

Controller Synthesis in Timed Büchi Automata: Robustness and Punctual Guards

Benoît Barbot¹, Damien Busatto-Gaston¹,
Catalin Dima¹, and Youssef Oualhadj¹

Univ Paris Est Creteil, LACL, F-94010 Creteil, France

`benoit.barbot@u-pec.fr`

`damien.busatto-gaston@u-pec.fr`

`catalin.dima@u-pec.fr`

`youssef.oualhadj@u-pec.fr`

Abstract. We consider the synthesis problem on timed automata with Büchi objectives, where delay choices made by a controller are subjected to small perturbations. Usually, the controller needs to avoid punctual guards, such as testing the equality of a clock to a constant. In this work, we generalize to a robustness setting that allows for punctual transitions in the automaton to be taken by controller with no perturbation. In order to characterize cycles that resist perturbations in our setting, we introduce a new structural requirement on the reachability relation along an accepting cycle of the automaton. This property is formulated on the region abstraction, and generalizes the existing characterization of winning cycles in the absence of punctual guards. We show that the problem remains within PSPACE despite the presence of punctual guards.

Keywords: Timed systems · Robustness · Controller synthesis.

1 Introduction

The design and verification of reactive systems usually requires the support of formal techniques to ensure the correction of the underlying model. The controller synthesis approach ensures a correct-by-design system since it supplies the designer of the system with formal tools ensuring that the system behaves according to a given specification. These formal tools consist in a game theoretic modeling where the reactive system is decomposed into controllable and uncontrollable behaviors. Ensuring the correctness of the system boils down to designing a sequence of controllable moves such that the specification holds under these moves regardless of the uncontrollable move that might be taken by the system. Thanks to the game theoretic toolbox, the sequence of controllable moves is automatically computed as a strategy in this game that ensures the correction of the designed systems.

When designing a reactive system with timing constraints, the formalism of choice is the one of *timed automata*. In a nutshell, the reactive system is modelled as a timed automaton over which two entities will compete. The first

entity (the controller) aims at enforcing an already agreed upon specification while the second one aims at preventing the controller from achieving it. This is known as timed games [8,2,6].

That being said, timed automata are nothing but a mere mathematical tool that might exhibit unrealistic behaviors. They may allow models that rely on infinitely precise behaviors to satisfy a specification, which can sometimes be an unrealistic assumption. A series of works focused on integrating some degree of robustness in the semantics of timed automata, i.e. enforcing behaviors resisting small perturbations in the evolution of the clock variables [3,10,14,7,16].

In particular, the controller synthesis problem for timed automata under *conservative semantics of perturbations* and with a Büchi objective is known to be PSPACE-complete [14,12] (see also [5] for a symbolic approach and [9] for a probabilistic extension). In this setting, a game is played between Controller and Pertubator on a timed automaton, with Controller choosing delays and transitions to follow, and Pertubator modifying every delay by a small amount. In particular, Controller must make sure that any transition he picks is still available after the deviated time elapse. The controller synthesis problem asks for the existence of a Controller strategy in this game that guarantees the objective against any decisions made by Pertubator.

The perturbation semantics naturally abstract imprecision in the measuring of time and the difficulty of enforcing a precise passage of time between consecutive events. To this end, the work in [14] rules out the possibility for Controller to choose *punctual* transitions, i.e. transitions which can be taken after a unique delay, since any perturbation to this delay would disable the transition. Such transitions correspond to guards in which at least one clock must be tested to have some exact value, for example $x = 2$.

However exact constraints might still occur in models of timed systems [11], e.g. when some controllers might have sufficiently reliable internal clocks. In such systems, some degree of unreliability in the measurement of time is still needed to model other uncontrollable components incorporating clocks that might be subjected to perturbations. A natural problem would then be to synthesize controllers in a system described as the synchronous composition of reliable components where delays and clock measurements are considered exact and unreliable components where perturbations are applied. This does not fit the setting of [14], as some transitions would not be subjected to perturbations. At the heart of the issue lie punctual transitions derived from reliable components of the system, that could make their way to the resulting timed automaton despite being forbidden under classical perturbation semantics. In this paper, we forgo the compositional framework and directly allow Controller to choose punctual transitions in the robust controller synthesis problem, in which case his choice of delay is not subjected to perturbation. Our setting, in which edges of the automaton are either reliable and punctual, or unreliable and non-punctual, paves the way towards the more general setting where even some non-punctual transitions could be considered reliable.

We show that this mixing of exact transitions and transitions under perturbation can be analyzed using the so-called (folded) orbit graphs formalism from [3,10,14,15]. Our algorithm that checks the existence of a robust controller for a Büchi objective involves finding, in the region automaton, a reachable cyclic path satisfying the Büchi condition whose folded orbit graph is a *cluster graph*, i.e., a disjoint union of complete graphs. The proof that such reachable cyclic paths can be robustly repeated involves, similarly with [14,9], showing that, starting from any clock valuation ν , and no matter what valuation ν' is reached as a cycle under perturbation is followed, Controller can enforce a run from ν' to a neighborhood of ν . Therefore, Controller ensures the robustness of his behaviour as he can compensate for any deviation.

In particular, this involves asking the reachability relation from valuation to valuation along a cyclic path of the automaton to be complete on sets of valuations. But, unlike in the case of [14], where these sets form the standard region partition, in our case they will form a finer partition of each region into subsets that we call *slices*.

In this paper, we start by defining in Section 2 our perturbation semantics in the presence of punctual guards, then in Section 3 we recall the folded orbit graphs that represent the reachability relation along a cycle. In Section 4, we show that robustly reaching a state by following some path of the automaton can be guaranteed under a simple syntactic assumption on the guards that are traversed. In Section 5, we define a class of folded orbit graphs (so-called cluster graphs), and state our main result: the cycles that can be iterated under our perturbed semantics exactly correspond to those in this class. In Section 6, we introduce the slice partition induced by a cluster folded orbit graph in order to represent sets of valuations with a complete reachability relation. Finally, in Section 7 we prove our main result by adapting the reasoning of [14] to slices in the presence of punctual transitions.

2 Partial robustness semantics

Timed automata. Given a finite set of clocks \mathcal{X} , we call *valuations* the elements of $\mathbb{R}_{\geq 0}^{\mathcal{X}}$. For a subset $R \subseteq \mathcal{X}$ and a valuation ν , $\nu[R := 0]$ is the valuation defined by $\nu[R := 0](x) = \nu(x)$ for $x \in \mathcal{X} \setminus R$ and $\nu[R := 0](x) = 0$ for $x \in R$. Given $d \in \mathbb{R}_{\geq 0}$ and a valuation ν , the valuation $\nu + d$ is called a time-successor of ν and is defined by $(\nu + d)(x) = \nu(x) + d$ for all $x \in \mathcal{X}$. We extend these operations to sets of valuations in the obvious way, and write $\mathbf{0}$ for the valuation that assigns 0 to every clock. We will consider the usual d_∞ metric on $\mathbb{R}^{\mathcal{X}}$, defined as $d_\infty(\nu, \nu') = \max_{x \in \mathcal{X}} |\nu(x) - \nu'(x)|$, and the Manhattan norm $\|\mathbf{v}\|_1$ of a vector $\mathbf{v} = (v_1, \dots, v_k)$, defined as $\sum_{i=1}^k v_i$. We say that a valuation ν is bounded by $\mathbf{B} > \mathbf{0}$ if $\nu \in [0, \mathbf{B}]^{\mathcal{X}}$.

An atomic clock constraint is a formula of the form $k \preceq x \preceq' l$ or $k \preceq x - y \preceq' l$ where $x, y \in \mathcal{X}$, $k, l \in \mathbb{Z} \cup \{-\infty, \infty\}$ and $\preceq, \preceq' \in \{<, \leq\}$. A *guard* g is a conjunction of atomic clock constraints. A valuation ν satisfies g , denoted $\nu \models g$, if all constraints are satisfied when each $x \in \mathcal{X}$ is replaced with $\nu(x)$.

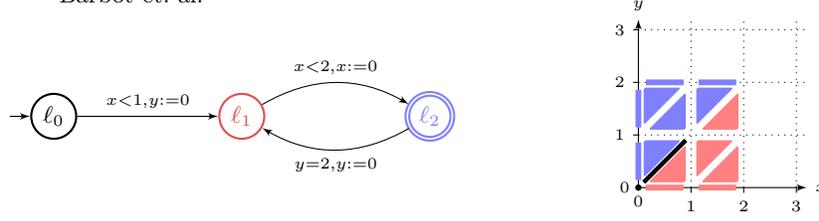


Fig. 1. A timed automaton, and some of its reachable regions.

We write $\text{Guards}(\mathcal{X})$ for the set of guards built on \mathcal{X} . A guard g is called *non-punctual* if there are at least two valuations ν and $\nu + d$ with $d \in \mathbb{R}_{\geq 0}$ that satisfy g . Conversely, a guard g is *punctual* if $\nu \models g \wedge d > 0$ implies $\nu + d \not\models g$.

A (bounded) *timed automaton* \mathcal{A} is a tuple $(\mathcal{L}, \mathcal{X}, \mathbf{B}, \ell_0, E)$, where \mathcal{L} is a finite set of locations, \mathcal{X} is a finite set of clocks, $\mathbf{B} \in \mathbb{N}_{>0}$ is an upper bound for clocks, $E \subseteq \mathcal{L} \times \text{Guards}(\mathcal{X}) \times 2^{\mathcal{X}} \times \mathcal{L}$ is a set of edges, and $\ell_0 \in \mathcal{L}$ is the initial location.

An edge $e = (\ell, g, R, \ell')$ is also written as $\ell \xrightarrow{g, R} \ell'$.

A configuration is a pair $q = (\ell, \nu) \in \mathcal{L} \times \mathbb{R}_{\geq 0}^{\mathcal{X}}$. The set of possible behaviors of a timed automaton can be described by the set of its runs, as follows. An infinite *run* of \mathcal{A} is a sequence $q_1 e_1 q_2 e_2 \dots$ where $q_i \in \mathcal{L} \times [0, \mathbf{B}]^{\mathcal{X}}$, and writing $q_i = (\ell, \nu)$, either $e_i \in \mathbb{R}_{\geq 0}$, in which case $q_{i+1} = (\ell, \nu + e_i)$ and e_i is called a delay transition, or $e_i = (\ell, g, R, \ell') \in E$, in which case $\nu \models g$, $q_{i+1} = (\ell', \nu[R := 0])$ and e_i is called an edge transition. A delay transition is sometimes denoted $(\ell, \nu) \xrightarrow{d} (\ell, \nu + d)$, and an edge transition is sometimes denoted $(\ell, \nu) \xrightarrow{g, R} (\ell', \nu[R := 0])$. A finite run of length k is a sequence $q_1 e_1 \dots q_k e_k q_{k+1}$ defined similarly. Whenever ρ is a finite run that ends in (ℓ, ν) and ρ' is a finite or infinite run starting from (ℓ, ν) , we let $\rho\rho'$ denote the concatenations of the two.

Game semantics. In order to define the robust controller synthesis problem, we introduce a game played on a timed automaton \mathcal{A} . Two players called **Controller** and **Pertubator** compete in an arena induced by \mathcal{A} and a parameter $\delta > 0$ in a two-player zero-sum game $\mathcal{G}_\delta(\mathcal{A})$. The interaction between **Controller** and **Pertubator** in $\mathcal{G}_\delta(\mathcal{A})$ follows the following rules to build an infinite run: given an initial configuration (ℓ, ν) , or a finite run ending in (ℓ, ν) ,

- either **Controller** picks a delay $d \geq 0$ and an edge $e = (\ell, g, R, \ell')$ with a *punctual* guard g to extend the run so that $\nu + d \models g$, in which case the run is extended with delay d and edge e without any perturbation,
- or **Controller** picks a delay $d \geq \delta$ and an edge $e = (\ell, g, R, \ell')$ with a *non-punctual* guard g to extend the run, so that g is satisfied after any delay in the set $[d - \delta, d + \delta]$. In this case, **Pertubator** chooses an actual delay $d' \in [d - \delta, d + \delta]$ to extend the run, after which the edge e is taken.

It is then again **Controller's** turn to play from the new configuration. We say that a run is *well-formed* if it is compatible with these semantics, so that it starts with a delay transition, alternates between delay transitions and edge transitions, and ends with an edge transition. Note that the concatenation of well-formed runs is well-formed as well, and that a well-formed run can be decomposed as the concatenation of atomic well-formed runs $(\ell, \nu) \xrightarrow{d} (\ell, \nu') \xrightarrow{g, R} (\ell', \nu'')$.

Formally, the state space of $\mathcal{G}_\delta(\mathcal{A})$ is partitioned into the states where it is Controller's turn to make a move $\mathcal{L} \times [0, \mathbf{B})^{\mathcal{X}}$, and the states where it is Pertubator's turn to make a move $\mathcal{L} \times [0, \mathbf{B})^{\mathcal{X}} \times \mathbb{R}_{\geq 0} \times E$. A play over the arena induced by $\mathcal{G}_\delta(\mathcal{A})$ is an infinite alternation between states of Controller and states of Pertubator that forms a run of the automaton. A strategy for Controller in $\mathcal{G}_\delta(\mathcal{A})$ is a function $\sigma_\delta^{\text{Cont}}$ that assigns to every finite play ending in a state of Controller a pair (d, e) where d is a delay and e an edge of \mathcal{A} . A strategy for Pertubator in $\mathcal{G}_\delta(\mathcal{A})$ is function $\sigma_\delta^{\text{Pert}}$ that assigns to every finite play ending in a state of Pertubator a perturbation in $[-\delta, \delta]$. Once a pair of strategies $(\sigma_\delta^{\text{Cont}}, \sigma_\delta^{\text{Pert}})$ is fixed, we denote by $\text{Outcome}(\sigma_\delta^{\text{Cont}}, \sigma_\delta^{\text{Pert}})$ the unique run induced over \mathcal{A} that starts from $(\ell_0, \mathbf{0})$ and follows them. Notice that this run is not necessarily infinite, since some Controller states may have no legal moves.

A Büchi objective is given by a subset of location Buchi $\subseteq \mathcal{L}$. We say that an infinite run ρ in \mathcal{A} satisfies a Büchi objective Buchi if the set of locations visited infinitely often along ρ contains states from Buchi. We say that Controller wins $\mathcal{G}_\delta(\mathcal{A})$ for the objective Buchi if Controller has a winning strategy $\sigma_\delta^{\text{Cont}}$ so that for any strategy $\sigma_\delta^{\text{Pert}}$ played by Pertubator, $\text{Outcome}(\sigma_\delta^{\text{Cont}}, \sigma_\delta^{\text{Pert}})$ is an infinite run that satisfies Buchi.

Finally, we define the *robust controller synthesis problem* as follows: given a timed automaton \mathcal{A} equipped with a Büchi objective Buchi, does there exists an amplitude $\delta > 0$ small enough so that Controller wins the resulting game $\mathcal{G}_\delta(\mathcal{A})$?

3 Preliminaries

The region abstraction. With respect to the set \mathcal{X} of clocks and an upper bound $\mathbf{B} \in \mathbb{N}_{>0}$ on clocks, we partition the set $[0, \mathbf{B})^{\mathcal{X}}$ of valuations into finitely many regions [1]. We denote by $\text{Regs}(\mathcal{X}, \mathbf{B})$ this set of (bounded) regions. Each region is characterised by a pair (ι, β) where $\iota: \mathcal{X} \rightarrow [0, \mathbf{B}) \cap \mathbb{N}$ and β is an ordered partition of \mathcal{X} into subsets $\beta_0 \uplus \beta_1 \uplus \dots \uplus \beta_m$ (with $m \geq 0$), where β_0 can be empty but $\beta_i \neq \emptyset$ for $1 \leq i \leq m$. A valuation ν of $[0, \mathbf{B})^{\mathcal{X}}$ belongs to the region characterised by (ι, β) if:

- for all $x \in \mathcal{X}$, $\iota(x) = \lfloor \nu(x) \rfloor$, where $\lfloor \nu(x) \rfloor$ is the integral part of $\nu(x)$;
- for all $x \in \beta_0$, $\text{fract}(\nu(x)) = 0$, where $\text{fract}(\nu(x)) = \nu(x) - \lfloor \nu(x) \rfloor$ is the fractional part of $\nu(x)$;
- for all $1 \leq i < j \leq m$, for all $x, y \in \beta_i$ and all $z \in \beta_j$, $0 < \text{fract}(\nu(x))$, $\text{fract}(\nu(x)) = \text{fract}(\nu(y))$ and $\text{fract}(\nu(x)) < \text{fract}(\nu(z))$.

A region r characterized by $(\iota, \beta_0 \uplus \dots \uplus \beta_m)$ is said to be of dimension m .

Remark 1. Intuitively, a region of dimension m is a polytope of dimension m containing every valuation ν such that the integer part of its coordinates is the integer valuation ι , and such that the fractional part of its coordinate is ordered according to the sequence $\beta_0 < \beta_1 < \dots < \beta_m$, with clocks in the same β_i having the same fractional part and those in β_0 having fractional part 0.

A region r can be described by a system of equations that forms a guard g of $\text{Guards}(\mathcal{X})$. Conversely, a guard intersected with $[0, \mathbf{B}]^{\mathcal{X}}$ can be seen as a finite union of regions in $\text{Regs}(\mathcal{X}, \mathbf{B})$. We say that a region r is non-punctual if it contains at least two distinct time-successor valuations. It is punctual otherwise.

The set $\text{Reg}(\mathcal{X}, \mathbf{B})$ partitions $[0, \mathbf{B}]^{\mathcal{X}}$ into regions that represent equivalence classes of the time-abstract bisimulation [1].

A region r' is a time-successor of a region r if for each valuation $\nu \in r$ there is a delay $d \geq 0$ so that $\nu + d \in r'$. Similarly, given a region r and a set of clocks R , the region $r[R := 0]$ is the region such that for all valuations $\nu \in r$, $\nu[R := 0] \in r[R := 0]$. Finally, we write $r \models g$ if every valuation in r satisfies g .

A pair (ℓ, r) with $\ell \in L$ and r a region is called a region state. A *region path* π is a finite or infinite sequence of delay transitions $(\ell, r) \xrightarrow{\text{delay}} (\ell, r')$ such that r' is a time-successor of r and edge transitions $(\ell, r) \xrightarrow{g, R} (\ell', r[R := 0])$ such that there is an edge $\ell \xrightarrow{g, R} \ell'$ in E with $r \models g$. We define the length of a finite region path and the concatenation of region paths as expected. Similarly, a region path is well-formed if it is compatible with the game semantics, so that it starts with a delay transition, alternates between delays and edges, and ends with an edge transition. A run ρ is said to follow a region path π if it contains the same sequence of delay and edge transitions, and every valuation visited along ρ belongs to the corresponding region at the same step in π .

A *region cycle* around a region state (ℓ, r) is a finite region path π that starts and ends in the same region state (ℓ, r) . We often omit ℓ and say that π is a cycle around r . A *region lasso* $\pi_0 \pi^\omega$ around a region state (ℓ, r) is an infinite region path described as a finite region path π_0 that ends in (ℓ, r) , followed by a region cycle π around (ℓ, r) that is iterated forever. In this definition, we allow for π_0 to be empty (so that π^ω is a valid lasso), but π has non-zero length.

In the absence of perturbations, finding a winning run for a Büchi objective Buchi amounts to finding a well-formed region lasso $\pi_0 \pi^\omega$ around some region state (ℓ, r) where ℓ is in Buchi [1]. We call such lassos *winning lassos*.

Corners. The *corners* of a region r of dimension m characterized by (ι, β) with $\beta = \beta_0 \uplus \dots \uplus \beta_m$ are the following $m+1$ valuations: for each $0 \leq i \leq m$, let c_i be the corner such that $\forall 0 \leq j \leq m-i, \forall x \in \beta_j, c_i(x) = \iota(x)$, and $\forall m-i < j \leq m, \forall x \in \beta_j, c_i(x) = \iota(x) + 1$. Let \mathcal{C} denote the set of corners $\{c_0, \dots, c_m\}$.

Remark 2. Intuitively, \mathcal{C} contains the valuations of integer coordinates that form the corners of the polyhedron r . In particular, they are totally ordered in Manhattan norm: $\|c_0\|_1 < \|c_1\|_1 < \dots < \|c_m\|_1$.

Let ν be a valuation in r of dimension m . From [10], we know that ν can always be expressed as a unique convex combination of the corners $\{c_0, \dots, c_m\}$ of r , so that there is a unique weight vector $\lambda = (\lambda_0, \dots, \lambda_m) \in (0, 1]^{m+1}$ with $\|\lambda\|_1 = 1$ such that $\nu = \sum_{i=0}^m \lambda_i c_i$. Conversely, every combination of corners $\sum_{i=0}^m \lambda_i c_i$ where $\lambda = (\lambda_0, \dots, \lambda_m) \in (0, 1]^{m+1}$ and $\|\lambda\|_1 = 1$ is a valuation in r .

Definition 1. A valuation ν in a region r of dimension m is said to have corner weights λ if $\lambda = (\lambda_0, \dots, \lambda_m) \in (0, 1]^{m+1}$, $\|\lambda\|_1 = 1$ and $\nu = \sum_{i=0}^m \lambda_i c_i$.

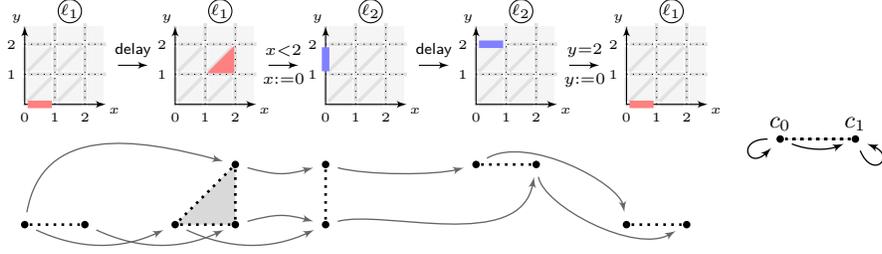


Fig. 2. The orbit graph of a region cycle in the automaton of Fig. 1, and its folding.

Orbit graphs. Let $(\text{step}_i)_{i \in \mathbb{N}}$ be a sequence of distinct fresh symbols. An orbit graph γ is a finite acyclic graph defined on a sequence of regions r_0, \dots, r_k , so that the vertices of the graph are the pairs (step_i, c) where $0 \leq i \leq k$ and c ranges over the corners of r_i , and so that every edge in γ is of the shape $(\text{step}_i, c) \rightarrow (\text{step}_{i+1}, c')$. Given an orbit graph γ defined on a sequence of regions r_0, \dots, r_k , and an orbit graph γ' defined on a sequence of regions r'_0, r'_1, \dots, r'_l with $r_k = r'_0$, the concatenation of γ and γ' is the orbit graph $\gamma\gamma'$ defined on the sequence of regions $r_0, \dots, r_k, r'_1, \dots, r'_l$ so that every edge $(\text{step}_i, c) \rightarrow (\text{step}_{i+1}, c')$ of γ is in $\gamma\gamma'$ and so that every edge $(\text{step}_i, c) \rightarrow (\text{step}_{i+1}, c')$ of γ' is in $\gamma\gamma'$ as $(\text{step}_{i+k}, c) \rightarrow (\text{step}_{i+k+1}, c')$. In the context of an orbit graph γ defined on a sequence of regions r_0, \dots, r_k , we let start and end denote the symbols step_0 and step_k , respectively, so that a path going through all of γ starts in a vertex (start, c) and ends in a vertex (end, c') .

Let $t = (\ell, r) \xrightarrow{\text{delay}} (\ell', r')$ be a delay transition from a region r of corners \mathcal{C} to a time-successor region r' of corners \mathcal{C}' . The orbit graph of t is the orbit graph defined on r, r' that contains every edge $(\text{start}, c) \rightarrow (\text{end}, c')$ such that c' is a time-successor of c . Let $t = (\ell, r) \xrightarrow{g, R} (\ell', r')$ be an edge transition from a region r of corners \mathcal{C} to a region r' of corners \mathcal{C}' with $r \models g$ and $r' = r[R := 0]$. The orbit graph of t is the orbit graph defined on r, r' that contains every edge $(\text{start}, c) \rightarrow (\text{end}, c')$ such that $c' = c[R := 0]$.

Let π be a finite region path. The *orbit graph* of π , denoted $\gamma(\pi)$, is defined as the concatenation of the orbit graphs of every delay transition and edge transition in π . In Fig. 2, we depict a region path and its associated orbit graph.

Remark 3. Intuitively, paths from corners to corners in $\gamma(\pi)$ represent the reachability relation along π of valuations arbitrarily close to these corners. As every valuation in a region r can be seen as a convex combination of the corners of r , $\gamma(\pi)$ represents as a finite graph the entire reachability relation along π : If $\gamma(\pi)$ is interpreted as an interval Markov chain (the set of all Markov chains with this graph as support), then the runs that follow π can be described as a Markov chain refining $\gamma(\pi)$, such that the corner weights of the starting valuation in the run describe the initial-state distribution, and the transition probabilities encode the flow of these corner weights along the run. This is formalised in Appendix A.

Folded orbit graphs. Let π be a region cycle around a region r , *i.e.* a finite region path that starts and ends in the same region state (ℓ, r) . let \mathcal{C} be the corners of r . The folded orbit graph of π is a graph $\Gamma(\pi)$ of vertices \mathcal{C} . For every corner c of r and every corner c' of r , there is an edge from c to c' in $\Gamma(\pi)$ if there is a path from (start, c) to (end, c') in the orbit graph $\gamma(\pi)$.

It follows by the orbit graph construction that $\Gamma(\pi)$ represents the reachability relation from corner to corner along π .

Graph terminology. We will use classical directed graph theory to describe the properties of folded orbit graphs: A graph is said to be strongly connected if every vertex is reachable from every other vertex. The strongly connected components (SCCs) of a graph form a partition into subgraphs that are strongly connected. Likewise, a graph is said to be weakly connected if its underlying undirected graph is strongly connected. The weakly connected components of a graph also form a partition into subgraphs that are weakly connected. The disjoint union of graphs is constructed by making the vertex set of the result be the disjoint union of the vertex sets of the given graphs, and by making the edge set of the result be the disjoint union of the edge sets of the given graphs. A graph is complete if every pair of vertices is connected by an edge.¹

Finally, a *cluster graph* is a disjoint union of complete graphs, *i.e.* a graph where every weakly connected component is a complete SCC.

4 Robustness of a finite path

We define robust paths, as a notion meant to represent finite region paths that Controller can traverse no matter what Pertubator does:

Definition 2. *We say that a well-formed finite region path π is robust if every edge transition $(\ell, r) \xrightarrow{g, R} (\ell', r')$ in π with a non-punctual guard g satisfies that r is non-punctual as well.*

Intuitively, this prevents situations where any non-zero perturbation would force the run out of π : the region r is reached right after a delay transition that includes a perturbation in this case, so it cannot be punctual if we wish for Controller to guarantee reaching it. Note that the concatenation of robust paths is robust.

In order to show that this notion characterize robustness, we recall classical data structures and the notion of controllable predecessors of a set of valuations.

Zones and Difference Bound Matrices. A (rational) Difference Bound Matrix (DBM) over a set of clocks $\mathcal{X} = \{x_1, \dots, x_n\}$ is a matrix $(M_{i,j})_{1 \leq i, j \leq n}$ so that each entry $M_{i,j}$ is a pair $(\preceq_{i,j}, m_{i,j})$ with $\preceq_{i,j} \in \{\leq, <\}$ and $m_{i,j} \in \mathbb{Q} \cup \{+\infty\}$. It represents a set of valuations z called a zone, so that $\nu \in z$ if and only if for each $1 \leq i, j \leq n$, $\nu \models x_i - x_j \preceq_{i,j} m_{i,j}$. These matrices can be used as efficient ways to represent guards or regions, and many useful operations over sets of valuations can be described as polynomial matrix operations on DBMs.

¹ including self-loops $v \rightarrow v$ for every vertex v .

Definition 3 ([4]). *If z is a zone, g is a guard, R is a set of clocks and \mathbf{B} is a bound on clocks, we define the following operations for any rational $\delta \geq 0$:*

- $\text{PreTime}_{\geq \delta}(z) = \{\nu \in [0, \mathbf{B}]^{\mathcal{X}} \mid \exists d \geq \delta, \nu + d \in z\}$,
- $g \cap z = \{\nu \in z \mid \nu \models g\}$,
- $\text{Unreset}_R(z) = \{\nu \in [0, \mathbf{B}]^{\mathcal{X}} \mid \nu[R := 0] \in z\}$, and
- $\text{Shrink}_{\delta}(z) = \{\nu \in [0, \mathbf{B}]^{\mathcal{X}} \mid \forall \varepsilon \in [-\delta, +\delta], \nu + \varepsilon \in z\}$.

In fact, if z is encoded as a DBM, the output of these operations is a DBM where entries are of the shape $(\preceq, m - \delta p)$ with $p \in \mathbb{N}$.

In order to make symbolic computations that abstract the perturbation value, one can use shrunk DBMs [13], a parametric DBM where δ is a parameter. Shrunk DBMs are pairs (M, P) where M is a DBM and P is a shrinking matrix of the shape $(p_{i,j})_{1 \leq i,j \leq n}$ so that every entry $p_{i,j}$ is in \mathbb{N} . Then, for any value of δ , $M - \delta P$ represents the DBM of entry $(\preceq_{i,j}, m_{i,j} - \delta p_{i,j})$. The DBM operations of Definition 3 can all be implemented on shrunk DBMs as parametric computations with an arbitrarily small $\delta > 0$.

Controllable predecessors. We recall the definition of the CPre operation:

Definition 4 ([12]). *Let $\pi = (\ell, r) \xrightarrow{\text{delay}} (\ell, r') \xrightarrow{g, R} (\ell', r'')$ be an atomic robust region path (with $r' \models g$), let $\delta \geq 0$ be some amplitude of perturbation and s be a set of valuations in r'' . If g is a non-punctual guard, then we let*

$$\text{CPre}_{\pi}^{\delta}(s) = \{\nu \in r \mid \exists d \geq \delta, \forall d' \in [d - \delta, d + \delta], \nu + d' \in r' \wedge (\nu + d')[R := 0] \in s\}.$$

If g is a punctual guard, then we let

$$\text{CPre}_{\pi}^{\delta}(s) = \text{CPre}_{\pi}^0(s) = \{\nu \in r \mid \exists d \geq 0, \nu + d \in r' \wedge (\nu + d)[R := 0] \in s\}.$$

If $\pi = \pi' \pi''$, then $\text{CPre}_{\pi}^{\delta}(s)$ is defined inductively as $\text{CPre}_{\pi'}^{\delta}(\text{CPre}_{\pi''}^{\delta}(s))$.

Overall, $\nu \in \text{CPre}_{\pi}^{\delta}(s)$ if and only if Controller has a strategy that ensure reaching s when starting from ν and following the path π , against any Perturbator strategy of maximal perturbation δ . In particular $\delta = 0$, $\text{CPre}_{\pi}^0(s)$ is the standard predecessor operator over region paths, *i.e.* the set of valuations ν so that there exists a well-formed run from ν to some $\nu' \in s$ that follows π without perturbations. Thus, if we set $s = r'$, the last region of π , it holds that $\text{CPre}_{\pi}^0(r')$ equals r , the first region of π , because regions represent equivalence classes of the time-abstract bisimulation relation [1]. We note the following inclusion properties: if $\delta \leq \delta'$, then $\text{CPre}_{\pi}^{\delta'}(s) \subseteq \text{CPre}_{\pi}^{\delta}(s)$, and if $s \subseteq s'$, then $\text{CPre}_{\pi}^{\delta}(s) \subseteq \text{CPre}_{\pi}^{\delta}(s')$.

Let π be a region path from r to r' , and let z be a zone in r' represented as a DBM (or a shrunk DBM), then $\text{CPre}_{\pi}^{\delta}(z)$ can be computed as a DBM (or as a shrunk DBM): if π is an atomic robust path with a non-punctual guard, then $\text{CPre}_{\pi}^{\delta}(z) = r \cap \text{PreTime}_{\geq \delta}(\text{Shrink}_{\delta}(g \cap \text{Unreset}_R(z)))$, and if π is an atomic robust path with a punctual guard, then $\text{CPre}_{\pi}^{\delta}(z) = r \cap \text{PreTime}_{\geq 0}(g \cap \text{Unreset}_R(z))$.

Proposition 5. *Let π be a well-formed region path from (ℓ, r) to (ℓ', r') . Then, there exists $\delta > 0$ so that $\text{CPre}_{\pi}^{\delta}(r') \neq \emptyset$ if and only if π is robust.*

In particular, if π is a robust region path that starts from (ℓ_0, r_0) and ends in (ℓ, r) with r_0 the region $\{\mathbf{0}\}$, then $\mathbf{0} \in \text{CPre}_\pi^\delta(r)$, so that **Controller** can guarantee reaching (ℓ, r) from $(\ell_0, \mathbf{0})$ for a small enough δ . As a result, the notion of robust region paths is sufficient to enforce a reachability objective. Winning for a Büchi objective, however, requires enforcing an infinite path, that will stay within a lasso forever. If π is a robust region cycle around a region r , we know from the inclusion properties of CPre_π^δ that for all $k \in \mathbb{N}_{>0}$, $r \supseteq \text{CPre}_\pi^\delta(r) \supseteq \text{CPre}_{\pi^2}^\delta(r) \supseteq \dots \supseteq \text{CPre}_{\pi^k}^\delta(r)$. For any fixed $\delta > 0$, this decreasing sequence of sets $\text{CPre}_{\pi^k}^\delta(r)$ may become empty for k big enough, meaning there is no strategy of **Controller** that is able to iterate the cycle π forever.

We will show that for a specific class of cycles π this scenario does not happen, as we will find some set s in r so that $s \subseteq \text{CPre}_\pi^\delta(s) \subseteq \text{CPre}_{\pi^2}^\delta(s) \subseteq \dots$, so that $\text{CPre}_{\pi^k}^\delta(r) \neq \emptyset$ for all $k \geq 1$, *i.e.* **Controller** can iterate π forever.

5 Robustness of an infinite path

As previously explained, for a cycle in the region abstraction, being a robust path does not imply that it can be iterated forever, so that solving for Büchi objectives requires a finer notion. We define robustly iterable cycles to characterise cycles that **Controller** can iterate forever, no matter what **Pertubator** does:

Definition 6. *We say that a well-formed region cycle π is robustly iterable if π is a robust path and $\Gamma(\pi)$ is a cluster graph. We say that a lasso $\pi_0\pi^\omega$ is robustly iterable if π_0 is either empty or a robust path, and if π is robustly iterable.*

The cluster graph condition represents a requirement on the reachability relation along π that generalises the notion of aperiodic cycle from [12,3,15] (such cycles had an iterate with a complete $\Gamma(\pi)$, which is in particular a cluster graph). This notion is stable through iterations of cycles, so that if π is a robustly iterable cycle, then for every $l \geq 1$, the cycle π^l has the same folded orbit graph as π (by decomposition into concatenated orbit graphs $\gamma(\pi) \dots \gamma(\pi)$), and therefore is robustly iterable as well. On the other hand, it is possible for a cycle π that is not robustly iterable to become robustly iterable after iterating it multiple times. Let us detail what can happen as a cycle is iterated:

Lemma 7. *Let π be a region cycle around a region r of dimension m , so that π is a robust path. Then only one of the following cases must hold:*

- *Either there exists $1 \leq k \leq m(m+1)!$ so that π^k is robustly iterable,*
- *or there exists $1 \leq k \leq (m+1)!$ so that $\Gamma(\pi^k)$ contains a weakly connected component that is not strongly connected.*

We will show that **Controller** can ensure staying in π forever in the first case of Lemma 7, but that in the second case and for any fixed $\delta > 0$, **Pertubator** can prevent him from staying in π forever.

Theorem 8. *Given an instance of the robust controller synthesis problem of automaton \mathcal{A} and objective **Buchi**, there exists $\delta > 0$ so that **Controller** wins $\mathcal{G}_\delta(\mathcal{A})$ if and only if there exists a winning lasso that is robustly iterable.*

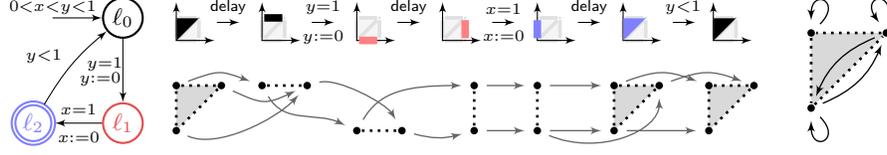


Fig. 3. A timed automaton with a robustly iterable cycle $\ell_0\ell_1\ell_2$ is on the left. The corresponding orbit graph is displayed in the center and its folded orbit graph is depicted on the right. The folded orbit graph is a cluster graph with two complete SCCs.

This characterisation based on robustly iterable lassos can be naturally extended to a non-deterministic algorithm using polynomial space, which implies membership to PSPACE for our problem. The algorithm consists in guessing a winning lasso that is robustly iterable, computing its folded orbit graph and checking that it is indeed a cluster graph. The details are somewhat involved but classical for this kind of problems (see *e.g.* the procedure in [12, Section 8.7] for a different characterisation based on orbit graphs), as the region path needs to be guessed transition after transition, so that the folded orbit graph is built by composition in an online fashion, and does not require the entire cycle to be stored in memory. The length of the cycle that is guessed can be exponentially bounded by combinatorial reasoning (on the number of folded orbit graphs that can be built from a given timed automaton). The PSPACE-hardness of our problem straightforwardly follows from [14] where the robust controller synthesis problem without punctual guards was shown to be PSPACE-complete. Hence:

Theorem 9. *The robust controller synthesis problem is PSPACE-complete.*

The remainder of this paper is devoted to the proof of the correctness of our characterization, *i.e.* Theorem 8.

6 Slicing regions

Given a single region r , we define a partition of r into sets of valuations called slices according to a partition of the corners of r .

Definition 10. *Let r be a region of dimension m and corner \mathcal{C} . Let $0 \leq k \leq m$. A corner partition of r is a partition $\mathcal{C}_0 \uplus \dots \uplus \mathcal{C}_k$ of \mathcal{C} into $k+1$ colors, such that a corner $c \in \mathcal{C}$ is said to have color $0 \leq i \leq k$ if $c \in \mathcal{C}_i$.*

Recall that the folded orbit graph of a robustly iterable region cycle is a cluster graph, *i.e.* a disjoint union of complete graphs.

Definition 11. *Let π be a robustly iterable region cycle around a region r , of folded orbit graph $\Gamma(\pi)$. The cluster corner partition of r induced by $\Gamma(\pi)$ is the corner partition of r so that every color corresponds to a complete SCC of $\Gamma(\pi)$.*

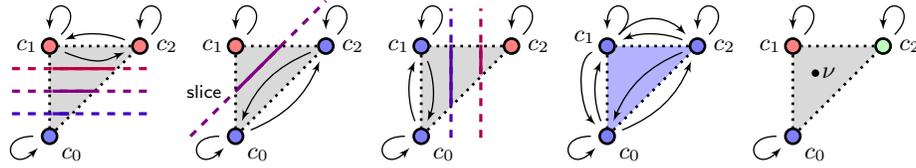


Fig. 4. Cluster corner partitions and some slices of a region of dimension 2.

In other words, corners c and c' have the same color in the cluster corner partition if and only if they are in the same complete SCC of $\Gamma(\pi)$.

Definition 12. Let $C_0 \uplus \dots \uplus C_k$ be a corner partition of a region r of dimension m into $k+1$ colors, ordered from 0 to k w.l.o.g.. A color weight vector is a weight vector $\mathbf{w} = (w_0, \dots, w_k) \in (0, 1]^{k+1}$ so that $\|\mathbf{w}\|_1 = \sum_{i=0}^k w_i = 1$. In this context, we say that \mathbf{w} gives color $0 \leq i \leq k$ the weight w_i .

Definition 13 (Slice partition). Let $C_0 \uplus \dots \uplus C_k$ be a corner partition of a region r of dimension m into $k+1$ colors. A slice w.r.t. $C_0 \uplus \dots \uplus C_k$ is a set of valuations defined by a color weight vector $\mathbf{w} = (w_0, \dots, w_k)$. Let ν be a valuation in r of corner weights $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_m)$. Then, ν belongs to the slice defined by \mathbf{w} , denoted $\text{slice}_{C_0 \uplus \dots \uplus C_k}^r(\mathbf{w})$, if for all $0 \leq j \leq k$, $\sum_{\substack{0 \leq i \leq m \\ \text{s.t. } c_i \in C_j}} \lambda_i = w_j$.

Whenever the partition $C_0 \uplus \dots \uplus C_k$ is the cluster corner partition of a region r induced by $\Gamma(\pi)$, where π is a robustly iterable region cycle around r , we write $\text{slice}_\pi(\mathbf{w})$ instead of $\text{slice}_{C_0 \uplus \dots \uplus C_k}^r(\mathbf{w})$.

Remark 4. Intuitively, \mathbf{w} gives a total weight for each color, such that the slice defined by \mathbf{w} contains the convex combinations of corners compatible with these total weights when summing every corner weight of the same color. As \mathbf{w} ranges over every color weight vector, the set $\{\text{slice}_\pi(\mathbf{w})\}$ partitions the entire region, so that each valuation in the region belongs to a unique slice. Slices are convex polyhedra, but may not be zones.² However, slices defined w.r.t. the cluster corner partition of a robustly iterable cycle are always zones (cf. Appendix D).

Example 1. Let r be the region $0 < x < y < 1$ of dimension 2 of corners c_0 , c_1 and c_2 . Fig. 4 depicts three examples of corner partitions of r into 2 colors $C_0 \uplus C_1$, and examples of folded orbit graphs that induce these partitions as cluster corner partitions. A few slices are depicted in each case for different color weight vectors. In particular, the partition displayed in the second example in Fig. 4 is the cluster corner partition induced by the cycle π from Fig. 3, and the slice drawn in this example is the slice $\text{slice}_\pi(\mathbf{w})$ for $\mathbf{w} = (\frac{1}{2}, \frac{1}{2})$. The two rightmost examples represent partitions with 1 and 3 colors that result in one slice equal to r , or singleton slices that contain a single valuation ν , respectively.

As detailed in Appendix E, the slices induced by a robustly iterable cycle π represent equivalence classes of the reachability relation, so that there is a run from ν to ν' following π if and only if ν and ν' belong to the same slice.

² On automata with 3 clocks, some corner partitions may induce slices described by linear equations of the shape $x + y - z = w$.

7 Proof of Theorem 8

We now proceed with the proof of our characterization in term of cluster graphs. This is formalized in the following propositions.

Proposition 14. *If there exists a winning lasso that is robustly iterable, then Controller wins $\mathcal{G}_\delta(\mathcal{A})$ for some $\delta > 0$.*

Proof (Sketch). Recall that, as described in the end of Section 4, in order to win, Controller has to exhibit a cycle that can be iterated despite the perturbation. We will actually show that a *robustly iterable lasso* can be iterated forever against any perturbation inflicted by *Pertubator*. The full proof can be found in Appendix E, here we sketch the essential steps of this proof and explain how we use the different notions introduced so far.

Let $\pi_0\pi^\omega$ be a robustly iterable winning lasso around the region r . In Fig. 5, we depict the essential steps of our proof from left to right. The first step of the proof, presented in Fig. 5.1, consists in showing that one can build a DBM N such that any shrinking of r contains N , i.e., for any shrinking matrix P and δ small enough, $N \subseteq r - \delta P$. The construction of this DBM is detailed in Appendix E.

The second step of the proof (Fig. 5.2) consists in showing that for any slice induced by the cluster corner partition that intersects N , there exists a *shrinking matrix* P such that $(r - \delta P) \cap \text{slice} = \text{CPre}_\pi^\delta(N \cap \text{slice})$. This is established in Lemma 28 of Appendix E. To obtain this latter equation we use two crucial properties of any slice induced by the cluster corner partition:

- The fact that the folded orbit graph of π is a cluster graph implies that the reachability relation along π is *complete* over the slice.
- Slices induced by a folded orbit graph are always *zones*, cf. Proposition 23 of Appendix D, and are thus compatible with (shrunk) DBM operations.

Finally, the first two step entail that, as displayed in Fig. 5.3:

$$N \cap \text{slice} \subseteq \text{CPre}_\pi^\delta(N \cap \text{slice}).$$

This result implies that Controller wins from any valuation ν in $N \cap \text{slice}$, as he can always enforce staying within $N \cap \text{slice}$. \square

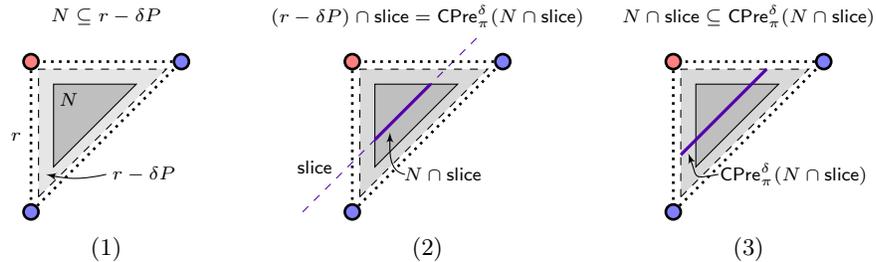


Fig. 5. Proof schema of Proposition 14 applied to the corner partition of Fig. 3.

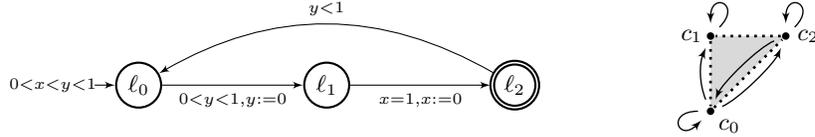


Fig. 6. A cycle that is not robustly iterable, and its folded orbit graph.

Moreover, we note that a cycle that only contains punctual transitions can trivially be iterated forever by Controller as there are no perturbations happening. However, such cycles always have fully-partitioned folded orbit graphs such as the rightmost example of Fig. 4, and therefore are always robustly iterable.

Proposition 15. *If there is no winning lasso that is robustly iterable in the region automaton of \mathcal{A} , then Controller cannot win $\mathcal{G}_\delta(\mathcal{A})$ for any $\delta > 0$.*

Proof (Sketch). To establish the above proposition, we first show that if Controller tries to win by repeating the same accepting region cycle then this iteration will not be possible forever. Indeed, let π be a region cycle and assume that $\Gamma(\pi)$ is not a cluster graph, then there must exist at least two weakly connected SCCs I and J such that J is reachable from I but not vice versa, cf. Fig. 6 where $I = \{c_0, c_2\}$ and $J = \{c_1\}$. Let ν be a valuation in the region r from Fig. 6. We show that Controller cannot iterate π forever when starting from ν .

Let $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ be the corner weights of ν . We first use a result from [14] that exhibits a precise strategy for Pertubator which inflicts a well chosen perturbation ε that pushes any run away from $\{c_0, c_2\}$ and towards $\{c_1\}$. This entails the following behavior: after each visit of r , one reaches a valuation ν' of corner weights $\lambda' = (\lambda'_0, \lambda'_1, \lambda'_2)$ such that $\lambda_0 + \lambda_2 > \lambda'_0 + \lambda'_2 + \varepsilon$. A direct corollary of this observation is that under this strategy of Pertubator, any run from ν that tries to visit r repeatedly will ultimately drift out of the region r . This is detailed in Appendix F. In order to conclude, we still need to argue that Controller cannot win by switching between different region cycles. But since there exists only a finite number of folded orbit graphs, one can use *Ramsey-like* arguments to factorize any infinite run into a finite prefix and factors that all share the same folded orbit graph. Assuming that no such folded orbit graph is a cluster graph, together with the previous intuition, entails Proposition 15. \square

8 Conclusion

This work is a first technical step towards a more general setting where timing measurements in components of the system could either be reliable or disruptable. As future work, we plan to investigate the following directions. Extend the current work by allowing some non-punctual transitions to be reliable (*i.e.* not under perturbation in the game semantics), and define criteria for repairing timed automata that are not robustly controllable in the classical sense by transforming them into timed automata with exact components that are robust in the setting addressed in this paper.

References

1. Alur, R., Dill, D.L.: A theory of timed automata. *Theoretical Computer Science* **126**(2), 183–235 (1994)
2. Asarin, E., Maler, O., Pnueli, A., Sifakis, J.: Controller synthesis for timed automata. In: *Proceedings of IFAC Symposium on System Structure and Control*. pp. 469–474. Elsevier (1998)
3. Basset, N., Asarin, E.: Thin and thick timed regular languages. In: Fahrenberg, U., Tripakis, S. (eds.) *Formal Modeling and Analysis of Timed Systems*. Lecture Notes in Computer Science, vol. 6919, pp. 113–128. Springer (2011)
4. Bengtsson, J., Yi, W.: Timed automata: Semantics, algorithms and tools. In: *Advanced Course on Petri Nets*. pp. 87–124. Springer (2003)
5. Busatto-Gaston, D., Monmege, B., Reynier, P., Sankur, O.: Robust controller synthesis in timed büchi automata: A symbolic approach. In: Dillig, I., Tasiran, S. (eds.) *Computer Aided Verification - 31st International Conference, CAV 2019, New York City, NY, USA, July 15-18, 2019, Proceedings, Part I*. Lecture Notes in Computer Science, vol. 11561, pp. 572–590. Springer (2019). https://doi.org/10.1007/978-3-030-25540-4_33
6. Cassez, F., David, A., Fleury, E., Larsen, K.G., Lime, D.: Efficient on-the-fly algorithms for the analysis of timed games. In: Abadi, M., de Alfaro, L. (eds.) *CONCUR 2005 - Concurrency Theory, 16th International Conference, CONCUR 2005, San Francisco, CA, USA, August 23-26, 2005, Proceedings*. Lecture Notes in Computer Science, vol. 3653, pp. 66–80. Springer (2005). https://doi.org/10.1007/11539452_9
7. Gupta, V., Henzinger, T.A., Jagadeesan, R.: Robust timed automata. In: Maler, O. (ed.) *Hybrid and Real-Time Systems, International Workshop. HART'97, Grenoble, France, March 26-28, 1997, Proceedings*. Lecture Notes in Computer Science, vol. 1201, pp. 331–345. Springer (1997). <https://doi.org/10.1007/BFB0014736>
8. Maler, O., Pnueli, A., Sifakis, J.: On the synthesis of discrete controllers for timed systems (an extended abstract). In: Mayr, E.W., Puech, C. (eds.) *STACS 95, 12th Annual Symposium on Theoretical Aspects of Computer Science, Munich, Germany, March 2-4, 1995, Proceedings*. Lecture Notes in Computer Science, vol. 900, pp. 229–242. Springer (1995). https://doi.org/10.1007/3-540-59042-0_76
9. Oualhadj, Y., Reynier, P., Sankur, O.: Probabilistic robust timed games. In: Baldan, P., Gorla, D. (eds.) *CONCUR 2014 - Concurrency Theory - 25th International Conference, CONCUR 2014, Rome, Italy, September 2-5, 2014, Proceedings*. Lecture Notes in Computer Science, vol. 8704, pp. 203–217. Springer (2014). https://doi.org/10.1007/978-3-662-44584-6_15
10. Puri, A.: Dynamical properties of timed automata. *Discret. Event Dyn. Syst.* **10**(1-2), 87–113 (2000). <https://doi.org/10.1023/A:1008387132377>
11. Rodríguez-Navas, G., Proenza, J.: Using timed automata for modeling distributed systems with clocks: Challenges and solutions. *IEEE Trans. Software Eng.* **39**(6), 857–868 (2013). <https://doi.org/10.1109/TSE.2012.73>
12. Sankur, O.: Robustness in timed automata : analysis, synthesis, implementation. (Robustesse dans les automates temporisés : analyse, synthèse, implémentation). Ph.D. thesis, École normale supérieure de Cachan, Paris, France (2013), <https://tel.archives-ouvertes.fr/tel-00910333>
13. Sankur, O., Bouyer, P., Markey, N.: Shrinking timed automata. *Inf. Comput.* **234**, 107–132 (2014). <https://doi.org/10.1016/J.IC.2014.01.002>

14. Sankur, O., Bouyer, P., Markey, N., Reynier, P.A.: Robust controller synthesis in timed automata. In: D'Argenio, P.R., Melgratti, H. (eds.) Proceedings of the 24th International Conference on Concurrency Theory (CONCUR'13). Lecture Notes in Computer Science, vol. 8052, pp. 546–560. Springer (2013). https://doi.org/10.1007/978-3-642-40184-8_38
15. Stainer, A.: Frequencies in forgetful timed automata. In: Jurdzinski, M., Nickovic, D. (eds.) Formal Modeling and Analysis of Timed Systems - 10th International Conference, FORMATS 2012, London, UK, September 18-20, 2012. Proceedings. Lecture Notes in Computer Science, vol. 7595, pp. 236–251. Springer (2012). https://doi.org/10.1007/978-3-642-33365-1_17
16. Wulf, M.D., Doyen, L., Markey, N., Raskin, J.: Robustness and implementability of timed automata. In: Lakhnech, Y., Yovine, S. (eds.) Formal Techniques, Modelling and Analysis of Timed and Fault-Tolerant Systems, Joint International Conferences on Formal Modelling and Analysis of Timed Systems, FORMATS 2004 and Formal Techniques in Real-Time and Fault-Tolerant Systems, FTRTFT 2004, Grenoble, France, September 22-24, 2004, Proceedings. Lecture Notes in Computer Science, vol. 3253, pp. 118–133. Springer (2004). https://doi.org/10.1007/978-3-540-30206-3_10

A Classical results on regions and orbit graphs.

Lemma 16 ([1]). *Let (ℓ_1, ν_1) and (ℓ_2, ν_2) be configurations of a timed automaton \mathcal{A} and r_1, r_2 be regions so that $\nu_1 \in r_1$ and $\nu_2 \in r_2$. The following are equivalent:*

- *there is a run from (ℓ_1, ν_1) to (ℓ_2, ν_2) in \mathcal{A} ,*
- *for all $\nu'_1 \in r_1$ there exists $\nu'_2 \in r_2$ and a run from (ℓ_1, ν'_1) to (ℓ_2, ν'_2) , and*
- *for all $\nu'_2 \in r_2$ there exists $\nu'_1 \in r_1$ and a run from (ℓ_1, ν'_1) to (ℓ_2, ν'_2) .*

As such, we will abstract sets of runs that go through the same regions and the same edges of \mathcal{A} as paths from region to region in a finite automaton called the region abstraction of \mathcal{A} .

Lemma 17 ([10]). *Let π be a region cycle around a region r of corners \mathcal{C} . For every corner $c \in \mathcal{C}$, there is a corner $c' \in \mathcal{C}$ so that there is an edge from c to c' in $\Gamma(\pi)$. Similarly, for every corner $c' \in \mathcal{C}$, there is a corner $c \in \mathcal{C}$ so that there is an edge from c to c' in $\Gamma(\pi)$.*

Lemma 18. *Let π be a well-formed region path from a region state (ℓ, r) of dimension m to a region state (ℓ', r') of dimension m' , of respective corners $\mathcal{C} = \{c_0, \dots, c_m\}$ and $\mathcal{C}' = \{c'_0, \dots, c'_{m'}\}$. Let $\nu \in r$ and $\nu' \in r'$ be valuations of respective corner weights $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_m)$ and $\boldsymbol{\lambda}' = (\lambda'_0, \dots, \lambda'_{m'})$. For each $0 \leq i \leq m$, let $\text{post}_i \subseteq \mathcal{C}'$ be the corners c'_j so that there is a path in $\gamma(\pi)$ from (start, c_i) to (end, c'_j) . Conversely, for each $0 \leq j \leq m'$ let $\text{pre}_j \subseteq \mathcal{C}$ be the corners c_i so that there is a path in $\gamma(\pi)$ from (start, c_i) to (end, c'_j) .*

Then, there is a run from (ℓ, ν) to (ℓ', ν') that follows π if and only if there is for each $c_i \in \mathcal{C}$ a probability distribution $\mathbf{p}_i : \text{post}_i \rightarrow [0, 1]$ so that for each $0 \leq j \leq m'$, $\lambda'_j = \sum_{c_i \in \text{pre}_j} \lambda_i \mathbf{p}_i(c'_j)$.

The above lemma entails the following characterization of the reachability relation between the valuations of the same region.

Corollary 19 ([10]). *Let π be a well-formed region cycle around a region state (ℓ, r) of dimension m , of corners $\mathcal{C} = \{c_0, \dots, c_m\}$. Let $\nu \in r$ and $\nu' \in r'$ be valuations of respective corner weights $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_m)$ and $\boldsymbol{\lambda}' = (\lambda'_0, \dots, \lambda'_{m'})$. For each $0 \leq i \leq m$, let $\text{post}_i \subseteq \mathcal{C}$ be the corners c_j so that there is an edge in $\Gamma(\pi)$ from c_i to c_j . Conversely, for each $0 \leq j \leq m$ let $\text{pre}_j \subseteq \mathcal{C}$ be the corners c_i so that there is an edge in $\Gamma(\pi)$ from c_i to c_j .*

Then, there is a run from ν to ν' that follows π if and only if there is for each $c_i \in \mathcal{C}$ a probability distribution $\mathbf{p}_i : \text{post}_i \rightarrow [0, 1]$ so that for each $0 \leq j \leq m$, $\lambda'_j = \sum_{c_i \in \text{pre}_j} \lambda_i \mathbf{p}_i(c_j)$.

B Proof of Lemma 7

Lemma 7. *Let π be a region cycle around a region r of dimension m , so that π is a robust path. Then only one of the following cases must hold:*

- *Either there exists $1 \leq k \leq m(m+1)!$ so that π^k is robustly iterable,*
- *or there exists $1 \leq k \leq (m+1)!$ so that $\Gamma(\pi^k)$ contains a weakly connected component that is not strongly connected.*

This property also holds if we replace π^k by the concatenation $\pi_1\pi_2\dots\pi_k$, where π_1, \dots, π_k are finite robust paths so that $\Gamma(\pi) = \Gamma(\pi_1) = \dots = \Gamma(\pi_k)$.

Proof. We start by showing that the two cases are mutually exclusive. If $\Gamma(\pi^k)$ is a cluster graph, then for every $l \geq 1$, the cycle π^{kl} has the same folded orbit graph as π^k (by decomposition of $\gamma(\pi^{kl})$ into concatenated orbit graphs $\gamma(\pi^k)\dots\gamma(\pi^k)$), and therefore is a cluster graph as well. Similarly, if $\Gamma(\pi^{k'})$ contains a weakly connected component that is not strongly connected, then for every $l' \geq 1$, $\Gamma(\pi^{k'l'})$ also has a weakly connected component that is not strongly connected. Indeed, if there is an SCC I in $\Gamma(\pi^{k'})$, a corner $c \in I$ and a corner $c' \notin I$, so that there is an edge from c to c' in $\Gamma(\pi^{k'})$ but no path from c' to c , then by decomposition of $\gamma(\pi^{k'l'})$ into concatenated orbit graphs $\gamma(\pi^{k'})\dots\gamma(\pi^{k'})$, there is some corner c'' reached from c' in $\gamma(\pi^{k'})$ by a path of length $l' - 1$, so that there is an edge from c to c'' in $\Gamma(\pi^{k'l'})$ but no path from c'' to c . Thus, if by contradiction both conditions were true for k and k' , respectively, then $\Gamma(\pi^{kk'})$ would be a cluster graph with a weakly connected component that is not strongly connected, which is impossible.

Now, assume that for every $1 \leq k \leq (m+1)!$, every weakly connected component of $\Gamma(\pi^k)$ is also strongly connected, *i.e.* $\Gamma(\pi^k)$ is a disjoint union of SCCs. Let us show that $\Gamma(\pi^{m(m+1)!})$ must be a cluster graph. First, we show that there is a self-loop on every corner in $\Gamma(\pi^{(m+1)!})$. Indeed, $\Gamma(\pi)$ is a disjoint union of SCCs, and by Lemma 17 every corner has an outgoing edge in $\Gamma(\pi)$, so that every corner must belong to some cycle of $\Gamma(\pi)$ of length $1 \leq l \leq m+1$. Then, there is a path from (start, c) to (end, c) in $\gamma(\pi^l)$, and thus in $\gamma(\pi^{(m+1)!})$ as $(m+1)!$ is a multiple of l . Thus, for all c there is a self-loop from c to c in $\Gamma(\pi^{(m+1)!})$, and by the starting assumption $\Gamma(\pi^{(m+1)!})$ is also a disjoint union of SCCs. Let c and c' be corners in the same SCC of $\Gamma(\pi^{(m+1)!})$. Then, there exists in $\Gamma(\pi^{(m+1)!})$ a path from c to c' of length $l \leq m$, and a path of length $m-l$ from c' to c' (by repeating the self loop on c'), so that there is a path of length m from c to c' in $\Gamma(\pi^{(m+1)!})$. It follows that there is an edge from c to c' in $\Gamma(\pi^{m(m+1)!})$ for all c, c' in the same SCC of $\Gamma(\pi^{(m+1)!})$. Moreover, for any c, c' in disjoint (and thus independent) SCCs of $\Gamma(\pi^{(m+1)!})$, there can be no edge from c to c' in $\Gamma(\pi^{m(m+1)!})$, so that overall $\Gamma(\pi^{m(m+1)!})$ is a cluster graph where every SCC of $\Gamma(\pi^{(m+1)!})$ is independent and complete.

Finally, we note that replacing π^k by the concatenation $\pi_1\pi_2\dots\pi_k$ of paths with the same folded orbit graph does not affect the previous proof, that entirely relies on the edges of $\Gamma(\pi)$ and $\Gamma(\pi^k)$. Indeed, we note by definition of folded orbit graphs that if $\Gamma(\pi) = \Gamma(\pi_1) = \dots = \Gamma(\pi_k)$ then $\Gamma(\pi_1\dots\pi_k) = \Gamma(\pi^k)$. \square

C Proofs of Section 4

Let M, N be DBMs. We say that $N \sqsubseteq M$ if $N \subseteq M$ and the constants in M and N are equal. In other words, N is obtained from M by making some large inequalities strict, but has the same interior. In particular M is a DBM that encodes a region r , it holds that $N \sqsubseteq M$ and $N \neq \emptyset$ implies $N = r$, as every constraint that encodes a region is either strict or an equality constraint that cannot be made strict without becoming empty.

The next two results are proven in [12, Lemma 8.6.1] and [12, Lemma 8.6.2] in a setting without punctual edges or regions. We prove them by induction on the length of π , by noting that they are stable by composition, so that only well-formed atomic paths need to be considered. In particular, CPre_π^δ with π an atomic robust path with a non-punctual edge can be decomposed into the DBM operations of Definition 3, and each of these operations satisfies the two results individually by [12]. If π is an atomic robust path with a punctual edge, then by definition of CPre in this case we can also decompose CPre_π^δ into the same elementary DBM operations, and conclude similarly.

Lemma 20. *Let π be a robust region path from r to r' , and let $\delta > 0$. Let M and M' be DBMs such that $M \neq \emptyset$ and $M' \neq \emptyset$. Moreover, assume that $M = \text{CPre}_\pi^0(M')$. Then, for any DBM $N' \sqsubseteq M'$ and for any shrinking matrix P' , there exists a DBM $N \sqsubseteq M$ and a shrinking matrix P such that $N - \delta P = \text{CPre}_\pi^\delta((N' - \delta P'))$.*

Lemma 21. *Let π be a robust region path from r to r' . Let N be a DBM so that there exists $\nu \in r'$ and $\varepsilon > 0$ with $(\text{Ball}_{d_\infty}(\nu, \varepsilon) \cap r') \subseteq N$. Then, $\text{CPre}_\pi^\delta(N)$ is non-empty for a small enough $\delta > 0$, and in fact contains $\text{Ball}_{d_\infty}(\nu', \varepsilon') \cap r$ for some $\nu' \in r$ and $\varepsilon' > 0$.*

Proposition 5. *Let π be a well-formed region path from (ℓ, r) to (ℓ', r') . Then, there exists $\delta > 0$ so that $\text{CPre}_\pi^\delta(r') \neq \emptyset$ if and only if π is robust.*

Proof. If π is not robust, then it contains some sub-path $\pi' = (\ell, r) \xrightarrow{\text{delay}} (\ell, r') \xrightarrow{g, R} (\ell', r'')$ where r' is punctual and g is non-punctual. Then, $\text{CPre}_{\pi'}^\delta(s) = \{\nu \in r \mid \exists d \geq \delta, \forall d' \in [d - \delta, d + \delta], \nu + d' \in r' \wedge (\nu + d')[R := 0] \in s\} = \emptyset$ as by definition of punctuality $\nu \in r'$ implies $\nu + d' \notin r'$ for $d' \neq 0$. Then, $\text{CPre}_\pi^\delta(s) = \emptyset$ by composition.

If π is a robust region path from (ℓ, r) to (ℓ', r') , we apply Lemma 21 on $N = r'$ to justify $\text{CPre}_\pi^\delta(r') \neq \emptyset$ for δ small enough. In particular, the existence of ν, ε such that $\text{Ball}_{d_\infty}(\nu, \varepsilon) \cap r' \subseteq N$ is trivial as $(A \cap r') \subseteq r'$ for any set A . \square

D Properties of slices

Lemma 22. *Let $C_0 \uplus \dots \uplus C_k$ be a corner partition of a region r into $k+1$ colors. The set of slices $\text{slice}_{C_0 \uplus \dots \uplus C_k}^r(\mathbf{w})$, where \mathbf{w} ranges over every color weight vector, partitions r into non-empty slices.*

Proof. For a given corner partition $C_0 \uplus \dots \uplus C_k$, for any valuation ν in r of corner weights $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_m)$ there is a unique weight vector $\mathbf{w} = (w_1, \dots, w_k)$ such that $\nu \in \text{slice}_{C_0 \uplus \dots \uplus C_k}^r(\mathbf{w})$: the one defined by $w_j = \sum_{\substack{0 \leq i \leq m \\ \text{s.t. } c_i \in C_j}} \lambda_i$ for all $0 \leq j \leq k$.

Conversely, for every color weight vector $\mathbf{w} = (w_1, \dots, w_k)$, $\text{slice}_{C_0 \uplus \dots \uplus C_k}^r(\mathbf{w})$ is not empty: it contains at least the valuation ν of corner weights $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_m)$, so that for each $0 \leq i \leq m$, if $c_i \in C_j$ in the corner partition then $\lambda_i = \frac{w_j}{|C_j|}$. \square

We show that slices defined from a cluster corner partition are zones.

Proposition 23. *Let π be a robustly iterable region cycle around a region r of dimension m . Then, for any rational color weight vector \mathbf{w} , the slice $\text{slice}_\pi(\mathbf{w})$ is a DBM. More precisely, $\text{slice}_\pi(\mathbf{w})$ is equal to the intersection of r and a set of atomic constraints of the shape $x = c$ or $x - y = c$, with $x, y \in \mathcal{X}$ and rational constants c .*

In order to prove Proposition 23, we need to describe cluster corner partitions in more details:

Lemma 24. *Let π be a robustly iterable region cycle around a region r of dimension m . Let $\mathcal{C} = \{c_0, \dots, c_m\}$ be the corners of r , so that $\|c_0\|_1 < \|c_1\|_1 < \dots < \|c_m\|_1$. Let $C_0 \uplus \dots \uplus C_k$ be the cluster corner partition of r induced by $\Gamma(\pi)$. There exists a set of splitting positions $P = \{p_0, \dots, p_k\} \subseteq \{0, \dots, m\}$ with $p_0 < \dots < p_k$, so that $C_0 \uplus \dots \uplus C_k$ can be ordered such that*

- $C_0 = \{c_i \mid i \in [0, p_0] \cup (p_k, m]\}$, and
- for all $1 \leq j \leq k$, $C_j = \{c_i \mid i \in (p_{j-1}, p_j]\}$

Proof. We show this splitting position-based characterisation of the cluster corner partition by generalizing it to region paths that may not be cycles, then by showing this generalisation by induction on the length of the path, and finally by applying it on a cycle.

Let π be a robust region path from a region r of dimension m to a region r' of dimension m' . Let $\mathcal{C} = \{c_0, \dots, c_m\}$ (resp. $\mathcal{C}' = \{c'_0, \dots, c'_{m'}\}$) be the corners of r (resp. r'), so that $\|c_0\|_1 < \|c_1\|_1 < \dots < \|c_m\|_1$ and $\|c'_0\|_1 < \|c'_1\|_1 < \dots < \|c'_{m'}\|_1$. The weakly connected components of $\gamma(\pi)$ naturally induce a partition of its vertices into $k+1$ colors, so that every vertex (step_i, c) has color $0 \leq j \leq k$ if it belongs to the j -th weakly connected component. This partition of $\gamma(\pi)$ naturally induces two corner partitions $C_0 \uplus \dots \uplus C_k$ and $C'_0 \uplus \dots \uplus C'_k$ of r and r' , respectively, based on the colors of **start** and **end** vertices.

We show by induction on the length of π that there exists a set of splitting positions $P = \{p_0, \dots, p_k\} \subseteq \{0, \dots, m\}$ with $p_0 < \dots < p_k$, so that $C_0 \uplus \dots \uplus C_k$ can be ordered such that

- $C_0 = \{c_i \mid i \in [0, p_0] \cup (p_k, m]\}$, and
- for all $1 \leq j \leq k$, $C_j = \{c_i \mid i \in (p_{j-1}, p_j]\}$

and there exists a set of splitting positions $P' = \{p'_0, \dots, p'_k\} \subseteq \{0, \dots, m'\}$ with $p'_0 < \dots < p'_k$, so that $C'_0 \uplus \dots \uplus C'_k$ can be ordered such that

- $C'_0 = \{c'_i \mid i \in [0, p'_0] \cup (p'_k, m']\}$, and
- for all $1 \leq j \leq k$, $C'_j = \{c'_i \mid i \in (p'_{j-1}, p'_j]\}$

We start by considering the case where π is a single delay transition from r to some time-successor r' . We note the following properties of $\gamma(\pi)$:

- every corner in r has a unique time-successor in r' , except for one corner that can have c'_0 and $c'_{m'}$ as successors if r' is non-punctual, and
- every corner in r' has a unique time-predecessor in r , except for one that can have c_0 and c_m as predecessors if r is non-punctual.

Then, the weakly connected components of $\gamma(\pi)$ induce partitions that are either singletons, $\{c_0, c_m\}$ or $\{c'_0, c'_{m'}\}$. These partitions can thus be described with consecutive splitting positions for the singletons, $p_0 = 0$ and $p_k = m - 1$ for $\{c_0, c_m\}$ if r is non-punctual, $p_0 = 0$ and $p_k = m$ if r is punctual, and similarly for r' .

We now consider the case where π is a single edge transition from r to r' of reset set R . We note the following properties of $\gamma(\pi)$:

- every corner c in r has a unique successor $c[R := 0]$ in r' ,
- if a corner c' in r' has two predecessors c_i and c_j in r with $\|c_i\|_1 < \|c_j\|_1$, then every corner c so that $\|c_i\|_1 < \|c\|_1 < \|c_j\|_1$ is also a predecessor of c' .

This last property holds by definition of corners as $c_i[R := 0] = c_j[R := 0]$ implies that R contains every clock that differentiates them in the clock order of the region, so that every corner c in between is also reset to c' as it only differs from c_i and c_j on the same clocks of R . Then, the weakly connected components of $\gamma(\pi)$ induce partitions that are all singletons in r' , and either singletons or intervals of consecutive corners in r . These partitions can thus be described with splitting positions, with $p_k = m$ and $p'_k = m'$.

Assume now that the inductive property holds on two paths that can be concatenated. Whenever a region path π is concatenated to another path π' , the operation can merge weakly connected components of their orbit graphs, thus reducing the number of colors. This happens at the junction of $\gamma(\pi)$ and $\gamma(\pi')$, where colors of each side that intersect merge to become a single color. These new colors are obtained as unions of intervals of corners of index $(p_{j-1}, p_j]$ and $(p'_{j'-1}, p'_{j'}]$ that intersect, which can always be expressed as new intervals of corners of the shape $(p''_{j''-1}, p''_{j''}]$.

The case of the corners of index in $[0, p_0] \cup (p_k, m]$ leads to the same conclusion, as this interval shape is also stable by union of intersecting intervals. This concludes the inductive proof.

Finally, assume π is a robustly iterable cycle. Then, In order to obtain the cluster corner partition induced by $\Gamma(\pi)$ from the partitions $C_0 \uplus \dots \uplus C_k$ and $C'_0 \uplus \dots \uplus C'_k$ induced by $\gamma(\pi)$ as described above, one needs to do one last merge of colors, with colors that intersect if we merge **start** and **end** vertices of $\gamma(\pi)$ merging to become colors of $\Gamma(\pi)$. This operation obeys the same principle as the concatenation of paths described above, and maintains the shape of the color partition as characterised by a set of splitting positions. \square

Proof (Proof of Proposition 23). Let π be a robustly iterable region cycle around a region r . Let r be of dimension m , with $\mathcal{C} = \{c_0, \dots, c_m\}$ the set of its corners, so that $\|c_0\|_1 < \|c_1\|_1 < \dots < \|c_m\|_1$. Let $C_0 \uplus \dots \uplus C_k$ be the cluster corner partition induced by $\Gamma(\pi)$. Assume w.l.o.g. that the smallest corner c_0 is in C_0 . For any given weight vector \mathbf{w} , let us detail a system of equations that encodes $\text{slice}_\pi(\mathbf{w})$.

Let r be a region. If r is a singleton that contains a single valuation ν (r is a region of dimension 0), then r is a closed set, r has only one corner, the only corner partition of r has one set ($k = 1$), and the only slice is equal to r . In the following, we assume that r has dimension at least 1.

The region r of dimension $m > 0$ is characterized by (ι, β) where β is a partition of \mathcal{X} into $\beta_0 \uplus \beta_1 \uplus \dots \uplus \beta_m$ with $\forall 1 \leq j \leq m, \beta_j \neq \emptyset$. Recall that the corners of r are the $m + 1$ valuations $c_0 \dots c_m$ so that for each $0 \leq i \leq m$, c_i is the corner such that $\forall 0 \leq j \leq m - i, \forall x \in \beta_j, c_i(x) = \iota(x)$, and $\forall m - i < j \leq m, \forall x \in \beta_j, c_i(x) = \iota(x) + 1$. In particular, c_0 is the smallest corner of r and c_m the largest, with $\|c_0\|_1 < \|c_1\|_1 < \dots < \|c_m\|_1$.

For each $1 \leq j \leq m$, fix x_j a clock in β_j , picked arbitrarily.

Let ν be a valuation in r , with $\lambda_0, \dots, \lambda_m \in (0, 1]$ its corner weights, so that $\sum_{i=0}^m \lambda_i = 1$ and such that $\nu = \sum_{i=0}^m \lambda_i c_i$. Then, note that for each $0 \leq j \leq m$, it holds that $c_i(x_j) = \iota(x_j)$ for all $0 \leq i \leq m$ with $i \leq m - j$ and $c_i(x_j) = \iota(x_j) + 1$ for all $0 \leq i \leq m$ with $i > m - j$. Then, for each $1 \leq j \leq m$, we have:

$$\begin{aligned}
\nu(x_j) - \iota(x_j) &= \left(\sum_{i=0}^m \lambda_i c_i(x_j) \right) - \iota(x_j) \\
&= \iota(x_j) \left(\sum_{i=0}^{m-j} \lambda_i \right) + (\iota(x_j) + 1) \left(\sum_{i=m-j+1}^m \lambda_i \right) - \iota(x_j) \left(\sum_{i=0}^m \lambda_i \right) \\
&= (\iota(x_j) + 1) \left(\sum_{i=m-j+1}^m \lambda_i \right) - \iota(x_j) \left(\sum_{i=m-j+1}^m \lambda_i \right) \\
&= \sum_{i=m-j+1}^m \lambda_i
\end{aligned}$$

For each $1 \leq j \leq m$, let $f_j = \nu(x_j) - \iota(x_j)$ be the fractional part of clock x_j . Moreover, we extend the notation to $j = 0$ and $j = m + 1$ by setting $f_0 = 0$ and $f_{m+1} = 1$. Then, it follows that

- first, $\lambda_0 = 1 - (\sum_{i=1}^m \lambda_i) = f_{m+1} - f_m$,
- then for each $0 < i < m$, $\lambda_i = (\sum_{l=i}^m \lambda_l) - (\sum_{l=i+1}^m \lambda_l) = f_{m-i+1} - f_{m-i}$,
- and finally, $\lambda_m = (\sum_{i=m}^m \lambda_i) - 0 = f_1 - f_0$.

So that for all $0 \leq i \leq m$, it holds that $\lambda_i = f_{m-i+1} - f_{m-i}$. Thus, for all $0 \leq a \leq b \leq m$, $\sum_{i=a}^b \lambda_i = f_{m-a+1} - f_{m-b}$.

Now, let $C_0 \uplus \dots \uplus C_k$ be the cluster corner partition of r into $k + 1$ colors induced by $\Gamma(\pi)$, characterized by the splitting positions p_0, \dots, p_k by Lemma 24. If $k = 0$, there is only one color ($C_0 = \mathcal{C}$), and the only slice of r is equal to r . We assume $k > 0$ in the following.

Let $\mathbf{w} = (w_0, \dots, w_k) \in (0, 1]^{k+1}$ be a color weight vector, so that $\|\mathbf{w}\|_1 = 1$, that defines a slice $\text{slice}_\pi(\mathbf{w})$. Let ν be a valuation in r of corner weights $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_m)$. Then, ν belongs to the slice $\text{slice}_\pi(\mathbf{w})$, if

- $w_0 = \sum_{i \in [0, p_0] \cup (p_k, m]} \lambda_i = f_{m+1} - f_{m-p_0} + f_{m-p_k} - f_0$, and
- for all $1 \leq j \leq k$, $w_j = \sum_{i \in (p_{j-1}, p_j]} \lambda_i = f_{m-p_{j-1}} - f_{m-p_j}$.

Then, for any $0 \leq j \leq k$, let \mathcal{E}_j be a clock constraint over \mathcal{X} encoding these conditions, so that

- if $p_k < m$: \mathcal{E}_0 is $x_{m-p_0} - x_{m-p_k} = 1 + \iota(x_{m-p_0}) - \iota(x_{m-p_k}) - w_0$,
- If $p_k = m$: \mathcal{E}_0 is $x_{m-p_0} = 1 + \iota(x_{m-p_0}) - w_0$, and
- for all $1 \leq j \leq k$, \mathcal{E}_j is $x_{m-p_j} - x_{m-p_{j-1}} = \iota(x_{m-p_j}) - \iota(x_{m-p_{j-1}}) - w_j$.

Note that these constraint \mathcal{E}_j are all equality constraints of the shape $x - y = c - w_i$, with c a constant in \mathbb{N} , except for \mathcal{E}_0 in the case where $p_k = m$ (*i.e.* when the corner partition is non-diagonal), which is a constraint of the shape $x = c - w_0$. This concludes the proof of Proposition 23. \square

E Proof of Proposition 14

Lemma 25 ([12, Lemma 8.6.4]). *Let r be a region. There exists a DBM N such that there exists $\nu \in r$ and $\varepsilon > 0$ with $(\text{Ball}_{d_\infty}(\nu, \varepsilon) \cap r) \subseteq N$, and such that for any shrinking matrix Q so that there exists $\delta > 0$ with $r - \delta Q \neq \emptyset$, we have $\exists \delta' > 0$, $N \subseteq (r - \delta' Q)$.*

Valuations that can reach each other must belong to the same slice:

Lemma 26. *Let π be a robustly iterable region cycle. Let \mathbf{w} and \mathbf{w}' be weight vectors, and ν, ν' be valuations of r so that $\nu \in \text{slice}_\pi(\mathbf{w})$ and $\nu' \in \text{slice}_\pi(\mathbf{w}')$. If there is a run from ν to ν' that follows π , then $\mathbf{w} = \mathbf{w}'$.*

Proof. By Corollary 19, since there is a run from ν to ν' there is for each $c_i \in \mathcal{C}$ a probability distribution $\mathbf{p}_i : \text{post}_i \rightarrow [0, 1]$ so that for each $0 \leq j \leq m$, $\lambda'_j = \sum_{c_i \in \text{pre}_j} \lambda_i \mathbf{p}_i(c_j)$. Since $\Gamma(\pi)$ is a cluster graph, the set pre_j (the immediate predecessors of c_j) is equal to the set of every corner that has the same color as c_j , and similarly for post_i . This means that for every color k ,

$$\sum_{c_j \in C_k} \lambda'_j = \sum_{c_j \in C_k} \sum_{c_i \in C_k} \lambda_i \mathbf{p}_i(c_j) = \sum_{c_i \in C_k} \lambda_i \left(\sum_{c_j \in C_k} \mathbf{p}_i(c_j) \right) = \sum_{c_i \in C_k} \lambda_i.$$

Then, as $\nu \in \text{slice}_\pi(\mathbf{w})$ and $\nu' \in \text{slice}_\pi(\mathbf{w}')$ imply that for every color k , $\sum_{c_j \in C_k} \lambda'_j = w_k$ and $\sum_{c_i \in C_k} \lambda_i = w'_k$, respectively, we conclude that $\mathbf{w} = \mathbf{w}'$. \square

On the other hand, the reachability relation is full on any given slice.

Lemma 27. *Let π be a robustly iterable region cycle around a region r . Let \mathbf{w} be a weight vector. Then, for any pair $\nu, \nu' \in \text{slice}_\pi(\mathbf{w})$, there exists a run from ν to ν' following π .*

Proof. Let ν and ν' be two valuations in the same slice. By Corollary 19, it is possible to re-distribute the corner weights of ν on the edges of the folded orbit graph in order to match the corner weights of ν' : indeed, they both have the same total weight on the corners of each individual color, and the folded orbit graph is complete on each color. The way to build the probability distributions \mathbf{p}_i for corners of a complete SCC is the same as in [3] for corners in a complete folded orbit graph. \square

We note that whenever Lemma 27 applies, we have by Lemma 26 that for all $\nu, \nu' \in r$, there is a run from ν to ν' following π if and only if there is a weight vector \mathbf{w} so that $\nu, \nu' \in \text{slice}_\pi(\mathbf{w})$.

Lemma 28. *Let π be a region cycle around a region r . Let \mathbf{w} be a weight vector. For any shrinking matrix Q so that there exists $\delta > 0$ such that $\text{slice}_\pi(\mathbf{w}) - \delta Q \neq \emptyset$, we have for every $\delta > 0$ that $\text{slice}_\pi(\mathbf{w}) - \delta Q = (r - \delta Q) \cap \text{slice}_\pi(\mathbf{w})$.*

Proof. First, since $\text{slice}_\pi(\mathbf{w}) \subseteq r$ we have $\text{slice}_\pi(\mathbf{w}) - \delta Q = (r \cap \text{slice}_\pi(\mathbf{w})) - \delta Q$. Then, by Proposition 23, every slice can be described as the intersection of r and a set of linear equalities of the shape $x - y = c$ or $x = c$. These constraints cannot be shrunk without making $\text{slice}_\pi(\mathbf{w}) - \delta Q$ empty. Therefore, any non-zero entry in the shrinking matrix Q must happen on a constraint of r , so that $(r \cap \text{slice}_\pi(\mathbf{w})) - \delta Q = (r - \delta Q) \cap \text{slice}_\pi(\mathbf{w})$, wich lets us conclude. \square

Proposition 14. *If there exists a winning lasso that is robustly iterable, then Controller wins $\mathcal{G}_\delta(\mathcal{A})$ for some $\delta > 0$.*

Proof. Given a robust region lasso $\pi_0\pi^\omega$, so that π_0 starts from $(\ell_0, \{\mathbf{0}\})$, π is a cycle around a region state (ℓ, r) with ℓ a winning state for the Büchi objective, π_0 is robust and π is robustly iterable, we argue that:

1. Fix N, ν_N, ε by Lemma 25 so that $\text{Ball}_{d_\infty}(\nu_N, \varepsilon) \cap r \subseteq N$ and for any shrinking matrix Q so that there exists $\delta > 0$ with $r - \delta Q \neq \emptyset$, we have $\exists \delta' > 0$, $N \subseteq r - \delta' Q$.
2. Let us show that N is winning for Controller. It is enough to show that for all $\nu \in N$, there is a set $\mathcal{N}_\nu \subseteq r$ so that $\nu \in \mathcal{N}_\nu$ and $\mathcal{N}_\nu \subseteq \text{CPre}_\pi^\delta(\mathcal{N}_\nu)$, *i.e.* Controller has a strategy that remains in \mathcal{N}_ν when starting from an valuation in \mathcal{N}_ν , so that it can iterate π forever from ν .
3. Fix any valuation $\nu \in N$. Let \mathbf{w} be the weight vector so that $\nu \in \text{slice}_\pi(\mathbf{w})$ by Lemma 22. Let $\mathcal{N}_\nu = N \cap \text{slice}_\pi(\mathbf{w})$. It is a DBM by Proposition 23.
4. We show that for small enough $\delta > 0$, $\mathcal{N}_\nu \subseteq \text{CPre}_\pi^\delta(\mathcal{N}_\nu)$.
 - (a) Use Lemma 27, $\nu \in \text{slice}_\pi(\mathbf{w})$ and $\nu \in \mathcal{N}_\nu$ to show that:

$$\text{slice}_\pi(\mathbf{w}) = \text{CPre}_\pi^0(\mathcal{N}_\nu)$$

- (b) Use Lemma 20 to show that there exists a shrinking matrix Q so that:

$$\forall \delta, \text{CPre}_\pi^\delta(\mathcal{N}_\nu) = \text{slice}_\pi(\mathbf{w}) - \delta Q$$

- (c) Use Lemma 21 to show $\text{slice}_\pi(\mathbf{w}) - \delta Q \neq \emptyset$
- (d) By Lemma 28, we have $\text{slice}_\pi(\mathbf{w}) - \delta Q = (r - \delta Q) \cap \text{slice}_\pi(\mathbf{w})$.
- (e) By definition of N , for small enough $\delta > 0$, $N \subseteq r - \delta Q$, so that

$$N \cap \text{slice}_\pi(\mathbf{w}) \subseteq (r - \delta Q) \cap \text{slice}_\pi(\mathbf{w})$$

- (f) Conclude that there exists $\delta > 0$ such that

$$\mathcal{N}_\nu \subseteq \text{CPre}_\pi^\delta(\mathcal{N}_\nu)$$

This means that starting from $\nu \in \mathcal{N}_\nu$, Controller has a strategy that always remain inside \mathcal{N}_ν , at each iteration of π . Therefore, controller can robustly follow π^ω when starting from any valuation ν in N . Moreover, by Lemma 21, $\text{CPre}_{\pi_0}^\delta(N)$ contains the only valuation in the initial region $\mathbf{0}$. Hence, Controller has a winning strategy that robustly follows the lasso $\pi_0\pi^\omega$. \square

F Proof of Proposition 15

Let ν be a valuation in r of dimension m such that ν has corner weights $\lambda = (\lambda_0, \dots, \lambda_m)$ in r . Let I be a subset of corner of r . We define the function

$$L_I : \bar{r} \rightarrow \mathbb{R}_{\geq 0},$$

$$\nu \mapsto \sum_{c_i \in I} \lambda_i c_i.$$

We will study the dynamics of L_I for region cycles whose folded orbit graph is not a cluster graph.

For a fixed $\delta > 0$, we define the strategy σ_δ^P for Pertubator as follows: After each delay $\nu \xrightarrow{d} \nu'$ played by Controller, consider a region r such that $\nu' + [\alpha, \beta] \subseteq r$ for some $0 < \alpha < \beta < \delta$ where $\beta - \alpha \geq \frac{\delta}{|\mathcal{X}|+1}$. Such a region must exist by definition of the game, otherwise Controller played an illegal move. Pertubator then applies a perturbation of $\frac{(\beta-\alpha)}{2}$. This strategy guarantees a time progress of at least $\varepsilon = \frac{\delta}{2(|\mathcal{X}|+1)}$.

A close inspection of the proof of Lemma 8.5.15 of [12] shows that both these observations hold in our case. We restate this lemma in our setting in the following lemma:

Lemma 29. [14] *Let ρ be a prefix of a run in $\text{Outcome}(-, \sigma_\delta^P)$ such that:*

- π the projection of ρ over regions is a cycle with at least one non-punctual edge,
- $\Gamma(\pi)$ contains at least two weakly connected SCCs I and J ,
- ρ starts from ν and ends in ν' ,

then the following equation holds true:

$$L_I(\nu') \leq L_I(\nu) - \frac{\varepsilon^2}{2} ,$$

The above lemma entails the intuition that if Controller tries to iterate over π to enforce ρ , then they will eventually be forced to propose delays smaller than δ which is not allowed by definition of our game semantics. This is formalized in the following corollary.

Corollary 30. *Let ρ be a run such that its projection over regions is a lasso $\pi_0 \pi^\omega$ and assume that for any $k > 0$, $\Gamma(\pi^k)$ is not robustly iterable, then ρ cannot be in $\text{Outcome}(-, \sigma_\delta^P)$.*

Proof. Assume towards a contradiction that ρ is in $\text{Outcome}(-, \sigma_\delta^P)$. By Lemma 7, there exists $k > 0$ such that $\Gamma(\pi^k)$ contains at least two weakly connected SCCs. Let I be an initial SCCs in $\Gamma(\pi^k)$, and let (ℓ, ν) be a state visited by ρ such that ν is in the region spanned by the corners of $\Gamma(\pi)$. Now denote ν_n the valuation along ρ after n iteration of the cycle described by $\Gamma(\pi^k)$ by Lemma 29 we have:

$$0 \leq L_I(\nu_n) < L_I(\nu) - n \frac{\varepsilon^2}{2} .$$

By definition of σ_δ^P , n can be chosen such that:

$$n \frac{\varepsilon^2}{2} > 1 .$$

This contradicts the non negativity of $L_I(\nu_n)$. □

In order to conclude we still need to argue that **Controller** cannot win by switching between different non robustly iterable cycles. To see that, take any infinite run ρ . Since there exists a finite number of folded orbit graphs, using Ramsey like arguments (c.f. Theorem 4.5.2 of [12]) we get that the projection over regions of the run ρ can be decomposed as $\pi_0 \pi_1 \pi_2 \dots \pi_n \dots$ such that $\Gamma(\pi_1) = \Gamma(\pi_2) = \Gamma(\pi_n) = \dots$. By assumption, $\Gamma(\pi_1)$ cannot be robustly iterable, hence Corollary 30 implies that ρ cannot be in $\text{Outcome}(-, \sigma_\delta^P)$. In other words, for any **Controller** strategy $\sigma_\delta^{\text{Cont}}$, $\text{Outcome}(\sigma_\delta^{\text{Cont}}, \sigma_\delta^P)$ cannot be at the same time an infinite run and a run that visits locations in Buchi infinitely often, hence $\sigma_\delta^{\text{Cont}}$ is not winning for the Büchi objective.