

BERGMAN LOCAL ISOMETRIES ARE BIHOLOMORPHISMS

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ABSTRACT. We prove that a proper holomorphic local isometry between bounded domains with respect to the Bergman metrics is necessarily a biholomorphism. The proof relies on a new method grounded in Information Geometry theories.

1. INTRODUCTION

Let Ω_1 and Ω_2 denote bounded domains in \mathbb{C}^n equipped with the Bergman metrics g_{B1} and g_{B2} , respectively. If a biholomorphism $f: \Omega_1 \rightarrow \Omega_2$ exists, then according to the transformation formula for the Bergman kernels (2.3), it is well-established that f induces an isometry with respect to the Bergman metric, i.e., $f^*g_{B2} = g_{B1}$. This article explores the converse implication. The main theorem is the following.

Theorem A. *Let Ω_1 and Ω_2 be bounded domains in \mathbb{C}^n . For a proper holomorphic map $f: \Omega_1 \rightarrow \Omega_2$, if $f^*g_{B2} = \lambda g_{B1}$ holds on an open subset $U \subset \Omega_1$ for some constant $\lambda > 0$, then f is a biholomorphism.*

Theorem A extends renowned Lu's uniformization theorem ([6]), which asserts that if a bounded domain Ω in \mathbb{C}^n admits the complete Bergman metric with constant holomorphic sectional curvature, then Ω is biholomorphic to the unit ball. To elucidate this, we briefly outline a proof of Lu's theorem. For a detailed exposition, we refer readers to (Theorem 4.2.2, [5]). Assuming the holomorphic sectional curvature is negatively constant by Myers' theorem, the universal cover $\tilde{\Omega}$ of Ω is biholomorphic to the unit ball, and the *covering map* $f: \mathbb{B}^n \rightarrow \Omega$ constitutes a *Bergman local isometry*. This implies that f is a \mathbb{C} -linear map in the Bergman representative coordinates at some point $p \in \mathbb{B}^n$. In other words, the following diagram

$$\begin{array}{ccc} \mathbb{B}^n & \xrightarrow{\text{rep}_p} & \mathbb{C}^n \\ f \downarrow & & \downarrow A \\ \Omega & \xrightarrow{\text{rep}_{f(p)}} & \mathbb{C}^n \end{array}$$

commutes locally on a neighborhood of $p \in \mathbb{B}^n$, where A represents a \mathbb{C} -linear map. Consequently, a left inverse $g: \Omega \rightarrow \mathbb{B}^n$ of f exists, implying injectivity of f . The significance of the unit ball \mathbb{B}^n lies in its properties: (1) $\text{rep}_p: \mathbb{B}^n \rightarrow \mathbb{C}^n$ is \mathbb{C} -linear (and thus invertible) and (2) \mathbb{B}^n is bounded (to apply the Riemann removable singularity theorem). Building upon this idea, S. Yoo ([9]) extended Lu's theorem by demonstrating that for a bounded domain $\Omega \subset \mathbb{C}^n$ admitting a Bochner "pole", a holomorphic Bergman

local isometry f from Ω onto a complex manifold M is necessarily a biholomorphism. We emphasize that Theorem A not only escapes the unit ball but also imposes no additional conditions (e.g., pseudoconvexity or Bergman completeness) apart from boundedness.

In a similar vein, M. Skwarczynski (IV.17. Theorem [8]) established that if f respects the transformation rule of the Bergman kernel (2.3), then f becomes a biholomorphism under the additional assumption that Ω_1 and Ω_2 are complete with respect to the Skwarczynski distance ρ defined in his thesis ([8]). Furthermore, he demonstrated that $\rho(z, w) = \rho(f(z), f(w))$ implies the injectivity of f . Notably, these assumptions concerns the Bergman kernel, which implies the assumption of Theorem A through differentiation. Hence his results follow from Theorem A immediately without any completeness assumption.

The proof of Theorem A relies on a novel method grounded in Information Geometry theories, particularly the Factorization Theorem (Theorem 2.1) and the result (Theorem 2.5) established by G. Cho and the author.

The Acknowledgments. The author would like to express his gratitude to Prof. Kang-Tae Kim for his insightful suggestion to approach Lu's theorem from a statistical perspective. The author extends his thanks to Prof. Kyeong-Dong Park, Sungmin Yoo, and Ye-Won Luke Cho for their valuable comments and words of encouragement. Additionally, the author acknowledges Hoseob Seo for engaging in profound discussions.

The author is supported by Learning & Academic research institution for Master's · PhD students, and Postdocs(LAMP) Program of the National Research Foundation of Korea(NRF) grant funded by the Ministry of Education(No. RS-2023-00301974).

2. PRELIMINARIES

2.1. Information Geometry. We briefly introduce the basic definitions of Information Geometry. For more details, we refer the readers to [1] and [2].

Let $\Xi \subset \mathbb{R}^m$ be a domain and dV be the standard Lebesgue measure on \mathbb{R}^m (more generally a set Ξ can be an arbitrary measurable space, but in this paper, we focus on a domain $\Xi \subset \mathbb{R}^m$ with the Borel σ -algebra). Let $\mathcal{P}(\Xi)$ be the space of all probability measures dominated by dV . In general, the space $\mathcal{P}(\Xi)$ is infinite-dimensional and a subset of the Banach space of all signed measures on Ξ with the total variation. From this Banach space, one can induce a smooth structure on $\mathcal{P}(\Xi)$. Then a triple (Ω, Ξ, Φ) (or $\Phi : \Omega \hookrightarrow \mathcal{P}(\Xi)$), where $\Omega \subset \mathbb{R}^n$ is domain and $\Phi : \Omega \rightarrow \mathcal{P}(\Xi)$ is a smooth embedding, is called a *statistical manifold* (or *statistical model*). We call Ω a *parameter space* and Ξ a *sample space*, respectively. Note that the dimensions n and m of the two spaces are independent.

On $\mathcal{P}(\Xi)$, there exists a natural pseudo-Riemannian metric g_F called the Fisher information metric. By using Φ as a (global) chart with a coordinate system (x_1, \dots, x_n) , the Fisher information metric $g_F(x) = \sum_{\alpha, \beta=1}^n g_{\alpha\beta}(x) dx_\alpha \otimes dx_\beta$ restricted on $\Phi(\Omega)$ can be written as

$$g_{\alpha\beta}(x) := \int_{\Xi} (\partial_\alpha \log P(x, \xi)) (\partial_\beta \log P(x, \xi)) P(x, \xi) dV(\xi),$$

where $\Phi(x) = P(x, \cdot)dV(\cdot)$ is a probability measure on Ξ , and $\partial_\alpha := \frac{\partial}{\partial x_\alpha}$, $\partial_\beta := \frac{\partial}{\partial x_\beta}$ for $\alpha, \beta = 1, \dots, n$.

One easy (but remarkable) example of a statistical manifold and the Fisher information metric is the set \mathcal{N} of all Gaussian normal distributions on \mathbb{R} . Since each element in \mathcal{N} is uniquely characterized by the mean $\mu \in \mathbb{R}$ and the standard deviation $\sigma > 0$, \mathcal{N} can be parametrized by the upper-half space \mathbb{H} in \mathbb{R}^2 , i.e., $\mathbb{H} \hookrightarrow \mathcal{N} \subset \mathcal{P}(\mathbb{R})$. Then \mathcal{N} with the Fisher information metric g_F becomes the hyperbolic space with negatively constant (Gaussian) curvature.

2.2. Sufficient Statistics. Let $\Phi_1 : \Omega_1 \hookrightarrow \mathcal{P}(\Xi_1)$ be a statistical manifold. For domains $\Xi_1 \subset \mathbb{R}^{m_1}$ and $\Xi_2 \subset \mathbb{R}^{m_2}$, a (Borel)-measurable function $f : \Xi_1 \rightarrow \Xi_2$ is called a *statistic*. Given a surjective statistic $f : \Xi_1 \rightarrow \Xi_2$, one can induce the natural map $\kappa : \mathcal{P}(\Xi_1) \rightarrow \mathcal{P}(\Xi_2)$ defined by the *measure push-forward* of f , i.e., for $\mu \in \mathcal{P}(\Xi_1)$,

$$(2.1) \quad \kappa(\mu)(B) := \mu(f^{-1}(B))$$

for each Borel subset $B \subset \Xi_2$. Then this induces a map $\kappa \circ \Phi_1 : \Omega_1 \rightarrow \mathcal{P}(\Xi_2)$ as follows.

$$(2.2) \quad \begin{array}{ccc} \Omega_1 & \xrightarrow{\Phi_1} & (\mathcal{P}(\Xi_1), g_{F_1}) \\ & \searrow_{\kappa \circ \Phi_1} & \downarrow \kappa \\ & & (\mathcal{P}(\Xi_2), g_{F_2}) \end{array}$$

Now, we compare two Fisher information metrics g_{F_1} and g_{F_2} of $\mathcal{P}(\Xi_1)$ and $\mathcal{P}(\Xi_2)$, respectively. Interestingly, the Fisher information metric always satisfies the monotone decreasing property, that is,

$$(\kappa \circ \Phi_1)^* g_{F_2}(X, X) \leq \Phi_1^* g_{F_1}(X, X)$$

for all $p \in \Omega_1$ and $X \in T_p(\Omega_1)$. When the inequality becomes equality, a statistic $f : \Xi_1 \rightarrow \Xi_2$ is called *sufficient* for $\Phi_1(\Omega_1)$. In other words, sufficient statistics preserve the geometric structure of statistical manifolds.

The following characterization theorem for sufficiency is the first ingredient for the proof of Theorem A.

Theorem 2.1 (cf. Theorem 2.1, [1]). (*Factorization Theorem*) *The following are equivalent.*

- (1) *A statistic $f : \Xi_1 \rightarrow \Xi_2$ is sufficient for $\Phi_1(\Omega_1)$.*
- (2) *The measurable function*

$$r(x, \xi) := \frac{P(x, \xi)}{Q(x, f(\xi))} \quad dV(\xi) - a.e.$$

does not depend on $x \in \Omega$, where $P(x, \xi)dV(\xi) \in \Phi_1(\Omega_1)$ and $\kappa(P(x, \xi)dV(\xi))(x, \zeta) := Q(x, \zeta)\kappa(dV)(\zeta)$.

- (3) *For each $x \in \Omega_1$, there exist functions $s(x, \cdot) \in L^1(\Xi_2, \kappa(dV))$ and $t \in L^1(\Xi_1, dV)$ such that*

$$P(x, \xi) = s(x, f(\xi))t(\xi) \quad dV(\xi) - a.e.,$$

where $P(x, \xi)dV(\xi) \in \Phi_1(\Omega_1)$.

For the proof, we refer the readers to [2]. In [2], Proposition 5.5 and Proposition 5.6 show that (3) \Leftrightarrow (2) and (2) \Leftrightarrow (1), respectively.

Remark 2.2. In condition (2) of Theorem 2.1, for each $x \in \Omega_1$, automatically $Q(x, \cdot) \in L^1(\Xi_2, \kappa(dV))$ by the definition of the measure push-forward and Radon–Nikodym theorem. Also, one can show that $r(x, \cdot) \in L^1(\Xi_1, dV)$ as follows.

$$\begin{aligned} \int_{\Omega_1} r(x, \xi) dV(\xi) &= \int_{\Omega_1} \frac{P(x, \xi)}{Q(x, f(\xi))} dV(\xi) = \int_{f(\Omega_1)} \frac{1}{Q(x, \zeta)} \kappa(P(x, \xi) dV(\xi)) \\ &= \int_{f(\Omega_1)} \kappa(dV) = \int_{\Omega_1} dV(\xi) < \infty. \end{aligned}$$

Remark 2.3. It is easy to see that an injective statistic f is always sufficient from condition (3) of Theorem 2.1: choose $s(x, \zeta) = P(x, f^{-1}(\zeta))$ and $t(\xi) = 1$.

2.3. Bergman Geometry. Let Ω be a bounded domain in \mathbb{C}^n and $A^2(\Omega)$ be the set of all L^2 holomorphic functions on Ω . Then $A^2(\Omega)$ is a separable Hilbert space with the inner product given by

$$\langle f, g \rangle := \int_{\Omega} f(z) \overline{g(z)} dV(z).$$

The *Bergman kernel* function $\mathcal{B} : \Omega \times \Omega \rightarrow \mathbb{C}$ is defined by

$$\mathcal{B}(z, \xi) := \sum_{j=0}^{\infty} s_j(z) \overline{s_j(\xi)},$$

where $\{s_j\}_{j=0}^{\infty}$ is a complete orthonormal basis for $A^2(\Omega)$. Note that the definition is independent of the choice of an orthonormal basis. The *Bergman metric* $g_B(z) = \sum_{\alpha, \beta=1}^n g_{\alpha\bar{\beta}}(z) dz_{\alpha} \otimes d\bar{z}_{\beta}$ on Ω is defined by

$$g_{\alpha\bar{\beta}}(z) := \frac{\partial^2}{\partial z_{\alpha} \partial \bar{z}_{\beta}} \log \mathcal{B}(z, z),$$

provided that $\mathcal{B}(z, z) > 0$ on Ω . For a bounded domain Ω , it is well-known that g_B is well-defined and positive-definite.

If $f : \Omega_1 \rightarrow \Omega_2$ is a biholomorphism between bounded domains $\Omega_1, \Omega_2 \subset \mathbb{C}^n$, the following transformation formula for the Bergman kernels holds:

$$(2.3) \quad \mathcal{B}_1(z, \xi) = J_{\mathbb{C}} f(z) \cdot \mathcal{B}_2(f(z), f(\xi)) \cdot \overline{J_{\mathbb{C}} f(\xi)},$$

where \mathcal{B}_1 and \mathcal{B}_2 are the Bergman kernels of Ω_1 and Ω_2 , respectively, and $J_{\mathbb{C}} f$ is the determinant of the complex Jacobian matrix of f . Moreover, from (2.3), f becomes an isometry with respect to the Bergman metric, which is one of the most important property of the Bergman metric.

2.4. Bounded domains as statistical manifolds. For a bounded domain $\Omega_1 \subset \mathbb{C}^n$, G. Cho and the author ([4]) constructed the map $\Phi_1 : \Omega_1 \rightarrow \mathcal{P}(\Omega_1)$ defined by

$$\Phi_1(z) := P_1(z, \xi) dV(\xi) := \frac{|\mathcal{B}_1(z, \xi)|^2}{\mathcal{B}_1(z, z)} dV(\xi).$$

Then they proved that $\Phi_1 : \Omega_1 \rightarrow \mathcal{P}(\Omega_1)$ is indeed a statistical manifold and the pull-back of the Fisher information metric on $\mathcal{P}(\Omega_1)$ is the same as the Bergman metric on Ω_1 . We call this $\Phi_1 : \Omega_1 \hookrightarrow \mathcal{P}(\Omega_1)$ a *Bergman statistical manifold*. They also present interesting other results in this framework.

For two bounded domains $\Omega_1, \Omega_2 \subset \mathbb{C}^n$, let $f : \Omega_1 \rightarrow \Omega_2$ be a proper holomorphic map. Note that a proper holomorphic map f is surjective (Proposition 15.1.5, [7]). It is known by R. Remmert that $V := \{f(z) : J_{\mathbb{C}}f(z) = 0\}$ is a complex variety in Ω_2 and

$$f : f^{-1}(\Omega_2 \setminus V) \rightarrow (\Omega_2 \setminus V) \text{ is an } m\text{-sheeted holomorphic covering map}$$

for some $m \in \mathbb{N}$. In this case, the measure push-forward κ of f defined by (2.1) can be explicitly written as

$$(2.4) \quad \kappa(P_1(z, \xi) dV(\xi))(z, \zeta) = \sum_{k=1}^m \frac{|\mathcal{B}_1(z, f_k^{-1}(\zeta))|^2 |J_{\mathbb{C}}f_k^{-1}(\zeta)|^2}{\mathcal{B}_1(z, z)} dV(\zeta)$$

for all $z \in \Omega_1$ and $\zeta \in \Omega_2 \setminus V$, where f_k^{-1} is a local inverse of f and $J_{\mathbb{C}}f_k^{-1}(\zeta)$ is the determinant of the complex Jacobian matrix of f_k^{-1} . For a Bergman statistical manifold, the diagram (2.2) becomes

$$\begin{array}{ccc} (\Omega_1, g_{B_1}) & \xrightarrow{\Phi_1} & (\mathcal{P}(\Omega_1), g_{F_1}) \\ & \searrow \kappa \circ \Phi_1 & \downarrow \kappa \\ & & (\mathcal{P}(\Omega_2), g_{F_2}) \end{array}$$

Remark 2.4. The way of the factorization (3) of Theorem 2.1 is not unique. For example, in a Bergman statistical manifold, if $f : \Omega_1 \rightarrow \Omega_2$ is a biholomorphism,

$$P_1(z, \xi) = \frac{|\mathcal{B}_1(z, \xi)|^2}{\mathcal{B}_1(z, z)} = \frac{|\mathcal{B}_1(z, f^{-1}(f(\xi)))|^2}{\mathcal{B}_1(z, z)} \cdot 1$$

or

$$P_1(z, \xi) = \frac{|\mathcal{B}_1(z, \xi)|^2}{\mathcal{B}_1(z, z)} = \frac{|\mathcal{B}_2(f(z), f(\xi))|^2 |J_{\mathbb{C}}f(z)|^2}{\mathcal{B}_1(z, z)} \cdot |J_{\mathbb{C}}f(\xi)|^2.$$

In the second factorization, the transformation formula for the Bergman kernel (2.3) is applied.

Although an injective statistic f is sufficient (Remark 2.3), the converse is not true in general. The following theorem proved by G. Cho and the author says that the converse is also true for a Bergman statistical manifold, which is our second main ingredient for the proof of Theorem A.

Theorem 2.5 (Corollary 4.9, [4]). *Assume that $f : \Omega_1 \rightarrow \Omega_2$ is a proper holomorphic map. Then f is injective if and only if f is sufficient for $\Phi_1(\Omega_1)$.*

For the readers' convenience, we briefly summarize the proof of Theorem 2.5.

They first proved that f is sufficient, i.e., $\Phi_1^* g_{F_1} = (\kappa \circ \Phi_1)^* g_{F_2}$ on Ω_1 , if and only if

$$(2.5) \quad \partial_\alpha \log \mathcal{B}_1(z, f_1^{-1}(\zeta)) = \cdots = \partial_\alpha \log \mathcal{B}_1(z, f_m^{-1}(\zeta))$$

for all $z \in \Omega_1$, $\zeta \in \Omega_2 \setminus V$ and $\alpha = 1, \dots, n$ by calculating the Fisher information metrics using (2.4).

Now suppose, for the sake of contradiction, that there exist $p \neq q \in f^{-1}(\Omega_2 \setminus V)$ such that $f(p) = f(q)$. Then (2.5) implies

$$(2.6) \quad \partial_\alpha \log \mathcal{B}_1(p, p) = \partial_\alpha \log \mathcal{B}_1(p, q).$$

Then, using a special orthonormal basis $\{s_j\}_{j=0}^\infty$ for $A^2(\Omega_1)$ with respect to p , they showed that, from (2.6), if $s(p) = 0$ then $s(q) = 0$ for all $s \in A^2(\Omega_1)$. This contradicts the fact that $A^2(\Omega_1)$ separates points, that is,

$$\text{for all } p \neq q \in \Omega_1, \exists s \in A^2(\Omega_1) \text{ such that } s(p) = 0 \text{ and } s(q) \neq 0.$$

Therefore they concluded that $f : f^{-1}(\Omega_2 \setminus V) \rightarrow (\Omega_2 \setminus V)$ is a injective map, and the properness of f (cf. Theorem 15.1.9, [7]) implies that in fact $f : \Omega_1 \rightarrow \Omega_2$ is injective. \square

3. PROOF OF THEOREM A

Since the surjectivity of f follows from that f is proper and holomorphic (Proposition 15.1.5, [7]), we only need to show that f is injective. By Theorem 2.1 and Theorem 2.5, it is enough to show that

$$P_1(z, \xi) := \frac{|\mathcal{B}_1(z, \xi)|^2}{\mathcal{B}_1(z, z)}$$

satisfies the condition (3) in Theorem 2.1.

From the condition $f^* g_{B_2} = \lambda g_{B_1}$, we have

$$\partial \bar{\partial} \log \mathcal{B}_1(z, z) - \lambda \partial \bar{\partial} \log \mathcal{B}_2(f(z), f(z)) = 0$$

for all $z \in \Omega_1$. Then there exist a (simply-connected) open neighborhood $U \subset \Omega_1$ and a holomorphic function φ on U such that

$$(3.1) \quad \log \mathcal{B}_1(z, z) - \lambda \log \mathcal{B}_2(f(z), f(z)) = \varphi(z) + \overline{\varphi(z)}$$

for all $z \in U$ and hence

$$\log \mathcal{B}_1(z, \xi) - \lambda \log \mathcal{B}_2(f(z), f(\xi)) = \varphi(z) + \overline{\varphi(\xi)}$$

for all $z, \xi \in U$ (by shrinking U if necessary). Furthermore,

$$(3.2) \quad \log P_1(z, \xi) - \lambda \log P_2(f(z), f(\xi)) = \varphi(\xi) + \overline{\varphi(\xi)}$$

for all $z, \xi \in U$. Denote the left side of (3.1) (after replacing the z variable by ξ) and the left side of (3.2) by $A(\xi)$ and $B(z, \xi)$, respectively. Then both $\exp(A(\xi))$ and

$\exp(B(z, \xi))$ are well-defined on the whole domain $\Omega_1 \times \Omega_1$. Since they are real-analytic functions and coincide on $U \times U$, we have

$$\begin{aligned} P_1(z, \xi) &= P_2(f(z), f(\xi))^\lambda \frac{\mathcal{B}_1(\xi, \xi)}{\mathcal{B}_2(f(\xi), f(\xi))^\lambda} \\ &= e^{-\lambda \mathcal{D}_2(f(z), f(\xi))} \cdot \mathcal{B}_1(\xi, \xi) \end{aligned}$$

for all $z, \xi \in \Omega_1$, where $\mathcal{D}_2(w, \zeta) := \log \frac{\mathcal{B}_2(w, w) \mathcal{B}_2(\zeta, \zeta)}{|\mathcal{B}_2(w, \zeta)|^2}$ is the Calabi's diastasis function ([3]) for Ω_2 . Hence, the condition (3) in Theorem 2.1 is satisfied for $s(z, \zeta) := e^{-\lambda \mathcal{D}_2(f(z), \zeta)}$ and $t(\xi) := \mathcal{B}_1(\xi, \xi)$ and the proof is completed. \square

In contrast that L^1 conditions are imposed for the functions s and t in the condition (3) of Theorem 2.1, our $t(\xi) := \mathcal{B}_1(\xi, \xi)$ is not a L^1 function on Ω_1 in general. However, the L^1 conditions are not needed in our situation because we have the explicit formula (2.4) for the measure push-forward. We end the paper by giving a proof for (3) \Rightarrow (2), which is mainly from Proposition 5.5 in [2].

Proposition 3.1. *Suppose that, for each $z \in \Omega_1$, there exist measurable functions $s(z, \cdot) : \Xi_2 \rightarrow \mathbb{R}$ and $t : \Xi_1 \rightarrow \mathbb{R}$ such that*

$$P_1(z, \xi) = s(z, f(\xi))t(\xi)$$

almost everywhere for $\xi \in \Omega_1$. Then the function

$$r(z, \xi) := \frac{P_1(z, \xi)}{Q(z, f(\xi))}$$

does not depend on $z \in \Omega_1$, where $\kappa(P_1(z, \xi)dV(\xi))(z, \zeta) := Q(z, \zeta)\kappa(dV)(\zeta)$.

Proof. From the assumption,

$$\begin{aligned} \kappa(P_1(z, \xi)dV(\xi))(z, \zeta) &= \sum_{k=1}^m s(z, f(f_k^{-1}(\zeta)))t(f_k^{-1}(\zeta))|J_{\mathbb{C}}f_k^{-1}(\zeta)|^2 dV(\zeta) \\ &= s(z, \zeta) \sum_{k=1}^m t(f_k^{-1}(\zeta))|J_{\mathbb{C}}f_k^{-1}(\zeta)|^2 dV(\zeta) \end{aligned}$$

and

$$Q(z, \zeta) := \frac{\kappa(P_1(z, \xi)dV(\xi))}{\kappa(dV)} = s(z, \zeta) \frac{\sum_{k=1}^m t(f_k^{-1}(\zeta))|J_{\mathbb{C}}f_k^{-1}(\zeta)|^2}{\sum_{k=1}^m |J_{\mathbb{C}}f_k^{-1}(\zeta)|^2}.$$

Therefore, $\frac{P_1(z, \xi)}{Q(z, f(\xi))}$ depends only on $\xi \in \Omega_1$. \square

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