On Rings of MAL'CEV–NEUMANN Series

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Dedicated to the memory of Professor Mohammad H. Fahmy

Abstract. In this paper, we investigate the conditions for the Mal'cev–Neumann series ring $\Lambda = R((G;\sigma;\tau))$ to be left fusible and an SA-ring. Also, we show that: if G is a quasitotally ordered group and U a Σ-compatible semiprime ideal of R, then $R((G;\sigma;\tau))$ is a $\Sigma_{U((G;\,\sigma;\,\tau))}$-zip ring if and only if R is a Σ_U -zip ring.

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1. Introduction

Throughout this paper R denotes an associative ring with identity, $r_R(X)$ the right annihilator of X in R for any subset X of a ring R, $nil(R)$ the set of all nilpotent elements of R and for any two nonempty subsets U and V of R, let $(U:V) = \{x \in R \mid Vx \subseteq U\}$. It is easy to see that if U and V are two right ideals of R, then $(U:V)$ is an ideal of R and such ideal is usually called the quotient of U by V. Section 2, is devoted to recall some background concerning the structure of the ring $\Lambda = R((G;\sigma;\tau))$ of Mal'cev–Neumann series. In section 3, we focus on a property of nonzero zero divisor elements related to the fact that the sum of two zero-divisors need not be a zero-divisor. So, the set of left zero-divisors in a ring R is not a left ideal. Therefore, there exists a left zero-divisor which can be expressed as the sum of a left zero-divisor and a non-left zero-divisor in R. This leads Ghashghaei & McGovern [7] to introduce a class of rings in which every element can be written as the sum of a left zero-divisor and left regular element. They called this class of rings left fusible. This leads us to extend the left fusible property of $R[x,\sigma]$ and $R[\![x,\sigma]\!]$ [7, Proposition 2.9] to the ring $\Lambda = R((G;\sigma;\tau))$ of Mal'cev–Neumann series in Proposition 3.2.In section 4, we discuss a class of rings introduced by Birkenmeier et al [2] called right SA-ring, it is exactly the class of rings for which the lattice of right annihilator ideals is a sub lattice of the lattice of ideals. This class includes all quasi-Baer (hence all Baer) rings and all right IN-rings (hence all right self injective rings). They showed that this class is closed under direct products, full and upper triangular matrix rings, certain polynomial rings, and two-sided rings of quotients. This drives us to extend the SA property of $R[x]$ [2, Theorem 3.2] to the ring $\Lambda = R((G;\sigma;\tau))$of Mal'cev–Neumann series in Theorem 4.5. In section 5, we discuss the class of right zip rings introduced by Faith [6] and its generalizations. A ring R is called right zip provided that if the right annihilator $r_R(X)$ of a subset X of R is zero, then there exists a finite subset $Y \subseteq X$ such that $r_R(Y) = 0$. R is zip if it is both right and left zip. The concept of zip rings was initiated by Zelmanowitz [18] and appeared in enormous papers [see,1, 4, 5, 6, 8 and 15] and references therein. Then Ouyang [13] generalized this concept through the introduction of the notion of right weak zip rings (i.e., rings provided that if the right weak annihilator of a subset X of R, $N_R(X) \subseteq nil(R)$, then there exists a finite subset $X_0 \subseteq X$ such that $N_R(X_0) \subseteq nil(R)$, where $N_R(X) = \{a \in R \mid xa \in nil(R) \text{ for each } x \in X\}$. Ouyang studied the transfer of the right (left) weak zip property between the base ring R and some of its extensions such as the ring of upper triangular matrices $T_n(R)$ and Ore extension $R[x, \sigma, \delta]$, where σ is an endomorphism and δ is a σ-derivation. Also, in [14] Ouyang et al continued the study of zip rings and introduced the notion of a Σ-zip ring and investigated some of its properties. For a proper ideal U of R, R is called Σ_U-zip provided that for any subset X of R with $X \nsubseteq U$, if $(U{:}X) = U$, then there exists a finite subset $Y \subseteq X$ such that $(U{:}Y) = U$. Clearly, if $U = 0$, then for any subset X of R, we have $(U{:}X) = r_R(X)$, and so R is Σ_0-zip if and only if R is right zip. If R is an NI ring (i.e., a ring in which $nil(R)$forms an ideal) and$U = nil(R)$. Then for any subset X of R, we have $(nil(R){:}X) = N_R(X)$, and so R is $\Sigma_{\mathrm{nil(R)}}$-zip if and only if R is weak zip. So, both right zip rings and weak zip rings are special cases of Σ-zip rings. This caused us to pay attention to prove that, R is Σ_U -zip ring if and only if $\Lambda = R((G;\sigma;\tau))$ is a $\Sigma_{U((G;\,\sigma;\,\tau))}$-zip ring.

2. Rings of Mal'cev–Neumann Series

The Mal'cev-Neumann construction appeared for the first time in the latter part of 1940's (the Laurent series ring, a particular case of Mal'cev-Neumann rings, was used before by Hilbert). Using them, Mal'cev and Neumann independently showed (in 1948 and 1949 resp.) that the group ring of an ordered group over a division ring can be embedded in a division ring. Since then, the construction has appeared in many papers, mainly in the study of various properties of division rings and related topics. For in-stance, Makar-Limanov in [10] used a skew Laurent series division ring to prove that the skew field of fractions of the first Weyl-algebra contains a free noncommutative subalgebra. Other results on Mal'cev-Neumann rings can be found in Lorenz [9], Musson and Stafford [11], Sonin [17] and Zhao and Liu. [19]. A pair $(G, \leq)$, where G is a group and $\leq$ an order relation, is called quasitotally if the order $\leq$ can be refined to a total order $\preccurlyeq$ on G. Let $(G, \leq)$ be a quasitotally ordered group, and σ a map from G into the group of automorphisms of $R(Aut(R))$, which assigns to each $x \in G, \sigma_x \in Aut(R)$. Suppose also that we are given a map τ from $G \times G$ to $U(R)$, the group of invertible elements of R. Let $\Lambda = R((G; \sigma; \tau))$denotes the set of all formal sums $f = \sum_{x \epsilon G} r_x \overline{x}$ with$r_x \in R$ such that $supp(f) = \{ x \in G \mid r_x \neq 0\}$ (the support of f) is a artinian and narrow subset of G, with component wise addition and multiplication defined by: $(\sum_{x \epsilon G} a_x \bar{x})(\sum_{y \epsilon G} b_y \bar{y}) = \sum_{z \epsilon G} \left(\sum_{\{x, y | xy = z\}} a_x \sigma_x \left(b_y\right) \tau(x, y)\right) \bar{z}$. In order to insure associativity

of Λ, it is necessary to impose two additional conditions on σ and τ, namely that for all $x, y, z \in G$,

(i) $\tau(xy; z) \sigma_x(\tau(x; y)) = \tau(x; yz) \tau(y; z)$;

(ii) $\sigma_y \sigma_z = \sigma_{yz} \eta(y; z)$;

where $\eta(y; z)$ denotes the automorphism of R induced by the unit $\tau(y; z)$. It is now routine to check that $\Lambda = R((G; \sigma; \tau))$ is a ring which is called the Mal'cev-Neumann series ring, that has as an R-basis, the set $\bar{G}$ (a copy of G)

and contains R as a subring where the embedding of R into Λ is given by $r \to r\bar{1}$. It is easy to see that the identity element of Λ is of the form $1 = u\bar{1}$ for some $u \in U(R)$, unlike group rings, crossed product also rings of Mal'cev–Neumann series do not have a natural basis. Indeed if $d: G \to U(R)$ assigns to each element $g \in G$ a unit d_g, then $\widetilde{G} = \{\tilde{g} = gd_g | \ g \in G\}$ yields an alternative R-basis for Λ which still exhibit the basic Mal'cev–Neumann structure and this is called a diagonal change of basis. Thus, via diagonal change of basis we can still assume that $1 = \bar{1}$.

In [14] Ouyang called an ideal U semiprime if for any $a \in R, a^n \in U$ implies $a \in U$.We denote $U((G;\sigma;\tau))$ the subset of Λ consisting of those elements whose coefficients lie in U, that is, $U((G;\sigma;\tau)) = \{f = \sum_{x\epsilon G} a_x\, x \in R((G;\sigma;\tau)) | a_x \in U, x \in suppf\}$ For each $f \in \Lambda$, let $C(f)$be the content of f, i.e. $C(f) = \{a_x \mid x \in supp(f)\}$.

For $f = (\sum_{x\epsilon G} a_x\, x)$ and g =$(\sum_{y\epsilon G} b_y\, y) \in \Lambda$ we define $X_w(f,g) = \{(x_i, y_j) \in G \times G |\ x_i y_j = w$ where $x_i \in supp(f)$ and $y_j \in supp(g)\}$. It is well known that $X_w(f,g)$ is a finite subset. Let R be a ring and G a quasitotally ordered group, R is called a G-Armendariz ring if whenever $f = \sum_{x\epsilon G} a_x\, x$ and $g = \sum_{y\epsilon G} b_y\, y \in \Lambda = R((G;\sigma;\tau))$ such that $fg = 0$ implies $a_x b_y = 0$ for each $x \in supp(f)$ and $y \in supp(g)$.

3. Fusible rings of Mal'cev–Neumann Series.

It is well known that an element $a \in R$ is a left zero-divisor if there is $0 \neq r \in R$ with $ar = 0$ and an element which is not a left zero-divisor is called a non-left zero-divisor. An element $a \in R$ is regular if it is neither a left zero-divisor nor a right zero-divisor. Let $Z_\ell(R)$ (respectively, $Z_\ell^*(R)$) denote the set of left zero-divisors (respectively, non-left zero-divisors) of R. Similarly, $Z_r(R)$ (respectively, $Z_r^*(R)$) denote the set of right zero-divisors (respectively, non-right zero-divisors) of R.

In this section, we study the left fusible and right nonsingular rings of Mal'cev–Neumann series.

Definition 3.1. A ring R is said to be left fusible if every element can be expressed as a sum of a left zero divisor and a non-left zero divisor (left regular).

Proposition 3.2. Let $(G, \leq)$ be a quasitotally ordered group, $\sigma: G \to Aut(R)$ and $\tau: G \times G \to U(R)$, the group of units in R. If R is a σ- compatible and left fusible ring, then $R((G; \sigma; \tau))$ is left fusible.

Proof. Let R be a left fusible ring and $0 \neq f \in R\big((G;\sigma;\tau)\big)$. Since, the order $\leq$ on G can be refined to a total ordered $\preccurlyeq$ on G, then there exists $0 \neq s_0 = \pi(f) \in G$ a minimal element in $supp\ f$ with respect to $\preccurlyeq$. Since R is a left fusible ring, then there exists $a \in Z_\ell(R)$ and $b \in Z_\ell^*(R)$ such that $f(s_0) = a + b$. Since $a \in Z_\ell(R)$, then there exists $d \in R$ such that $ad = 0$. Now consider $g, h \in R\big((G;\sigma;\tau)\big)$ such that

$g(s) = \begin{cases} a & \text{if} \quad s = s_0 \\ 0 & \text{otherwise} \end{cases}$ and $h(s) = \begin{cases} \mathrm{b} & \text{if} \quad s = s_0 \\ f(s) & \text{if} \quad s \neq s_0 \end{cases}$, consequently $f = g + h$.Since R is σ-compatible and $\tau(1, s_0)$ is an invertible element it follows that $0 = ad = g(s_0)d = g(s_0)\sigma_{s_0}(d) = g(s_0)\sigma_{s_0}(d)\tau(1, s_0) =$ $(gd\bar{1})(s_0)$. Hence $g \in Z_\ell(R((G;\sigma;\tau)))$. Now we prove that $h \in Z_\ell^*(R((G;\sigma;\tau)))$. To the contrary suppose that $h \in Z_\ell(R((G;\sigma;\tau)))$, so there exists $k \in R((G;\sigma;\tau))$ such that $hk = 0$. By hypothesis the order $\leq$ can be refined to a total ordered $\preccurlyeq$ on G. So, let $s' = \pi(h)$ be the minimal element in $supp\, h$. Hence, $0 = (hk)(s_0 + s') = h(s_0)\sigma_{s_0}k(s')\tau(s_0, s') = h(s_0)k(s') = bk(s')$, since R is σ-compatible and $\tau(s_0, s')$ is an invertible element which contradicts the fact that $b \in Z_\ell^*(R)$. Hence $h \in Z_\ell^*(R((G;\sigma;\tau)))$ and we get that $f = g + h$ where $g \in Z_\ell(R((G;\sigma;\tau)))$ and $h \in Z_\ell^*(R((G;\sigma;\tau)))$. Therefore $R((G;\sigma;\tau))$ is left fusible. A right ideal I of a ring R is said to be essential (or large), denoted

by $I \leq_e R_R$, if for every right ideal L of R, $I \cap L = 0$ implies that

$L = 0$. Following [7], the right singular ideal of a ring R is denoted by $Sing(R_R) = \{x \in R | r(x) \leq_e R\}$ where $r(x)$ denotes the right annihilator of x. A ring R is called right nonsingular if $Sing(R_R) = 0$. Similarly, we can define $Sing\big(R((G;\sigma;\tau))_{R((G;\sigma;\tau))}\big)$.

Corollary 3.3. Let $(G, \leq)$ a quasitotally ordered group, $\sigma: G \to Aut(R)$ and $\tau: G \times G \to U(R)$, the group of units in R. If R is a σ-compatible and left fusible ring, then $R((G;\sigma;\tau))$ is a right nonsingular ring.

Proof. By proposition 2.1, $R((G;\sigma;\tau))$ is a left fusible ring yielding that $R((G;\sigma;\tau))$is a right nonsingular ring by [7, proposition 2.11].

4. SA rings of Mal'cev–Neumann Series.

In this section, we study the right *IN* and right *SA* on rings of Mal'cev–Neumann series.

Definition 4.1. A ring R is said to be a right *SA*-ring, if for any two ideals I, J of R there is an ideal K of R such that $r(I) + r(J) = r(K)$, where $r(I)$ denotes the right annihilator of I.

Definition 4.2. A ring R is called a right *Ikeda–Nakayama* (for short, a right IN-ring) if the left annihilator of the intersection of any two right ideals is the sum of the left annihilators; that is, if $\ell(I \cap J) = \ell(I) + \ell(J)$for all right ideals $I, J\ of\ R$; and we say R is an IN-ring if R is a left and a right IN-ring (see [3],[12]).

Lemma 4.3. Let R be a ring, $(G, \leq)$ a quasitotally ordered group and I and J right ideals in R. Then $I((G;\sigma;\tau))$ and $J((G;\sigma;\tau))$ are right ideals of

$R((G;\sigma;\tau))$ such that $\ell_{R((G;\sigma;\tau))}\big(I((G;\sigma;\tau)) \cap J((G;\sigma;\tau))\big) = \ell_{R((G;\sigma;\tau))}\big((I \cap J)((G;\sigma;\tau))\big) = \ell_R(I \cap J)((G;\sigma;\tau))$.

Proof. It can be easily shown that $I\big((G;\sigma;\tau)\big) \cap J\big((G;\sigma;\tau)\big) = (I \cap J)\big((G;\sigma;\tau)\big)$. Hence, using [16, lemma 2.1] the Lemma follows.

Proposition 4.4. Let R be a σ- compatible ring, $(G, \leq)$ a totally ordered group. If $\Lambda = R((G;\sigma;\tau))$ is a right IN-ring, then so is R**.**

Proof. Let I and J be right ideals of R. Then $I((G;\sigma;\tau))$ and $J((G;\sigma;\tau))$ are right ideals of $\Lambda = R((G;\sigma;\tau))$, so by hypothesis, $\ell_\Lambda\big(I((G;\sigma;\tau)) \cap J((G;\sigma;\tau))\big) = \ell_\Lambda\big(I((G;\sigma;\tau))\big) + \ell_\Lambda\big(J((G;\sigma;\tau))\big)$. Hence $\ell_R(I \cap J)$ $((G;\sigma;\tau)) = \ell_\Lambda\big(I((G;\sigma;\tau))\big) + \ell_\Lambda\big(J((G;\sigma;\tau))\big)$ [16, lemma 2.1]. We need to prove that $\ell_R(I \cap J) = \ell_R(I) + \ell_R(J)$. It is clear that $\ell_R(I) + \ell_R(J) \subseteq \ell_R(I \cap J)$. Now let $a \in \ell_R(I \cap J)$.

Then $f = a\overline{1} \in \ell_{\Lambda}\big((I \cap J)((G;\sigma;\tau))\big)$. Hence by hypothesis there exists $h = \sum_{y\in G} b_y\,\overline{y} \in \ell_{\Lambda}\big(I((G;\sigma;\tau))\big) = \ell_R(I)((G;\sigma;\tau))$ by Lemma 2.3 and $g \in \sum_{z\in G} c_z\overline{z} \in \ell_{\Lambda}\big(J((G;\sigma;\tau))\big) = \ell_R(J)((G;\sigma;\tau))$ such that $f = a\overline{1} = h + g$. Hence there exist b and c such that $a = b + c$ where $b \in \ell_R(I)$ and $c \in \ell_R(J)$; thus $a \in \ell_R(I) + \ell_R(J)$ and consequently $\ell_R(I \cap J) = \ell_R(I) + \ell_R(J)$.

Theorem 4.5. The following statements hold:

(i) If $\Lambda = R((G;\sigma;\tau))$ is a σ- compatible and right SA-ring, then R is a right SA-ring;

(ii) If R is a G-Armendariz ring, then R is a right SA-ring if and only if $\Lambda = R((G;\sigma;\tau))$ is a right SA-ring.

Proof. (i) Let I and J be right ideals of R. Then $I((G;\sigma;\tau))$ and $J((G;\sigma;\tau))$ are right ideals of $R((G;\sigma;\tau))$. So there exist a right ideal K of $R((G;\sigma;\tau))$ such that $r_{\Lambda}\big(I((G;\sigma;\tau))\big) + r_{\Lambda}\big(J((G;\sigma;\tau))\big) = r_{\Lambda}(K)$. Now let $K_0 = \bigcup_{f\in K} C(f)$, then it follows that K_0 is a right ideal of R. We prove that $r_R(I) + r_R(J) = r_R(K_0)$. Suppose that $a \in r_R(I)$ and $b \in r_R(J)$. Then $f = a\overline{1} \in r_{\Lambda}\big(I((G;\sigma;\tau))\big) = r_R(I)((G;\sigma;\tau))$ and $g = b\overline{1} \in r_{\Lambda}\big(J((G;\sigma;\tau))\big) = r_R(J)((G;\sigma;\tau))$, so by hypothesis $f + g \in r_{\Lambda}(K)$. Then for each $h \in K$, $h\big(a\overline{1} + b\overline{1}\big) = 0$. So, $a + b = r(C(h)) \subseteq r_R(K_0)$. Therefore, $r_R(I) + r_R(J) \subseteq r_R(K_0)$. Now let, $c \in r_R(K_0)$. Then, $h = c\overline{1} \in r_{\Lambda}(K)$. By assumption there exists $f = \sum_{x\in G} a\,\overline{x} \in r_R(I)((G;\sigma;\tau))$ and $g = \sum_{y\in G} b\,\overline{y} \in r_R(J)((G;\sigma;\tau))$ such that, $f + g = h$. Hence there exist $a_x \in r_R(I)$ and $b_y \in$

$r_R(J)$ such that $a_x + b_y = c \in r_R(I) + r_R(J)$. So, $r_R(K_0) \subseteq r_R(I) + r_R(J)$. Consequently $r_R(K_0) = r_R(I) + r_R(J)$. Hence, R is a right SA-ring.

(ii) The necessity is evident by (i). Now, let R be a G-Armendariz and right SA-ring and I and J are right two ideals of $R((G;\sigma;\tau))$.

Let, $I_0 = \bigcup_{f\in I} C(f)$ and $J_0 = \bigcup_{f\in J} C(f)$ are right two ideals of R, where $C(f\ )$ denotes the content of f. Then, there exists an ideal K in R such that

$r_R(K) = r_R(I_0) + r_R(J_0)$. Now we prove that $r_\Lambda(I) + r_\Lambda(J) = r_\Lambda\big(K((G;\sigma;\tau))\big)$. It is sufficient to show that $r_\Lambda(I) + r_\Lambda(J) \subseteq r_\Lambda\big(K((G;\sigma;\tau))\big)$. Let $f = \sum_{x\in G} a_x\,\overline{x} \in r_\Lambda(I)$ and $g = \sum_{y\in G} b_y\,\overline{y} \in r_\Lambda(J)$, Then for each $h \in I$ and $\rho \in J$, $hf = 0$ and $\rho g = 0$. Since, R is G-Armendariz, then $h_k.a_x = 0$ and $\rho_z.b_y = 0$ for all $k \in supp(h)$, $x \in supp(f)$, $z \in supp(\rho)$ and $y \in supp(g)$. Therefore, $a_x \in r(I_0)$ and $b_y \in r(J_0)$.So $a_x + b_y \in r_R(I_0) + r_R(J_0) = r_R(K)$, i.e., there exists $c \in r_R(K)$ such that $a_x + b_y = c$, then for all $m \in G$, $\sum_{m\in G} a_x\overline{m} + \sum_{m\in G} b_y\overline{m} = \sum_{m\in G} c\,\overline{m}$ which implies that $f + g \in r_\Lambda\big(K((G;\sigma;\tau))\big)$. Thus $r_\Lambda(I) + r_\Lambda(J) \subseteq r_\Lambda\big(K((G;\sigma;\tau))\big)$. Therefore $r_\Lambda(I) + r_\Lambda(J) = r_\Lambda\big(K((G;\sigma;\tau))\big)$.

5. Σ-Zip Rings of Mal'cev–Neumann series

In this section, we investigate Σ_U –zip property in the ring Λ of Mal'cev- Neumann series.

Definition 5.1 [14, Definition 4.1]. Let $\sigma: S \to End(R)$ be a monoid homomorphism and U an ideal of R. We say that U is Σ-compatible if for each $a, b \in R$ and each $s \in S$, $ab \in U \leftrightarrow a\sigma_s(b) \in U$.

Definition 5.2 [14]. An ideal U of a ring R is called semiprime if for any $a \in R$, $a^n \in U$ implies $a \in U$.

Lemma 5.3 [14, Lemma 4.2]. Let $\sigma: S \to End(R)$ be a monoid homomorphism and U an ideal of R. If U is Σ-compatible, then for each $a, b \in R$ and each $s \in S$, $ab \in U \leftrightarrow \sigma_s(a)b \in U$.

Theorem 5.4. Let R be a ring, U a Σ-compatible semiprime ideal of R and G a totally ordered group, Then, R is Σ_U -zip ring if and only if $\Lambda = R((G;\sigma;\tau))$ is a $\Sigma_{U((G;\,\sigma;\,\tau))}$-zip ring.

Proof. Suppose that $\Lambda = R((G;\sigma;\tau))$is a $\Sigma_{U((G;\,\sigma;\,\tau))}$-zip and $Y \subseteq R$ such that $Y \nsubseteq U$ and $(U\colon Y) = U$. Let $\bar{Y} = \{y\bar{1} \,|\, y \in Y\} \subseteq R((G;\sigma;\tau))$. (we need to show that there exists a finite subset $Y_0 \subseteq Y$ such that $(U\colon Y_0) = U$). It is clear that $U((G;\sigma;\tau)) = \{f \in R((G;\sigma;\tau)) \,|\, f(y) \in U$ for each $y \in supp\, f\}$, is such that $U\big((G;\sigma;\tau)\big) \subseteq (U((G;\sigma;\tau))\colon \bar{Y})$ and it is sufficient to show that $(U((G;\sigma;\tau))\colon \bar{Y}) \subseteq U((G;\sigma;\tau))$. So, let $f \in (U\big((G;\sigma;\tau)\big)\colon \bar{Y})$, therefore $(\bar{y}f)(s) = y\bar{1}f(s) = y\sigma_1 f(s)\tau(1,s) = yf(s)\tau(1,s) \in U$, since $\tau(1,s)$ is an invertible element in R we conclude that $yf(s) \in U$. Hence, $f(s) \in U$ for each $s \in supp\, f$. Consequently $f \in U((G;\sigma;\tau))$ and it follows that $U\big((G;\sigma;\tau)\big) = (U((G;\sigma;\tau))\colon \bar{Y})$

Since $R((G;\sigma;\tau))$ is $\Sigma_{U((G;\,\sigma;\,\tau))}$-zip, it follows that there exists a finite subset $\bar{Y}_0 \subseteq \bar{Y}$ such that $(U((G;\sigma;\tau)){:}\,\bar{Y}_0) = U((G;\sigma;\tau))$. So, $Y_0 = \{y \in Y \mid y\bar{1} \in \bar{Y}_0\}$ is a finite subset of Y and we get $(U\big((G;\sigma;\tau)\big){:}\,\bar{Y}_0) \cap R = U((G;\sigma;\tau)) \cap R = U$. Therefore, R is a Σ_U -zip ring.

Conversely, suppose that R is a Σ_U -zip ring and $X \subseteq R((G;\sigma;\tau))$ such that $X \nsubseteq U((G;\sigma;\tau))$ and $(U((G;\sigma;\tau)){:}\,X) = U((G;\sigma;\tau))$. Let $C_X = \cup_{f\in X} C_f = \cup_{f\in X} \{f(s) | s \in supp\ f\} \subseteq R$ be the content of all element of X.

We need to show that $(U{:}\,C_X) = U$. It is clear that $U \subseteq (U{:}\,C_X)$. So, it is sufficient to show that $(U{:}\,C_X) \subseteq U$. Let $r \in (U{:}\,C_X)$. Then, $ar \in U$ for each $a \in C_X$. Since U is a Σ-compatible ideal, then $a\sigma_s(r) \in U$

for each $s \in G$. Hence for each $f \in X$ and $s \in G, (fr\bar{1})(s) = f(s)\sigma_s(r)\tau(1,s) \in U$. Therefore, $r\bar{1} \in (U((G;\sigma;\tau)){:}\,X) = U((G;\sigma;\tau))$. Hence, $r \in U$ and $(U{:}\,C_X) = U$ follows. Since R is a Σ_U -zip ring, then there exists a finite subset C_{X_0} of C_X such that $(U{:}\ C_{X_0}) = U$. Let, $X_0 = \{f \in X \mid f(s) \in C_{X_0}$ for some $s \in supp\ f\}$ be a minimal subset of X and it is clear that X_0 is finite. We need to show that $(U((G;\sigma;\tau)){:}\,X_0) = U((G;\sigma;\tau))$. It is clear that $U\big((G;\sigma;\tau)\big) \subseteq (U((G;\sigma;\tau)){:}\,X_0)$. So, it is sufficient to show that $(U((G;\sigma;\tau)){:}\,X_0) \subseteq U((G;\sigma;\tau))$. So, let $g \in (U((G;\sigma;\tau)){:}\,X_0)$, with v be the minimal element of $supp\ g$. Then for each $f \in X_0$, with the minimal element u of $supp\,f$, $(fg)(uv) = f(u)\sigma_u\big(g(v)\big)\tau(u,v) \in U$. Therefore, $f(u)\sigma_u\big(g(v)\big)\tau(u,v) = f(u)\sigma_u\left(g(v)\sigma_u^{-1}\big(\tau(u,v)\big)\right) \in U$.Since, $U$is a semiprime and Σ-compatible ideal, then $f(u)\left(g(v)\sigma_u^{-1}\left(\tau(u,v)\right)\right)$ and $\sigma_s\left(g(v)\sigma_u^{-1}\left(\tau(u,v)\right)\right)f(u) \in U$ for each $s \in G$.

Now, suppose that $w \in G$ is such that for each $u \in supp\ f$ and each $v \in supp\ g$, such that $uv < w$, $\sigma_s\left(g(v)\sigma_u^{-1}\left(\tau(u,v)\right)\right)f(u) \in U$ for each $s \in G$. Using transfinite induction, we need to show that $f(u_i)\sigma_{u_i}g(v_i)\tau(u_i,v_i) \in U$ for each $uv = w$. Since,$X_w(f,g) = \{(u,v) \in G \times G | uv = w, u \in suppf$ and $v \in supp\ g\}$is a finite set and G totally ordered then, let $\{u_i, v_i, i = 1,2,3,\dots,n\}$ be such that$u_1 < u_2 \dots\dots < u_n$ and $v_n < v_{n-1} \dots\dots < v_1$. Hence, $(fg)(w) = \sum_{\substack{(u,v)\in X_w(f,g)\\ i=1}}^{n} f(u_i)\sigma_{u_i}\left(g(v_i)\right)\tau(u_i,v_i) =$

$$f(u_1)\sigma_{u_1}\left(g(v_1)\right)\tau(u_1,v_1) + \cdots + f(u_n)\sigma_{u_n}(g(v_n))\tau(u_n,v_n) = a_1 \in U (1)$$

Since, for each$u_i, i \geq 2$, $u_1v_i < u_iv_i = w$, then using induction hypothesis, we have $\sigma_{u_i}\left(g(v_i)\sigma_{u_i}^{-1}\left(\tau(u_i,v_i)\right)\right)f(u_1) \in U$. Multiplying (1) on the right by$f(u_1)$,then we have $fg(w)f(u_1) = f(u_1)\sigma_{u_1}\left(g(v_1)\right)\ \tau(u_1,v_1)f(u_1) + f(u_2)\sigma_{u_2}\left(g(v_2)\sigma_{u_2}^{-1}(\tau(u_2,v_2)\right)f(u_1) + \cdots +$
$f(u_n)\sigma_{u_n}\left(g(v_n)\sigma_{u_n}^{-1}\left(\tau(u_n,v_n)\right)\right)f(u_1) = a_1 f(u_1) \in U$. Thus, we obtain $f(u_1)\sigma_{u_1}\left(g(v_1)\sigma_{u_1}^{-1}\left(\tau(u_1,v_1)\right)\right)f(u_1) \in U$. Since U is semiprime it follows that $f(u_1)\sigma_{u_1}(g(v_1))\ \tau(u_1,v_1) \in U$. Now, subtract $f(u_1)\sigma_{u_1}(g(v_1))\ \tau(u_1,v_1)$ and multiply by $f(u_2)$ from the right of both sides of (1), it follows that $(fg)(w)f(u_2) - f(u_1)\sigma_{u_1}(g(v_1))\ \tau(u_1,v_1)f(u_2) = f(u_2)\sigma_{u_2}(g(v_2))\ \tau(u_2,v_2)f(u_2) + \cdots + f(u_n)\sigma_{u_n}(g(v_n))\ \tau(u_n,v_n)f(u_2)$=

$a_1 f(u_2) - f(u_1)\sigma_{u_1}(g(v_1))\,\tau(u_1, v_1)f(u_2) \in U$. Using the same argument as above we obtain $f(u_2)\sigma_{u_2}(g(v_2))\,\tau(u_2, v_2) \in U$. Continuing this process, we can show that $f(u_i)\sigma_{u_i}(g(v_i))\,\tau(u_i, v_i) \in U$ for each $i = 1,2,3, \dots, n$ such that $u_i v_i = w$. Thus, $f(u)\sigma_u(g(v))\tau(u, v) \in U$ for each

$u \in suppf$ and $v \in suppg$. Hence $g \in U((G; \sigma; \tau))$ and it follows that $(U((G; \sigma; \tau)): X_0) \subseteq U((G; \sigma; \tau))$. Consequently, we deduce that $R((G; \sigma; \tau))$is a $\Sigma_{U((G;\,\sigma;\,\tau))}$-zip ring.

In the following we give some examples of Σ_{U}- zip ring

Example 5.5. Let $R = Z_4$ be the ring of integer modulo 4 and $U =< 2 >$ the ideal generated by 2, then R is a Σ_{U}- zip ring since $(U: X) = U$ for each subset $X \nsubseteq U$. If $U = \{0\}$, then R is Σ_0- zip as well as zip.

Example 5.6. Let $R = T(Z_4, Z_4) \cong \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, a, b \in Z_4\right\}$ the trivial extension of Z_4. We can write the proper ideal of R as $U = \left\{\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}: m \in Z_4\right\}$ let X be any subset of R with $X \nsubseteq U$, then R is a Σ_{U}- zip ring since $(U: X) = U$ and for any subset X_0 of X, we have $(U: \mathrm{X}_0) = U$ by a routine computation. So R is Σ_{U}- zip.

References

[1] Beachy J. A. and Blair W. D. (1975). Rings whose faithful left ideals are cofaithful. *Pacific J. Math*. 58, 1–13.

[2] Birkenmeier G. F., Ghirati M. and Taherifar A. (2015).When is a Sum of Annihilator Ideals an Annihilator Ideal, Comm. Algebra 43, 2690-2702.

[3] Camillo V., Nicholson W. K. and Yousif M. F. (2000). Ikeda- Nakayama rings. J. Algebra 266, 1001-1010.

[4] Cedo F. (1991).Zip rings and Mal’cev domains. Comm. Algebra 19, 1983–1991.

[5] Cortes W. (2008). Skew polynomial extensions over zip rings. Int. J. Math. Sci. 10, 1–8.

[6] Faith C. (1989). Rings with zero intersection property on annihilators: zip rings. Publ. Math. 33, 329–332.

[7] Ghashghaei E. and Mcgovern W. W. (2017). Fusible rings. Comm. Algebra 45, 1151–1165.

[8] Hong C. Y., Kim N. K., Kwak T. K. and Lee Y. (2005).Extension of zip rings. J. Pure Appl. Algebra 195, 231–242.

[9] Lorenz E. M. (1983). Division algebras generated by finitely generated nilpotent groups. J. Algebra 85,368–381.

[10] Makar L.-Limanov(1983). The skew field of fractions of the Weyl algebra contains a free noncommutative subalgebra, Comm. Algebra 11, 2003–2006.

[11] Musson I. M. and Stafford K. (1993). Mal’cev-Neumann group rings. Comm. Algebra 21, 2065–2075.

[12] Nicholson W. K. and Yousif M. F. (2003). Quasi-Frobenius Rings, Cambridge University Press.

[13] Ouyang L.(2009). Ore extensions of weak zip rings, Glasg. Math. J. 51, 525–537.

[14] Ouyang L., Zhou Q. and Jinfang Wu. (2017). Extensions of Σ-Zip rings. Int. Elec. J. Algebra 21,1-22.

[15] Salem R. M. (2013). Generalized power series over zip and weak zip rings. Southeast Asian Bull. Math. 37,259–268.

[16] Salem R. M. (2012).Mal'cev-Neumann series over zip and Weak zip rings. Asian- European J. Math. 4,1250058-1250066.

[17] Sonin C. (1998). Krull dimension of Mal'cev-Neumann rings, Comm. Algebra 26,2915–2931.

[18] Zelmanowitz J. M. (1976).The finite intersection property on annihilator right ideals. Proc. Amer. Math. Soc. 57,213–216.

[19] Zhao R.and Liu Z. (2008). On some properties of Mal'cev-Neumann modules. Bull. Korean Math. Soc. 45,445–456.