NO COMPACT SPLIT LIMIT RICCI FLOW OF TYPE II FROM THE BLOW-DOWN

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ABSTRACT. By Perelman's \mathcal{L} -geodesic theory, we study the blow-down solutions on a noncompact κ -noncollapsed steady gradient Ricci soliton (M^n, g) $(n \ge 4)$ with nonnegative curvature operator and positive Ricci curvature away from a compact set of M. We prove that any (n - 1)dimensional compact split ancient solution from the blow-down of (M, g)is of type I. The result is a generalization of our previous work from n = 4to any dimension.

0. INTRODUCTION

Let (M^n, g, f) $(n \ge 4)$ be a complete noncompact κ -noncollapsed steady gradient Ricci soliton with curvature operator $\operatorname{Rm} \ge 0$ away from a compact set K of M. Let $g(\cdot, t) = \phi_t^*(g)$ $(t \in (-\infty, \infty))$ be an induced ancient Ricci flow of (M, g), where ϕ_t is a family of transformations generated by the gradient vector field $-\nabla f$. For any sequence of $p_i \in M$ $(\to \infty)$, we consider the (normally) rescaled Ricci flows $(M, g_{p_i}(t); p_i)$, where

(0.1)
$$g_{p_i}(t) = r_i^{-1} g(\cdot, r_i t),$$

 $r_i R(p_i) = 1$. By a version of Perelman's compactness theorem for ancient κ -solutions [26, Proposition 1.3] (also see Proposition 1.1), we know that $(M, g_{p_i}(t); p_i)$ converge subsequently to a splitting flow $(N \times \mathbb{R}, \bar{g}(t); p_{\infty})$ in the Cheeger-Gromov sense, where

(0.2)
$$\bar{g}(t) = h(t) + ds^2$$
, on $N \times \mathbb{R}$,

and h(t) $(t \in (-\infty, 0])$ is an ancient κ -solution on an (n-1)-dimensional N. For simplicity, we call (N, h(t)) a split limit flow (arising from the blow-down of (M, g)).

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Recall that an ancient κ -solution (N, h(t)) $(t \in (-\infty, 0])$ is of type I if it satisfies

$$\sup_{N\times(-\infty,0]}(-t)|R(x,t)|<\infty.$$

Otherwise, it is called type II, if it satisfies

N

$$\sup_{t \times (-\infty,0]} (-t) |R(x,t)| = \infty.$$

In this paper, we prove

Theorem 0.1. Let (M^n, g) $(n \ge 4)$ be a noncompact κ -noncollapsed steady gradient Ricci soliton which satisfies

(0.3)
$$\operatorname{Rm} \ge 0 \text{ and } \operatorname{Ric} > 0 \text{ on } M \setminus K$$

Then any (n-1)-dimensional compact split ancient solution (N, h(t)) in (0.2) from the blow-down of (M, g) is of type I. In the other words, there is no compact split ancient solution of type II from the blow-down of (M, g).

Since any compact ancient κ -solution of type I is a gradient shrinking Ricci soliton, which has been classified in [10, Theorem 7.34] and [6, 20] (also see [26, Proposition 4.1]), Theorem 0.1 actually gives a classification of all (n-1)-dimensional compact split blow-down solutions of (M, g), which satisfies (0.3).

As an application of Theorem 0.1, we prove the following alternative principle.

Corollary 0.2. Let (M^n, g) be a steady gradient Ricci soliton in Theorem 0.1. Then either all split blow-down solutions (N, h(t)) of (M, g) in (0.2) are (n-1)-dimensional compact ancient κ -solution of type I, or (n-1)-dimensional noncompact ancient κ -solution.

Corollary 0.2 confirms a conjecture [26, Conjecture 4.5] and is a generalization of [26, Theorem 0.2] hereby from n = 4 to any dimension. We would like to mention that our previous proof in the case n = 4 depends highly on a deep classification result for 3*d* compact κ -solutions of type II proved by Brendle-Daskalopoulos-Sesum [5].

In order to follow the argument in [26] to prove Theorem 0.1, we use Perelman's \mathcal{L} -geodesic theory in [22, Section 7, Section 9] to construct a limit gradient shrinking Ricci soliton through a sequence of normally rescaled Ricci flows (cf. Section 4, 5). Inspired by Bamler-Chan-Ma-Zhang in their recent work [2], we are able to choose a suitable base point in each level set of the potential function f such that the corresponding ℓ -length is uniformly bounded (cf. Section 2). Unfortunately, we cannot use Perelman's result directly to get the gradient estimate for the (rescaled) reduced distance

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function $\ell_i(x, \tau)$ since the nonnegative curvature condition just holds outside a compact set of M (see Remark 4.3). We will do the curvature decay estimates to overcome the difficulty (see Section 3).

To see some examples of steady gradient Ricci solitons (M, g) which satisfy the condition (0.3) in Theorem 0.1, we refer the reader to [7, 1, 18], etc.

According to the proof of Theorem 0.1, we can get the following explicit curvature decay estimate.

Theorem 0.3. Let (M, g) be a steady gradient Ricci soliton in Theorem 0.1. Suppose that there exists a sequence of rescaled Ricci flows $(M, g_{p_i}(t); p_i)$, which converges subsequently to a splitting Ricci flow $(N \times \mathbb{R}, \bar{g}(t); p_{\infty})$ as in (0.2) for some compact ancient κ -solution (N, h(t)). Then the scalar curvature of (M, g) decays to zero linearly. Namely, there exist two positive constants C_1 and C_2 such that

(0.4)
$$\frac{C_1}{\rho(x)} \le R(x) \le \frac{C_2}{\rho(x)}.$$

Theorem 0.3 is an improvement of [26, Lemma 2.2]. We notice that complete noncompact κ -noncollapsed steady gradient Ricci solitons with nonnegative curvature under the condition (0.4) have been classified by Deng-Zhu [11, 13, 14].

The paper is organized as follows. In Section 1, we first review a compactness theorem for normally rescaled Ricci flows of (M,g) in [26] (cf. Proposition 1.1), then we recall the Perelman's \mathcal{L} -geodesic theory and translate it for a steady gradient Ricci soliton. In Section 2, we use a method in [2] to prove the existence of ℓ -centers in each level set of the potential function f. In Section 3, we do the curvature estimates for ℓ -centers (cf. Lemma 3.2 and Proposition 3.4). In Section 4, we get the gradient estimate for the (rescaled) reduced distance function $\ell_i(x,\tau)$ and then construct limit shrinking Ricci solitons through the normally rescaled Ricci flows of (M,g) (cf. Proposition 4.2 and Proposition 4.4). Both of Theorem 0.1 and Theorem 0.3 will be proved in Section 5.

Since 3d complete noncompact κ -noncollapsed steady gradient Ricci solitons have been classified by Brendle motivated by a conjecture of Perelman [3], we will always assume the dimension $n \geq 4$ in this paper.

1. Preliminaries

A complete Riemannian metric g on M is called a gradient Ricci soliton if there exists a smooth function f (which is called a potential function) on M such that

(1.1)
$$R_{ij}(g) + \rho g_{ij} = \nabla_i \nabla_j f,$$

where $\rho \in \mathbb{R}$ is a constant. The gradient Ricci soliton is called expanding, steady and shrinking according to $\rho > =, < 0$, respectively. These three types of Ricci solitons correspond to three different blow-up solutions of Ricci flow [16].

In the case of steady Ricci solitons, we can rewrite (1.1) as

(1.2)
$$2\operatorname{Ric}(g) = L_X g,$$

where L_X is the Lie operator along the gradient vector field (VF) $X = \nabla f$ generalized by f. Let $\{\phi_t^*\}_{t \in (-\infty,\infty)}$ be a 1-ps of transformations generated by -X. Then $g(t) = \phi_t^*(g)$ ($t \in (-\infty,\infty)$) is a solution of Ricci flow. Namely, g(t) satisfies

(1.3)
$$\frac{\partial g}{\partial t} = -2\operatorname{Ric}(g), \ g(0) = g.$$

For simplicity, we call g(t) the soliton Ricci flow of (M, g).

By (1.2), we have

(1.4)
$$\langle \nabla R, \nabla f \rangle = -2 \operatorname{Ric}(\nabla f, \nabla f),$$

where R is the scalar curvature of g. It follows

$$R + |\nabla f|^2 = \text{Const.}$$

Since R is alway positive ([25, 8]), the above equation can be normalized by

$$(1.5) R + |\nabla f|^2 = 1.$$

In this paper we always assume that (M^n, g) $(n \ge 4)$ is a noncompact κ -noncollapsed steady gradient Ricci soliton with nonnegative curvature operator $\operatorname{Rm} \ge 0$ away from a compact set K of M.

The following splitting theorem was proved in [26, Proposition 1.2], which can be regarded as a version of Perelman's compactness theorem for higher dimensional ancient κ -solutions [22, 17].

Proposition 1.1. Let (M^n, g) $(n \ge 4)$ be a noncompact κ -noncollapsed steady gradient Ricci soliton with $\operatorname{Rm} \ge 0$ on $M \setminus K$. Let $p_i \to \infty$ and $(M, g_{p_i}(t); p_i)$ a sequence of rescaled flows with $R_{p_i}(p_i, 0) = 1$ as in (0.1). Then $(M, g_{p_i}(t); p_i)$ subsequently converge to a splitting flow $(N \times \mathbb{R}, \overline{g}(t) =$ $h(t) + ds^2; p_{\infty})$ as in (0.2), where (N, g(t)) is an (n-1)-dimensional ancient κ -solution. Moreover, for n = 4, $\operatorname{Rm} \ge 0$ can be weakened to the sectional curvature $\operatorname{Km} \ge 0$ on $M \setminus K$.

By Proposition 1.1, we have the following Laplace estimate for the scalar curvature of (M, g).

Lemma 1.2. Let (M,g) be the κ -noncollapsed steady gradient Ricci soliton with $\operatorname{Rm} \geq 0$ on $M \setminus K$. Then there exists a uniform constant C > 0, such that

(1.6)
$$\frac{|\Delta R(p,t)|}{R^2(p,t)} \le C, \ \forall (p,t) \in M \times (-\infty,0].$$

Proof. On the contrary, if (1.6) is not true, there is a sequence of p_i and t_i such that

(1.7)
$$\frac{|\Delta R(p_i, t_i)|}{R^2(p_i, t_i)} \to \infty$$

Then we consider rescaled flows $(M, g_i(t); p_i)$, where

$$g_i(t) = R(p_i, t_i)g(R^{-1}(p_i, t_i)t + t_i).$$

By the isometry $(M, g(t_i); p_i) \cong (M, g, \phi_{t_i}(p_i))$, it is easy to see

(1.8) $(M, g_i(t); p_i) \cong (M, g_{\phi_{t_i}(p_i)}(t); \phi_{t_i}(p_i)),$

where

$$g_{\phi_{t_i}(p_i)}(t) = R_g(\phi_{t_i}(p_i))g(R_g^{-1}(\phi_{t_i}(p_i))t).$$

We may assume that $\phi_{t_i}(p_i) \to \infty$, otherwise,

$$\frac{|\Delta R(p_i, t_i)|}{R^2(p_i, t_i)} = \frac{|\Delta_g R_g(\phi_{t_i}(p_i))|_g}{R_g^2(\phi_{t_i}(p_i))} \le C_0,$$

which contradicts to (1.7).

By Proposition 1.1, the scaled Ricci flows $(M, g_{\phi_{t_i}(p_i)}(t); \phi_{t_i}(p_i))$ converge in the Cheeger-Gromov sense. Thus

$$\left| \operatorname{Rm} \left(q, g_{\phi_{t_i}(p_i)}(t) \right) \right| \le C_1, \forall (q, t) \in B \left(\phi_{t_i}(p_i), 1; g_{\phi_{t_i}(p_i)}(-1) \right) \times [-1, 0].$$

By the Shi's estimate [24], we get

$$\left| \Delta R \left(q, g_{\phi_{t_i}(p_i)}(t) \right) \right| \le C_2, \forall (q, t) \in B \left(\phi_{t_i}(p_i), \frac{1}{2}; g_{\phi_{t_i}(p_i)}(-1) \right) \times [-\frac{1}{2}, 0].$$

In particular,

$$\left| \Delta R \left(\phi_{t_i}(p_i), g_{\phi_{t_i}(p_i)}(0) \right) \right| = \frac{|\Delta_g R_g(\phi_{t_i}(p_i))|_g}{R_g^2(\phi_{t_i}(p_i))} = \frac{|\Delta R(p_i, t_i)|}{R^2(p_i, t_i)} \le C_2,$$

which is a contradiction with (1.7). The lemma is proved.

We note that an ancient κ -solution is a κ -noncollapsed solution of Ricci flow (1.3) with $R_m(\cdot, t) \geq 0$ defined for any $t \in (-\infty, 0]$. The purpose of paper is to use Perelman's \mathcal{L} -geodesic theory to study the geometry of split ancient κ -solution (N, g(t)) in Proposition 1.1.

1.1. \mathcal{L} -length and \mathcal{L} -geodesic. Let $(M, \hat{g}(t))$ $(t \in [0, \infty))$ be a backward Ricci flow on M, namely, $\hat{g}(t)$ satisfies

(1.9)
$$\frac{\partial \hat{g}}{\partial t} = 2\operatorname{Ric}(\hat{g}), \ \hat{g}(0) = g.$$

For any $\tau > 0$ and any piecewisely smooth curve $\Gamma : [0, \tau] \to M$ with $\Gamma(0) = o$ and $\Gamma(\tau) = p$, \mathcal{L} -length of Γ is defined by (cf. Perelman [22, Section 7]),

(1.10)
$$\mathcal{L}(\Gamma) := \int_0^\tau \sqrt{s} \left(R_{\hat{g}_s} + |\dot{\Gamma}|_{\hat{g}_s}^2 \right) (\Gamma(s)) ds.$$

Then for any pair (x, τ) , we define a function (called \mathcal{L} -distance function) by

$$L(x,\tau) := \inf_{\Gamma} \mathcal{L}(\Gamma),$$

The above infimum is taken for any piecewise smooth curve $\Gamma : [0, \tau] \to M$ with $\Gamma(0) = o$ and $\Gamma(\tau) = x$. By Perelman [22, Section 7], the infimum can be attained by a smooth curve which is called a minimal \mathcal{L} -geodesic for the pair (x, τ) .

Set

(1.11)
$$\ell(x,\tau) := \frac{1}{2\sqrt{\tau}}L(x,\tau),$$

which is called the reduced distance of backward Ricci flow $(M, \hat{g}(t))$. By [22, Section 7.1], we know that for any $\tau > 0$, there exists a point $x \in M$ such that $\ell(x, \tau) \leq n/2$. Any such point x is called an ℓ -center of $(M, \hat{g}(t))$ at time τ .

The following lemmas was proved in Perelman's paper [22].

Lemma 1.3. ([22], also see [17, Section 18]) Let $\Gamma(s)$ $(s \in [0, \tau])$ be a minimal \mathcal{L} -geodesic with $\Gamma(0) = o$ and $\Gamma(\tau) = x$ on $(M, \hat{g}(t))$ $(t \in [0, \infty))$. Let $Y(\tau) = \frac{d\Gamma}{ds}(\tau)$. Then we have

(1.12)
$$\begin{aligned} |\nabla L|^2(x,\tau) &= 4\tau |Y(\tau)|^2(x) \\ &= -4\tau R(x,-\tau) + 4\tau (R(x,-\tau) + |Y(\tau)|^2(x)) \end{aligned}$$

and

(1.13)
$$\tau^{\frac{3}{2}} \left(R(x, -\tau) + |Y(\tau)|^2(x) \right) = -K(x, \tau) + \frac{1}{2}L(x, \tau),$$

where

(1.14)
$$K(x,\tau) = \int_0^\tau s^{\frac{3}{2}} \left(-\frac{\partial R}{\partial \tau} - 2\langle Y, \nabla R \rangle + 2\operatorname{Ric}(Y,Y) - \frac{1}{s}R\right)(\Gamma(s))ds.$$

Lemma 1.4. ([22], also see [21, Lemma 2.19])

(1.15)
$$\frac{\partial\ell}{\partial\tau} - \frac{R}{2} + \frac{|\nabla\ell|^2}{2} + \frac{\ell}{2\tau} = 0,$$

(1.16)
$$\frac{\partial \ell}{\partial \tau} - \Delta \ell + |\nabla \ell|^2 - R + \frac{n}{2\tau} \ge 0,$$

and

(1.17)
$$\Delta \ell - \frac{|\nabla \ell|^2}{2} + \frac{R}{2} + \frac{\ell - n}{2\tau} \le 0.$$

Moreover, (1.16) becomes an equality at a point if and only if (1.17) becomes an equality at that point.

1.2. \mathcal{L} -length associated to steady Ricci soliton. By a variable change $t = -\tau$, $\hat{g}(\tau) = g(-t)$ ($t \in (-\infty, 0]$) becomes a backward Ricci flow on $[0, \infty) \times M$. Then the length $\mathcal{L}(\Gamma)$ and $\ell(x, \tau)$ associated to $(M, \hat{g}(\tau))$ can be translated into ones for the soliton Ricci flow (M, g(t)). Actually, we have

(1.18)
$$\mathcal{L}(\Gamma) = \int_0^\tau \sqrt{s} \left(R_{g(-s)} + |\dot{\Gamma}|^2_{g(-s)} \right) (\Gamma(s)) ds.$$

Since the scalar curvature $R(\cdot)$ is always positive by a result of Chen [8], the integral function in (1.18) is positive.

By the isometry $(\Gamma(s), g(-s))$ with $(\phi_{-s}(\Gamma(s)), g = g(0))$ for all $s \in [0, \tau]$, (1.18) becomes

$$\mathcal{L}(\Gamma) = \int_0^\tau \sqrt{s} \left(R_g + |(\phi_{-s})_*(\dot{\Gamma}(s))|_g^2 \right) (\phi_{-s}(\Gamma(s))) ds.$$

Let $\gamma(s) = \phi_{-s}(\Gamma(s))$, then

$$\dot{\gamma}(s) = \nabla f|_{\gamma} + (\phi_{-s})_*(\dot{\Gamma}(s)).$$

It follows

(1.19)
$$\mathcal{L}(\Gamma) = \int_0^\tau \sqrt{s} \left(R_g + |\dot{\gamma}(s) - \nabla f|_g^2 \right) (\gamma(s)) ds.$$

Hence, the integral function in (1.19) is just for $\gamma(s)$ -curve in M with the fixed soliton metric g. Without confusion, we also call $\gamma(s)$ ($s \in [0, \tau]$) a minimal \mathcal{L} -geodesic with $\gamma(0) = o$ and $p = \gamma(\tau) = \phi_{-\tau}(\Gamma(\tau))$ as long as $(\Gamma(s), g(-s))$ is a minimal \mathcal{L} -geodesic with $\Gamma(0) = o$ and $x = \Gamma(\tau)$.

As in [2], we write the reduced distance $\ell(x, \tau)$ as

(1.20)
$$\lambda(p,\tau) = \ell(\phi_{\tau}(p),\tau).$$

Thus

$$\lambda(p,\tau) = \lambda(\phi_{-\tau}(x),\tau) = \ell(x,\tau) \le \frac{n}{2},$$

if x is an ℓ -center at time $-\tau$. In the following we always use $\lambda(p,\tau)$ (or $\ell(x,\tau)$) to study the location of p instead of ℓ -center x. Without confusion, we also call p a ℓ -center as long as

(1.21)
$$\lambda(p,\tau) \le A_0$$

where $A_0 \ge \frac{n}{2}$ is a fixed constant, which will be determined in next section. By a parameter change $u = \sqrt{s}$, (1.19) can be also written as

(1.22)
$$\mathcal{L}(\Gamma) := \int_0^{\sqrt{\tau}} \left(2u^2 R + \frac{1}{2} |\dot{\gamma} - 2u\nabla f|^2 \right) du,$$

where $\bar{\gamma}(u) = \gamma(u^2)$. (1.22) will be used often in next sections.

2. Location of ℓ -center

In this section, we study the location of ℓ -center p_{τ} which satisfies (1.21) by the method in [2]. From this section, we always assume that the potential function f of steady Ricci soliton (M, g) satisfies

(2.1)
$$c_1 \rho(x) \le f(x) \le c_2 \rho(x),$$

where c_1 and c_2 are two constants. Then we may further assume f(o) = 0and $f(p) \ge 0$ for any $p \in M$. For each $\tau > 0$, we define a level set of f by

$$\Sigma_{\tau} = \{ p \in M | f(p) = \tau \}.$$

Clearly, Σ_{τ} is compact. In the last section, we will verify the condition (2.1) to prove the main results in Introduction 0.

We first observe

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Lemma 2.1. $\lambda(o,\tau) \geq \frac{\tau}{3}$, for any $\tau > 0$.

Proof. We note that all Γ -curves are loops at o associated to the length $\lambda(o, \tau)$. Then by (1.22) together with (1.5), we get

$$\begin{aligned} \int_{0}^{\tau} \sqrt{s} \left(R + |\dot{\gamma} - \nabla f|^{2} \right) &= \int_{0}^{\sqrt{\tau}} \left(\frac{1}{2} |\dot{\tilde{\gamma}} - 2u\nabla f|^{2} + 2u^{2}R(\tilde{\gamma}(u)) \right) du \\ &= \int_{0}^{\sqrt{\tau}} \left(\frac{1}{2} |\dot{\tilde{\gamma}}|^{2} - 2u(f \circ \tilde{\gamma} - f(o))' + 2u^{2} \right) du \\ &= \frac{2}{3}\tau^{3/2} + \int_{0}^{\sqrt{\tau}} \left(\frac{1}{2} |\dot{\tilde{\gamma}}|^{2} + 2f \circ \tilde{\gamma}(u) \right) du \\ &\geq \frac{2}{3}\tau^{3/2}. \end{aligned}$$

$$2.2)$$

Thus the lemma comes from (1.11) immediately.

The following lemma is due to [2, Lemma 2.2 and Lemma 2.3].

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Lemma 2.2. There is a universal constant $\alpha \in (0,1)$ such that for any $\tau \gg 1$ and any p' with $\lambda(p',\tau) \leq \frac{n}{2}$, it holds

(2.3)
$$\operatorname{dist}_g(o, p') \ge \alpha \tau.$$

Proof. Let $\gamma_1 : [0, \tau] \to M$ be a minimizing \mathcal{L} -geodesic from o to p', and $\tilde{\gamma}_2 : [\sqrt{\tau}, (1+\delta)\sqrt{\tau}] \to M$ be a minimizing g-geodesic from p' to o such that $|\frac{d}{ds}\tilde{\gamma}_2|_g = \delta$. Set $\gamma_2 : [\tau, (1+\delta)^2\tau] \to M$ by $\gamma_2(s) = \tilde{\gamma}_2(\sqrt{s})$. Then $\gamma_1 \cup \gamma_2$ is a special loop at o with parameter $s \in [0, (1+\delta)^2\tau]$. By (1.19) and (1.22), we compute

$$\begin{split} & L\left(o,(1+\delta)^{2}\tau\right) \\ & \leq \int_{0}^{\tau}\sqrt{s}\left(R+|\dot{\gamma}_{1}-\nabla f|^{2}\right)ds + \int_{\tau}^{(1+\delta)^{2}\tau}\sqrt{s}\left(R+|\dot{\gamma}_{2}-\nabla f|^{2}\right)ds \\ & = L(p',\tau) + \int_{\sqrt{\tau}}^{(1+\delta)\sqrt{\tau}}\left(\frac{1}{2}\left|\dot{\tilde{\gamma}}_{2}\right|^{2}+2u\left|\dot{\tilde{\gamma}}_{2}\right|\left|\nabla f\right|+2u^{2}\right)du \\ & \leq L(p',\tau) + \int_{\sqrt{\tau}}^{(1+\delta)\sqrt{\tau}}\left(\left|\dot{\tilde{\gamma}}_{2}\right|^{2}+4u^{2}\right)du \\ & \leq L(p',\tau) + \frac{\text{dist}^{2}(o,p')}{\delta\sqrt{\tau}} + 4\frac{(1+\delta)^{3}-1}{3}\tau^{3/2} \\ & \leq L(p',\tau) + \frac{\text{dist}^{2}(o,p')}{\delta\sqrt{\tau}} + 10\delta\tau^{3/2}. \end{split}$$

Note that $\lambda(p', \tau) \leq \frac{n}{2}$. Thus for $\tau \gg 1$, we get

(2.4)

$$\lambda(o, (1+\delta)^{2}\tau) \leq \frac{1}{1+\delta}\lambda(p', \tau) + \frac{\operatorname{dist}^{2}(o, p')}{2\delta(1+\delta)\tau} + 5\frac{\delta}{1+\delta}\tau$$

$$\leq \frac{\operatorname{dist}^{2}(o, p')}{2\delta\tau} + 10\delta\tau.$$

By Lemma 2.1, it follows

$$\frac{(1+\delta)^2\tau}{3} \le \lambda(o, (1+\delta)^2\tau) \le \frac{\operatorname{dist}^2(o, p')}{2\delta\tau} + 10\delta\tau.$$

Hence by taking $\delta \ll 1$, we obtain (2.3) for some small α .

By Lemma 2.2, we give the following location estimate of ℓ -center p_{τ} in the level set Σ_{τ} .

Proposition 2.3. For any $\tau \gg 1$, there is a $p_{\tau} \in \Sigma_{\tau}$, and $\tau_0 \in [c\tau, C\tau]$ such that $\lambda(p_{\tau}, \tau_0) \leq A_0$, where c, C and A_0 are uniform constants.

Proof. Let c_1 be the constant in (2.1). Let $\gamma : [0, \frac{\tau}{c_1 \alpha}] \to M$ be a minimizing \mathcal{L} -geodesic from o to $p = \gamma(\frac{\tau}{c_1 \alpha})$ such that

$$\lambda(p, \frac{\tau}{c_1 \alpha}) \le \frac{n}{2}.$$

Then by Lemma 2.2, we see that

$$f(p) \ge c_1 \operatorname{dist}_g(p, o) \ge \tau.$$

 Set

$$\tau_0 := \sup\{s \in [0, \frac{\tau}{c_1 \alpha}] : f(\gamma(s)) \le \tau\}, \quad p_\tau := \gamma(\tau_0).$$

Thus $\tau_0 \leq \frac{\tau}{c_1 \alpha} = C \tau$. We need to estimate the lower bound of τ_0 . Define $\tilde{\gamma} : [0, \sqrt{\frac{\tau}{c_1 \alpha}}] \to M$ by $\tilde{\gamma}(u) = \gamma(u^2)$. Then as is (2.2), we have

$$\begin{split} L(p,\frac{\tau}{c_1\alpha}) &\geq \int_0^{\sqrt{\tau_0}} \left(\frac{1}{2}|\dot{\tilde{\gamma}}|^2 - 2u(f\circ\tilde{\gamma} - f(o))'\right) du\\ &= \int_0^{\sqrt{\tau_0}} \left(\frac{1}{2}|\dot{\tilde{\gamma}}|^2 - 2u\frac{d}{du}f(\tilde{\gamma}(u))\right) du\\ &= \int_0^{\sqrt{\tau_0}} \left(\frac{1}{2}|\dot{\tilde{\gamma}}|^2 + 2f(\tilde{\gamma}(u))\right) du - 2\sqrt{\tau_0}\tau\\ &\geq \int_0^{\sqrt{\tau_0}} \frac{1}{2}|\dot{\tilde{\gamma}}|^2 du - 2\sqrt{\tau_0}\tau. \end{split}$$

It follows

$$\frac{1}{2}\int_0^{\sqrt{\tau_0}} |\dot{\tilde{\gamma}}|^2 \le L(p, \frac{\tau}{c_1 \alpha}) + 2\sqrt{\tau_0}\tau.$$

Consequently,

$$\frac{1}{2} \int_0^{\sqrt{\tau_0}} |\dot{\tilde{\gamma}}|^2 \le n \sqrt{\frac{\tau}{c_1 \alpha}} + 2\sqrt{\tau_0} \tau.$$

Thus

$$\begin{aligned} \frac{c_1^2}{2}\tau^2 &= \frac{c_1^2}{2}f^2(p_{\tau}) \\ &\leq \frac{1}{2}\operatorname{dist}(o, p_{\tau})^2 \leq \frac{1}{2}\left(\int_0^{\sqrt{\tau_0}} |\dot{\tilde{\gamma}}|\right)^2 \leq \frac{1}{2}\sqrt{\tau_0}\int_0^{\sqrt{\tau_0}} |\dot{\tilde{\gamma}}|^2 \\ &\leq n\sqrt{\frac{\tau_0\tau}{c_1\alpha}} + 2\tau_0\tau \leq \frac{c_1^2}{4}\tau^2 + 2\tau_0\tau. \end{aligned}$$

This proves

(2.5)
$$\tau_0 \ge \frac{c_1^2}{8}\tau = c\tau$$

for some dimensional constant c > 0.

Note that $\gamma : [0, \tau_0] \to M$ is also a minimizing \mathcal{L} -geodesic from o to p_{τ} . Thus, by (2.5), we get

$$\lambda\left(p_{\tau},\tau_{0}\right) \leq \frac{\sqrt{\tau/c_{1}\alpha}}{\sqrt{\tau_{0}}}\lambda\left(p,\frac{\tau}{c_{1}\alpha}\right) \leq \frac{n}{2\sqrt{cc_{1}\alpha}} = A_{0}.$$

3. Curvature estimates for ℓ -center

In this section, we give the curvature estimates for ℓ -center p_r determined in Proposition 2.3.

Let $\gamma(s): [0, \tau_0] \to M$ be a minimal \mathcal{L} -geodesic with $\gamma(0) = o$ and $\gamma(\tau_0) = p_{\tau}$, where p_{τ} is an ℓ -center as in Proposition 2.3. Since $p_{\tau} \in \Sigma_{\tau}$, we see that $\gamma(s) \cap K \neq \emptyset$ when $\tau \gg 1$. Set

(3.1)
$$\tau' = \inf\{t | \gamma(s) \subset M \setminus K, \forall s > t\}$$

and $o'_{\tau} = \gamma(\tau')$. Then the restricting $\gamma(s)$ on $[\tau', \tau_0]$, denoted by $\gamma_2(s)$, is a minimal \mathcal{L} -geodesic between o'_{τ} and p_{τ} . Thus it satisfies that $\operatorname{Rm} \geq 0$, in particular Ric ≥ 0 on $\gamma_2(s)$. As in (1.20), we denote a λ -function starting from o'_{τ} by $\lambda_{o'_{\tau}}$.

Lemma 3.1. Let τ' and $\lambda_{o'_{\tau}}$ defined as above for $\tau' \gg 1$. Then

(3.2)
$$\frac{\tau'}{\tau_0} \to 0, \text{ as } \tau_0 \to \infty.$$

and $\lambda_{o'_{\tau}}(p_{\tau}, \tau_0 - \tau') \leq C_0$ for some uniform constant $C_0 > 0$.

Proof. We first prove the first assertion (3.2). Suppose that (3.2) fails. Then there exists a constant $\epsilon > 0$ such that

(3.3)
$$\tau' \ge \epsilon \tau_0.$$

Since $\lambda(p_{\tau}, \tau_0) \leq A_0$, by (1.19), we have

$$L(p_{\tau},\tau_0) = \int_0^{\tau_0} \sqrt{s} \left(R_g + |\dot{\gamma}(s) - \nabla f|_g^2 \right) (\gamma(s)) ds \le 2A_0 \sqrt{\tau_0}.$$

By (1.5), it follows

$$L(p_{\tau},\tau_{0}) = \int_{0}^{\sqrt{\tau_{0}}} \left(2u^{2}R + \frac{1}{2} |\dot{\bar{\gamma}} - 2u\nabla f|^{2} \right) du$$

$$= \int_{0}^{\sqrt{\tau_{0}}} \left(2u^{2} + \frac{1}{2} |\dot{\bar{\gamma}}|^{2} - 2u \frac{d}{du} f(\bar{\gamma}(u)) \right) du$$

$$= \int_{0}^{\sqrt{\tau_{0}}} \left(2u^{2} + \frac{1}{2} |\dot{\bar{\gamma}}|^{2} + 2f(\bar{\gamma}(u)) \right) du - 2\sqrt{\tau_{0}} f(p_{\tau})$$

$$(3.4) \leq 2A_{0}\sqrt{\tau_{0}}.$$

We divide $\gamma(s)$ into two paths γ_1 and γ_2 such that $\gamma_1 : [0, \tau']$ with $\gamma_1(0) = o$ and $\gamma_1(\tau') = \sigma_{o_{\tau'}}$, and $\gamma_2 : [\tau', \tau_0]$ with $\gamma_2(\tau') = \sigma_{0_{\tau'}}$ and $\gamma_2(\tau_0) = p_{\tau}$. Then $L(p_{\tau}, \tau_0) = \mathcal{L}(\gamma_1) + \mathcal{L}(\gamma_2),$

where

(3.5)
$$\mathcal{L}(\gamma_1) = \int_0^{\sqrt{\tau'}} \left(2u^2 + \frac{1}{2} |\dot{\gamma}|^2 + 2f(\bar{\gamma}(u)) \right) du - 2\sqrt{\tau'} f(o'_{\tau}),$$

and

$$\mathcal{L}(\gamma_2) = \int_{\sqrt{\tau'}}^{\sqrt{\tau_0}} \left(2u^2 + \frac{1}{2} |\dot{\bar{\gamma}}|^2 + 2f(\bar{\gamma}(u)) \right) du + 2\sqrt{\tau'} f(o_\tau') - 2\sqrt{\tau_0} f(p_\tau).$$

Since o'_{τ} lies in ∂K , there exists a uniform constant $C_2 > 0$, such that $f(o'_{\tau}) \leq C_2$. Thus by (3.4), we obtain

$$\mathcal{L}(\gamma_2) \le 2A_0\sqrt{\tau_0} - \mathcal{L}(\gamma_1)$$

= $2A_0\sqrt{\tau_0} - \int_0^{\sqrt{\tau'}} \left(2u^2 + \frac{1}{2}|\dot{\gamma}|^2 + 2f(\bar{\gamma}(u))\right) du + 2\sqrt{\tau'}f(o'_{\tau})$
(3.6) $\le 2(A_0 + C_2)\sqrt{\tau_0}.$

By (3.5) and (3.3), we also have

(3.7)

$$\mathcal{L}(\gamma_1) \ge \int_0^{\sqrt{\tau'}} 2u^2 - 2C_2\sqrt{\tau_0}$$

$$\ge \frac{2}{3}\tau'^{\frac{3}{2}} - 2C_2\sqrt{\tau_0}$$

$$\ge \frac{2}{3}\epsilon^{\frac{3}{2}}\tau_0^{\frac{3}{2}} - 2C_2\sqrt{\tau_0}.$$

Note that

(3.8)
$$\mathcal{L}(\gamma_1) \le L(p_\tau, \tau_0) \le 2A_0 \sqrt{\tau_0}.$$

Clearly, (3.7) is a contradiction with (3.8) when $\tau_0 >> 1$. Thus (3.2) must be true.

By (3.6), we get

$$\lambda_{o'_{\tau}}(p_{\tau}, \tau_0 - \tau') = \frac{1}{2\sqrt{\tau_0 - \tau'}} \mathcal{L}(\gamma_2) = \frac{2(A_0 + C_2)\sqrt{\tau_0}}{2\sqrt{\tau_0 - \tau'}}.$$

By (3.2), it follows

$$\lambda_{o'_{\tau}}(p_{\tau}, \tau_0 - \tau') \le 2(A_0 + C_2)$$

as long as $\tau_0 \gg 1$. Thus we prove the lemma.

By Lemma 3.1, we know that for any $\tau \gg 1$ there exists a $o'_{\tau} \in \partial K$ and $\tau'_0 = \tau_0 - \tau'$, such that

(3.9)
$$\lambda_{o'_{\tau}}(p_{\tau},\tau'_0) \le C_0,$$

and

(3.10)
$$\frac{c}{2}\tau \le \tau_0' \le 2C\tau$$

By (3.9) and (3.10), we use the Perelman's argument [22, Section 7] to derive the upper bound estimate of curvature for the ℓ -center p_{τ} in the following. Namely, we prove

Lemma 3.2. Let p_r be the ℓ -center determined in Proposition 2.3. Then there exists a uniform constant C_0 such that for any $\tau \gg 1$ it holds

$$(3.11) R(p_{\tau}) \le \frac{C_0}{\tau}$$

Proof. Let $\Gamma(s) : [0, \tau_0]$ be a minimal \mathcal{L} -geodesic with $\Gamma(\tau_0) = \phi_{\tau_0}(p_{\tau}) = x_{\tau_0}$ and $\Gamma(0) = o$. Let $Y(s) = \frac{d\Gamma}{ds}$ and ℓ the corresponding reduced length. Then by (1.12) and (1.13) in Lemma 1.3 (also see [17, Section 25]), for $x = \Gamma(\hat{\tau})$ with $\hat{\tau} \in [0, \tau_0]$, we have

$$4\hat{\tau}|\nabla \ell|^{2}(x,\hat{\tau}) = -4\tau R(x,-\hat{\tau}) + 4\ell(x,\hat{\tau}) - \frac{4}{\sqrt{\hat{\tau}}} \int_{0}^{\tau} s^{\frac{3}{2}} H(Y(\Gamma(s))) ds$$

$$(3.12) \qquad + \frac{4}{\sqrt{\hat{\tau}}} \int_{0}^{\hat{\tau}} \sqrt{s} R(\Gamma(s),-s) ds,$$

where

$$H(Y) = -\frac{\partial R}{\partial s} - 2\langle Y, \nabla R \rangle + 2\operatorname{Ric}(Y, Y).$$

By the isometry, $(\phi_{(-\hat{\tau})^*}(Y), g = g(0)) = (Y, g(-\hat{\tau}))$, we see

$$R_{\hat{\tau}}(x,-\hat{\tau}) = \frac{\partial R}{\partial \hat{\tau}}(x,-\hat{\tau}) = \langle \nabla R, \nabla f \rangle_g(\phi_{-\hat{\tau}}(x)),$$

$$\langle Y, \nabla R \rangle(x, \hat{\tau}) = \langle Y, \nabla R \rangle_{g(-\hat{\tau})}(x) = \langle \phi_{(-\hat{\tau})^*}(Y), \nabla R \rangle_g(\phi_{-\hat{\tau}}(x))$$

and

$$Ric(Y,Y)(x,\hat{\tau}) = Ric_{g(-\hat{\tau})}(Y,Y)(x) = Ric_{g}(\phi_{(-\hat{\tau})^{*}}(Y),\phi_{(-\hat{\tau})^{*}}(Y))(\phi_{-\hat{\tau}}(x)).$$

Then

$$\begin{split} H(Y)(x,\hat{\tau}) &= -\langle \nabla R, \nabla f \rangle_g(\phi_{-\hat{\tau}}(x)) - 2\langle \phi_{(-\hat{\tau})^*}(Y), \nabla R \rangle_g(\phi_{-\hat{\tau}}(x)) \\ &+ 2\operatorname{Ric}(\phi_{(-\hat{\tau})^*}(Y), \phi_{(-\hat{\tau})^*}(Y))(\phi_{-\hat{\tau}}(x)) \\ &= 2\operatorname{Ric}(\nabla f, \nabla f)(\phi_{-\hat{\tau}}(x)) + 4\operatorname{Ric}(\phi_{(-\hat{\tau})^*}(Y), \nabla f)(\phi_{-\hat{\tau}}(x)) \\ &+ 2\operatorname{Ric}(\phi_{(-\hat{\tau})^*}(Y), \phi_{(-\hat{\tau})^*}(Y))(\phi_{-\hat{\tau}}(x)) \\ &= 2\operatorname{Ric}(\phi_{(-\hat{\tau})^*}(Y) + \nabla f, \phi_{(-\hat{\tau})^*}(Y) + \nabla f)(\phi_{-\hat{\tau}}(x)). \end{split}$$

Moreover, by (1.20), we have

(3.13)
$$|\nabla \ell|^2_{g(-\hat{\tau})}(x,\hat{\tau}) = |\nabla \lambda|^2_g(\phi_{-\hat{\tau}}(x),\hat{\tau}).$$

Thus by (3.12), we get

$$\begin{aligned} 4\hat{\tau}|\nabla\lambda|^2(p,\hat{\tau}) &= -4\hat{\tau}R_g(p) + 4\lambda(p,\hat{\tau}) \\ &\quad -\frac{8}{\sqrt{\hat{\tau}}}\int_0^{\hat{\tau}} s^{\frac{3}{2}} 2\operatorname{Ric}_g(\phi_{(-s)^*}(Y) + \nabla f, \phi_{(-s)^*}(Y) + \nabla f)(\gamma(s))ds \\ (3.14) &\quad +\frac{4}{\sqrt{\hat{\tau}}}\int_0^{\hat{\tau}} \sqrt{s}R_g(\gamma(s))ds, \end{aligned}$$

where $p = \phi_{-\hat{\tau}}(x)$ and $\gamma(\hat{\tau}) = \phi_{-\hat{\tau}}(\Gamma(\hat{\tau}))$.

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Recall that the minimal \mathcal{L} -geodesic $\gamma_2(s) = \phi_{-s}(\Gamma(s))$ $(s \in [\tau', \tau_0])$ is contained in $M \setminus K$. Then

(3.15)
$$\operatorname{Ric}(\phi_{(-s)^*}(Y) + \nabla f, \phi_{(-s)^*}(Y) + \nabla f)(\gamma_2(s)) \ge 0.$$

Thus for the λ -function starting from $o'_{\tau} = \phi_{-\tau'}(\Gamma(\tau'))$, we get by (3.14),

$$\begin{aligned} 4\tau_0' |\nabla \lambda_{o_\tau'}|^2 (p_\tau, \tau_0') &\leq -4\tau_0' R_g(p_\tau) + 4\lambda_{o_{\tau'}}(p_\tau, \tau_0') \\ &+ \frac{4}{\sqrt{\tau_0'}} \int_0^{\tau_0'} \sqrt{s} R_g(\gamma_2(s)) ds \\ &\leq -4\tau_0' R_g(p_\tau) + 4\lambda_{o_\tau'}(p_\tau) + 8\lambda_{o_\tau'}(p_\tau) \\ &= -4\tau_0' R_g(p_\tau) + 12\lambda_{o_\tau'}(p_\tau), \end{aligned}$$

where $\tau'_0 = \tau_0 - \tau'$ and the second inequality comes from (1.19). It follows

(3.16)
$$R(p_{\tau}) \le \frac{3\lambda_{o_{\tau}'}}{\tau_0'}$$

Hence, by (3.10), we obtain (3.11).

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To get lower bound estimate of scalar curvature, we need

Lemma 3.3. Let (M, g) be a κ -noncollapsed steady gradient Ricci soliton with Ric ≥ 0 on $M \setminus K$. Suppose that (2.1) holds. Then there are compact set K' with $K \subset K'$ and constant $c_0 > 0$ such that

$$(3.17) 1 - R(x) \ge c_0 > 0, \ \forall \ x \in M \setminus K'.$$

Proof. On the contrary, we suppose that (3.17) fails. Then there exists a sequence of points $p_i \to \infty$ such that

$$R(p_i) \ge 1 - \epsilon,$$

where $\epsilon > 0$ is a small constant to be determined lately. Thus by (1.5), we get

$$(3.18) \qquad |\nabla f|(p_i) \le \sqrt{\epsilon}.$$

Denote $\sigma_i(t)$ to be the unit speed minimal geodesic with $\sigma_i(0) = o$ and $\sigma_i(s_i) = p_i$. We may assume that $p_i \in M \setminus K$. Let

$$s'_i = \inf\{t \mid \sigma_i(u) \subset M \setminus K, \ \forall u > t\}$$

and $p'_i = \sigma_i(s'_i)$. Since K is compact, there exists a constant D > 0 such that $s'_i \leq D$ for all i and $f(p'_i) \leq D$ for all i. Thus by (3.18), we get

$$\langle \nabla f, \sigma'_i \rangle(p_i) \le |\nabla f|(p_i) \le \sqrt{\epsilon}.$$

Hence, for any $t \in [s'_i, s_i]$, we obtain

$$\begin{split} \langle \nabla f, \sigma'_i \rangle (\sigma_i(t)) &= \langle \nabla f, \sigma'_i \rangle (p_i) - \int_t^{s_i} \frac{d}{du} \langle \nabla f, \sigma'_i \rangle (\sigma_i(u)) du \\ &= \langle \nabla f, \sigma'_i \rangle (p_i) - \int_t^{s_i} \operatorname{Ric}(\sigma'_i, \sigma'_i) (\sigma_i(u)) du \\ &\leq \langle \nabla f, \sigma'_i \rangle (p_i) \\ &\leq \sqrt{\epsilon}. \end{split}$$

Consequently,

(3.19)
$$f(p_i) = f(p'_i) + \int_{s'_i}^{s_i} \frac{d}{du} f(\sigma_i(u)) du$$
$$= f(p'_i) + \int_{s'_i}^{s_i} \langle \nabla f, \sigma'_i \rangle (\sigma_i(u)) du$$
$$\leq D + \sqrt{\epsilon} (s_i - s'_i)$$
$$\leq 10\sqrt{\epsilon} \rho(p_i), \ \forall i \gg 1.$$

By choosing $\epsilon \leq \frac{c_1^2}{10000}$, (3.19) contradicts to (2.1). Therefore, we prove the lemma.

By Lemma 3.3 and Lemma 1.2, we prove

Proposition 3.4. Let (M,g) be a κ -collapsed steady gradient Ricci soliton with $\operatorname{Rm} \geq 0$ on $M \setminus K$. Suppose that (2.1) holds. Then there exists a constant C > 0 such that

$$(3.20) R(p) \ge \frac{C}{\rho(p)}$$

for all $\rho(p) > r_0$.

Proof. We use the argument in the proof of [12, Proposition 4.3]. By Lemma 3.3, we can choose two compact sets \hat{K} and K' of M with $K \subset K' \subset \hat{K}$ such that

$$(3.21) 1 - R(x) \ge c_0 > 0, \ \forall \ x \in \tilde{K} \setminus K',$$

where c_0 is a small constant. By a result of Chen [8], we may also assume

(3.22)
$$R(x) \ge c_0, \ \forall \ x \in \hat{K} \setminus K'.$$

For any $p \in M \setminus \hat{K}$, we have

$$\frac{d}{dt}R(\phi_t(p)) = \operatorname{Ric}(\nabla f, \nabla f) \ge 0, \ \forall t \le 0.$$

It follows

$$0 \le R(\phi_t(p)) \le R(p), \ \forall t \le 0.$$

Since

$$\frac{d}{dt}f(\phi_t(p)) = -|\nabla f|(\phi_t(p)), \ \forall t \le 0,$$

by (1.5), we get

$$1 - R(p) \le -\frac{d}{dt} f(\phi_t(p)) \le 1, \ \forall t \le 0.$$

Hence

(3.23)
$$(1 - R(p))|t| \le f(p) - f(\phi_t(p)) \le |t|, \ \forall t \le 0.$$

By Lemma 1.2 and the evolution equation of scalar curvature, we have

$$\left|\frac{\partial}{\partial t}R^{-1}(p,t)\right| \le \frac{|\Delta R(p,t)|}{R^2(p,t)} + \frac{2|\operatorname{Ric}(p,t)|^2}{R^2(p,t)} \le C_0 + 2,$$

and consequently,

(3.24)
$$R(p,t)|t| \ge \frac{|t|}{(C_0+2)|t| + R(p,0)^{-1}} \ge \frac{1}{2(C_0+2)}$$

for all $|t| \gg 1$. Since for any $p \in M \setminus \hat{K}$, there exists $x_p \in \hat{K} \setminus K'$ and $t_p < 0$ such that $\phi_{t_p}(x_p) = p$. Thus by the first inequality in (3.24) together with the first inequality in (3.23), we get

(3.25)

$$R(p) \geq \frac{1}{|t_p|} \cdot \frac{1}{(C_0 + 2) + (R(x_p)|t_p|)^{-1}} \geq \frac{1 - R(x_p)}{f(p) - f(x_p)} \cdot \frac{1}{(C_0 + 2) + (R(x_p)|t_p|)^{-1}} \geq \frac{1 - R(x_p)}{2(f(p) - f(o))} \cdot \frac{1}{(C_0 + 2) + (R(x_p)|t_p|)^{-1}}$$

Note that $f(x_p) \leq C'_0$ for $x_p \in \hat{K} \setminus K'$. Then by the second inequality in (3.23) together with (3.22), we have

$$|t_p| \ge f(p) - f(x_p) \ge c_0 \frac{f(p) - C'_0}{c_0} \ge \frac{1}{R(x_p)}$$

as long as $\rho(p) >> 1$. Hence, by inserting the above relation into (3.25) together with (3.21), we obtain

$$R(p) \ge \frac{c_0}{2c_2(C_0+3)\rho(p)},$$

where c_2 is the constant in (2.1).

4. Construction of shrinking Ricci solitons

In this section, we construct the shrinking Ricci soliton via the blow-down around p_{τ} by the estimates in Section 2, 3.

Let $\{p_i\} \to \infty$ be any sequence in M. Set $\tau_i = f(p_i)$. Then by Proposition 2.3, there are $q_i \in \Sigma_{\tau_i}$ and $\tau'_i \in [c\tau_i, C\tau_i]$ such that

(4.1)
$$\lambda(q_i, \tau_i') \le A_0,$$

where c, C and A_0 are uniform constants. Equivalently, we have

(4.2)
$$\ell(x_i, \tau_i') \le A_0$$

where $x_i = \phi_{\tau'_i}(q_i)$.

By Lemma 3.2 and Proposition 3.4, we apply Proposition 1.1 to prove

Lemma 4.1. Let x_i , τ'_i defined as above. Then the sequence of rescaled Ricci flows $(M, \tau'_i^{-1}g(-\tau'_i + \tau'_it); x_i)$ converges to $(N' \times \mathbb{R}, g'_{\infty} = h'(t) + ds^2; x_{\infty}), t \in (-\infty, 0]$, where (N', h'(t)) is an non-flat ancient κ -solution.

Proof. By the isometry

(4.3)
$$(g(-\tau'_i), x_i) \stackrel{\phi_{-\tau'_i}}{\cong} (g = g(0), q_i),$$

the rescaled Ricci flow $(M, \tau_i'^{-1}g(-\tau_i'+\tau_i't); x_i)$ is isometric to $(M, \tau_i'^{-1}g(\tau_i't); q_i)$. By Lemma 3.2 and Proposition 3.4, we see that there exists a constant C > 0 such that

(4.4)
$$C^{-1} \le \tau_i' R(q_i) \le C.$$

Thus the limit of rescaled Ricci flows $(M, \tau_i'^{-1}g(\tau_i't); q_i)$ is isometric to one of $(M, R(q_i)g(R(q_i)^{-1}t); q_i)$. Moreover, by Proposition 1.1, the limit of $(M, \tau_i'^{-1}g(-\tau_i'+\tau_i't); x_i)$ is a split flow $(g'_{\infty} = h'(t) + ds^2; x_{\infty})$ on $N' \times \mathbb{R}$. Since $R_{g'_{\infty}}(x_{\infty}) \geq C^{-1}$ by (4.4), (N', h'(t)) is a non-flat ancient κ -solution. \Box

Let $g_i(t) = \tau_i^{\prime-1}g(\tau_i^{\prime}t)$ and $\ell_i(x,\tau) = \ell(x,\tau_i^{\prime}\tau)$, where $\tau = -t$. Then $\ell_i(x,\tau)$ is the reduced distance from (o,0) w.r.t. the backward flow $\hat{g}_i(\tau) = g_i(-\tau)$. Moreover, by the scale invariant property and (4.2), we have

(4.5)
$$\ell_i(x_i, 1) = \ell_o(x_i, \tau_i) \le A_0.$$

On the other hand, by the convergence in Lemma 4.1, for any fixed radius D > 0, we have

$$(4.6) R(p,t) \le C(D),$$

where $(p,t) \in B_{g_i(-1)}(x_i, D) \times [-10, -1]$ when $i \gg 1$. By (4.5) and (4.6), we do the derivative estimate of $\ell_i(x, \tau)$ in the following.

Proposition 4.2. Let $\ell_i(x,\tau)$ be a sequence of reduced distance functions defined as above. Then for any fixed D > 0, there exists uniform constant $\overline{C}(D) > 0$ such that

(4.7)
$$0 \le \ell_i(x,\tau) \le \bar{C}(D), \ \forall (\tau,t) \in B_{g_i(-2)}(x_i,D) \times [-10,-2]$$

and

$$(4.8) \quad \left|\frac{\partial \ell_i}{\partial \tau}(x,\tau)\right| + \left|\nabla \ell_i(x,\tau)\right| \le \bar{C}(D), \ \forall (\tau,t) \in B_{g_i(-2)}(x_i,D) \times [-8,-2]$$

for all $i \gg 1$, where $\tau = -t$.

Proof. The nonnegativity of ℓ_i follows from the definition (1.10) and the nonnegativity of scalar curvature for ancient solutions. In the following, we always denote C, \overline{C}, C_i to be uniform constants only depending on D.

Let $\Gamma_i(\tau)$ be the minimal \mathcal{L} -geodesics between (o, 0) and $(x_i, 1)$. Then by (4.5), we have

$$\mathcal{L}(\Gamma_i(\tau)) \le 2A_0.$$

Since the Ricci curvature of $g_i(t)$ is nonnegative, by Harnack inequality, it holds

(4.9)
$$B_{g_i(t)}(x_i, D) \subseteq B_{g_i(-1)}(x_i, D), \ \forall \ t \le -1.$$

On the other hand, by (4.6) and the distance distortion estimate we also have

(4.10)
$$B_{g_i(-1)}(x_i, D) \subseteq B_{g_i(t)}(x_i, C_0(D)), \ \forall \ t \in [-10, -2].$$

Now we fix $\tau \in [2, 10]$ and let $\sigma_i(s)$, $s \in [1, \tau]$, be the minimal geodesic from x_i to x w.r.t. $g_i(-\tau)$, where $x \in B_{g_i(-\tau)}(x_i, D) \subseteq B_{g_i(-2)}(x_i, D) \subseteq B_{g_i(-1)}(x_i, D)$. Then $\sigma_i(s) \subseteq B_{g_i(-1)}(x_i, D)$. Thus by (4.6) and the distance distortion estimate, we obtain

$$|\sigma_i'(s)|_{g_i(-s)}^2 \le e^{2nC(D)|t|} |\sigma_i'(s)|_{g_i(-\tau)}^2 \le C(D) |\sigma_i'(s)|_{g_i(t)}^2$$

for all $s \in [1, \tau]$. Consequently,

$$L_{i}(x,\tau) \leq \mathcal{L}(\Gamma_{i}) + \int_{1}^{\tau} \sqrt{s} (R_{g_{i}(-s)} + |\sigma'(s)|^{2}_{g_{i}(-s)}) ds$$

$$\leq \mathcal{L}(\Gamma_{i}) + \int_{1}^{\tau} \sqrt{s} (R_{g_{i}(-s)} + C(D)|\sigma'(s)|^{2}_{g_{i}(-\tau)}) ds$$

$$\leq 2A_{0} + 2\sqrt{\tau}C(D) + \frac{2}{3}C(D)\frac{D^{2}(\tau^{\frac{3}{2}} - 1)}{(\tau - 1)^{2}}$$

$$\leq 2A_{0} + C'(D).$$

Hence, (4.7) follows from (4.11) together with (4.10).

Next we prove (4.8). Let $\tau \in [2, 8]$ and $\overline{\Gamma}_i(s)$ be a minimal \mathcal{L} -geodesic between (o, 0) and (x, τ) , where $x \in B_{g_i(-\tau)}(x_i, D)$. Set

$$\bar{\tau}'_i = \inf\{t \mid \bar{\Gamma}_i(s) \subset B_{g_i(-\tau)}(x_i, D), \ \forall s > t\}$$

and $q_i = \bar{\Gamma}_i(\bar{\tau}'_i)$. By a change of variable $u = \sqrt{s}$, we write $\hat{\Gamma}_i(u) = \bar{\Gamma}_i(u^2)$. Then

$$\mathcal{L}(\hat{\Gamma}_{i}(u)) = \int_{0}^{\sqrt{\tau}} 2u^{2}R_{g_{i}(-u^{2})} + \frac{1}{2}|\hat{\Gamma}_{i}'(u)|_{g_{i}(-u^{2})}^{2}du.$$

It follows

(4.)

(4.12)
$$\int_{\sqrt{\bar{\tau}'_i}}^{\sqrt{\tau}} \frac{1}{2} |\hat{\Gamma}'_i(u)|^2_{g_i(-u^2)} du \le \mathcal{L}(\hat{\Gamma}_i(u)) = 2\sqrt{\tau}\ell_i(x,\tau) \le C_1(D),$$

where the last inequality follows from (4.7). On the other hand, by the distance distortion estimates, we have

$$D^{2} = d_{g_{i}(-\tau)}^{2}(x,q_{i}) \leq \left(\int_{\sqrt{\tau_{i}'}}^{\sqrt{\tau}} |\hat{\Gamma}_{i}'(u)|_{g_{i}(-\tau)} du\right)^{2}$$

$$\leq C(D) \left(\int_{\sqrt{\tau_{i}'}}^{\sqrt{\tau}} |\hat{\Gamma}_{i}'(u)|_{g_{i}(-u^{2})} du\right)^{2}$$

$$\leq C(D) \left(\sqrt{\tau} - \sqrt{\tau_{i}'}\right)^{2} \int_{\sqrt{\tau_{i}'}}^{\sqrt{\tau}} |\hat{\Gamma}_{i}'(u)|_{g_{i}(-u^{2})}^{2} du.$$

Thus by (4.12), we get

(4.13)
$$\sqrt{\tau} - \sqrt{\bar{\tau}_i'} \ge C_3(D)$$

for all $x \in B_{g_i(-\tau)}(x_i, D)$ and $i \gg 1$.

We notice that $\hat{\Gamma}'_i(u)$ satisfies the minimal \mathcal{L} -geodesic equation,

$$\nabla_{\hat{\Gamma}'_i(u)}\hat{\Gamma}'_i(u) - 2u^2\nabla R + 4u\operatorname{Ric}(\hat{\Gamma}'_i(u)) = 0, \forall \ u \in [\sqrt{\bar{\tau}'_i}, \sqrt{\tau}].$$

Then by Shi's estimates and the fact that $\operatorname{Ric} \geq 0$ on $B_{g_i(-1)}(x_i, D)$ for all $i \gg 1$, we have

$$\frac{d}{du} |\hat{\Gamma}'_{i}(u)|^{2}_{g_{i}(-u^{2})} = 4u^{2} \langle \nabla R, \hat{\Gamma}'_{i} \rangle - 4u \operatorname{Ric}(\hat{\Gamma}'_{i}, \hat{\Gamma}'_{i}) \\
\leq 4u^{2} |\langle \nabla R, \hat{\Gamma}'_{i} \rangle|_{g_{i}(-u^{2})} \\
\leq C_{4}(D) |\hat{\Gamma}'_{i}(u)|_{g_{i}(-u^{2})} \\
\leq 4C_{4}(D)(1 + |\hat{\Gamma}'_{i}(u)|^{2}_{g_{i}(-u^{2})}).$$

Integrating the above inequality and by (4.12), we obtain

$$\begin{aligned} |\hat{\Gamma}'_{i}(\sqrt{\tau})|^{2}_{g_{i}(-\tau)} &- |\hat{\Gamma}'_{i}(u)|^{2}_{g_{i}(-u^{2})} \\ &= \int_{u}^{\sqrt{\tau}} \frac{d}{dv} |\hat{\Gamma}'_{i}(v)|^{2}_{g_{i}(-v^{2})} dv \\ &\leq 4\sqrt{\tau}C_{4}(D) + 8C_{4}(D) \int_{\sqrt{\tau}'_{i}}^{\sqrt{\tau}} \frac{1}{2} |\hat{\Gamma}'_{i}(u)|^{2}_{g_{i}(-u^{2})} du \\ &\leq C_{5}(D). \end{aligned}$$

Consequently,

(4.14)
$$|\hat{\Gamma}'_{i}(u)|^{2}_{g_{i}(-u^{2})} \geq |\hat{\Gamma}'_{i}(\sqrt{\tau})|^{2}_{g_{i}(-\tau)} - C_{5}(D), \ \forall u \in [\sqrt{\overline{\tau}'_{i}}, \sqrt{\tau}].$$

By (4.13) and (4.14), we have

$$\begin{aligned} \mathcal{L}(\bar{\Gamma}_{i}) &\geq \int_{\sqrt{\tau_{i}'}}^{\sqrt{\tau}} (\frac{1}{2} |\hat{\Gamma}_{i}'(u)|_{g_{i}(-u^{2})}^{2} + 2u^{2}R_{g_{i}(-u^{2})}) du \\ &\geq \int_{\sqrt{\tau_{i}'}}^{\sqrt{\tau}} \frac{1}{2} |\hat{\Gamma}_{i}'(u)|_{g_{i}(-u^{2})}^{2} du \\ &\geq C_{6}(D)(\sqrt{\tau} - \sqrt{\tau_{i}'}) |\hat{\Gamma}_{i}'(\sqrt{\tau})|_{g_{i}(-\tau)}^{2} - C_{6}(D)C_{5}(D) \\ &\geq C_{7}(D) |\hat{\Gamma}_{i}'(\sqrt{\tau})|_{g_{i}(-\tau)}^{2} - C_{8}(D). \end{aligned}$$

On other hand, by (4.7), we have

$$\mathcal{L}(\bar{\Gamma}_i) \le 2\sqrt{\tau}\ell_i(x,\tau) \le 10C(D).$$

Thus combining above two inequalities, we obtain

$$|\hat{\Gamma}'_i(\sqrt{\tau})|^2_{g_i(-\tau)} \le C_9(D).$$

By (1.12), it follows

$$|\nabla \ell_i(x,\tau)|^2 = |\bar{\Gamma}'_i(\tau)|^2 = \frac{|\hat{\Gamma}'_i(\sqrt{\tau})|^2}{4\tau} \le \frac{C_9(D)}{8}.$$

Moreover, by (1.15) and (4.7), we deduce

$$2\left|\frac{\partial}{\partial\tau}\ell_i(x,\tau)\right| \le |\nabla\ell_i(x,\tau)|^2 + R + \frac{\ell_i(x,\tau)}{\tau} \le \frac{C_9(D)}{8} + C(D).$$

Hence, the above two relations give (4.8). The proposition is proved.

Remark 4.3. Since the Ricci curvature of (M, g) is just nonnegative outside the compact set K, we cannot use the global Harnack inequality directly in [22, Section 7] to get the gradient estimates for $\ell_i(x, \tau)$ -function at any spacetime (x, τ) . We shall restrict the corresponding minimal \mathcal{L} -geodesic on $M \setminus K$ and do the time length estimate for the restricted minimal \mathcal{L} -geodesic, see Lemma 3.1 and (4.13).

By Lemma 4.1 and Proposition 4.2, we are able to construct a limit nonflat shrinking Ricci soliton via the rescaled Ricci flows of (M, g(t)) by following the strategy in [22, Section 9].

Proposition 4.4. Let x_i , τ'_i chosen as in Lemma 4.1. Then the sequence of rescaled Ricci flows $(M, \tau'_i)^{-1}g(\tau'_it); x_i)$ converges to $(N' \times \mathbb{R}, g'_{\infty} = h'(t) + ds^2; x_{\infty}), t \in (-\infty, -1],$ where (N', h'(t)) is an non-flat shrinking Ricci soliton.

Proof. By Lemma 4.1, we know that the sequence of rescaled Ricci flows $(M, \tau_i'^{-1}g(\tau_i't); x_i)$ converges to $(M_{\infty} = N' \times \mathbb{R}, h'(t) + ds^2; x_{\infty}), (-\infty, -1],$ where (N', h'(t)) is an non-flat ancient κ -solution. We only need to show (N', h'(t)) is indeed a shrinking Ricci soliton.

By Proposition 4.2, the sequence of functions $\{\ell_i\}$ converges subsequently to a function ℓ_{∞} on $M_{\infty} \times [2, 8]$ in the C_{loc}^{α} sense for any $\alpha \in (0, 1)$. Since ℓ_{∞} is also locally Lipschitz, which is an element in $W_{loc}^{1,2}(M_{\infty} \times [2, 8])$, we may assume that $\{\ell_i\}$ converges weakly in $W_{loc}^{1,2}$ to ℓ_{∞} . Moreover, according to [19, Section 9], by the monotonicity of reduced volume, one can show the inequalities (1.16) and (1.17) in Lemma 1.4 for ℓ_{∞} becomes equalities simultaneously. Namely, ℓ_{∞} satisfies

$$2\frac{\partial l_{\infty}}{\partial \tau} + |\nabla l_{\infty}|^2 - R_{g'_{\infty}} + \frac{l_{\infty}}{\tau} = 0$$

and

(4.15)
$$2\Delta_{g'_{\infty}}l_{\infty} - |\nabla l_{\infty}|^2 + R_{g'_{\infty}} + \frac{l_{\infty} - n}{\tau} = 0$$

in the distributional sense.

Let

$$u = (4\pi\tau)^{\frac{n}{2}} e^{\ell_{\infty}} > 0.$$

Then u satisfies the conjugate heat equation

(4.16)
$$\frac{\partial u}{\partial \tau} - \Delta_{g'_{\infty}} u + R_{g'_{\infty}} u = 0$$

in the distributional sense. Thus the standard regularity theory gives smoothness of ℓ_{∞} . On the other hand, if we let

$$v = \left(\tau \left(2\Delta_{g'_{\infty}}l_{\infty} - |\nabla l_{\infty}|^2 + R_{g'_{\infty}}\right) + l_{\infty} - n\right)u,$$

then by (4.15), we have v = 0. Following the Perelman's computation [22, Proposition 9.1], we get by (4.16),

$$\left(\frac{\partial}{\partial \tau} - \Delta_{g'_{\infty}} + R_{g'_{\infty}}\right)v = -2\tau \left|\operatorname{Ric}_{g'_{\infty}} + \nabla^2 l_{\infty} - \frac{1}{2\tau}g'_{\infty}(\tau)\right|^2 u$$

on $M_{\infty} \times [2, 8]$. Thus by u > 0, we obtain

$$\operatorname{Ric}_{g'_{\infty}} + \nabla^2 l_{\infty} - \frac{1}{2\tau} g'_{\infty}(\tau) = 0$$

on $M_{\infty} \times [2, 8]$. This implies that $(M_{\infty}, g'_{\infty}(t); x_{\infty})$ $(t \in [-8, -2])$ is a shrinking Ricci soliton. By the uniqueness of Ricci flow and the splitting structure $M_{\infty} = N' \times \mathbb{R}$, we prove the theorem immediately.

By (4.3), we actually prove

Theorem 4.5. Let q_i , τ'_i be the sequences chosen as in (4.1). Then the sequence of rescaled Ricci flows $(M, g_{q_i}(t); q_i)$ converges to $(N' \times \mathbb{R}, h'(t) + ds^2; q_{\infty}), t \in (-\infty, 0]$ in the Cheeger-Gromov sense, where (N', h'(t)) is a non-flat shrinking Ricci soliton.

5. PROOFS OF MAIN THEOREMS

In this section, we prove main results in Introduction 0 by following the strategy in [26]. Let us first recall the following definition introduced by Perelman (cf. [23, 18, 26], etc.).

Definition 5.1. For any $\epsilon > 0$, we say a pointed Ricci flow $(M_1, g_1(t); p_1), t \in [-T, 0]$ (T may be the infinity), is ϵ -close to another pointed Ricci flow $(M_2, g_2(t); p_2), t \in [-T, 0]$, if there is a diffeomorphism onto its image $\bar{\phi} : B_{g_2(0)}(p_2, \epsilon^{-1}) \to M_1$, such that $\bar{\phi}(p_2) = p_1$ and $\|\bar{\phi}^*g_1(t) - g_2(t)\|_{C^{[\epsilon^{-1}]}} < \epsilon$ for all $t \in [-\min\{T, \epsilon^{-1}\}, 0]$, where the norms and derivatives are taken with respect to $g_2(0)$.

By the compactness result for rescaled Ricci flows in Proposition 1.1, we know that for any $\epsilon > 0$, there exists a compact set $D(\epsilon) > 0$, such that for any $p \in M \setminus D$, $(M, g_p(t); p)$ is ϵ -close to a splitting flow $(h_p(t)+ds^2; p)$, where $h_p(t)$ is an (n-1)-dimensional ancient κ -solution. Since the ϵ -close splitting flow $(h_p(t) + ds^2; p)$ may not be unique for a point p, we may introduce a function on M for each ϵ as in [18, 26],

(5.1)
$$F_{\epsilon}(p) = \inf_{h_p} \{ \operatorname{Diam}(h_p(0)) \in (0, \infty) \}.$$

For simplicity, we always omit the subscribe ϵ in the function $F_{\epsilon}(p)$ below.

5.1. Proofs of Theorem 0.1 and Corollary 0.2. As in [26, Theorem 0.2], we use the argument by contradiction to prove Theorem 0.1. Suppose that there exists a sequence of normally scaled Ricci flows $(M, g_{p_i}(t); p_i)$ $(p_i \to \infty)$, which converges to a limit Ricci flow $(N \times \mathbb{R}, h(t) + ds^2; p_{\infty})$, where $(N, h(t), p_{\infty})$ is a compact ancient κ -solution of type II. Then by [26, Lemma 2.2] under the condition $\operatorname{Rm} \geq 0$ and $\operatorname{Ric} > 0$ on $M \setminus K$, the scalar curvature of (M, g) decays to zero uniformly, i.e.

(5.2)
$$\lim_{x \to \infty} R(x) = 0.$$

Thus by the normalization identity (1.5), it follows

(5.3)
$$|\nabla f(x)| \to 1 \text{ as } \rho(x) \to \infty.$$

Moreover, by [14, Lemma 2.2] (or [9, Theorem 2.1]), f satisfies (2.1). Thus all results in the above sections 1-4 hold.

Proof of Theorem 0.1. Choose a sequence of $t_k \to -\infty$. Then by [26, Lemma 4.3],

(5.4)
$$\lim_{k \to \infty} \operatorname{Diam}(h(t_k)) R_h^{\frac{1}{2}}(p_{\infty}, t_k) \ge \lim_{k \to \infty} \operatorname{Diam}(h(t_k)) R_{h,\min}^{\frac{1}{2}}(t_k) = \infty.$$

Thus there is $t_0 \in \{t_k\}$ for some k_0 such that

(5.5)
$$\operatorname{Diam}(h(t_0))R_h^{\frac{1}{2}}(p_{\infty}, t_0) > 100C_0A,$$

where the large constants C_0 and A will be determined lately.

Set $T = t_0 R^{-1}(p_i)$ and choose $\epsilon < -\frac{1}{100t_0}$. Then $(M, g_{p_i}(t); p_i)$ is ϵ -close to $(N \times \mathbb{R}, h(t) + ds^2; p_{\infty})$ when i >> 1. Thus by (5.5) and the convergence of $g_{p_i}(t)$, we get

(5.6)
$$\operatorname{Diam}(\bar{g}(T))R^{\frac{1}{2}}(p_i,T) = \operatorname{Diam}(\bar{g}_{p_i}(t_0))R^{\frac{1}{2}}_{p_i}(p_i,t_0) > 90C_0A,$$

as long as i >> 1, where $\bar{g}(T)$ is the induced metric of g on $f^{-1}(f(\phi_T(p_i)))$ and $\bar{g}_{p_i}(t)$ is the induced metric of $g_{p_i}(t)$ on $f^{-1}(f(p_i))$, respectively. It follows

(5.7)

$$\operatorname{Diam}(R(\phi_T(p_i))\overline{g}(T)) = \operatorname{Diam}(\overline{g}(T))R^{\frac{1}{2}}(\phi_T(p_i))$$

$$= \operatorname{Diam}(\overline{g}(T))R^{\frac{1}{2}}(p_i,T)$$

$$> 90C_0A, \ i >> 1.$$

Let $(\tilde{N}, \tilde{h}(t))$ be an (n-1)-dimensional split ancient flow for the rescaled flow $(M, g_{(\phi_T(p_i))}(t); \phi_T(p_i))$ as in Proposition 1.1. Then as in the proof of [26, Proposition 3.6], one can show that $\text{Diam}(\tilde{N}, \tilde{h}(0))$ is closed to $\text{Diam}(N, R_h(p_{\infty}, t_0)h(t_0))$, where R_h denote the scalar curvature of (N, h(x, t)). In fact, it is a limit of hypersurfaces

$$(f^{-1}(f(\phi_T(p_i))), R(\phi_T(p_i))\bar{g}(T); \phi_T(p_i)).$$

In particular, $(\tilde{N}, \tilde{h}(0))$ is compact. Thus by (5.7), we get

 $\operatorname{Diam}(\tilde{N}, \tilde{h}(0)) \ge 90C_0A.$

By the Definition 5.1, it follows

(5.8) $F(\phi_T(p_i)) > 50C_0A, \ i >> 1.$

On the other hand, by Proposition 2.3, on each level set $\Sigma_{f(\phi_T(p_i))}$, there exists $q_i \in \Sigma_{f(\phi_T(p_i))}$ and $\tau_i \in [cf(\phi_T(p_i)), Cf(\phi_T(p_i))]$ such that $\lambda(q_i, \tau_i) \leq A_0$, where c, C and A_0 are uniform constants in Proposition 2.3. Clearly, $\tau_i \to \infty$ since $f(\phi_T(p_i)) \to \infty$ when $i \to \infty$. Thus we can apply Theorem 4.5 to see that the sequence of normally scaled flows $(M, g_{q_i}(t); q_i)$ converges to a nonflat shrinking gradient Ricci soliton $(N' \times \mathbb{R}, h'(t) + ds^2, q_\infty)$. It follows that (N', h'(t)) is also a non-flat shrinking soliton with $\operatorname{Rm} \geq 0$. Furthermore, by [26, Lemma 2.6 and Remark 2.8], (N', h'(t)) is compact as same as $(\tilde{N}, \tilde{h}(t))$. Hence, by the classification result of compact κ -noncollapsed shrinking Ricci solitons with $\operatorname{Rm} \geq 0$ (cf. [10, Theorem 7.34], [6], [20]), there exists a large constant A such that

and

$$R_{h'(0)} \equiv 1.$$

By the convergence of $(M, g_{q_i}(t); q_i)$ and (5.9), we see

(5.10)
$$\operatorname{Diam}(R(q_i)\bar{g}(T)) = \operatorname{Diam}(\bar{g}(T))R^{\frac{1}{2}}(q_i) \le 2A$$

Then by [26, Lemma 1.3 and Lemma 2.6], there exists a large constant C_0 such that

(5.11)
$$\frac{1}{C_0} R(x) \le R(q_i) \le C_0 R(x), \ \forall x \in \Sigma_{f(\phi_T(p_i))}$$

for all $i \gg 1$. Thus combining (5.10) and (5.11), we get

(5.12)

$$\operatorname{Diam}(R(\phi_T(p_i))\overline{g}(T)) = \operatorname{Diam}(\overline{g}(T))R^{\frac{1}{2}}(\phi_T(p_i))$$

$$\leq C_0\operatorname{Diam}(\overline{g}(T))R^{\frac{1}{2}}(q_i)$$

$$\leq 2C_0A,$$

which contradicts to (5.8). Hence, the theorem is proved.

Proof of Corollary 0.2. By Theorem 0.1, (n-1)-dimensional compact split limit ancient flow (N, h(t)) of type II cannot occur from the blow-down of (M, g). Thus we need only to show that the compact split ancient flows of type I and noncompact split ancient flows cannot occur simultaneously from the blow-down of (M, g). In fact, the latter is true by [27, Theorem 1.2]. The corollary is proved.

5.2. Proof of Theorem 0.3. As in Theorem 0.1, we see that (5.2), (5.3) and (2.1) are all satisfied. Thus all results in the above sections 1-4 hold.

Proof of Theorem 0.3. The first inequality in (0.4) has been proved in Proposition 3.4. We only need to prove the second inequality. Suppose that it is not true, then there exists a sequence of points $p'_i \to \infty$ such that

(5.13)
$$R(p'_i) \ge \frac{i}{\rho(p'_i)}.$$

Let $s_i = f(p'_i)$. Then by (2.1), there exists a uniform constant $C_1 > 0$ such that

(5.14)
$$C_1^{-1}\rho(p'_i) \le s_i \le C_1\rho(p'_i).$$

By Proposition 2.3 Lemma 3.2, for each $i \gg 1$ there exists $q_i \in \Sigma_{s_i}$ such that

$$(5.15) R(q_i) \le \frac{C_2}{s_i}$$

for some uniform constant $C_2 > 0$. On the other hand, by Corollary 0.2, we know that (n-1)-dimensional split limit solution (N', h'(t)) of rescaled Ricci flows $(M, g_{p'_i}(t); p'_i)$ is a compact ancient κ -solution of type I. Thus as in (5.9), we have

$$\operatorname{Diam}(N', h'(0)) \le A.$$

It follows that there exists uniform constant $C_0 > 0$ as in (5.11) such that

(5.16)
$$C_0^{-1}R(q_i) \le R(p'_i) \le C_0R(q_i), \ \forall \ i \gg 1.$$

Hence, combining (5.14), (5.15) and (5.16), we obtain

$$(5.17) R(p_i') \le \frac{C_3}{\rho(p_i')}$$

for some uniform constant $C_3 > 0$. But this is a contradiction with (5.13) when $i \gg 1$. The theorem is proved.

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