}\!\} \in \text{HA}_{\emptyset, \{\!\{ E \}\!\}} \cup \text{St}_{\text{fv}(t_0), \{\!\{ E \}\!\}}$, so applying congruence reduction rules accordingly to reach the redex $(\lambda x. \mathbf{t}) \{\!\{ \phi^l \}\!\}$, we can reduce this subterm with rule DB-STABLE*, so that $(\lambda x. \mathbf{t}) \{\!\{ \phi^l \}\!\} \xrightarrow{\text{db}, \emptyset, \{\!\{ E \}\!\}, \text{fv}(t_0), \{\!\{ E \}\!\}, \mu} t[x/\{\!\{ \phi^l \}\!\}]$.

On the other hand, the step $(*)$ holds applying rule ES-L-DIST, as $x \notin \text{fv}(\{\!\{ \pi \}\!\})$ and $x \notin \text{fv}(\{\!\{ D \}\!\})$ holds by α -conversion.

2. If $(D, x, \phi^l : \pi, E_1 : [x/v^\Delta] : E_2) \rightsquigarrow_{ue} (D, v^\alpha, \phi^l : \pi, E_1 : [x/v^\Delta] : E_2)$ then

$$\begin{aligned}
 \llbracket (D, x, \phi^l : \pi, E_1[x/v^\Delta]E_2) \rrbracket &= \llbracket F_{(D, x, \phi^l : \pi, E_1[x/v^\Delta]E_2)} \rrbracket \langle x \rangle \\
 &= (\llbracket E_1[x/v^\Delta]E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \phi^l : \pi \rrbracket \rangle \rangle) \langle x \rangle \\
 &= ((\llbracket E_2 \rrbracket \langle \diamond \llbracket E_1[x/v^\Delta] \rrbracket \rangle) \langle \llbracket D \rrbracket \langle \llbracket \phi^l : \pi \rrbracket \rangle \rangle) \langle x \rangle \\
 &= (\llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \phi^l : \pi \rrbracket \rangle \llbracket E_1[x/v^\Delta] \rrbracket \rangle) \langle x \rangle \\
 &= (\llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle \diamond \llbracket \phi^l \rrbracket \rangle \llbracket E_1[x/v^\Delta] \rrbracket \rangle \rangle) \langle x \rangle \\
 &= \llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle x \llbracket \phi^l \rrbracket \rangle \rangle \llbracket E_1[x/v^\Delta] \rrbracket \rangle \\
 &\xrightarrow{\text{lsv}, \diamond, \text{fv}(t_0), \diamond} \llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle t \llbracket \phi^l \rrbracket \rangle \rangle \llbracket E_1[x/v^\Delta] \rrbracket \rangle \quad (\star) \\
 &= (\llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle \diamond \llbracket \phi^l \rrbracket \rangle \rangle \llbracket E_1[x/v^\Delta] \rrbracket \rangle) \langle t \rangle \\
 &= (\llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \phi^l : \pi \rrbracket \rangle \llbracket E_1[x/v^\Delta] \rrbracket \rangle) \langle t \rangle \\
 &= (\llbracket E_2 \rrbracket \langle \diamond \llbracket E_1[x/v^\Delta] \rrbracket \rangle \langle \llbracket D \rrbracket \langle \llbracket \phi^l : \pi \rrbracket \rangle \rangle) \langle t \rangle \\
 &= (\llbracket E_1[x/v^\Delta]E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \phi^l : \pi \rrbracket \rangle \rangle) \langle t \rangle \\
 &= \llbracket F_{(D, v^\alpha, \phi^l : \pi, E_1[x/v^\Delta]E_2)} \rrbracket \langle t \rangle \\
 &= \llbracket (D, t^\alpha, \phi^l : \pi, E_1[x/v^\Delta]E_2) \rrbracket
 \end{aligned}$$

The step (\star) holds since $\llbracket E_2 \rrbracket \langle \llbracket D \rrbracket \langle \llbracket \pi \rrbracket \langle x \llbracket \phi^l \rrbracket \rangle \rangle \llbracket E_1[x/v^\Delta] \rrbracket \rangle \in \text{Stable}_{\diamond, \text{fv}(t_0)}$ by item 1 of Lemma D.18. So applying congruence reduction rules and rule LSV^\bullet accordingly to reach the redex $x \llbracket \phi^l \rrbracket$, we can reduce this subterm with rule APPL^\bullet , so that the reduction $x \llbracket \phi^l \rrbracket \xrightarrow{\text{sub}(x, v), \diamond \llbracket E_1E_2 \rrbracket \cup \{x\}, \text{fv}(t_0) \llbracket E_1E_2 \rrbracket, \mu} v \llbracket \phi^l \rrbracket$, is derived from applying rule SUB^\bullet :

$$x \xrightarrow{\text{sub}(x, v), \diamond \llbracket E_1E_2 \rrbracket \cup \{x\}, \text{fv}(t_0) \llbracket E_1E_2 \rrbracket, \diamond} v.$$

3. If $(D, \mathbf{t} \mathbf{u}, \pi, E) \rightsquigarrow_{c_1} (D : (\mathbf{t}, \pi), \mathbf{u}, \epsilon, E)$ then $\llbracket (D, \mathbf{t} \mathbf{u}, \pi, E) \rrbracket = \llbracket (D : (\mathbf{t}, \pi), \mathbf{u}, \epsilon, E) \rrbracket$, is proved as in case \rightsquigarrow_{c_1} in Lemma D.18.
4. If $(D : (\mathbf{t}, \pi), \lambda x. \mathbf{u}, \epsilon, E) \rightsquigarrow_{c_2} (D, \mathbf{t}, (\lambda x. \mathbf{u})^\Delta : \pi, E)$ then $\llbracket (D : (\mathbf{t}, \pi), \lambda x. \mathbf{u}, \epsilon, E) \rrbracket = \llbracket (D, \mathbf{t}, (\lambda x. \mathbf{u})^\Delta : \pi, E) \rrbracket$, is proved as in case \rightsquigarrow_{c_2} in Lemma D.18.
5. If $(D : (\mathbf{t}, \pi), x, \pi', E) \rightsquigarrow_{c_3} (D, \mathbf{t}, (x, \pi')^\S : \pi, E)$, with $x \notin \text{dom}(E)$, then $\llbracket (D : (\mathbf{t}, \pi), x, \pi', E) \rrbracket = \llbracket (D, \mathbf{t}, (x, \pi')^\S : \pi, E) \rrbracket$, is proved as in case \rightsquigarrow_{c_3} in Lemma D.18.
6. If $(D : (\mathbf{t}, \pi), x, \pi', E_1 : [x/\phi^\S] : E_2) \rightsquigarrow_{c_4} (D, \mathbf{t}, (x, \pi')^\S : \pi, E_1 : [x/\phi^\S] : E_2)$ then $\llbracket (D : (\mathbf{t}, \pi), x, \pi', E_1[x/\phi^\S]E_2) \rrbracket = \llbracket (D, \mathbf{t}, (x, \pi')^\S : \pi, E_1[x/\phi^\S]E_2) \rrbracket$ is proved as in case \rightsquigarrow_{c_4} in Lemma D.18.
7. If $(D : (\mathbf{t}, \pi), x, \epsilon, E_1 : [x/\mathbf{u}^\Delta] : E_2) \rightsquigarrow_{c_5} (D, \mathbf{t}, x^\Delta : \pi, E_1 : [x/\mathbf{u}^\Delta] : E_2)$ then $\llbracket (D : (\mathbf{t}, \pi), x, \epsilon, E_1[x/\mathbf{u}^\Delta]E_2) \rrbracket = \llbracket (D, \mathbf{t}, x^\Delta : \pi, E_1[x/\mathbf{u}^\Delta]E_2) \rrbracket$ is proved as in case \rightsquigarrow_{c_5} in Lemma D.18. □

We are now ready to give the main results of this section. On the first hand, we have the high-level implementation result, stating the following:

THEOREM 6.3 (HIGH-LEVEL IMPLEMENTATION). *Let t be a pure term (without ESSs) and $S = \text{fv}(t)$. If $t \xrightarrow{\beta_f^n} t'$ then there exists s such that $t \xrightarrow{k} s$ where $s^\Downarrow = t'$ and $k \in O(|t| \cdot (n^2 + 1))$.*

PROOF. Let $t \xrightarrow{\beta_f^n} t'$. By Thm. 8, Thm. 9 and Coro. 1 in [2], there is a sequence of p transitions $s \rightsquigarrow^p s'$ in the GLAMoUr, where s is an initial state such that $\llbracket s \rrbracket = t$ and $p \in O(|t| \cdot (n^2 + 1))$. By Lemmas 6.1 and 6.2 there exists a term t'' such that $t = \llbracket s \rrbracket \xrightarrow{k} t'' \equiv \llbracket s' \rrbracket$ where $k \leq p$, so $k \in O(|t| \cdot (n^2 + 1))$. To conclude, we are left to show that $t''^\Downarrow = t'$. It is easy to see that $t'' \equiv \llbracket s' \rrbracket$ implies $t''^\Downarrow = \llbracket s' \rrbracket^\Downarrow$, so it suffices to show that $\llbracket s' \rrbracket^\Downarrow = t'$. This is a consequence of Thm. 3, Def. 1, and Thm. 8 in [2]. □

Next we move on the low-level implementation part, which expresses:

THEOREM 6.4 (LOW-LEVEL IMPLEMENTATION). *Let t be a pure term (without ESSs). If $t \xrightarrow{\beta_f^n} t'$ with t' in normal form and s is an initial state such that $\llbracket s \rrbracket = t$ then $s \rightsquigarrow^k s'$ where $\llbracket s' \rrbracket$ is structurally equivalent to t' and $k \in O(|t| \cdot (n + 1))$.*

PROOF. First we claim that the GLAMoUr terminates when starting from the initial state s . Indeed, \rightsquigarrow can be written as the union of $\rightsquigarrow_{\text{um}, \text{ue}}$ and $\rightsquigarrow_{c_1, \dots, c_5}$. The relation $\rightsquigarrow_{c_1, \dots, c_5}$ is known to be terminating from [2], so an infinite reduction $s \rightsquigarrow \dots$ must

contain an infinite number of $\rightsquigarrow_{\text{um,ue}}$ steps. By Lemmas 6.1 and 6.2 this means that there must exist an infinite $\xrightarrow{\Delta}$ reduction starting from $\{\{s\}\} = t$. This is impossible because t is known to have a normal form and $\xrightarrow{\Delta}$ has the diamond property.

Now let $s \rightsquigarrow^k s'$ be a reduction to normal form in the GLAMoUr containing m multiplicative, e exponential, and c administrative steps, so $k = m + e + c$. By Lemmas 6.1 and 6.2 we have $t = \{\{s\}\} \xrightarrow{\Delta}^{m+e} t' \equiv \{\{s'\}\}$. Moreover $t \xrightarrow{\Delta}^n t'$ by hypothesis. Since s' is \rightsquigarrow -normal, we know by Lemma 6.1 that $\{\{s'\}\}$ is $\xrightarrow{\Delta}$ -normal, so t' is also $\xrightarrow{\Delta}$ -normal. But $\xrightarrow{\Delta}$ has the diamond property, so $t' = t''$ and $n = m + e$. To conclude, we are left to show that $k \in O(|t|(n+1))$. By Lemma 6 in [2] we know that $c \in O(|t|(e+1))$. Since $n = m + e$, finally we have that $k = m + e + c \in O(|t|(n+1))$. \square

E PROOFS OF SECTION 7 “A QUANTITATIVE INTERPRETATION”

In this section we show the results concerning the typing system \mathcal{U} . We start with general lemmas and remarks used through this section, and then we show soundness and completeness in Appendix E.1 and Appendix E.2 respectively.

LEMMA 7.1 (RELEVANCE). *If $\Gamma \vdash^{(m,e)} t : \mathcal{T}$ then $\text{rv}(t) \subseteq \text{dom}(\Gamma) \subseteq \text{fv}(t)$.*

PROOF. By induction on the derivation of the judgment $\Gamma \vdash^{(m,e)} t : \mathcal{T}$.

1. VAR. Then $x : \mathcal{T} \vdash^{(0, \text{ta}(\mathcal{T}))} x : \mathcal{T}$, where $\Gamma = x : \mathcal{T}$ and $m = 0$ and $e = \text{ta}(\mathcal{T})$ and $t = x$. Therefore $\text{dom}(x : \mathcal{T}) = \{x\} = \text{rv}(x) = \text{fv}(x)$.
2. ABS. Then

$$\frac{(\Gamma_i; x : \mathcal{S}_i^? \vdash^{(m_i, e_i)} s : \mathcal{R}_i)_{i \in I}}{+_{i \in I} \Gamma_i \vdash^{(+_{i \in I} m_i, +_{i \in I} e_i)} \lambda x. s : [\mathcal{S}_i^? \rightarrow \mathcal{R}_i]_{i \in I}} \text{ABS}$$

where $\Gamma = +_{i \in I} \Gamma_i$ and $m = +_{i \in I} m_i$ and $e = +_{i \in I} e_i$ and $t = \lambda x. s$ and $\mathcal{T} = [\mathcal{S}_i^? \rightarrow \mathcal{R}_i]_{i \in I}$. We can apply *i.h.* on s , yielding $\text{rv}(s) \subseteq \text{dom}(\Gamma_i; x : \mathcal{S}_i^?) \subseteq \text{fv}(s)$ for all $i \in I$. We need to separate in cases, for each $i \in I$, depending on whether $\mathcal{S}_i^? = \perp$ or not. Since both cases are analogous, we only focus on the case in which $\mathcal{S}_i^? \neq \perp$. So we have $\bigcup_{i \in I} \text{rv}(s) \subseteq \bigcup_{i \in I} \text{dom}(\Gamma_i; x : \mathcal{S}_i^?) \subseteq \bigcup_{i \in I} \text{dom}(\Gamma_i) \cup \{x\} \subseteq \text{fv}(s)$. By removing x from the inequalities, we obtain:

$$\text{rv}(\lambda x. s) = \emptyset \subseteq \text{dom}(+_{i \in I} \Gamma_i) = \bigcup_{i \in I} \text{dom}(\Gamma_i) \subseteq \text{fv}(s) \setminus \{x\} = \text{fv}(\lambda x. s)$$

3. APPP. Then

$$\frac{\Gamma_1 \vdash^{(m_1, e_1)} s : \mathfrak{s} \quad \Gamma_2 \vdash^{(m_2, e_2)} u : \mathfrak{t}}{\Gamma_1 + \Gamma_2 \vdash^{(m_1 + m_2, e_1 + e_2)} s u : \mathfrak{s}} \text{APPP}$$

where $\Gamma = \Gamma_1 + \Gamma_2$ and $m = m_1 + m_2$ and $e = e_1 + e_2$ and $t = s u$ and $\mathcal{T} = \mathfrak{s}$. By *i.h.* on both s and u we have that $\text{rv}(s) \subseteq \text{dom}(\Gamma_1) \subseteq \text{fv}(s)$ and $\text{rv}(u) \subseteq \text{dom}(\Gamma_2) \subseteq \text{fv}(u)$ respectively. We conclude that $\text{rv}(s u) \subseteq \text{dom}(\Gamma_1 + \Gamma_2) = \text{dom}(\Gamma_1) \cup \text{dom}(\Gamma_2) \subseteq \text{fv}(s u)$.

4. APPC. Analogous to the previous case.
5. ES. Then

$$\frac{\Gamma_1; x : \mathcal{S}^? \vdash^{(m_1, e_1)} s : \mathcal{T} \quad \mathcal{S}^? \triangleleft \mathcal{S} \quad \Gamma_2 \vdash^{(m_2, e_2)} u : \mathcal{S}}{\Gamma_1 + \Gamma_2 \vdash^{(m_1 + m_2, e_1 + e_2)} s[x/u] : \mathcal{T}} \text{ES}$$

where $\Gamma = \Gamma_1 + \Gamma_2$ and $m = m_1 + m_2$ and $e = e_1 + e_2$ and $t = s[x/u]$. By *i.h.* on both s and u we have that $\text{rv}(s) \subseteq \text{dom}(\Gamma_1; x : \mathcal{S}^?) \subseteq \text{fv}(s)$ and $\text{rv}(u) \subseteq \text{dom}(\Gamma_2) \subseteq \text{fv}(u)$ respectively. By removing x from the inequations we obtain $\text{rv}(s) \setminus \{x\} \subseteq \text{dom}(\Gamma_1) \subseteq \text{fv}(s) \setminus \{x\}$, so we conclude $\text{rv}(s[x/u]) = (\text{rv}(s) \setminus \{x\}) \cup \text{rv}(u) \subseteq \text{dom}(\Gamma_1 + \Gamma_2) = \text{dom}(\Gamma_1) \cup \text{dom}(\Gamma_2) \subseteq (\text{fv}(s) \setminus \{x\}) \cup \text{fv}(u) = \text{fv}(s[x/u])$. \square

Some (simple) properties of the notion of *appropriateness*, defined in Section 7, are the following:

Remark E.1.

1. If $\text{appropriate}_{\mathcal{A}}(x : \mathcal{T})$ and $\mathcal{T}^? \triangleleft \mathcal{T}$ then $\text{appropriate}_{\mathcal{A}}(x : \mathcal{T}^?)$.
2. If $\text{appropriate}_{\mathcal{A}}(\Gamma)$ and for all $x \in \mathcal{A}'$, $x \notin \text{dom}(\Gamma)$ then $\text{appropriate}_{\mathcal{A} \cup \mathcal{A}'}(\Gamma)$.
3. If $\text{appropriate}_{\mathcal{A}}(\Gamma)$ and $\text{appropriate}_{\mathcal{A}}(\Delta)$ then $\text{appropriate}_{\mathcal{A}}(\Gamma + \Delta)$.

LEMMA E.2 (TYPES OF HEREDITARY ABSTRACTIONS). *Let $\Gamma \vdash^{(m,e)} t : \mathcal{T}$ where $t \in \text{HA}_{\mathcal{A}}$ and $\text{appropriate}_{\mathcal{A}}(\Gamma)$. Then $\mathcal{T} \neq \mathfrak{s}$.*

PROOF. By induction on the derivation of $\Gamma \vdash^{(m,e)} t : \mathcal{T}$.

1. VAR. Let $x : \mathcal{T} \vdash^{(0,n)} x : \mathcal{T}$, with $n = \text{ta}(\mathcal{T})$. Moreover, $x \in \text{HA}_{\mathcal{A}}$ and $\text{appropriate}_{\mathcal{A}}(x : \mathcal{T})$. By the premise of rule H-VAR, we know that $x \in \mathcal{A}$, and thus $\mathcal{T} \neq \mathfrak{s}$ by definition of $\text{appropriate}_{\mathcal{A}}(x : \mathcal{T})$.
2. ABS. This case is immediate since the judgment is $+_{i \in I} \Gamma_i \vdash^{(+_{i \in I} m_i, +_{i \in I} e_i)} \lambda x. t : [\mathcal{T}_i^? \rightarrow \mathcal{S}_i]_{i \in I}$, whose type is not equal to \mathfrak{s} .
3. APPP. This case is not possible, since the term is of the form $s u$, and the hypothesis $s u \in \text{HA}_{\mathcal{A}}$ does not hold.
4. APPC. Analogous to the previous case.
5. ES. Then

$$\frac{\Gamma_1; x : \mathcal{S}^? \vdash^{(m_1, e_1)} s : \mathcal{T} \quad \mathcal{S}^? \triangleleft \mathcal{S} \quad \Gamma_2 \vdash^{(m_2, e_2)} u : \mathcal{S}}{\Gamma_1 + \Gamma_2 \vdash^{(m_1+m_2, e_1+e_2)} s[x/u] : \mathcal{T}} \text{ES}$$

where $\Gamma = \Gamma_1 + \Gamma_2$, $m = m_1 + m_2$, $e = e_1 + e_2$ and $t = s[x/u]$. Moreover, $s[x/u] \in \text{HA}_{\mathcal{A}}$, which can be derived either by rule H-SUB₁ or rule H-SUB₂:

- 5.1 H-SUB₁. Then $s \in \text{HA}_{\mathcal{A}}$ and $x \notin \mathcal{A}$. Given that $\Gamma = \Gamma_1 + \Gamma_2$, then $\text{appropriate}_{\mathcal{A}}(\Gamma_1; x : \mathcal{S}^?)$. We can apply *i.h.* on s , yielding $\mathcal{T} \neq \mathfrak{s}$.
- 5.2 H-SUB₂. Then $s \in \text{HA}_{\mathcal{A} \cup \{x\}}$, $x \notin \mathcal{A}$ and $u \in \text{HA}_{\mathcal{A}}$. Given that $\Gamma = \Gamma_1 + \Gamma_2$, then $\text{appropriate}_{\mathcal{A}}(\Gamma_2)$. We can apply *i.h.* on u , yielding $\mathcal{S} \neq \mathfrak{s}$. Since $\mathcal{S}^? \triangleleft \mathcal{S}$, we can conclude that $\mathcal{S}^? \neq \mathfrak{s}$ so $\text{appropriate}_{\mathcal{A} \cup \{x\}}(\Gamma_1; x : \mathcal{S}^?)$ holds. By *i.h.* on s , we have $\mathcal{T} \neq \mathfrak{s}$. □

LEMMA E.3 (SPLITTING / MERGING). *Let $\mathcal{T}_1, \mathcal{T}_2$ be two types such that their sum $\mathcal{T}_1 + \mathcal{T}_2$ is well-defined. Then the following are equivalent:*

1. $\Gamma \vdash^{(m, e)} v : \mathcal{T}_1 + \mathcal{T}_2$
2. There exist $\Gamma_1, \Gamma_2, m_1, e_1, m_2, e_2$ such that:
 - (a) $\Gamma_1 \vdash^{(m_1, e_1)} v : \mathcal{T}_1$
 - (b) $\Gamma_2 \vdash^{(m_2, e_2)} v : \mathcal{T}_2$
 - (c) $\Gamma = \Gamma_1 + \Gamma_2$ and $m = m_1 + m_2$ and $e = e_1 + e_2$.

PROOF. We first show that $(1 \Rightarrow 2)$. The judgment can be derived using rule VAR or rule ABS:

- VAR. Then $\Gamma \vdash^{(0, e)} x : \mathcal{T}_1 + \mathcal{T}_2$, with $e = \text{ta}(\mathcal{T}_1 + \mathcal{T}_2)$. Since the sum of the types is well defined, there are two subcases:
 1. $\mathcal{T}_1 = \mathcal{T}_2 = \mathfrak{s}$ so that $\mathcal{T}_1 + \mathcal{T}_2 = \mathfrak{s}$, hence $\text{ta}(\mathcal{T}_1 + \mathcal{T}_2) = 0$. Taking $\Gamma_1 = x : \mathfrak{s}$, $\Gamma_2 = x : \mathfrak{s}$, $m_1 = 0$, $m_2 = 0$, $e_1 = 0$, $e_2 = 0$, it's easy to check that the three conditions hold.
 2. $\mathcal{T}_1 = \mathcal{M}_1$ and $\mathcal{T}_2 = \mathcal{M}_2$ so that $\mathcal{T}_1 + \mathcal{T}_2 = \mathcal{M}_1 \uplus \mathcal{M}_2$. We can write $\text{ta}(\mathcal{M}_1 + \mathcal{M}_2)$ as $\text{ta}(\mathcal{M}_1) + \text{ta}(\mathcal{M}_2)$. Taking $\Gamma_1 = x : \mathcal{M}_1$, $\Gamma_2 = x : \mathcal{M}_2$, $m_1 = 0$, $m_2 = 0$, $e_1 = \text{ta}(\mathcal{M}_1)$, $e_2 = \text{ta}(\mathcal{M}_2)$, it's easy to check the that three conditions hold.
- ABS. Then

$$\mathcal{D} := \left(\frac{(\Gamma_i; x : \mathcal{R}_i^? \vdash^{(m_i, e_i)} t : \mathcal{S}_i)_{i \in I}}{+_{i \in I} \Gamma_i \vdash^{(+_{i \in I} m_i, +_{i \in I} e_i)} \lambda x. t : [\mathcal{R}_i^? \rightarrow \mathcal{S}_i]_{i \in I}} \text{ABS} \right)$$

with $[\mathcal{R}_i^? \rightarrow \mathcal{S}_i]_{i \in I} = \mathcal{T}_1 + \mathcal{T}_2$. Since the sum of types is well defined, there are two subcases:

1. $\mathcal{T}_1 = \mathcal{T}_2 = \mathfrak{s}$ so that $\mathcal{T}_1 + \mathcal{T}_2 = \mathfrak{s}$. This case is not possible because the term has the non-idempotent type $[\mathcal{R}_i^? \rightarrow \mathcal{S}_i]_{i \in I}$.
2. $\mathcal{T}_1 = \mathcal{M}_1$ and $\mathcal{T}_2 = \mathcal{M}_2$ so that $\mathcal{T}_1 + \mathcal{T}_2 = \mathcal{M}_1 \uplus \mathcal{M}_2$. Then we can write I as $I = J \uplus K$, in such a way that $\mathcal{T}_1 = \mathcal{M}_1 = [\mathcal{R}_j^? \rightarrow \mathcal{S}_j]_{j \in J}$ and $\mathcal{T}_2 = \mathcal{M}_2 = [\mathcal{R}_k^? \rightarrow \mathcal{S}_k]_{k \in K}$. In particular, $[\mathcal{R}_i^? \rightarrow \mathcal{S}_i]_{i \in I} = [\mathcal{R}_j^? \rightarrow \mathcal{S}_j]_{j \in J} + [\mathcal{R}_k^? \rightarrow \mathcal{S}_k]_{k \in K}$. Taking $\Gamma_1 = +_{j \in J} \Gamma_j$, $\Gamma_2 = +_{k \in K} \Gamma_k$, $m_1 = +_{j \in J} m_j$, $m_2 = +_{k \in K} m_k$, $e_1 = +_{j \in J} e_j$, $e_2 = +_{k \in K} e_k$, we can check that the three conditions hold:
 - (a) $+_{j \in J} \Gamma_j \vdash^{(+_{j \in J} m_j, +_{j \in J} e_j)} \lambda x. t : [\mathcal{R}_j^? \rightarrow \mathcal{S}_j]_{j \in J}$ by ABS rule, since $(\Gamma_j; x : \mathcal{T}_j^? \vdash^{(m_j, e_j)} t : \mathcal{S}_j)_{j \in J}$ holds, as they are j premises of \mathcal{D}
 - (b) $+_{k \in K} \Gamma_k \vdash^{(+_{k \in K} m_k, +_{k \in K} e_k)} \lambda x. t : [\mathcal{R}_k^? \rightarrow \mathcal{S}_k]_{k \in K}$ by ABS rule, since $(\Gamma_k; x : \mathcal{T}_k^? \vdash^{(m_k, e_k)} t : \mathcal{S}_k)_{k \in K}$ holds, as they are k premises of \mathcal{D}
 - (c) $\Gamma = +_{i \in I} \Gamma_i = +_{j \in J} \Gamma_j + +_{k \in K} \Gamma_k = \Gamma_1 + \Gamma_2$ and $m = m_{i \in I} = +_{j \in J} m_j + +_{k \in K} m_k = m_1 + m_2$ and $e = e_{i \in I} = +_{j \in J} e_j + +_{k \in K} e_k = e_1 + e_2$

Now we can show that $(2 \Rightarrow 1)$. First, note that v is either a variable or an abstraction, so both judgments are derived using rule VAR or rule ABS.

1. VAR. Then $x : \mathcal{T}_1 \vdash^{(0, e_1)} x : \mathcal{T}_1$, with $e_1 = \text{ta}(\mathcal{T}_1)$ and we also have $x : \mathcal{T}_2 \vdash^{(0, e_2)} x : \mathcal{T}_2$, with $e_2 = \text{ta}(\mathcal{T}_2)$. Using the third condition, we can conclude $x : \mathcal{T}_1 + \mathcal{T}_2 \vdash^{(0, e_1 + e_2)} x : \mathcal{T}_1 + \mathcal{T}_2$ by applying rule VAR, since $e_1 + e_2 = \text{ta}(\mathcal{T}_1) + \text{ta}(\mathcal{T}_2) = \text{ta}(\mathcal{T}_1 + \mathcal{T}_2)$.

2. ABS. The following conditions hold:

(a)

$$\mathcal{D}_1 := \left(\frac{(\Gamma_j; x : \mathcal{R}_j^? \vdash^{(m_j, e_j)} t : \mathcal{S}_j)_{j \in J}}{+_{j \in J} \Gamma_j \vdash^{(+_{j \in J} m_j, +_{j \in J} e_j)} \lambda x. t : [\mathcal{R}_j^? \rightarrow \mathcal{S}_j]_{j \in J}} \text{ABS} \right)$$

with $\mathcal{T}_1 = [\mathcal{R}_j^? \rightarrow \mathcal{S}_j]_{j \in J}$, $m_1 = +_{j \in J} m_j$, $e_1 = +_{j \in J} e_j$

(b)

$$\mathcal{D}_2 := \left(\frac{(\Gamma_k; x : \mathcal{R}_k^? \vdash^{(m_k, e_k)} t : \mathcal{S}_k)_{k \in K}}{+_{k \in K} \Gamma_k \vdash^{(+_{k \in K} m_k, +_{k \in K} e_k)} \lambda x. t : [\mathcal{R}_k^? \rightarrow \mathcal{S}_k]_{k \in K}} \text{ABS} \right)$$

with $\mathcal{T}_2 = [\mathcal{R}_k^? \rightarrow \mathcal{S}_k]_{k \in K}$, $m_2 = +_{k \in K} m_k$, $e_2 = +_{k \in K} e_k$.

(c) $\Gamma = \Gamma_1 + \Gamma_2 = +_{j \in J} \Gamma_j +_{k \in K} \Gamma_k$ and $m = m_1 + m_2 = +_{j \in J} m_j +_{k \in K} m_k$ and $e = e_1 + e_2 = +_{j \in J} e_j +_{k \in K} e_k$.

We can conclude $\Gamma \vdash^{(m, e)} \lambda x. t : \mathcal{T}_1 + \mathcal{T}_2$ by applying rule ABS, using the derivations \mathcal{D}_1 and \mathcal{D}_2 as premises. \square

Definition E.4 (Typing of substitution contexts). We extend the type system for typing substitution contexts, writing the typing judgments for substitution contexts as $\Gamma \Vdash^{(m, e)} L \triangleright \Delta$, and the new typing rules are:

$$\frac{}{\emptyset \Vdash^{(0,0)} \diamond \triangleright \emptyset} \text{EMPTYSUBSTX}$$

$$\frac{\Gamma_1; x : \mathcal{T}_1^? \Vdash^{(m_1, e_1)} L \triangleright \Delta \quad \mathcal{T}_1^? + \mathcal{T}_2^? \triangleleft \mathcal{T} \quad \Gamma_2 \vdash^{(m_2, e_2)} t : \mathcal{T}}{\Gamma_1 + \Gamma_2 \Vdash^{(m_1+m_2, e_1+e_2)} L[x/t] \triangleright \Delta; x : \mathcal{T}_2^?} \text{ADDSUBSTX}$$

Example E.5. Let $t = x[x/y]$ with $y : \mathcal{T} \vdash^{(0, \text{ta}(\mathcal{T}))} y : \mathcal{T}$:

$$\frac{\frac{}{x : \perp \Vdash^{(0,0)} \diamond \triangleright \emptyset} \text{EMPTYSUBSTX} \quad \perp + \mathcal{T} \triangleleft \mathcal{T} \quad \frac{}{y : \mathcal{T} \vdash^{(0, \text{ta}(\mathcal{T}))} y : \mathcal{T}} \text{VAR}}{y : \mathcal{T} \Vdash^{(0, \text{ta}(\mathcal{T}))} [x/y] \triangleright x : \mathcal{T}} \text{ADDSUBSTX}$$

Example E.6. Let $t = z[x/y]$ with $y : \mathbb{t} \vdash^{(0,0)} y : \mathbb{t}$:

$$\frac{\frac{}{x : \perp \Vdash^{(0,0)} \diamond \triangleright \emptyset} \text{EMPTYSUBSTX} \quad \perp + \perp \triangleleft \mathbb{t} \quad \frac{}{y : \mathbb{t} \vdash^{(0,0)} y : \mathbb{t}} \text{VAR}}{y : \mathbb{t} \Vdash^{(0,0)} [x/y] \triangleright x : \perp} \text{ADDSUBSTX}$$

LEMMA E.7 (RELEVANCE FOR TYPING OF SUBSTITUTION CONTEXTS). *If $\Gamma \Vdash^{(m, e)} L \triangleright \Delta$ then $\text{dom}(\Gamma) \subseteq \text{fv}(L)$ and $\text{dom}(\Delta) \subseteq \text{dom}(L)$.*

PROOF. By induction on the derivation of the judgment $\Gamma \Vdash^{(m, e)} L \triangleright \Delta$.

1. **EMPTYSUBSTX.** The judgment is $\emptyset \Vdash^{(0,0)} \diamond \triangleright \emptyset$, where $\Gamma = \Delta = \emptyset$ and $m = e = 0$ and $L = \diamond$. Then $\text{dom}(\emptyset) = \emptyset = \text{fv}(\diamond)$ and $\text{dom}(\emptyset) = \emptyset = \text{dom}(\diamond)$.
2. **ADDSUBSTX.** Then

$$\frac{\Gamma_1; x : \mathcal{T}_1^? \Vdash^{(m_1, e_1)} L' \triangleright \Delta' \quad \mathcal{T}_1^? + \mathcal{T}_2^? \triangleleft \mathcal{T} \quad \Gamma_2 \vdash^{(m_2, e_2)} t : \mathcal{T}}{\Gamma_1 + \Gamma_2 \Vdash^{(m_1+m_2, e_1+e_2)} L'[x/t] \triangleright \Delta'; x : \mathcal{T}_2^?} \text{ADDSUBSTX}$$

where $\Gamma = \Gamma_1 + \Gamma_2$ and $\Delta = \Delta'; x : \mathcal{T}_2^?$ and $m = m_1 + m_2$ and $e = e_1 + e_2$ and $L = L'[x/t]$. On the one hand, we have $\text{dom}(\Gamma_1; x : \mathcal{T}_1^?) \subseteq \text{fv}(L')$ by *i.h.*. Removing x from the inequation we obtain $\text{dom}(\Gamma_1; x : \mathcal{T}_1^?) \setminus \{x\} = \text{dom}(\Gamma_1) \subseteq \text{fv}(L') \setminus \{x\}$, hence having:

$$\begin{aligned} \text{dom}(\Gamma_1) \cup \text{dom}(\Gamma_2) &\subseteq (\text{fv}(L') \setminus \{x\}) \cup \text{dom}(\Gamma_2) \\ &\subseteq (\text{fv}(L') \setminus \{x\}) \cup \text{fv}(t) && \text{(By Lemma 7.1)} \\ &= \text{fv}(L'[x/t]) \end{aligned}$$

On the other hand

$$\begin{aligned} \text{dom}(\Delta'; x : \mathcal{T}_2^?) &\subseteq \text{dom}(\Delta') \cup \{x\} \\ &\subseteq \text{dom}(L') \cup \{x\} \quad (\text{By } i.h.) \\ &= \text{dom}(L'[x/t]) \end{aligned}$$

and we are done. □

LEMMA E.8 (COMPOSITION / DECOMPOSITION). *The following are equivalent:*

1. $\Gamma \vdash^{(m,e)} tL : \mathcal{T}$
2. *There exist $\Gamma_t, \Gamma_L, \Delta, m_t, e_t, m_L, e_L$ such that:*
 - (a) $\Gamma_L \Vdash^{(m_L, e_L)} L \triangleright \Delta$
 - (b) $\Gamma_t; \Delta \vdash^{(m_t, e_t)} t : \mathcal{T}$
 - (c) $\Gamma = \Gamma_L + \Gamma_t$ and $m = m_L + m_t$ and $e = e_L + e_t$.

Furthermore, in the $(1 \Rightarrow 2)$ direction, if $\text{inv}(\mathcal{A}, \mathcal{S}, tL)$ holds, then $\text{appropriate}_{\mathcal{A}}(\Gamma)$ implies $\text{appropriate}_{\mathcal{A}}(\Delta)$.

PROOF. Both directions of the proof are by induction on L .

$(1 \Rightarrow 2)$

- $L = \diamond$. The judgment is of the form $\Gamma \vdash^{(m,e)} t : \mathcal{T}$. Taking $\Gamma_t = \Gamma, \Gamma_L = \emptyset, \Delta = \emptyset, m_t = m, e_t = e, m_L = 0$ and $e_L = 0$ we obtain the following statements:
 - (a) $\emptyset \Vdash^{(0,0)} \diamond \triangleright \emptyset$, by rule `EMPTYSUBSCTX`
 - (b) $\Gamma \vdash^{(m,e)} t : \mathcal{T}$, by hypothesis
 - (c) $\Gamma = \emptyset + \Gamma$ and $m = 0 + m$ and $e = 0 + e$

Furthermore, it is immediate to conclude that $\text{appropriate}_{\mathcal{A}}(\emptyset)$ holds, so we are done.

- $L = L'[x/s]$. The judgment is of the form $\Gamma \vdash^{(m,e)} tL'[x/s] : \mathcal{T}$, which can only be derived using rule `ES`:

$$\frac{\Gamma_1; x : \mathcal{S}^? \vdash^{(m_1, e_1)} tL' : \mathcal{T} \quad \mathcal{S}^? \triangleleft \mathcal{S} \quad \Gamma_2 \vdash^{(m_2, e_2)} s : \mathcal{S}}{\Gamma_1 + \Gamma_2 \vdash^{(m_1+m_2, e_1+e_2)} tL'[x/s] : \mathcal{T}} \text{ES}$$

where $\Gamma = \Gamma_1 + \Gamma_2$ and $m = m_1 + m_2$ and $e = e_1 + e_2$ and $L = L'[x/s]$. By *i.h.* on L' , there exist $\Gamma_t, \Gamma'_t, \Delta, m_t, e_t, m_{L'}$ and $e_{L'}$ such that:

- (a') $\Gamma'_t \Vdash^{(m_{L'}, e_{L'})} L' \triangleright \Delta$
- (b') $\Gamma_t; \Delta \vdash^{(m_t, e_t)} t : \mathcal{T}$
- (c') $\Gamma_t; x : \mathcal{S}^? = \Gamma'_t + \Gamma_t$ and $m_t = m_{L'} + m_t$ and $e_t = e_{L'} + e_t$.

Furthermore, $\text{inv}(\mathcal{A}, \mathcal{S}, tL'[x/s])$ implies $\text{inv}(\tilde{\mathcal{A}}, \tilde{\mathcal{S}}, tL')$ with $\tilde{\mathcal{A}} = \mathcal{A} \cup \{x\}$ and $\tilde{\mathcal{S}} = \mathcal{S}$ as well as $\tilde{\mathcal{A}} = \mathcal{A}$ and $\tilde{\mathcal{S}} = \mathcal{S} \cup \{x\}$, so that $\text{appropriate}_{\tilde{\mathcal{A}}}(\Gamma_t; x : \mathcal{S}^?)$ implies $\text{appropriate}_{\tilde{\mathcal{A}}'}(\Delta)$. By statement (c'), we can write $\mathcal{S}^?$ as $\mathcal{S}_{L'}^? + \mathcal{S}_t^?$, so that $\Gamma'_t = \Gamma_{L'}; x : \mathcal{S}_{L'}^?$ and $\Gamma_t = \Gamma'_t; x : \mathcal{S}_t^?$. Moreover, $\mathcal{S}_{L'}^? + \mathcal{S}_t^? \triangleleft \mathcal{S}$. We then have $\Gamma_1 = \Gamma_{L'} + \Gamma'_t$. Taking $\Gamma_L = \Gamma_{L'} + \Gamma_2, \Gamma_t, \Delta, m_t, e_t, m_L = m_{L'} + m_2$ and $e_L = e_{L'} + e_2$ we have:

- (a) Applying rule `ADDSUBSCTX`, we obtain

$$\frac{\Gamma_{L'}; x : \mathcal{S}_{L'}^? \Vdash^{(m_{L'}, e_{L'})} L' \triangleright \Delta \quad \mathcal{S}_{L'}^? + \mathcal{S}_t^? \triangleleft \mathcal{S} \quad \Gamma_2 \vdash^{(m_2, e_2)} s : \mathcal{S}}{\Gamma_{L'} + \Gamma_2 \Vdash^{(m_{L'}+m_2, e_{L'}+e_2)} L'[x/s] \triangleright \Delta; x : \mathcal{S}_t^?} \text{ADDSUBSCTX}$$

- (b) $\Gamma_t; \Delta \vdash^{(m_t, e_t)} t : \mathcal{T}$, by condition (b')
- (c) $\Gamma = \Gamma_1 + \Gamma_2 = \Gamma_t + \Gamma_{L'} + \Gamma_2 = \Gamma_L + \Gamma_t$ and

$$m = m_1 + m_2 = m_{L'} + m_t + m_2 = m_L + m_t$$

$$e = e_1 + e_2 = e_{L'} + e_t + e_2 = e_L + e_t$$

Assume $\text{inv}(\mathcal{A}, \mathcal{S}, tL'[x/s])$ holds. Then we want to show that $\text{appropriate}_{\mathcal{A}}(\Gamma)$ implies $\text{appropriate}_{\mathcal{A}'}(\Delta)$, knowing already that $\text{appropriate}_{\mathcal{A}'}(\Delta)$ holds by the *i.h.* We consider two cases:

1. $\mathcal{A}^{L'[x/s]} = \mathcal{A}' \cup \{x\}$. This means $s \in \text{HA}_{\mathcal{A}}$. Moreover, $\text{appropriate}_{\mathcal{A}}(\Gamma_2)$ holds, as $\text{appropriate}_{\mathcal{A}}(\Gamma)$ and $\Gamma = \Gamma_1 + \Gamma_2$. Then $\mathcal{S} \neq s$ by Lemma E.2. Therefore $\mathcal{S}_{L'}^? + \mathcal{S}_t^? \triangleleft \mathcal{S}$ implies $\mathcal{S}_{L'}^? \neq s$, and we can conclude $\text{appropriate}_{\mathcal{A}'}(\Delta)$.
2. $\mathcal{A}^{L'[x/s]} = \mathcal{A}'$. Immediate by *i.h.*

$(2 \Rightarrow 1)$

- $L = \diamond$. Then there exist $\Gamma_t, \Gamma_\diamond, \Delta, m_t, e_t, m_\diamond, e_\diamond$ such that:

- (a) $\Gamma_\diamond \Vdash^{(m_\diamond, e_\diamond)} \diamond \triangleright \Delta$
- (b) $\Gamma_t; \Delta \vdash^{(m_t, e_t)} t : \mathcal{T}$
- (c) $\Gamma = \Gamma_\diamond + \Gamma_t$ and $m = m_\diamond + m_t$ and $e = e_\diamond + e_t$.

The judgment from condition (a) can only be derived by rule `EMPTYSUBSCtx`, hence $\Gamma_\diamond = \Delta = \emptyset$, and $m_\diamond = e_\diamond = 0$. By condition (b) and (c) it is immediate to conclude $\Gamma \vdash^{(m, e)} t : \mathcal{T}$.

- $L = L' [x/s]$. Then there exist $\Gamma_t, \Gamma_L, \Delta, m_t, e_t, m_L, e_L$ such that:

- (a) $\Gamma_L \Vdash^{(m_L, e_L)} L' [x/s] \triangleright \Delta$
- (b) $\Gamma_t; \Delta \vdash^{(m_t, e_t)} t : \mathcal{T}$
- (c) $\Gamma = \Gamma_L + \Gamma_t$ and $m = m_L + m_t$ and $e = e_L + e_t$.

The judgment from condition (a) can only be derived by rule `ADDSUBSCtx` so

$$\frac{\Gamma_{L'}; x : \mathcal{S}_1^? \Vdash^{(m_{L'}, e_{L'})} L' \triangleright \Delta' \quad \mathcal{S}_1^? + \mathcal{S}_2^? \triangleleft \mathcal{S} \quad \Gamma_s \vdash^{(m_s, e_s)} s : \mathcal{S}}{\Gamma_{L'} + \Gamma_s \Vdash^{(m_{L'} + m_s, e_{L'} + m_s)} L' [x/s] \triangleright \Delta'; x : \mathcal{S}_2^?} \text{ADDSUBSCtx}$$

where $\Gamma_L = \Gamma_{L'} + \Gamma_s$, $\Delta = \Delta'$; $x : \mathcal{S}_2^?$, $m_L = m_{L'} + m_s$, $e_L = e_{L'} + e_s$.

Then there exist $\Sigma_t = \Gamma_t; x : \mathcal{S}_2^?$, $\Sigma_{L'} = \Gamma_{L'}; x : \mathcal{S}_1^?$, Δ' , $m_t, e_t, m_{L'}, e_{L'}$ such that:

- (a') $\Sigma_{L'} \Vdash^{(m_{L'}, e_{L'})} L' \triangleright \Delta'$
- (b') $\Sigma_t; \Delta' \vdash^{(m_t, e_t)} t : \mathcal{T}$
- (c') $\Sigma = \Sigma_{L'} + \Sigma_t$ and $m' = m_{L'} + m_t$ and $e' = e_{L'} + e_t$.

We can apply *i.h.* on L' , yielding $\Sigma \vdash^{(m', e')} tL' : \mathcal{T}$. Finally, we apply rule `ES` and obtain

$$\frac{\Gamma_{L'} + \Gamma_t; x : (\mathcal{S}_1^? + \mathcal{S}_2^?) \vdash^{(m_{L'} + m_t, e_{L'} + e_t)} tL' : \mathcal{T} \quad \mathcal{S}_1^? + \mathcal{S}_2^? \triangleleft \mathcal{S} \quad \Gamma_s \vdash^{(m_s, e_s)} s : \mathcal{S}}{\Gamma \vdash^{(m, e)} tL' [x/s] : \mathcal{T}} \text{ES}$$

□

Definition E.9. Let X be a finite set of variables. We say that a context Γ is X -**tight** if for all $x \in X$, $\Gamma(x) = \perp$ or $\Gamma(x) = \mathbb{k}$.

LEMMA E.10. Let $\text{inv}(\mathcal{A}, \mathcal{S}, t)$. If $\Gamma \vdash^{(m, e)} t : \mathcal{T}$, where Γ is \mathcal{S} -tight and $t \in \text{St}_{\mathcal{S}}$. Then \mathcal{T} must be tight.

PROOF. By induction on the derivation of $t \in \text{St}_{\mathcal{S}}$.

1. **s-VAR.** Then $t = x$ and $x \in \mathcal{S}$ by premise of the rule `s-VAR`. The judgment $\Gamma \vdash^{(m, e)} x : \mathcal{T}$ can only be derived by rule `VAR`, so Γ is of the form $x : \mathcal{T}$. The context $x : \mathcal{T}$ is \mathcal{S} -tight by hypothesis, and we already know that $x \in \mathcal{S}$, hence $\Gamma(x)$ must be tight, and so we are done.
2. **s-APP.** Then $t = su$ and $s \in \text{St}_{\mathcal{S}}$ by premise of the rule `s-APP`. The judgment $\Gamma \vdash^{(m, e)} su : \mathcal{T}$ can be derived either by rule `APP` or rule `APPC`:
 - 2.1 **APP.** Then

$$\frac{\Gamma_1 \vdash^{(m_1, e_1)} s : \mathfrak{s} \quad \Gamma_2 \vdash^{(m_2, e_2)} u : \mathbb{k}}{\Gamma_1 + \Gamma_2 \vdash^{(m_1 + m_2, e_1 + e_2)} su : \mathfrak{s}} \text{APP}$$

where $\Gamma = \Gamma_1 + \Gamma_2$, $m = m_1 + m_2$, $e = e_1 + e_2$ and $\mathcal{T} = \mathfrak{s}$ which is tight, so we are done.

- 2.2 **APPC.** Then

$$\frac{\Gamma_1 \vdash^{(m_1, e_1)} s : [\mathcal{S}^? \rightarrow \mathcal{T}] \quad \mathcal{S}^? \triangleleft \mathcal{S} \quad \Gamma_2 \vdash^{(m_2, e_2)} u : \mathcal{S}}{\Gamma_1 + \Gamma_2 \vdash^{(1 + m_1 + m_2, e_1 + e_2)} su : \mathcal{T}} \text{APPC}$$

where $\Gamma = \Gamma_1 + \Gamma_2$, $m = 1 + m_1 + m_2$ and $e = e_1 + e_2$. Moreover $\text{inv}(\mathcal{A}, \mathcal{S}, su)$ implies $\text{inv}(\mathcal{A}, \mathcal{S}, s)$, and Γ_1 is \mathcal{S} -tight since Γ is \mathcal{S} -tight and $\Gamma = \Gamma_1 + \Gamma_2$. We have that $[\mathcal{S}^? \rightarrow \mathcal{T}]$ must be tight by *i.h.* on s , which is impossible and yields to a contradiction. Hence this case is not possible.

3. **s-SUB₁.** Then $t = s[x/u]$, and

$$\frac{s \in \text{St}_{\mathcal{S}} \quad x \notin \mathcal{S}}{s[x/u] \in \text{St}_{\mathcal{S}}} \text{s-SUB}_1$$

The judgment $\Gamma \vdash^{(m, e)} s[x/u] : \mathcal{T}$ can only be derived by rule `ES`, so

$$\frac{\Gamma_1; x : \mathcal{S}^? \vdash^{(m_1, e_1)} s : \mathcal{T} \quad \mathcal{S}^? \triangleleft \mathcal{S} \quad \Gamma_2 \vdash^{(m_2, e_2)} u : \mathcal{S}}{\Gamma_1 + \Gamma_2 \vdash^{(m_1 + m_2, e_1 + e_2)} s[x/u] : \mathcal{T}} \text{ES}$$

where $\Gamma = \Gamma_1 + \Gamma_2$, $m = m_1 + m_2$ and $e = e_1 + e_2$. Moreover $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ implies $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)$, and $\Gamma_1; x : \mathcal{S}^?$ is \mathcal{S} -tight since (1) Γ is \mathcal{S} -tight and $\Gamma = \Gamma_1 + \Gamma_2$, and (2) $x \notin \mathcal{S}$ by premise of rule s-SUB₁. And $s \in \text{St}_{\mathcal{S}}$ holds by premise of rule s-SUB₁. Applying *i.h.* on s , we conclude that \mathcal{T} is tight, so we are done.

4. s-SUB₂. Then $t = s[x/u]$, and

$$\frac{s \in \text{St}_{\mathcal{S} \cup \{x\}} \quad x \notin \mathcal{S} \quad u \in \text{St}_{\mathcal{S}}}{s[x/u] \in \text{St}_{\mathcal{S}}} \text{ s-SUB}_2$$

The judgment $\Gamma \vdash^{(m,e)} s[x/u] : \mathcal{T}$ can only be derived by rule ES, so

$$\frac{\Gamma_1; x : \mathcal{S}^? \vdash^{(m_1, e_1)} s : \mathcal{T} \quad \mathcal{S}^? \triangleleft \mathcal{S} \quad \Gamma_2 \vdash^{(m_2, e_2)} u : \mathcal{S}}{\Gamma_1 + \Gamma_2 \vdash^{(m_1+m_2, e_1+e_2)} s[x/u] : \mathcal{T}} \text{ ES}$$

where $\Gamma = \Gamma_1 + \Gamma_2$, $m = m_1 + m_2$ and $e = e_1 + e_2$.

By Barendregt's convention we may assume $x \notin \mathcal{A} \cup \mathcal{S}$. Moreover, $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ implies $\text{inv}(\mathcal{A}, \mathcal{S}, u)$. Also Γ_2 is \mathcal{S} -tight since Γ is \mathcal{S} -tight and $\Gamma = \Gamma_1 + \Gamma_2$. By premise of rule s-SUB₂, $u \in \text{St}_{\mathcal{S}}$ holds. We apply *i.h.* on u , yielding that \mathcal{S} is tight. On the other hand, $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ implies $\text{inv}(\mathcal{A}, \mathcal{S} \cup \{x\}, s)$, and $\Gamma_1; x : \mathcal{S}^?$ is $(\mathcal{S} \cup \{x\})$ -tight since (1) \mathcal{S} is tight and $\mathcal{S}^? \triangleleft \mathcal{S}$ by premise of rule ES, so $\mathcal{S}^?$ must be either \perp or \mathbb{L} , and (2) Γ is $(\mathcal{S} \cup \{x\})$ -tight and $\Gamma = \Gamma_1 + \Gamma_2$. By premise of rule s-SUB₂, $u \in \text{St}_{\mathcal{S}} \subseteq \text{St}_{\mathcal{S} \cup \{x\}}$ holds. Applying *i.h.* on s we conclude that \mathcal{T} is tight, so we are done. \square

LEMMA E.11. *Let $\text{inv}(\mathcal{A}, \mathcal{S}, t)$. Suppose that the following holds:*

1. $\Gamma; \Delta \vdash^{(m,e)} t : \mathbb{L}$
2. $\text{appropriate}_{\mathcal{A}}(\Gamma)$, with Γ tight
3. $\Delta \neq \emptyset$
4. $\text{dom}(\Delta) \subseteq \mathcal{A}$
5. For all $x \in \text{dom}(\Delta)$, we have that $\Delta(x)$ is a non-empty multiset.

Then $t \notin \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$.

PROOF. By induction on the derivation of the judgment $\Gamma; \Delta \vdash^{(m,e)} t : \mathbb{L}$. \square

LEMMA E.12. *Let $\text{inv}(\mathcal{A}, \mathcal{S}, t)$ and suppose the following hypothesis hold: (1) $t \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$; (2) $\Gamma \vdash^{(m,e)} t : \mathbb{L}$ is tight; (3) $\text{appropriate}_{\mathcal{A}}(\Gamma)$; (4) If $\mu = @$ then $\mathbb{L} = \mathfrak{s}$. Then $(m, e) = (0, 0)$.*

PROOF. By induction on the derivation of $t \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^\bullet$. \square

E.1 Soundness of \mathcal{U}

This subsection aims to give the main results regarding the soundness of the type system \mathcal{U} with respect to the UOCBV^\bullet strategy. In order to prove this result, stated in Theorem 7.4, we need to show that the subject reduction property (Proposition 7.3) holds. For doing that, we start by presenting the substitution lemma for \mathcal{U} in Lemma E.13.

LEMMA E.13 (SUBSTITUTION). *Suppose that the following conditions hold:*

- (a) $t \xrightarrow{\text{sub}_{(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu}} t'$
- (b) $\Gamma; x : \mathcal{T}^? \vdash^{(m,e)} t : \mathcal{S}$
- (c) $\text{appropriate}_{\mathcal{A} \cup \{x\}}(\Gamma; x : \mathcal{T}^?)$
- (d) If $\mu = @$ then either $\mathcal{S} = \mathfrak{s}$ or \mathcal{S} is a singleton, i.e. of the form $[\alpha]$.

Then there exist α and $\mathcal{T}_2^?$ such that $\mathcal{T}^? = [\alpha] + \mathcal{T}_2^?$ and such that, whenever $\Delta \vdash^{(m', e')} v : [\alpha]$, we have that $e > 0$ and $\Gamma + \Delta; x : \mathcal{T}_2^? \vdash^{(m+m', e+e'-1)} t' : \mathcal{S}$.

PROOF. By induction on the derivation of $t \xrightarrow{\text{sub}_{(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu}} t'$.

1. SUB[•]. The following conditions hold:

- (a) $x \xrightarrow{\text{sub}_{(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, @}} v$
- (b) $\Gamma; x : \mathcal{T}^? \vdash^{(m,e)} x : \mathcal{S}$
- (c) $\text{appropriate}_{\mathcal{A} \cup \{x\}}(\Gamma; x : \mathcal{T}^?)$
- (d) Since $\mu = @$, either $\mathcal{S} = \mathfrak{s}$ or \mathcal{S} is a singleton

The judgment of condition (b) can only be derived by the rule VAR , hence $\mathcal{T}^? = \mathcal{S}$, $\Gamma = \emptyset$, $m = 0$ and $e = \text{ta}(\mathcal{S})$. Furthermore, $\mathcal{S} \neq \mathfrak{s}$, since $\text{appropriate}_{\mathcal{A} \cup \{x\}}(x : \mathcal{S})$ holds. This implies by hypothesis that \mathcal{S} is a singleton $[\alpha]$ for some type α . Thus $e = 1$. Taking such α and $\mathcal{T}_2^? = \perp$, whenever $\Delta \vdash^{(m', e')} v : [\alpha]$, we can conclude that $\emptyset + \Delta \vdash^{(0+m', 1+e'-1)} v : [\alpha]$.

2. APPL^\bullet . The following conditions hold:

- (a) $s u \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s' u$, with $s \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, @} s'$.
- (b) $\Gamma; x : \mathcal{T}^? \vdash^{(m, e)} s u : \mathcal{S}$
- (c) $\text{appropriate}_{\mathcal{A} \cup \{x\}}(\Gamma; x : \mathcal{T}^?)$
- (d) If $\mu = @$ then either $\mathcal{S} = \mathfrak{s}$ or \mathcal{S} is a singleton

The judgment $\Gamma; x : \mathcal{T}^? \vdash^{(m, e)} s u : \mathcal{S}$ can be derived either by rule APP or rule APPC :

2.1 APP . Then

$$\mathcal{D} := \left(\frac{\Gamma_s; x : \mathcal{T}_s^? \vdash^{(m_s, e_s)} s : \mathfrak{s} \quad \Gamma_u; x : \mathcal{T}_u^? \vdash^{(m_u, e_u)} u : \mathfrak{t}}{\Gamma_s + \Gamma_u; x : (\mathcal{T}_s^? + \mathcal{T}_u^?) \vdash^{(m_s+m_u, e_s+e_u)} s u : \mathfrak{s}} \text{APP} \right)$$

with $\Gamma = \Gamma_s + \Gamma_u$, $\mathcal{T}^? = \mathcal{T}_s^? + \mathcal{T}_u^?$, $m = m_s + m_u$ and $e = e_s + e_u$. The following holds:

- (a') $s \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu_s} s'$, with $\mu_s = @$ by condition (a)
- (b') $\Gamma_s; x : \mathcal{T}_s^? \vdash^{(m_s, e_s)} s : \mathfrak{s}$, by condition (b)
- (c') $\text{appropriate}_{\mathcal{A} \cup \{x\}}(\Gamma_s; x : \mathcal{T}_s^?)$, since $\text{appropriate}_{\mathcal{A} \cup \{x\}}(\Gamma; x : \mathcal{T}^?)$ holds, and $\Gamma = \Gamma_s + \Gamma_u$, $\mathcal{T}^? = \mathcal{T}_s^? + \mathcal{T}_u^?$, and by condition (c)
- (d') Here $\mu_s = @$ and the type of s is \mathfrak{s} .

We can apply *i.h.* on s , yielding α and $\mathcal{T}_{s_2}^?$ such that $\mathcal{T}_s^? = [\alpha] + \mathcal{T}_{s_2}^?$ and such that, whenever $\Delta \vdash^{(m', e')} v : [\alpha]$, we have that $e_s > 0$ and $\Gamma_s + \Delta; x : \mathcal{T}_{s_2}^? \vdash^{(m_s+m', e_s+e'-1)} s' : \mathfrak{s}$. Then, taking this judgment and the second premise of \mathcal{D} , we can apply rule APP , yielding $\Gamma + \Delta; x : (\mathcal{T}_{s_2}^? + \mathcal{T}_u^?) \vdash^{(m+m', e+e'-1)} s' u : \mathfrak{s}$, with $e > 0$ because $e = e_s + e_u$, and $e_s > 0$ by *i.h.*

We take $\mathcal{T}_2^? = \mathcal{T}_{s_2}^? + \mathcal{T}_u^?$, and we can conclude that $\mathcal{T}^? = [\alpha] + \mathcal{T}_2^?$ holds.

2.2 APPC . Then

$$\mathcal{D} := \left(\frac{\Gamma_s; x : \mathcal{T}_s^? \vdash^{(m_s, e_s)} s : [\mathcal{R}^? \rightarrow \mathcal{S}] \quad \mathcal{R}^? \triangleleft \mathcal{R} \quad \Gamma_u; x : \mathcal{T}_u^? \vdash^{(m_u, e_u)} u : \mathcal{R}}{\Gamma_s + \Gamma_u; x : (\mathcal{T}_s^? + \mathcal{T}_u^?) \vdash^{(1+m_s+m_u, e_s+e_u)} s u : \mathcal{S}} \text{APPC} \right)$$

with $\Gamma = \Gamma_s + \Gamma_u$, $\mathcal{T}^? = \mathcal{T}_s^? + \mathcal{T}_u^?$, $m = 1 + m_s + m_u$ and $e = e_s + e_u$. The following holds:

- (a') $s \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu_s} s'$ with $\mu_s = @$ by condition (a)
- (b') $\Gamma_s; x : \mathcal{T}_s^? \vdash^{(m_s, e_s)} s : [\mathcal{R}^? \rightarrow \mathcal{S}]$, by condition (b)
- (c') $\text{appropriate}_{\mathcal{A} \cup \{x\}}(\Gamma_s; x : \mathcal{T}_s^?)$, since $\text{appropriate}_{\mathcal{A} \cup \{x\}}(\Gamma; x : \mathcal{T}^?)$ holds, and $\Gamma = \Gamma_s + \Gamma_u$, and by condition (c)
- (d') We have $\mu_s = @$ and the type of s is the singleton $[\mathcal{R}^? \rightarrow \mathcal{S}]$.

We can apply *i.h.* on s , yielding α and $\mathcal{T}_{s_2}^?$ such that $\mathcal{T}_s^? = [\alpha] + \mathcal{T}_{s_2}^?$ and such that, whenever $\Delta \vdash^{(m', e')} v : [\alpha]$, we have that $e_s > 0$ and $\Gamma_s + \Delta; x : \mathcal{T}_{s_2}^? \vdash^{(m_s+m', e_s+e'-1)} s' : [\mathcal{R}^? \rightarrow \mathcal{S}]$. Then, taking this judgment and the second premise of \mathcal{D} , we can apply rule APPC , yielding $\Gamma + \Delta; x : (\mathcal{T}_{s_2}^? + \mathcal{T}_u^?) \vdash^{(m+m', e+e'-1)} s' u : \mathcal{S}$, with $e > 0$ because $e = e_s + e_u$, and $e_s > 0$ by *i.h.* We take $\mathcal{T}_2^? = \mathcal{T}_{s_2}^? + \mathcal{T}_u^?$, and we can conclude $\mathcal{T}^? = [\alpha] + \mathcal{T}_2^?$ holds.

3. The remaining cases are similar. □

PROPOSITION 7.3 (SUBJECT REDUCTION). *Let $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'$ where $\rho \in \{\text{db}, \text{lsv}\}$ and $\Gamma \vdash^{(m, e)} t : \mathcal{T}$ and $\text{appropriate}_{\mathcal{A}}(\Gamma)$. Suppose moreover that, if $\mu = @$ then either $\mathcal{T} = \mathfrak{s}$ or \mathcal{T} is a singleton, i.e. of the form $[\alpha]$. Then $\Gamma \vdash^{(m', e')} t' : \mathcal{T}$, where, if $\rho = \text{db}$ we have that $m > 0$ and $(m', e') = (m - 1, e)$, and if $\rho = \text{lsv}$ we have that $e > 0$ and $(m', e') = (m, e - 1)$.*

PROOF. By induction on the derivation of $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'$.

1. DB^\bullet . The following conditions hold:

- (a) $(\lambda x. s) \text{L} u \xrightarrow{\text{db}, \mathcal{A}, \mathcal{S}, \mu} s[x/u] \text{L}$
- (b) $\Gamma \vdash^{(m, e)} (\lambda x. s) \text{L} u : \mathcal{T}$
- (c) $\text{appropriate}_{\mathcal{A}}(\Gamma)$
- (d) If $\mu = @$ then either $\mathcal{T} = \mathfrak{s}$ or \mathcal{T} is a singleton, i.e. of the form $[\alpha]$.

The judgment of condition (b) can be derived either by rule APPP or rule APPC.

1.1 APPP. Then

$$\frac{\Gamma_1 \vdash^{(m_1, e_1)} (\lambda x. s)L : \mathfrak{s} \quad \Gamma_2 \vdash^{(m_2, e_2)} u : \mathfrak{t}}{\Gamma_1 + \Gamma_2 \vdash^{(m_1+m_2, e_1+e_2)} (\lambda x. s)L u : \mathfrak{s}} \text{ APPP}$$

where $\Gamma = \Gamma_1 + \Gamma_2$, $m = m_1 + m_2$, $e = e_1 + e_2$, $t = (\lambda x. s)L u$ and $\mathcal{T} = \mathfrak{s}$. By Lemma E.8, there exist $\Gamma_{11}, \Gamma_{12}, \Delta, m_{11}, e_{11}, m_{12}, e_{12}$ such that $\Gamma_{12} \Vdash^{(m_{12}, e_{12})} L \triangleright \Delta$ and $\Gamma_{11}; \Delta \vdash^{(m_{11}, e_{11})} \lambda x. s : \mathfrak{s}$. This leads to a contradiction, since an abstraction cannot have type \mathfrak{s} . Then this case is not possible.

1.2 APPC. Then

$$\frac{\Gamma_1 \vdash^{(m_1, e_1)} (\lambda x. s)L : [\mathcal{S}^? \rightarrow \mathcal{T}] \quad \mathcal{S}^? \triangleleft \mathcal{S} \quad \Gamma_2 \vdash^{(m_2, e_2)} u : \mathcal{S}}{\Gamma_1 + \Gamma_2 \vdash^{(1+m_1+m_2, e_1+e_2)} (\lambda x. s)L u : \mathcal{T}} \text{ APPC}$$

where $\Gamma = \Gamma_1 + \Gamma_2$, $m = 1 + m_1 + m_2$, $e = e_1 + e_2$, and $t = (\lambda x. s)L u$. By Lemma E.8, there exist $\Gamma_{11}, \Gamma_{12}, \Delta, m_{11}, e_{11}, m_{12}, e_{12}$ such that $\Gamma_{11}; \Delta \vdash^{(m_{11}, e_{11})} \lambda x. s : [\mathcal{S}^? \rightarrow \mathcal{T}]$ and $\Gamma_{12} \Vdash^{(m_{12}, e_{12})} L \triangleright \Delta$, with $\Gamma_1 = \Gamma_{11} + \Gamma_{12}$, $m_1 = m_{11} + m_{12}$ and $e_1 = e_{11} + e_{12}$. The judgment for the term $\lambda x. s$ can only be derived by rule ABS:

$$\frac{\Gamma_{11}; \Delta; x : \mathcal{S}^? \vdash^{(m_{11}, e_{11})} s : \mathcal{T}}{\Gamma_{11}; \Delta \vdash^{(m_{11}, e_{11})} \lambda x. s : [\mathcal{S}^? \rightarrow \mathcal{T}]} \text{ ABS}$$

Then we build the following derivation:

$$\frac{\Gamma_{11}; \Delta; x : \mathcal{S}^? \vdash^{(m_{11}, e_{11})} s : \mathcal{T} \quad \mathcal{S}^? \triangleleft \mathcal{S} \quad \Gamma_2 \vdash^{(m_2, e_2)} u : \mathcal{S}}{\Gamma_{11} + \Gamma_2; \Delta \vdash^{(m_{11}+m_2, e_{11}+e_2)} s[x/u] : \mathcal{T}} \text{ ES}$$

By applying Lemma E.8 again, we can conclude that $\Gamma_1 + \Gamma_2 \vdash^{(m_1+m_2, e_1+e_2)} s[x/u]L : \mathcal{T}$. In this case $\rho = \text{db}$, and with $m > 0$, and $(m', e') = (m_1 + m_2, e_1 + e_2) = (m - 1, e)$.

2. LSV[•]. The following conditions hold:

(a)

$$\frac{s \xrightarrow{\text{sub}_{(x,v)}, \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s' \quad x \notin \mathcal{A} \cup \mathcal{S} \quad vL \in \text{HA}_{\mathcal{A}}}{s[x/vL] \xrightarrow{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} s'[x/v]L} \text{ LSV}^{\bullet}$$

(b) $\Gamma \vdash^{(m, e)} s[x/vL] : \mathcal{T}$

(c) $\text{appropriate}_{\mathcal{A}}(\Gamma)$

(d) If $\mu = @$ then either $\mathcal{T} = \mathfrak{s}$ or \mathcal{T} is a singleton, i.e. of the form $[\alpha]$.

The judgment of condition (b) can only be derived by the rule ES, so

$$\frac{\Gamma_1; x : \mathcal{S}^? \vdash^{(m_1, e_1)} s : \mathcal{T} \quad \mathcal{S}^? \triangleleft \mathcal{S} \quad \Gamma_2 \vdash^{(m_2, e_2)} vL : \mathcal{S}}{\Gamma_1 + \Gamma_2 \vdash^{(m_1+m_2, e_1+e_2)} s[x/vL] : \mathcal{T}} \text{ ES}$$

where $\Gamma = \Gamma_1 + \Gamma_2$, $m = m_1 + m_2$, $e = e_1 + e_2$, and $t = s[x/vL]$. By Lemma E.8, there exist $\Gamma_{21}, \Gamma_{22}, \Delta, m_{21}, e_{21}, m_{22}, e_{22}$ such that $\Gamma_{21}; \Delta \vdash^{(m_{21}, e_{21})} v : \mathcal{S}$ and $\Gamma_{22} \Vdash^{(m_{22}, e_{22})} L \triangleright \Delta$, with $\Gamma_2 = \Gamma_{21} + \Gamma_{22}$, $m_2 = m_{21} + m_{22}$ and $e_2 = e_{21} + e_{22}$.

Moreover, we can assume $x \notin \text{dom}(\Gamma)$ by α -conversion so that $\text{appropriate}_{\mathcal{A} \cup \{x\}}(\Gamma)$ also holds. Therefore $\text{appropriate}_{\mathcal{A} \cup \{x\}}(\Gamma_1; x : \mathcal{S}^?)$ holds since (1): $\text{appropriate}_{\mathcal{A} \cup \{x\}}(\Gamma)$ and Γ_1 is part of Γ , and (2): $vL \in \text{HA}_{\mathcal{A}}$ by condition (a), so $\mathcal{S} \neq \mathfrak{s}$ by Lemma E.2 and then $\mathcal{S}^? \triangleleft \mathcal{S}$ in condition (b) implies $\mathcal{S}^? \neq \mathfrak{s}$. Then, we have all the conditions to apply Lemma E.13, yielding α and $\mathcal{S}_2^?$ such that $\mathcal{S}^? = [\alpha] + \mathcal{S}_2^?$, hence $\mathcal{S}^? = \mathcal{S}$, thus $\mathcal{S} = [\alpha] + \mathcal{S}_2^?$. We have two cases, depending on whether $\mathcal{S}_2^?$ is \perp or not:

2.1 $\mathcal{S}_2^? = \perp$. Then, $\mathcal{S} = [\alpha] + \mathcal{S}_2^? = [\alpha] = [\alpha] + []$. Recall that $\Gamma_{21}; \Delta \vdash^{(m_{21}, e_{21})} v : [\alpha]$ holds and note that moreover $\emptyset \vdash^{(0,0)} v : []$. By Lemma E.13 we have that $e_1 > 0$ and $\Gamma_1 + (\Gamma_{21}; \Delta); x : \perp \vdash^{(m_1+m_{21}, e_1+e_{21}-1)} s' : \mathcal{T}$. Then we build the following derivation:

$$\frac{\Gamma_1 + (\Gamma_{21}; \Delta); x : \perp \vdash^{(m_1+m_{21}, e_1+e_{21}-1)} s' : \mathcal{T} \quad \perp \triangleleft [] \quad \emptyset \vdash^{(0,0)} v : []}{\Gamma_1 + \Gamma_{21}; \Delta \vdash^{(m_1+m_{21}, e_1+e_{21}-1)} s'[x/v] : \mathcal{T}} \text{ ES}$$

2.2 $\mathcal{S}_2^? \neq \perp$. Then $\mathcal{S}_2^? = \mathcal{S}_2$ so we can write $\mathcal{S} = [\alpha] + \mathcal{S}_2$. Hence we have $\Gamma_{21}; \Delta \vdash^{(m_{21}, e_{21})} v : [\alpha] + \mathcal{S}_2$. We can apply Lemma E.3, so that there exists $\Gamma'_{211} = \Gamma_{211}; \Delta_1, \Gamma'_{212} = \Gamma_{212}; \Delta_2, m_{211}, m_{212}, e_{211}, e_{212}$ such that $\Gamma'_{211} \vdash^{(m_{211}, e_{211})} v : [\alpha]$ and $\Gamma'_{212} \vdash^{(m_{212}, e_{212})} v : \mathcal{S}_2$, with $\Gamma_{21} = \Gamma'_{211} + \Gamma'_{212}$ and $\Delta = \Delta_1 + \Delta_2$, and $m_{21} = m_{211} + m_{212}$ and $e_{21} = e_{211} + e_{212}$. So by Lemma E.13,

whenever $\Gamma_{211}; \Delta_1 \vdash^{(m_{211}, e_{211})} v : [\alpha]$, we have that $e_1 > 0$ and $\Gamma_1 + (\Gamma_{211}; \Delta_1); x : \mathcal{S}_2 \vdash^{(m_1+m_{211}, e_1+e_{211}-1)} s' : \mathcal{T}$. Then we build the following derivation:

$$\frac{\Gamma_1 + (\Gamma_{211}; \Delta_1); x : \mathcal{S}_2 \vdash^{(m_1+m_{211}, e_1+e_{211}-1)} s' : \mathcal{T} \quad \mathcal{S}_2 \triangleleft \mathcal{S}_2 \quad \Gamma_{212}; \Delta_2 \vdash^{(m_{212}, e_{212})} v : \mathcal{S}_2}{\Gamma_1 + \Gamma_{21}; \Delta \vdash^{(m_1+m_{21}, e_1+e_{21}-1)} s'[x/v] : \mathcal{T}} \text{ES}$$

We can now apply Lemma E.8 using the judgments $\Gamma_{22} \Vdash^{(m_{22}, e_{22})} \mathbb{L} \triangleright \Delta$ and $\Gamma_1 + \Gamma_{21}; \Delta \vdash^{(m_1+m_{21}, e_1+e_{21}-1)} s'[x/v] : \mathcal{T}$, yielding $\Gamma \vdash^{(m, e-1)} s'[x/v] \mathbb{L} : \mathcal{T}$. In this case $\rho = \text{lsv}$ and it holds that $e > 0$ since $e_1 > 0$ and $e = e_1 + e_2$. Moreover, $(m', e') = (m, e - 1)$, so we are done.

3. The congruence cases are uninteresting and are omitted here. \square

THEOREM 7.4 (SOUNDNESS OF \mathcal{U}). *Let $\mathcal{S} = \text{fv}(t)$ and let $\Gamma \vdash^{(m, e)} t : \mathcal{T}$ be a tight derivation. Then there exists a $\xrightarrow{\text{top}, \mathcal{S}}$ -irreducible term s such that $t \xrightarrow{\text{top}, \mathcal{S}}^{m+e} s$ where m and e are respectively the number of db and lsv steps in the reduction.*

PROOF. By induction on $m + e$, separating in cases depending if either $t \in \text{NF}_{\emptyset, \text{fv}(t), \textcircled{\#}}^{\bullet}$ or not:

1. If $t \in \text{NF}_{\emptyset, \text{fv}(t), \textcircled{\#}}^{\bullet}$, we also have that the judgment $\Gamma \vdash^{(m, e)} t : \mathcal{T}$ is tight, $\text{appropriate}_{\emptyset}(\Gamma)$, and $\mu = \textcircled{\#}$. We can apply Lemma E.12, yielding $m = 0, e = 0$. Then we conclude with $s := t$.
2. If $t \notin \text{NF}_{\emptyset, \text{fv}(t), \textcircled{\#}}^{\bullet}$, then by Corollary B.11 there exist a term t' and a rule name ρ such that $t \xrightarrow{\rho, \emptyset, \text{fv}(t), \textcircled{\#}} t'$. By Subject reduction (Proposition 7.3), we have that $\Gamma \vdash^{(m', e')} t' : \mathcal{T}$, with $m > 0$ and $(m', e') = (m - 1, e)$ if $\rho = \text{db}$, and if $\rho = \text{lsv}$ then $e > 0$ and $(m', e') = (m, e - 1)$. Since the judgment is tight, we can apply *i.h.*, so there exists a term $s \in \text{NF}_{\emptyset, \text{fv}(t'), \textcircled{\#}}^{\bullet}$ such that $t' \xrightarrow{\rho_1, \emptyset, \text{fv}(t'), \textcircled{\#}} \dots \xrightarrow{\rho_n, \emptyset, \text{fv}(t'), \textcircled{\#}} s$, where $n = m + e - 1$ and

$$m' = \#\{i \mid 1 \leq i \leq n, \rho_i = \text{db}\} \quad e' = \#\{i \mid 1 \leq i \leq n, \rho_i = \text{lsv}\}$$

Then we have a reduction sequence. If we rename ρ by ρ_{n+1} , we can conclude that

$$t \xrightarrow{\rho_{n+1}, \emptyset, \text{fv}(t), \textcircled{\#}} t' \xrightarrow{\rho_1, \emptyset, \text{fv}(t), \textcircled{\#}} \dots \xrightarrow{\rho_n, \emptyset, \text{fv}(t), \textcircled{\#}} s$$

where $1 + n = m + e$ and if

- $\rho_{n+1} = \text{db}$:

$$\begin{aligned} m &= (m - 1) + 1 & e &= e' \\ &= m' + 1 & &= \#\{i \mid 1 \leq i \leq n, \rho_i = \text{lsv}\} \\ &= \#\{i \mid 1 \leq i \leq 1, \rho_i = \text{db}\} + 1 & &= \#\{i \mid 1 \leq i \leq 1 + n, \rho_i = \text{lsv}\} \\ &= \#\{i \mid 1 \leq i \leq 1 + n, \rho_i = \text{db}\} \end{aligned}$$

- $\rho_{n+1} = \text{lsv}$:

$$\begin{aligned} m &= m' & e &= (e - 1) + 1 \\ &= \#\{i \mid 1 \leq i \leq n, \rho_i = \text{db}\} & &= e' + 1 \\ &= \#\{i \mid 1 \leq i \leq 1 + n, \rho_i = \text{db}\} & &= \#\{i \mid 1 \leq i \leq n, \rho_i = \text{lsv}\} + 1 \\ & & &= \#\{i \mid 1 \leq i \leq 1 + n, \rho_i = \text{lsv}\} \end{aligned}$$

\square

E.2 Completeness of \mathcal{U}

In this subsection we give the main results regarding the completeness of the type system \mathcal{U} with respect to the uocbv^{\bullet} strategy. In order to prove this result, stated in Theorem 7.7, we need to show that the subject expansion property (Proposition 7.6) holds. For doing that, we start by presenting the anti-substitution lemma for \mathcal{U} in Lemma E.14. Moreover, given that we are working with a tight typing system, we also need to show that normal forms of uocbv^{\bullet} are tight typable, as stated in Proposition 7.5.

PROPOSITION 7.5 (NORMAL FORMS ARE TIGHT TYPABLE). *Let t be a term such that $t \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$ and $\text{inv}(\mathcal{A}, \mathcal{S}, t)$. Then there exists a tight type \mathbb{t} such that $\text{TEnv}(\mathcal{A}, \mathcal{S}, t) \vdash^{(0,0)} t : \mathbb{t}$. Moreover, if $t \in \text{HA}_{\mathcal{A}}$ then $\mathbb{t} = [\]$, and if $t \in \text{St}_{\mathcal{S}}$ then $\mathbb{t} = \mathfrak{s}$.*

PROOF. By induction on the derivation of $t \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}$.

1. NF-VAR[•]. Then

$$\frac{x \in \mathcal{A} \Rightarrow \mu = \emptyset}{x \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}} \text{NF-VAR}^{\bullet}$$

There are two possible cases depending on whether $x \in \mathcal{A}$ or $x \in \mathcal{S}$.

1.1 $x \in \mathcal{A}$. By definition $\text{TEnv}(\mathcal{A}, \mathcal{S}, x)(x) = []$, since $x \in \mathcal{A} \cap \text{rv}(x)$ and $\text{TEnv}(\mathcal{A}, \mathcal{S}, x)(y) = \perp$ for any other variable y .

Having $\text{ta}([]) = 0$, we apply rule VAR, yielding $x : [] \vdash^{(0,0)} x : []$, with $x \in \text{HA}_{\mathcal{A}}$ by rule H-VAR.

1.2 $x \in \mathcal{S}$. By definition $\text{TEnv}(\mathcal{A}, \mathcal{S}, x)(x) = s$, since $x \in \mathcal{S} \cap \text{rv}(x)$ and $\text{TEnv}(\mathcal{A}, \mathcal{S}, x)(y) = \perp$ for any other variable y .

Having $\text{ta}(s) = 0$, we apply rule VAR, yielding $x : s \vdash^{(0,0)} x : s$, with $x \in \text{St}_{\mathcal{S}}$ by rule s-VAR.

2. NF-LAM[•]. Then

$$\frac{}{\lambda x. s \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^{\bullet}} \text{NF-LAM}^{\bullet}$$

For any variable y , we have $\text{TEnv}(\mathcal{A}, \mathcal{S}, \lambda x. s)(y) = \perp$ since $\text{rv}(\lambda x. s) = \emptyset$. We apply rule ABS, yielding $\vdash^{(0,0)} \lambda x. s : []$, with $\lambda x. s \in \text{HA}_{\mathcal{A}}$ by rule H-LAM.

3. NF-APP[•]. Then

$$\frac{s \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^{\bullet} \quad u \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^{\bullet}}{s u \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}} \text{NF-APP}^{\bullet}$$

Since $\text{inv}(\mathcal{A}, \mathcal{S}, s u)$ implies $\text{inv}(\mathcal{A}, \mathcal{S}, s)$ and $\text{inv}(\mathcal{A}, \mathcal{S}, u)$, then by *i.h.* on s and u there exist tight types \mathbb{t}_1 and \mathbb{t}_2 such that $\text{TEnv}(\mathcal{A}, \mathcal{S}, s) \vdash^{(0,0)} s : \mathbb{t}_1$ and $\text{TEnv}(\mathcal{A}, \mathcal{S}, u) \vdash^{(0,0)} u : \mathbb{t}_2$. Moreover, $s \in \text{St}_{\mathcal{S}}$ by Lemma B.4, so $\mathbb{t}_1 = s$. We apply rule APP, yielding $\text{TEnv}(\mathcal{A}, \mathcal{S}, s) + \text{TEnv}(\mathcal{A}, \mathcal{S}, u) \vdash^{(0,0)} s u : s$, with $s u \in \text{St}_{\mathcal{S}}$ by rule s-APP. Notice that $\text{TEnv}(\mathcal{A}, \mathcal{S}, s) + \text{TEnv}(\mathcal{A}, \mathcal{S}, u) = \text{TEnv}(\mathcal{A}, \mathcal{S}, s u)$, so we are done.

4. NF-ESA[•]. Then

$$\frac{s \in \text{NF}_{\mathcal{A} \cup \{x\}, \mathcal{S}, \mu}^{\bullet} \quad u \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^{\bullet} \quad u \in \text{HA}_{\mathcal{A}}}{s[x/u] \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}} \text{NF-ESA}^{\bullet}$$

Since $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ implies $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)$ and $\text{inv}(\mathcal{A}, \mathcal{S}, u)$, then by *i.h.* on s and u there exist tight types \mathbb{t}_1 and \mathbb{t}_2 such that $\text{TEnv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s) \vdash^{(0,0)} s : \mathbb{t}_1$ and $\text{TEnv}(\mathcal{A}, \mathcal{S}, u) \vdash^{(0,0)} u : \mathbb{t}_2$. Moreover, $\mathbb{t}_2 = []$ since $u \in \text{HA}_{\mathcal{A}}$. Notice that $\text{TEnv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s) = \text{TEnv}(\mathcal{A}, \mathcal{S}, s); x : \text{TEnv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)(x)$. Moreover, $\text{TEnv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)(x)$ is $[]$ if $x \in \text{rv}(s)$, and \perp otherwise. We build the following derivation:

$$\frac{\frac{\text{(By i.h.)}}{\text{TEnv}(\mathcal{A}, \mathcal{S}, s); x : \text{TEnv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)(x) \vdash^{(0,0)} s : \mathbb{t}_1} \quad \frac{\text{(By i.h.)}}{\text{TEnv}(\mathcal{A}, \mathcal{S}, u) \vdash^{(0,0)} u : []}}{\text{TEnv}(\mathcal{A}, \mathcal{S}, s) + \text{TEnv}(\mathcal{A}, \mathcal{S}, u) \vdash^{(0,0)} s[x/u] : \mathbb{t}_1} \text{ES}$$

where $\text{TEnv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)(x) \triangleleft []$ necessarily holds since $\text{TEnv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)(x)$ is \perp or $[]$. Notice that $\text{TEnv}(\mathcal{A}, \mathcal{S}, s) + \text{TEnv}(\mathcal{A}, \mathcal{S}, u) = \text{TEnv}(\mathcal{A}, \mathcal{S}, s[x/u])$, so we are done.

5. NF-ESS[•]. Then

$$\frac{s \in \text{NF}_{\mathcal{A}, \mathcal{S} \cup \{x\}, \mu}^{\bullet} \quad u \in \text{NF}_{\mathcal{A}, \mathcal{S}, @}^{\bullet} \quad u \in \text{St}_{\mathcal{S}}}{s[x/u] \in \text{NF}_{\mathcal{A}, \mathcal{S}, \mu}^{\bullet}} \text{NF-ESS}^{\bullet}$$

Since $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/u])$ implies $\text{inv}(\mathcal{A}, \mathcal{S} \cup \{x\}, s)$ and $\text{inv}(\mathcal{A}, \mathcal{S}, u)$, then by *i.h.* on s and u there exist tight types \mathbb{t}_1 and \mathbb{t}_2 such that $\text{TEnv}(\mathcal{A}, \mathcal{S} \cup \{x\}, s) \vdash^{(0,0)} s : \mathbb{t}_1$ and $\text{TEnv}(\mathcal{A}, \mathcal{S}, u) \vdash^{(0,0)} u : \mathbb{t}_2$. Moreover, $\mathbb{t}_2 = s$ since $u \in \text{St}_{\mathcal{S}}$. Notice that $\text{TEnv}(\mathcal{A}, \mathcal{S} \cup \{x\}, s) = \text{TEnv}(\mathcal{A}, \mathcal{S}, s); x : \text{TEnv}(\mathcal{A}, \mathcal{S} \cup \{x\}, s)(x)$. Moreover, $\text{TEnv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)(x)$ is s if $x \in \text{rv}(s)$ and \perp otherwise. We build the following derivation:

$$\frac{\frac{\text{(By i.h.)}}{\text{TEnv}(\mathcal{A}, \mathcal{S}, s); x : \text{TEnv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)(x) \vdash^{(0,0)} s : \mathbb{t}_1} \quad \frac{\text{(By i.h.)}}{\text{TEnv}(\mathcal{A}, \mathcal{S}, u) \vdash^{(0,0)} u : s}}{\text{TEnv}(\mathcal{A}, \mathcal{S}, s) + \text{TEnv}(\mathcal{A}, \mathcal{S}, u) \vdash^{(0,0)} s[x/u] : \mathbb{t}_1} \text{ES}$$

where $\text{TEnv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)(x) \triangleleft s$ necessarily holds since $\text{TEnv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)(x)$ is \perp or s . Notice that $\text{TEnv}(\mathcal{A}, \mathcal{S}, s) + \text{TEnv}(\mathcal{A}, \mathcal{S}, u) = \text{TEnv}(\mathcal{A}, \mathcal{S}, s[x/u])$, so we are done. \square

LEMMA E.14 (ANTI-SUBSTITUTION). *Let $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, t)$. Consider a subset $\mathcal{A}_0 \subseteq \mathcal{A}$ and let \mathcal{B} be a set of variables disjoint from \mathcal{A} . Suppose also that the following conditions hold:*

- (a) $t \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} t'$
- (b) $\Gamma; x : \mathcal{T}^? \vdash^{(m,e)} t' : \mathcal{S}$
- (c) appropriate $_{\mathcal{A} \cup \mathcal{B} \cup \{x\}}(\Gamma; x : \mathcal{T}^?)$
- (d) If $\mu = @$ then either $\mathcal{S} = \mathfrak{s}$ or \mathcal{S} is a singleton, i.e. of the form $[\alpha]$.
- (e) $v \in \text{HA}_{\mathcal{A}_0 \cup \mathcal{B}}$

Then there exist $\Gamma_t, \Gamma_v, \alpha, m_t, e_t, m_v, e_v$ such that $\Gamma = \Gamma_t + \Gamma_v$ and $m = m_t + m_v$ and $e = e_t + e_v$; $\Gamma_t; x : \mathcal{T}^? + [\alpha] \vdash^{(m_t, e_t+1)} t : \mathcal{S}$; and $\Gamma_v \vdash^{(m_v, e_v)} v : [\alpha]$.

PROOF. By induction on the derivation of $t \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} t'$.

1. SUB * . Then the following conditions hold:

- (a) $x \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, @} v$, where $t = x, \mu = @$ and $t' = v$
- (b) $\Gamma; x : \mathcal{T}^? \vdash^{(m,e)} v : \mathcal{S}$
- (c) appropriate $_{\mathcal{A} \cup \mathcal{B} \cup \{x\}}(\Gamma; x : \mathcal{T}^?)$
- (d) Since $\mu = @$, so either $\mathcal{S} = \mathfrak{s}$ or \mathcal{S} is of the form $[\alpha]$.
- (e) $v \in \text{HA}_{\mathcal{A}_0 \cup \mathcal{B}}$

Recall that $x \notin \text{fv}(v)$ by the grammar of rule names. We analyze by cases the form of v :

1.1 $v = y$. Then $x \neq y$. The judgment of condition (b) can be derived only by the rule VAR, so it is of the form $y : \mathcal{S} \vdash^{(0, \text{ta}(\mathcal{S}))} y : \mathcal{S}$, with $\Gamma = y : \mathcal{S}, m = 0$ and $e = \text{ta}(\mathcal{S})$. Moreover, $\mathcal{T}^?$ must be \perp . And $y \in \mathcal{A}_0 \cup \mathcal{B}$, since the judgment of condition (e) can only be derived by the rule H-VAR. Then $(y : \mathcal{S})(y) \neq \mathfrak{s}$ by condition (c), as $\mathcal{A}_0 \cup \mathcal{B} \cup \{x\} \subseteq \mathcal{A} \cup \mathcal{B} \cup \{x\}$. Therefore $\mathcal{S} = [\alpha]$ for some type α , and thus $\text{ta}([\alpha]) = 1$ necessarily holds. Taking $\Gamma_t = \emptyset, \Gamma_v = y : [\alpha], \alpha, m_t = 0, e_t = 0, m_v = 0, e_v = 1$ the following statements hold:

- $\Gamma = y : [\alpha] = \Gamma_t + \Gamma_v$ and $m = 0 = m_t + m_v$ and $e = 1 = e_t + e_v$
- $x : [\alpha] \vdash^{(0,1)} x : [\alpha]$, by rule VAR
- $y : [\alpha] \vdash^{(0,1)} y : [\alpha]$, by condition (b).

1.2 $v = \lambda y. s$. The judgment of condition (b) can be derived only by the rule ABS. Moreover, we can derive the judgment $\lambda y. s \in \text{HA}_{\mathcal{A} \cup \mathcal{B} \cup \{x\}}$ by rule H-LAM, and along with condition (c) we conclude $\mathcal{S} \neq \mathfrak{s}$ by Lemma E.2. Then the only possible case is when \mathcal{S} is of the form $[\alpha]$ for some $\alpha = \mathcal{R}^? \rightarrow \mathcal{Q}$. Since $x \notin \text{fv}(\lambda y. s)$ it must be the case that $x \notin \text{dom}(\Gamma; x : \mathcal{T}^?)$, by Lemma 7.1, that is, $\mathcal{T}^? = \perp$. Then the following derivation is for the judgment of condition (b):

$$\frac{\Gamma; y : \mathcal{R}^? \vdash^{(m,e)} s : \mathcal{Q}}{\Gamma \vdash^{(m,e)} \lambda y. s : [\mathcal{R}^? \rightarrow \mathcal{Q}]} \text{ABS}$$

Taking $\Gamma_t = \emptyset, \Gamma_v = \Gamma, \alpha = \mathcal{R}^? \rightarrow \mathcal{Q}, m_t = 0, e_t = 0, m_v = m, e_v = e$ the following statements hold:

- $\Gamma = \Gamma_t + \Gamma_v$ and $m = m_t + m_v$ and $e = e_t + e_v$
- $x : [\mathcal{R}^? \rightarrow \mathcal{Q}] \vdash^{(0,1)} x : [\mathcal{R}^? \rightarrow \mathcal{Q}]$, by rule VAR
- $\Gamma_v \vdash^{(0,1)} \lambda y. s : [\mathcal{R}^? \rightarrow \mathcal{Q}]$, by condition (b).

2. APPL * . Then the following conditions hold:

(a)

$$\frac{s \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, @} s'}{s u \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu} s' u} \text{APPL}^*$$

where $t = s u$ and $t' = s' u$

- (b) $\Gamma; x : \mathcal{T}^? \vdash^{(m,e)} s' u : \mathcal{S}$
- (c) appropriate $_{\mathcal{A} \cup \mathcal{B} \cup \{x\}}(\Gamma; x : \mathcal{T}^?)$
- (d) If $\mu = @$ then either $\mathcal{S} = \mathfrak{s}$ or \mathcal{S} is of the form $[\alpha]$.
- (e) $v \in \text{HA}_{\mathcal{A}_0 \cup \mathcal{B}}$

Since $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s u)$ then in particular $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)$. The judgment of condition (b) can be derived either by rule APP or by rule APPC:

2.1 APP. Then

$$\frac{\Gamma_1; x : \mathcal{T}_1^? \vdash^{(m_1, e_1)} s' : \mathfrak{s} \quad \Gamma_2; x : \mathcal{T}_2^? \vdash^{(m_2, e_2)} u : \mathfrak{t}}{\Gamma_1 + \Gamma_2; x : \mathcal{T}_1^? + \mathcal{T}_2^? \vdash^{(m_1+m_2, e_1+e_2)} s' u : \mathfrak{s}} \text{APP}$$

where $\Gamma = \Gamma_1 + \Gamma_2, \mathcal{T}^? = \mathcal{T}_1^? + \mathcal{T}_2^?, m = m_1 + m_2, e = e_1 + e_2$ and $\mathcal{S} = \mathfrak{s}$. The following conditions hold:

- (a') $s \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, @} s'$, by condition (a)
- (b') $\Gamma_1; x : \mathcal{T}_1^? \vdash^{(m_1, e_1)} s' : \mathfrak{s}$, by condition (b)
- (c') appropriate $_{\mathcal{A} \cup \mathcal{B} \cup \{x\}}$ ($\Gamma_1; x : \mathcal{T}_1^?$), since $\Gamma = \Gamma_1 + \Gamma_2$ and $\mathcal{T}^? = \mathcal{T}_1^? + \mathcal{T}_2^?$, and by condition (c)
- (d') Here $\mu = @$ and the term is typed with \mathfrak{s} .
- (e') $v \in \text{HA}_{\mathcal{A}_0 \cup \mathcal{B}}$, by condition (e)

Then we can apply *i.h.* on s' , yielding $\Gamma_s, \Gamma_v, \alpha, m_s, e_s, m_v, e_v$ such that

- $\Gamma_1 = \Gamma_s + \Gamma_v$ and $m_1 = m_s + m_v$ and $e_1 = e_s + e_v$
- $\Gamma_s; x : \mathcal{T}_1^? + [\alpha] \vdash^{(m_s, e_s+1)} s : \mathfrak{s}$
- $\Gamma_v \vdash^{(m_v, e_v)} v : [\alpha]$

Taking $\Gamma_t = \Gamma_s + \Gamma_2, \Gamma_v, \alpha, m_t = m_s + m_2, e_t = e_s + e_2, m_v, e_v$ the following statements hold:

- $\Gamma = \Gamma_1 + \Gamma_2 = \Gamma_s + \Gamma_v + \Gamma_2 = \Gamma_t + \Gamma_v$ and $m = m_1 + m_2 = m_s + m_v + m_2 = m_t + m_v$ and $e = e_1 + e_2 = e_s + e_v + e_2 = e_t + e_v$
-

$$\frac{\Gamma_s; x : \mathcal{T}_1^? + [\alpha] \vdash^{(m_s, e_s+1)} s : \mathfrak{s} \quad \Gamma_2; x : \mathcal{T}_2^? \vdash^{(m_2, e_2)} u : \mathfrak{t}}{\Gamma_s + \Gamma_2; x : (\mathcal{T}_1^? + [\alpha] + \mathcal{T}_2^?) \vdash^{(m_s+m_2, e_s+1+e_2)} s u : \mathfrak{s}} \text{ APPP}$$

- $\Gamma_v \vdash^{(m_v, e_v)} v : [\alpha]$

2.2 APPC. Then

$$\frac{\Gamma_1; x : \mathcal{T}_1^? \vdash^{(m_1, e_1)} s' : [\mathcal{R}^? \rightarrow \mathcal{S}] \quad \mathcal{R}^? \triangleleft \mathcal{R} \quad \Gamma_2; x : \mathcal{T}_2^? \vdash^{(m_2, e_2)} u : \mathcal{R}}{\Gamma_1 + \Gamma_2; x : \mathcal{T}_1^? + \mathcal{T}_2^? \vdash^{(1+m_1+m_2, e_1+e_2)} s' u : \mathcal{S}} \text{ APPC}$$

where $\Gamma = \Gamma_1 + \Gamma_2, \mathcal{T}^? = \mathcal{T}_1^? + \mathcal{T}_2^?, m = 1 + m_1 + m_2$ and $e = e_1 + e_2$. The following conditions hold:

- (a') $s \xrightarrow{\text{sub}(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, @} s'$, by condition (a)
- (b') $\Gamma_1; x : \mathcal{T}_1^? \vdash^{(m_1, e_1)} s' : [\mathcal{R}^? \rightarrow \mathcal{S}]$, by condition (b)
- (c') appropriate $_{\mathcal{A} \cup \mathcal{B} \cup \{x\}}$ ($\Gamma_1; x : \mathcal{T}_1^?$), since $\Gamma = \Gamma_1 + \Gamma_2$ and $\mathcal{T}^? = \mathcal{T}_1^? + \mathcal{T}_2^?$, and by condition (c)
- (d') Here $\mu = @$ and the term is the singleton $[\mathcal{R}^? \rightarrow \mathcal{S}]$.
- (e') $v \in \text{HA}_{\mathcal{A}_0 \cup \mathcal{B}}$, by condition (e)

Then we can apply *i.h.* on s' , yielding $\Gamma_s, \Gamma_v, \alpha, m_s, e_s, m_v, e_v$ such that

- $\Gamma_1 = \Gamma_s + \Gamma_v$ and $m_1 = m_s + m_v$ and $e_1 = e_s + e_v$
- $\Gamma_s; x : \mathcal{T}_1^? + [\alpha] \vdash^{(m_s, e_s+1)} s : [\mathcal{R}^? \rightarrow \mathcal{S}]$
- $\Gamma_v \vdash^{(m_v, e_v)} v : [\alpha]$

Taking $\Gamma_t = \Gamma_s + \Gamma_2, \Gamma_v, \alpha, m_t = 1 + m_s + m_2, e_t = e_s + e_2, m_v, e_v$ the following statements hold:

- $\Gamma = \Gamma_1 + \Gamma_2 = \Gamma_s + \Gamma_v + \Gamma_2 = \Gamma_t + \Gamma_v$ and $m = 1 + m_1 + m_2 = 1 + m_s + m_v + m_2 = m_t + m_v$ and $e = e_1 + e_2 = e_s + e_v + e_2 = e_t + e_v$
-

$$\frac{\Gamma_s; x : \mathcal{T}_1^? + [\alpha] \vdash^{(m_s, e_s+1)} s : [\mathcal{R}^? \rightarrow \mathcal{S}] \quad \mathcal{R}^? \triangleleft \mathcal{R} \quad \Gamma_2; x : \mathcal{T}_2^? \vdash^{(m_2, e_2)} u : \mathcal{R}}{\Gamma_s + \Gamma_2; x : (\mathcal{T}_1^? + [\alpha] + \mathcal{T}_2^?) \vdash^{(1+m_s+m_2, e_s+1+e_2)} s u : \mathcal{S}} \text{ APPC}$$

- $\Gamma_v \vdash^{(m_v, e_v)} v : [\alpha]$

3. The remaining cases are similar. □

PROPOSITION 7.6 (SUBJECT EXPANSION). *Let $\text{inv}(\mathcal{A}, \mathcal{S}, t)$, and let $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'$ where $\rho \in \{\text{db}, \text{lsv}\}$ and $\Gamma \vdash^{(m', e')} t' : \mathcal{T}$ and appropriate $_{\mathcal{A}}(\Gamma)$. Suppose moreover that if $\mu = @$ then either $\mathcal{T} = \mathfrak{s}$ or \mathcal{T} is a singleton, i.e. of the form $[\alpha]$. Then $\Gamma \vdash^{(m, e)} t : \mathcal{T}$, where, if $\rho = \text{db}$ we have that $(m, e) = (m' + 1, e')$, and if $\rho = \text{lsv}$ we have that $(m, e) = (m', e' + 1)$.*

PROOF. By induction on the derivation of $t \xrightarrow{\rho, \mathcal{A}, \mathcal{S}, \mu} t'$.

1. **DB**. The following conditions hold:

- (a) $(\lambda x. s) \text{L } u \xrightarrow{\text{db}, \mathcal{A}, \mathcal{S}, \mu} s[x/u] \text{L}$
- (b) $\Gamma \vdash^{(m', e')} s[x/u] \text{L} : \mathcal{T}$
- (c) appropriate $_{\mathcal{A}}(\Gamma)$

(d) If $\mu = @$ then either $\mathcal{T} = \mathfrak{s}$ or \mathcal{T} is a singleton, i.e. of the form $[\alpha]$.

By Lemma E.8 there exist $\Gamma_{s[x/u]}, \Gamma_L, \Delta, m'_{s[x/u]}, e'_{s[x/u]}, m'_L$ and e'_L such that:

1. $\Gamma_L \Vdash^{(m'_L, e'_L)} L \triangleright \Delta$
2. $\Gamma_{s[x/u]}; \Delta \vdash^{(m'_{s[x/u]}, e'_{s[x/u]})} s[x/u] : \mathcal{T}$

3. $\Gamma = \Gamma_L + \Gamma_{s[x/u]}$, and $m' = m'_L + m'_{s[x/u]}$, and $e' = e'_L + e'_{s[x/u]}$
 The judgment of statement 2. can be derived only by rule ES:

$$\frac{\Sigma_1; \Delta_1; x : \mathcal{S}^? \vdash^{(n_1, f_1)} s : \mathcal{T} \quad \mathcal{S}^? \triangleleft \mathcal{S} \quad \Sigma_2; \Delta_2 \vdash^{(n_2, f_2)} u : \mathcal{S}}{\Sigma_1 + \Sigma_2; \Delta_1 + \Delta_2 \vdash^{(n_1+n_2, f_1+f_2)} s[x/u] : \mathcal{T}} \text{ES}$$

where $\Gamma_{s[x/u]} = \Sigma_1 + \Sigma_2$, $\Delta = \Delta_1 + \Delta_2$, $m'_{s[x/u]} = n_1 + n_2$ and $e'_{s[x/u]} = f_1 + f_2$. By α -conversion we may assume that the bound variables in $(\lambda x. s)L$ don't occur free in u . Since $\text{dom}(\Delta) = \text{dom}(\Delta_1 + \Delta_2) = \text{dom}(\Delta_1) \cup \text{dom}(\Delta_2)$ is a subset of $\text{dom}(L)$ by Lemma E.7, we may assume that $\text{dom}(\Delta) \cap \text{fv}(u) = \emptyset$, so in particular $\text{dom}(\Delta_2) \cap \text{fv}(u) = \emptyset$. We have $\text{dom}(\Sigma_2) \cup \text{dom}(\Delta_2) = \text{dom}(\Sigma_2; \Delta_2) \subseteq \text{fv}(u)$ by Lemma 7.1. Therefore $\Delta_2 = \emptyset$, with $\Delta = \Delta_1$. We apply rule ABS to the judgment derivation $\Sigma_1; \Delta_1; x : \mathcal{S}^? \vdash^{(n_1, f_1)} s : \mathcal{T}$, yielding $\Sigma_1; \Delta \vdash^{(n_1, f_1)} \lambda x. s : [\mathcal{S}^? \rightarrow \mathcal{T}]$. By Lemma E.8, we compose this judgment with the one in statement 1., yielding $\Gamma_L + \Sigma_1 \vdash^{(m'_L+n_1, e'_L+f_1)} (\lambda x. s)L : [\mathcal{S}^? \rightarrow \mathcal{T}]$. Now we apply rule APPC, taking this judgment and $\Sigma_2; \Delta_2 \vdash^{(n_2, f_2)} u : \mathcal{S}$ as premises:

$$\frac{\Gamma_L + \Sigma_1 \vdash^{(m'_L+n_1, e'_L+f_1)} (\lambda x. s)L : [\mathcal{S}^? \rightarrow \mathcal{T}] \quad \mathcal{S}^? \triangleleft \mathcal{S} \quad \Sigma_2 \vdash^{(n_2, f_2)} u : \mathcal{S}}{\Gamma_L + \Sigma_1 + \Sigma_2 \vdash^{(1+m'_L+n_1+n_2, e'_L+f_1+f_2)} (\lambda x. s)L u : \mathcal{T}} \text{APPC}$$

where $\Gamma_L + \Sigma_1 + \Sigma_2 = \Gamma_L + \Gamma_{s[x/u]} = \Gamma$. Moreover since $\rho = \text{db}$, we have that $(m, e) = (1 + m'_L + n_1 + n_2, e'_L + f_1 + f_2) = (1 + m'_L + m'_{s[x/u]}, e'_L + e'_{s[x/u]}) = (m' + 1, e')$.

2. LSV $^\bullet$. The following conditions hold:

(a)

$$\frac{s \xrightarrow{\text{sub}_{(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu}} s' \quad x \notin \mathcal{A} \cup \mathcal{S} \quad vL \in \text{HA}_{\mathcal{A}}}{s[x/vL] \xrightarrow{\text{lsv}, \mathcal{A}, \mathcal{S}, \mu} s'[x/v]L} \text{LSV}^\bullet$$

(b) $\Gamma \vdash^{(m', e')} s'[x/v]L : \mathcal{T}$

(c) $\text{appropriate}_{\mathcal{A}}(\Gamma)$

(d) If $\mu = @$ then either $\mathcal{T} = \mathfrak{s}$ or \mathcal{T} is a singleton, i.e. of the form $[\alpha]$.

By Lemma E.8 there exist $\Gamma_{s'[x/v]}$, Γ_L , Δ , $m'_{s'[x/v]}$, $e'_{s'[x/v]}$, m'_L and e'_L such that:

1. $\Gamma_L \Vdash^{(m'_L, e'_L)} L \triangleright \Delta$
2. $\Gamma_{s'[x/v]}; \Delta \vdash^{(m'_{s'[x/v]}, e'_{s'[x/v]})} s'[x/v] : \mathcal{T}$
3. $\Gamma = \Gamma_L + \Gamma_{s'[x/v]}$, and $m' = m'_L + m'_{s'[x/v]}$, and $e' = e'_L + e'_{s'[x/v]}$.

Furthermore, $\text{appropriate}_{\mathcal{A}}(\Gamma)$ by condition (c), and note that $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/vL])$ implies $\text{inv}(\mathcal{A}, \mathcal{S}, s'[x/v]L)$, since $\mathcal{A} \# \mathcal{S}$ and $\text{fv}(s'[x/v]L) = \text{fv}(s[x/vL]) \subseteq \mathcal{A} \cup \mathcal{S}$. Therefore $\text{appropriate}_{\mathcal{A}^L}(\Delta)$. The judgment of statement (b) can only be derived by rule ES:

$$\frac{\Sigma_1; \Delta_1; x : \mathcal{S}^? \vdash^{(n_1, f_1)} s' : \mathcal{T} \quad \mathcal{S}^? \triangleleft \mathcal{S} \quad \Sigma_2; \Delta_2 \vdash^{(n_2, f_2)} v : \mathcal{S}}{\Sigma_1 + \Sigma_2; \Delta_1 + \Delta_2 \vdash^{(n_1+n_2, f_1+f_2)} s'[x/v] : \mathcal{T}} \text{ES}$$

where $\Gamma_{s'[x/v]} = \Sigma_1 + \Sigma_2$, $\Delta = \Delta_1 + \Delta_2$, $m'_{s'[x/v]} = n_1 + n_2$ and $e'_{s'[x/v]} = f_1 + f_2$. Moreover, $\text{inv}(\mathcal{A}, \mathcal{S}, s[x/vL])$ implies $\text{inv}(\mathcal{A} \cup \{x\}, \mathcal{S}, s)$. The following conditions hold:

(a') $s \xrightarrow{\text{sub}_{(x,v), \mathcal{A} \cup \{x\}, \mathcal{S}, \mu}} s'$, by condition (a)

(b') $(\Sigma_1; \Delta_1); x : \mathcal{S}^? \vdash^{(n_1, f_1)} s' : \mathcal{T}$, by premise of the rule in condition (b)

(c') $\text{appropriate}_{\mathcal{A}^L \cup \{x\}}((\Sigma_1; \Delta_1); x : \mathcal{S}^?)$ since

- (1) $\text{appropriate}_{\mathcal{A}}(\Sigma_1)$, as $\Gamma = \Gamma_L + \Sigma_1 + \Sigma_2$, and $\text{appropriate}_{\mathcal{A}}(\Gamma)$ by condition (c). We conclude by Remark E.1 that $\text{appropriate}_{\mathcal{A}^L \cup \{x\}}(\Sigma_1)$.
- (2) $\text{appropriate}_{\mathcal{A}^L}(\Delta_1)$, as $\Delta = \Delta_1 + \Delta_2$, and $\text{appropriate}_{\mathcal{A}^L}(\Delta)$, justified before. We conclude by Remark E.1 that $\text{appropriate}_{\mathcal{A}^L \cup \{x\}}(\Delta_1)$.
- (3) $\text{appropriate}_{\{x\}}(x : \mathcal{S}^?)$. Indeed, $v \in \text{HA}_{\mathcal{A}^L}$ by Lemma B.19, and $\text{appropriate}_{\mathcal{A}^L}(\Sigma_2; \Delta_2)$, since (3.1): $\text{appropriate}_{\mathcal{A}^L}(\Sigma_2)$, given that $\Gamma = \Gamma_L + \Sigma_1 + \Sigma_2$, and $\text{appropriate}_{\mathcal{A}}(\Gamma)$ by condition (c); and (3.2): $\text{appropriate}_{\mathcal{A}^L}(\Delta_2)$, given $\text{appropriate}_{\mathcal{A}^L}(\Delta)$ mentioned before and $\Delta = \Delta_1 + \Delta_2$. Hence we can apply Lemma E.2 yielding $\mathcal{S} \neq \mathfrak{s}$. Then $\mathcal{S}^? \triangleleft \mathcal{S}$ implies $\mathcal{S}^? \neq \mathfrak{s}$. By Remark E.1, we conclude that $\text{appropriate}_{\mathcal{A}^L \cup \{x\}}(x : \mathcal{S}^?)$.

(d') If $\mu = @$ then either $\mathcal{T} = \mathfrak{s}$ or \mathcal{T} is a singleton, i.e. of the form $[\alpha]$, by condition (d).

(e') $v \in \text{HA}_{\mathcal{A}^L}$, by Lemma B.19

Therefore by Lemma E.14, we have that there exist $\Theta_s, \Theta_v, \alpha, n_s, f_s, n_v$ and f_v such that:

- 1'. $\Sigma_1; \Delta_1 = \Theta_s + \Theta_v$ and $n_1 = n_s + n_v$ and $f_1 = f_s + f_v$
- 2'. $\Theta_s; x : \mathcal{S}^? + [\alpha] \vdash^{(n_s, f_s+1)} s : \mathcal{T}$
- 3'. $\Theta_v \vdash^{(n_v, f_v)} v : [\alpha]$

By statement 1', we can write Θ_s as $\Sigma_{11}; \Delta_{11}$ and Θ_v as $\Sigma_{12}; \Delta_{12}$ with $\Sigma_1 = \Sigma_{11} + \Sigma_{12}$ and $\Delta_1 = \Delta_{11} + \Delta_{12}$. By α -conversion the bound variables in vL don't occur free in s ; in particular $\text{dom}(L) \cap \text{fv}(s) = \emptyset$. Since $\text{dom}(\Delta) = \text{dom}(\Delta_1 + \Delta_2) = \text{dom}(\Delta_1) \cup \text{dom}(\Delta_2)$ is a subset of $\text{dom}(L)$ by Lemma E.7, then $\text{dom}(\Delta) \cap \text{fv}(s) = \emptyset$, so in particular $\text{dom}(\Delta_2) \cap \text{fv}(s) = \emptyset$. We have $\text{dom}(\Theta_s; x : \mathcal{S}^? + [\alpha]) \subseteq \text{fv}(s)$ by Lemma 7.1. Therefore $\Delta_{11} = \emptyset$, with $\Delta_1 = \Delta_{12}$. We apply Lemma E.3, merging the judgment of statement 3' and the judgment $\Sigma_2; \Delta_2 \vdash^{(n_2, f_2)} v : \mathcal{S}$, so that we obtain $(\Sigma_{12}; \Delta_1) + (\Sigma_2; \Delta_2) \vdash^{(n_v+n_2, f_v+f_2)} v : [\alpha] + \mathcal{S}$. We compose this result with the judgment from statement 1, by Lemma E.8, yielding $\Gamma_L + \Sigma_{12} + \Sigma_2 \vdash^{(m'_L+n_v+n_2, e'_L+f_v+f_2)} vL : [\alpha] + \mathcal{S}$. We apply rule ES:

$$\frac{\Sigma_{11}; x : \mathcal{S}^? + [\alpha] \vdash^{(n_s, f_s+1)} s : \mathcal{T} \quad \mathcal{S}^? \triangleleft \mathcal{S} \quad \Gamma_L + \Sigma_{12} + \Sigma_2 \vdash^{(m'_L+n_v+n_2, e'_L+f_v+f_2)} vL : [\alpha] + \mathcal{S}}{\Sigma_{11} + \Gamma_L + \Sigma_{12} + \Sigma_2 \vdash^{(n_s+m'_L+n_v+n_2, f_s+1+e'_L+f_v+f_2)} s[x/vL] : \mathcal{T}} \text{ES}$$

where $\Sigma_{11} + \Gamma_L + \Sigma_{12} + \Sigma_2 = \Sigma_1 + \Gamma_L + \Sigma_2 = \Gamma_{s'[x/v]} + \Gamma_L = \Gamma$. Moreover, since $\rho = \text{lsv}$, we have that $(m, e) = (n_s + m'_L + n_v + n_2, f_s + 1 + e'_L + f_v + f_2) = (m'_L + n_1 + n_2, f_1 + f_2 + e'_L + 1) = (m'_L + m'_{s'[x/v]}, e'_{s'[x/v]} + e'_L + 1) = (m', e' + 1)$, so we are done.

3. The congruence cases are uninteresting and are omitted here. □

THEOREM 7.7 (COMPLETENESS OF \mathcal{U}). *Let $\mathcal{S} = \text{fv}(t)$ and consider a reduction sequence $t \xrightarrow{\bullet}_{\text{top}, \mathcal{S}}^n s$ where s is $\xrightarrow{\bullet}_{\text{top}, \mathcal{S}}$ -irreducible. Let $n = m+e$ where m and e are respectively are the number of db and lsv steps in the sequence. Then there exists a tight environment Γ and a tight type \mathbb{t} such that $\Gamma \vdash^{(m, e)} t : \mathbb{t}$.*

PROOF. By induction on n :

1. $n = 0$. Then $t \in \text{NF}_{\emptyset, \text{fv}(t), \emptyset}^\bullet$, and $m = e = 0$ by definition of m and e . By Proposition 7.5, there exists a tight type \mathbb{t} such that $\text{TEnv}(\emptyset, \text{fv}(t), t) \vdash^{(0, 0)} t : \mathbb{t}$, with $\text{TEnv}(\emptyset, \text{fv}(t), t)$ a tight environment, so we are done.
2. $n = n' + 1$. Then $t \notin \text{NF}_{\emptyset, \text{fv}(t), \emptyset}^\bullet$. The reduction sequence is then of the form:

$$t \xrightarrow{\bullet}_{\rho_1, \emptyset, \text{fv}(t), \emptyset} t' \xrightarrow{\bullet}_{\rho_2, \emptyset, \text{fv}(t), \emptyset} \dots \xrightarrow{\bullet}_{\rho_{n'+1}, \emptyset, \text{fv}(t), \emptyset} s$$

where $n' = m' + e'$ and

$$m' = \#\{i \mid 2 \leq i \leq n', \rho_i = \text{db}\} \quad e' = \#\{i \mid 2 \leq i \leq n', \rho_i = \text{lsv}\}$$

Since $\text{fv}(t) = \text{fv}(t')$, since the reduction is non-erasing when $\rho_1 \in \{\text{db}, \text{lsv}\}$, then we can apply *i.h.*, yielding that there exists a tight environment Γ and a tight type \mathbb{t} such that $\Gamma \vdash^{(m', e')} t' : \mathbb{t}$. By Subject Expansion, it holds that $\Gamma \vdash^{(m, e)} t : \mathbb{t}$, where if $\rho = \text{db}$ then $(m, e) = (m' + 1, e')$ and if $\rho = \text{lsv}$ then $(m, e) = (m', e' + 1)$, so we are done. □