

Slopes of fibrations with trivial vertical fundamental groups

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Abstract

Kodaira fibrations have non-trivial vertical fundamental groups and their slopes are all 12. In this paper, we show that 12 is indeed the sharp upper bound for the slopes of fibrations with trivial vertical fundamental groups. Precisely, for each $g \geq 3$ we prove the existence of fibrations of genus g with trivial vertical fundamental groups whose slopes can be arbitrarily close to 12. This gives a relative analogy of Roulleau-Urúza's work [RU15] on the slopes of surfaces of general type with trivial fundamental groups.

1 Introduction

We work over the complex number \mathbb{C} . Let X be a smooth minimal projective surface of general type. Denote by $c_1^2(X), c_2(X)$ the two Chern numbers of X . It is well-known that $c_1^2(X) = K_X^2$ is the self-intersection number of the canonical divisor K_X , and that $c_2(X) = \chi_{\text{top}}(X)$ is the Euler number of X . Moreover, they satisfy the following Noether equality:

$$c_1^2(X) + c_2(X) = K_X^2 + \chi_{\text{top}}(X) = 12\chi(\mathcal{O}_X), \quad (1.1)$$

where $\chi(\mathcal{O}_X)$ is the Euler characteristic of the structure sheaf \mathcal{O}_X . A fundamental problem in the study of the geography of surfaces of general type is: for which pair of integers (x, y) , there exists a smooth minimal projective surface X of general type such that $(c_1^2(X), c_2(X)) = (x, y)$? By (1.1), this is equivalent to describe all the possible values $(K_X^2, \chi(\mathcal{O}_X))$. There is a long history to this problem and we refer to [Pe81, Pe87] and [BHPV04, VII §8] for an introduction. First, both K_X^2 and $\chi(\mathcal{O}_X)$ are positive integers and satisfy the following restrictions (Noether's inequality and Miyaoka-Yau's inequality):

$$2\chi(\mathcal{O}_X) - 6 \leq K_X^2 \leq 9\chi(\mathcal{O}_X).$$

Sommese [So84] proved that every rational number between 2 and 9 can be realized as the quotient $K_X^2/\chi(\mathcal{O}_X)$ (will be called the slope of X for convenience) of some surface X .

The situation becomes subtle when the simple connectedness condition is imposed on the surface X . Persson [Pe81] constructed a series of examples showing that all the rational

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numbers between 2 and 8 occur as the slopes of simply connected surfaces. According to the characterization of the Miyaoka-Yau equality, any surface of general type with $K_X^2 = 9\chi(\mathcal{O}_X)$ is a ball quotient and hence can not be simply connected. Simply connected surfaces of general type with nonnegative index (i.e., its slope ≥ 8) seemed difficult to construct. In fact, the Watershed conjecture predicted that any simply connected surface of general type has negative index (i.e., its slope < 8). Nevertheless, Moishezon-Teicher [MT87] constructed the first example of simply connected surfaces with positive index, and Chen [Ch87] found simply connected surfaces with slopes up to 8.75. More recently in the beautiful work [RU15], Roulleau-Urzúa proved that the slopes of simply connected surfaces are dense in $[8, 9)$. In particular, the slopes of simply connected surfaces of general type can be arbitrarily close to 9.

We are interested in the analogy of the geography problem for surface fibrations. Let $f: X \rightarrow C$ be a relatively minimal surface fibration whose general fiber is of genus $g \geq 2$. We consider the following relative invariants:

$$\begin{cases} K_f^2 = K_X^2 - 8(g-1)(g(C)-1), \\ \chi_f = \chi(\mathcal{O}_X) - (g-1)(g(C)-1), \\ e_f = \chi_{\text{top}}(X) - 4(g-1)(g(C)-1). \end{cases}$$

These invariants are non-negative integers satisfying the following properties:

- (1). $K_f^2 = 0 \iff \chi_f = 0$, if and only if f is locally trivial.
- (2). $e_f = 0$ if and only if f is smooth.
- (3). The Noether equality holds: $12\chi_f = K_f^2 + e_f$.

Analogously, one may ask a similar geography problem for surface fibrations (cf. [AK02, § 1.1]): *for which pair of integers (x, y) , there exists a relatively minimal surface fibration $f: X \rightarrow C$ such that $(K_f^2, \chi_f) = (x, y)$?* For a locally non-trivial fibration f , the *slope* of f is defined as $\lambda_f = K_f^2/\chi_f$, and it satisfies:

$$\frac{4(g-1)}{g} \leq \lambda_f \leq 12.$$

The first inequality is the slope inequality [Xi87, CH88], and the equality can be reached by hyperelliptic fibrations; the second inequality follows from the non-negativity of e_f , and the equality can be reached by Kodaira fibrations whenever $g \geq 3$.

Xiao started to consider the above geography problem for surface fibrations around 1980s, where he got partial answers for fibrations of genus $g = 2$, cf. [Xi85, Theorem 2.9] and [Xi92, Theorem 4.3.5]. Motivated by Xiao's work, Chen [Ch87] generalized it to hyperelliptic fibrations, based on which Chen constructed many simply connected surfaces with positive index as mentioned before. Recently, Liu-Lu [LL24] proved that any rational number $r \in [4(g-1)/g, 12]$ can be realized as the slope of a fibration of genus g whenever $g > 3$, which gives an analogy of Sommese's result [So84] in the relative version.

We would like to take the fundamental group into consideration. Recall that the fundamental group of the fibered surface X can be divided into two parts with the following

exact sequence (cf. [Xi91]):

$$1 \longrightarrow \mathcal{V}_f \longrightarrow \pi_1(X) \longrightarrow \mathcal{H}_f \longrightarrow 1,$$

where \mathcal{V}_f is called the *vertical fundamental group* of the fibration f , see Section 2.3 for a more precise definition. Xiao [Xi87] proved that the vertical fundamental group \mathcal{V}_f is trivial if f is non-hyperelliptic with $\lambda_f < 4$. It is not difficult [Xi91] to construct surface fibrations with lower slope admitting a trivial vertical fundamental group. The situation is different if the slope λ_f is large. In the extreme case when $\lambda_f = 12$, the vertical fundamental group \mathcal{V}_f can never be trivial, cf. Lemma 2.3. Hence it is natural to wonder: is 12 the sharp upper bound for the slopes of fibrations with trivial vertical fundamental groups? Our main purpose is to answer this question affirmatively, which gives a relative analogy of Roulleau-Urzuía's work [RU15] on the slopes of surfaces of general type with trivial fundamental groups.

Theorem 1.1. *For each $g \geq 3$, there exists a sequence of fibrations $f_n: X_n \rightarrow C_n$ of genus g such that $\mathcal{V}_{f_n} = \{1\}$ and $\lim_{n \rightarrow \infty} \lambda_{f_n} = 12$.*

Different from the constructions (using cyclic coverings) in [Ch87, LL24], our construction is based on the moduli spaces and the Torelli map. As a byproduct of our method, we can construct more examples with given slopes in the case of $g = 3$.

Theorem 1.2. *For each rational number $r \in [\frac{8}{3}, 12)$, there exists a fibration f of genus 3 such that $\mathcal{V}_f = \{1\}$ and $\lambda_f = r$.*

We tend to believe this should be true for each $g > 3$; see Remark 3.2. Combining with the results in [LL24], we get the following.

Corollary 1.3. *For each $g \geq 2$ and each rational number $r \in [4(g-1)/g, \lambda_M(g)]$, there exists a fibration of genus g with slope r , where*

$$\lambda_M(g) := \begin{cases} 7, & \text{if } g = 2, \\ 12, & \text{if } g \geq 3. \end{cases}$$

Remark 1.4. (1). If f is hyperelliptic, it is known [Xi92] that $\lambda_f \leq \lambda_M^h(g)$ and the upper bounds can be reached (see [Mo98, LT13]), where

$$\lambda_M^h(g) := \begin{cases} 12 - \frac{8g+4}{g^2}, & \text{if } g \text{ is even,} \\ 12 - \frac{8g+4}{g^2-1}, & \text{if } g \text{ is odd.} \end{cases}$$

When $g = 2$, the fibration f is always hyperelliptic, and hence $\lambda_f \leq 7$ in this case. It is well-known that there exist Kodaira fibrations of genus g for every $g \geq 3$. Hence $\lambda_M(g)$ is indeed the sharp upper bound of the slopes of fibrations of genus g .

(2). The hyperelliptic fibrations constructed in [LL24] are in fact with trivial vertical fundamental group. In other words, these examples show that for each genus $g \geq 2$ and each rational number $r \in (4(g-1)/g, \lambda_M^h(g))$, there exists a fibration f of genus g satisfying that $\mathcal{V}_f = \{1\}$ and $\lambda_f = r$, see Remark 2.5. However, the vertical fundamental groups of non-hyperelliptic fibrations constructed in [LL24] seem far from being trivial.

2 Preliminaries

2.1 Moduli spaces and the Torelli map

Let \mathcal{M}_g (resp. \mathcal{H}_g) be the moduli space of smooth projective curves (resp. hyperelliptic curves) of genus $g \geq 2$, and \mathcal{A}_g be the moduli space of g -dimensional principally polarized abelian varieties. Recall that the Torelli morphism

$$j: \mathcal{M}_g \longrightarrow \mathcal{A}_g,$$

which associates to a curve its Jacobian with the canonical principal polarization. The Torelli morphism j is injective; in fact it is an immersion [OS79]. Let $\overline{\mathcal{M}}_g = \overline{\mathcal{M}}_g^{DM}$ be the Deligne-Mumford compactification of \mathcal{M}_g , and $\overline{\mathcal{A}}_g = \overline{\mathcal{A}}_g^S$ be the Satake compactification of \mathcal{A}_g . The boundary $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ consists of several divisors Δ_i ($0 \leq i \leq [g/2]$); while the boundary $\overline{\mathcal{A}}_g \setminus \mathcal{A}_g$ is of codimension at least two. In fact,

$$\overline{\mathcal{A}}_g \setminus \mathcal{A}_g = \bigcup_{0 \leq i \leq g-1} \mathcal{A}_i.$$

It turns out that the Torelli map j extends to a map from $\overline{\mathcal{M}}_g$ to $\overline{\mathcal{A}}_g$, which is still denoted by j . The extended Torelli map is no longer injective: it collapses the boundary divisors. When $g \geq 3$, the boundary $j(\overline{\mathcal{M}}_g) \setminus j(\mathcal{M}_g)$ is of codimension at least two, cf. [DM18].

In order to assure the representability, it is necessary to take certain level structure into consideration. Fixing $l \geq 3$ an integer, let $\mathcal{M}_{g,[l]}$ (resp. $\mathcal{A}_{g,[l]}$ and so on) be the corresponding moduli space with full level- l structure. No specific choice of the level $l (\geq 3)$ is made because it is only imposed to assure the representability, which plays no essential role in our study. The Torelli map j can be similarly defined over the moduli spaces with level structure. However, the Torelli map

$$j: \mathcal{M}_{g,[l]} \longrightarrow \mathcal{A}_{g,[l]}$$

is no longer an immersion, but a two-to-one map ramified exactly over the locus of hyperelliptic curves $\mathcal{H}_{g,[l]} \subseteq \mathcal{M}_{g,[l]}$, cf. [OS79]. As the level $l \geq 3$, there is a universal family of smooth curves (resp. stable curves) over \mathcal{M}_g (resp. $\overline{\mathcal{M}}_g$) with the following commutative diagram, cf. [Po77, Theorem 10.9].

$$\begin{array}{ccc} \mathcal{S}_{g,[l]} & \hookrightarrow & \overline{\mathcal{S}}_{g,[l]} \\ \downarrow f & & \downarrow \bar{f} \\ \mathcal{M}_{g,[l]} & \hookrightarrow & \overline{\mathcal{M}}_{g,[l]} \end{array}$$

The following lemma can be found in [LZ19, Lemma A.1].

Lemma 2.1. *There exists an involution σ_g (resp. τ_g) on $\mathcal{S}_{g,[l]}$ (resp. $\mathcal{M}_{g,[l]}$) such that the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{S}_{g,[l]} & \xrightarrow{\sigma_g} & \mathcal{S}_{g,[l]} \\ \downarrow f & & \downarrow f \\ \mathcal{M}_{g,[l]} & \xrightarrow{\tau_g} & \mathcal{M}_{g,[l]} \end{array}$$

Moreover, $j \circ \tau_g(x) = j(x)$ for any $x \in \mathcal{M}_{g,[l]}$, the fixed locus of τ_g is exactly the hyperelliptic locus $\mathcal{H}_{g,[l]} \subseteq \mathcal{M}_{g,[l]}$, and for $p \in \mathcal{H}_{g,[l]}$, $\sigma_g|_{F_p}: F_p \rightarrow F_p$ is the hyperelliptic involution of F_p , where $F_p \subseteq \mathcal{S}_{g,[l]}$ is the hyperelliptic curve over p .

2.2 Modular invariants

Let $f: X \rightarrow C$ be a surface fibration (or simply fibration), i.e., f is a proper surjective morphism from a smooth projective surface onto a smooth projective curve with connected fibers. Denote by g the genus of a general fiber of f . We will always assume that f is relatively minimal, i.e., there is no (-1) -curve contained in fibers of f . If $\pi: \tilde{C} \rightarrow C$ is a base change of degree d , then the *pullback fibration* $\tilde{f}: \tilde{X} \rightarrow \tilde{C}$ of f with respect to π is defined as the relatively minimal model of the desingularization of $X \times_C \tilde{C} \rightarrow \tilde{C}$. Denote by F the singular fiber of f over $p = f(F) \in C$. If π is totally ramified over p , then the fiber \tilde{F} of the pullback fibration \tilde{f} over $\pi^{-1}(p)$ is called *d -th root model* of F . If π is ramified over p and some non-critical points of f such that the fibers of \tilde{f} over $\pi^{-1}(p)$ are all semistable, then the *Chern numbers* of F are defined as follows (see [Ta94, Ta96]),

$$c_1^2(F) = K_f^2 - \frac{1}{d}K_{\tilde{f}}^2, \quad c_2(F) = e_f - \frac{1}{d}e_{\tilde{f}}, \quad \chi_F = \chi_f - \frac{1}{d}\chi_{\tilde{f}}.$$

The fibration f induces a moduli map $J: C \rightarrow \overline{\mathcal{M}}_g$ from C to the moduli space $\overline{\mathcal{M}}_g$. Let λ be the Hodge divisor class of $\overline{\mathcal{M}}_g$, δ be the boundary divisor class, and $\kappa = 12\lambda - \delta$ be the first Morita-Mumford class. Then there are three fundamental *modular invariants* of f defined as follows (see [Ta10]),

$$\kappa(f) = \deg J^* \kappa, \quad \lambda(f) = \deg J^* \lambda, \quad \delta(f) = \deg J^* \delta.$$

If f is semistable, then

$$\kappa(f) = K_f^2, \quad \delta(f) = e_f, \quad \lambda(f) = \chi_f. \quad (2.1)$$

These modular invariants satisfy the *base change property*, i.e., if \tilde{f} is the pullback fibration of f with respect to a base change of degree d , then

$$\kappa(\tilde{f}) = d \cdot \kappa(f), \quad \delta(\tilde{f}) = d \cdot \delta(f), \quad \lambda(\tilde{f}) = d \cdot \lambda(f). \quad (2.2)$$

It is proved in [Ta94, Ta96] that

$$\begin{cases} K_f^2 = \kappa(f) + \sum_{i=1}^s c_1^2(F_i), \\ e_f = \delta(f) + \sum_{i=1}^s c_2(F_i), \\ \chi_f = \lambda(f) + \sum_{i=1}^s \chi_{F_i}, \end{cases} \quad (2.3)$$

where F_1, \dots, F_s are all the singular fibers of f .

Example 2.2. Let F_g^h be the singular hyperelliptic fiber of genus g with the following dual graph, where $\begin{smallmatrix} n \\ \circ \\ -e \end{smallmatrix}$ denotes a smooth rational curve with self-intersection number $(-e)$ and multiplicity n in F_g^h . For convenience, we omit the subscript $(-e)$ whenever $e = 2$.

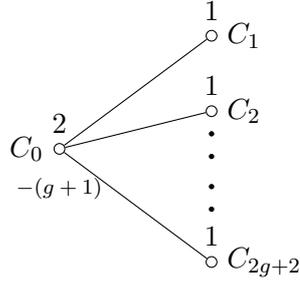


Figure 1: Hyperelliptic fiber F_g^h .

Then F_g^h is simply connected [Pe81, Lemma A], and the Chern numbers of F_g^h are as follows (see also [LL24, Example 2.4])

$$c_1^2(F_g^h) = 2g - 2, \quad \chi_{F_g^h} = \frac{g}{2}. \quad (2.4)$$

Let \tilde{F}_g^h be the d -th root model of F_g^h . Since

$$F_g^h = \sum_{i=0}^{2g+2} n_i C_i = 2C_0 + C_1 + \cdots + C_{2g+2}$$

is normal crossing, by [LuT13, Section 2.2], the multiplicity of the strict transform of C_i in \tilde{F}_g^h is $n_i / \gcd(n_i, d)$ and $\tilde{F}_g^h = F_g^h$ when d is odd.

2.3 The fundamental group

In this subsection, we recall some general facts about the fundamental group of a surface fibration. Let $f: X \rightarrow C$ be a fibration of genus g , and F be any general fiber. According to [Xi91, Lemma 1], the embedding of F in X induces a homomorphism

$$\alpha: \pi_1(F) \longrightarrow \pi_1(X),$$

whose image is a normal subgroup of $\pi_1(X)$ and independent of the choice of the general fiber F . The image of α , denoted by \mathcal{V}_f , is called *the vertical fundamental group* of X . Let $\mathcal{H}_f := \pi_1(X)/\mathcal{V}_f$ be the quotient group, then one obtains the following exact sequence:

$$1 \longrightarrow \mathcal{V}_f \longrightarrow \pi_1(X) \longrightarrow \mathcal{H}_f \longrightarrow 1.$$

The next property is more or less well-known.

Lemma 2.3. *Suppose that f is a Kodaira fibration of genus $g \geq 3$, then the vertical fundamental group \mathcal{V}_f is non-trivial.*

Proof. In this case, f admits no singular fiber, and hence it is a topological fiber bundle. It follows that the induced homomorphism from $\pi_1(F)$ to $\pi_1(X)$ is injective with the following exact sequence:

$$1 \longrightarrow \pi_1(F) \xrightarrow{\alpha} \pi_1(X) \longrightarrow \pi_1(C) \rightarrow 1.$$

In particular, the vertical fundamental group \mathcal{V}_f is non-trivial. \square

The next lemma, due to Xiao [Xi91, Lemma 3], provides a useful way to determine whether $\mathcal{V}_f = \{1\}$.

Lemma 2.4. *Let F_0 be any fiber of $f: X \rightarrow C$ and $\alpha_0: \pi_1(F_0) \rightarrow \pi_1(X)$ the induced homomorphism, then $\mathcal{V}_f \subseteq \text{Im}(\alpha_0)$. In particular, \mathcal{V}_f is trivial if f admits a simply connected fiber.*

Remark 2.5. It is shown in [LL24] that for each $g \geq 2$ and each rational number $r \in (4(g-1)/g, \lambda_M^h(g))$, there exists a hyperelliptic fibration $f_{g,r}$ of genus g with $\lambda_{f_{g,r}} = r$. By the construction of $f_{g,r}$ in the proof of [LL24, Theorem 1.3], we know that $f_{g,r}$ has at least one singular fiber F_g^h , which is simply connected by Example 2.2. Hence the vertical fundamental group of $f_{g,r}$ is trivial by Lemma 2.4.

3 The construction

In this section, we mainly aim to construct surface fibrations with trivial vertical fundamental groups whose slopes tend to 12.

Proof of Theorem 1.1. We use the notations and terminology introduced in Section 2. Let $L_1, L_2, \dots, L_{3g-4}$ be very ample divisors on the Satake compactification $\overline{\mathcal{A}}_{g,[l]} = \overline{\mathcal{A}}_{g,[l]}^S$, where the level $l \geq 3$. Since $\dim \mathcal{M}_g = 3g - 3$ and the boundary $j(\overline{\mathcal{M}}_g) \setminus j(\mathcal{M}_g)$ is of codimension at least two, the intersection $L_1 L_2 \cdots L_{3g-4}$ is a complete smooth curve $C \subseteq j(\mathcal{M}_g)$. Moreover, by the very ampleness of these L_i 's, we may assume that the curve C intersects $j(\mathcal{H}_{g,[l]})$ transversely with $C \cap j(\mathcal{H}_{g,[l]}) \neq \emptyset$.

Let $B = j^{-1}(C) \subseteq \mathcal{M}_{g,[l]}$ be the inverse image of C . Note that the Torelli morphism

$$j: \mathcal{M}_{g,[l]} \longrightarrow \mathcal{A}_{g,[l]}$$

is two-to-one and ramified exactly over the hyperelliptic locus $\mathcal{H}_{g,[l]}$. As C intersects $j(\mathcal{H}_{g,[l]})$ transversely with $C \cap j(\mathcal{H}_{g,[l]}) \neq \emptyset$, it follows that B is smooth and irreducible with $B \cap \mathcal{H}_{g,[l]} \neq \emptyset$. Let $h: X \rightarrow B$ be the family of smooth curves induced from the universal family $\mathfrak{f}: \mathcal{S}_{g,[l]} \rightarrow \mathcal{M}_{g,[l]}$. According to Lemma 2.1, there is an involution σ (resp. τ) on X (resp. B) such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ h \downarrow & & \downarrow h \\ B & \xrightarrow{\tau} & B \end{array}$$

Moreover, the quotient $B/\langle \tau \rangle \cong C$, and $\sigma|_{\Gamma_p}: \Gamma_p \rightarrow \Gamma_p$ is the hyperelliptic involution of Γ_p for any fixed point $p \in \text{Fix}(\tau) \subseteq B$, where $\Gamma_p = h^{-1}(p)$ is the fiber over p .

Let $f_0: X_0 \rightarrow C \cong B/\langle \tau \rangle$ be the relatively minimal model of the quotient fibred surface $X/\langle \sigma \rangle \rightarrow C \cong B/\langle \tau \rangle$. By our construction, the possible singular fibers of f_0 are those corresponding to Γ_p with $p \in \text{Fix}(\tau) \subseteq B$. As the restricted map

$$\sigma|_{\Gamma_p}: \Gamma_p \rightarrow \Gamma_p$$

is the hyperelliptic involution of Γ_p , its image $\Pi(\Gamma_p)$ is a rational curve with $2g + 2$ singularities of $X/\langle\sigma\rangle$ on $\Pi(\Gamma_p)$, where $\Pi: X \rightarrow X/\langle\sigma\rangle$ is the quotient map. By resolving the singularities, it follows that the corresponding fiber $F_{j(p)} = f_0^{-1}(j(p))$ in X_0 is isomorphic to F_g^h , which is described in Example 2.2 for any $p \in \text{Fix}(\tau)$. Let F_1, \dots, F_s be all the singular fibers of f_0 , then by (2.3) and (2.4) we have that

$$K_{f_0}^2 = \kappa(f_0) + \sum_{i=1}^s c_1^2(F_i) = \kappa(f_0) + (2g - 2)s,$$

$$\chi_{f_0} = \lambda(f_0) + \sum_{i=1}^s \chi_{F_i} = \lambda(f_0) + \frac{1}{2}gs.$$

Note that h is a Kodaira fibration. By (2.1) and (2.2),

$$\kappa(f_0) = \frac{1}{2}\kappa(h) = \frac{1}{2}K_h^2, \quad \text{and} \quad \lambda(f_0) = \frac{1}{2}\lambda(h) = \frac{1}{2}\chi_h.$$

Thus

$$\frac{\kappa(f_0)}{\lambda(f_0)} = \lambda_h = 12.$$

Let $p_1 = f_0(F_1), \dots, p_s = f_0(F_s)$. Then $\{p_1, \dots, p_s\}$ are just the ramification divisor of the induced double cover

$$j|_B: B \rightarrow C.$$

In particular, s is even and $s \geq 2$ since we have assume that $s \neq 0$. Let

$$\pi_n: C_n \rightarrow C$$

be a cyclic cover of degree $2n + 1$ branched exactly over $\{p_1, \dots, p_s\}$ with ramification indices being all equal to $2n + 1$. Such a cyclic cover exists. Indeed, take positive integers $0 < m_i < 2n + 1$, $1 \leq i \leq s$, such that

- (i). $\gcd(m_i, 2n + 1) = 1$ for any $1 \leq i \leq s$;
- (ii). $\sum_{i=1}^s m_i$ is a multiple of $2n + 1$.

Then the divisor $R := \sum_{i=1}^s m_i p_i$ is $(2n + 1)$ -divisible, i.e., there exists a line bundle L such that $\mathcal{O}(R) \sim L^{\otimes(2n+1)}$, where ‘ \sim ’ stands for the linear equivalence. Then the relation $\mathcal{O}(R) \sim L^{\otimes(2n+1)}$ defines a cyclic cover $\pi_n: C_n \rightarrow C$ satisfying our requirements. Using such a cyclic cover π_n to do the base change, let $f_n: X_n \rightarrow C_n$ be the pullback fibration of f_0 , and $F_{i,n} \subseteq X_n$ be the corresponding fiber of F_i for $1 \leq i \leq s$. Then by Example 2.2,

$$F_{i,n} \cong F_i \cong F_g^h, \quad \forall 1 \leq i \leq s.$$

Hence by (2.2), (2.3) and (2.4),

$$K_{f_n}^2 = \kappa(f_n) + \sum_{i=1}^s c_1^2(F_{i,n}) = (2n + 1)\kappa(f_0) + (2g - 2)s,$$

$$\chi_{f_n} = \lambda(f_n) + \sum_{i=1}^s \chi_{F_{i,n}} = (2n + 1)\lambda(f_0) + \frac{1}{2}gs.$$

Thus the slope of f_n is

$$\lambda_{f_n} = \frac{K_{f_n}^2}{\chi_{f_n}} = \frac{(2n+1)\kappa(f_0) + (2g-2)s}{(2n+1)\lambda(f_0) + \frac{1}{2}gs}.$$

When n goes to infinity, the slope of f_n tends to 12 as required. Moreover, the vertical fundamental group \mathcal{V}_{f_n} is trivial by Lemma 2.4, since f_n admits a singular fiber isomorphic to F_g^h , which is simply connected by Example 2.2. \square

Applying the above construction to the case $g = 3$, we can prove Theorem 1.2.

Proof of Theorem 1.2. We follow the notations introduced in the proof of Theorem 1.1. Take $g = 3$, and let $h: X \rightarrow B$ be the Kodaira fibration of genus $g = 3$ constructed in the proof of Theorem 1.1. Let $f_0: X_0 \rightarrow C \cong B/\langle\tau\rangle$ be the relatively minimal model of the quotient fibred surface $X/\langle\sigma\rangle \rightarrow C \cong B/\langle\tau\rangle$. Then it is known that

$$\begin{aligned} K_{f_0}^2 &= \kappa(f_0) + \sum_{i=1}^s c_1^2(F_i) = \kappa(f_0) + 4s, \\ \chi_{f_0} &= \lambda(f_0) + \sum_{i=1}^s \chi_{F_i} = \lambda(f_0) + \frac{3}{2}s, \end{aligned}$$

and that

$$\kappa(f_0) = \frac{1}{2}\kappa(h) = \frac{1}{2}K_h^2, \quad \lambda(f_0) = \frac{1}{2}\lambda(h) = \frac{1}{2}\chi_h, \quad \text{with } \frac{K_h^2}{\chi_h} = 12.$$

Here s is the number of singular fibers of f_0 . Let δ_i ($i = 0, 1$) be the divisor class of Δ_i of $\overline{\mathcal{M}}_3 \setminus \mathcal{M}_3$, and \mathfrak{h} be the divisor class of $\overline{\mathcal{H}}_3^{DM}$. By the formulas (see [HM82])

$$\kappa = \frac{1}{3}\delta_0 + 3\delta_1 + \frac{4}{3}\mathfrak{h}, \quad \lambda = \frac{1}{9}\delta_0 + \frac{1}{3}\delta_1 + \frac{1}{9}\mathfrak{h},$$

it follows that

$$K_h^2 = \kappa(h) = \frac{4}{3}s, \quad \chi_h = \lambda(h) = \frac{1}{9}s.$$

Hence

$$\kappa(f_0) = \frac{2}{3}s, \quad \lambda(f_0) = \frac{1}{18}s, \tag{3.1}$$

and

$$K_{f_0}^2 = \kappa(f_0) + 4s = \frac{14}{3}s, \quad \chi_{f_0} = \lambda(f_0) + \frac{3}{2}s = \frac{14}{9}s. \tag{3.2}$$

We claim that

Claim 3.1. For any rational number $r \in [3, 12)$, there exists a base change $\pi: \tilde{C} \rightarrow C$, such that

- (i). the slope of \tilde{f} is $\lambda_{\tilde{f}} = r$, where $\tilde{f}: \tilde{X} \rightarrow \tilde{C}$ is the pullback fibration under the base change π ;

- (ii). the fibration \tilde{f} admits singular fibers, and every singular fiber is isomorphic to F_3^h , where F_3^h is the singular hyperelliptic fiber of genus 3 as in Example 2.2.

We first prove Theorem 1.2 based on the above claim. By the above claim, for any given rational number $r \in [3, 12)$, one can construct a non-hyperelliptic fibration f_r of genus $g = 3$, such that the slope $\lambda_{f_r} = r$ and the vertical fundamental group is trivial by Lemma 2.4. Combining this with Remark 2.5, we prove Theorem 1.2. It remains to prove the above claim.

Proof of Claim 3.1. The main idea is more or less the same as that in the proof of Theorem 1.1. But we have to carefully choose the branched points of the base change instead of what we do in the proof of Theorem 1.1, where we simply take a base change totally ramified over the images of all singular fibers of f_0 .

First, if $r = 3$, then the fibration f_0 already satisfies our requirements by (3.2). Thus we will assume in the following that $3 < r < 12$. Let m be any odd integer satisfying $m > \frac{27r-72}{12-r}$, and let

$$d = \frac{28m(r-3)}{9(3r-8)(m-1)}.$$

By replacing f_0 by a base change unbranched over $\{p_1, \dots, p_s\}$, we may assume that ds is also an integer at least 2, where $p_i = f_0(F_i)$ is the image of the singular fiber F_i . Indeed, let $\pi_k : C_k \rightarrow C$ be any base change of degree k , whose branch locus does not contain any p_i , then the number of singular fibers as well as the invariants $K_{f_k}^2$ and χ_{f_k} are multiplied by k . Note that $0 < d < 1$ by definition. Let

$$\pi: \tilde{C} \rightarrow C$$

be a cyclic cover of degree m branched exactly over $\{p_1, \dots, p_{ds}\}$ with ramification indices being all equal to m . Such a cyclic cover exists as already showed in the proof of Theorem 1.1. Let \tilde{f} be the pullback fibration. Then the number of singular fibers of \tilde{f} is $ds+m(s-ds)$ and each singular fiber is isomorphic to F_3^h since m is assumed to be odd. Based on (3.1) and the formulas in Section 2.2, one computes the relative invariants of \tilde{f} as follows

$$\begin{aligned} K_{\tilde{f}}^2 &= m\kappa(f_0) + 4(ds + m(s - ds)) = \left(\frac{14}{3}m + 4d(1 - m)\right)s, \\ \chi_{\tilde{f}} &= m\lambda(f_0) + \frac{3}{2}(ds + m(s - ds)) = \left(\frac{14}{9}m + \frac{3}{2}d(1 - m)\right)s. \end{aligned}$$

So the slope of \tilde{f} is

$$\lambda_{\tilde{f}} = \frac{K_{\tilde{f}}^2}{\chi_{\tilde{f}}} = \frac{84m + 72(1 - m)d}{28m + 27(1 - m)d} = r.$$

The last equality follows from the definition of d above. Moreover, by construction \tilde{f} admits singular fibers and every singular fiber is isomorphic to F_3^h as required. This proves Claim 3.1, and hence completes the proof of Theorem 1.2. \square

Remark 3.2. Regarding our result in Theorem 1.2 for the case $g = 3$, we like to conjecture that, for each $g > 3$ and each rational number $r \in [4(g-1)/g, 12)$, there exists a fibration f of genus g such that $\lambda_f = r$ and $\mathcal{V}_f = \{1\}$.

References

- [AK02] T. Ashikaga, K. Konno: *Global and local properties of pencils of algebraic curves*, Algebraic Geometry 2000, Azumino, Advanced Studies in pure mathematics **36** (2002), 1-49.
- [BHPV04] W. P. Barth, K. Hulek, C. Peters, A. Van de Ven: *Compact complex surfaces*, second ed., Springer-Verlag, 2004.
- [Ch87] Z. Chen: *On the geography of surfaces. Simply connected minimal surfaces with positive index*, Math. Ann. **277** (1987), 141-164.
- [CH88] M. Cornalba, J. Harris: *Divisor classes associated to families of stable varieties, with applications to the moduli space of curves*, Ann. Scient. Ec. Norm. Sup. **21** (1988), 455-475.
- [DM18] R. Donagi, D. Morrison: *Conformal field theories and compact curves in moduli spaces*, J. High Energy Phys. **021** (5) (2018), front matter+6 pp.
- [HM82] J. Harris, D. Mumford: *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), 23-86.
- [LT13] X.-L. Liu, S.-L. Tan: *Families of hyperelliptic curves with maximal slopes*, Sci. China Math. **56** (9) (2013), 1743-1750.
- [LL24] X.-L. Liu, J. Lu: *On the geography of slopes of fibrations*, Publ. Res. Inst. Math. Sci., accepted.
- [LuT13] J. Lu, S.-L. Tan: *Inequalities between the Chern numbers of a singular fiber in a family of algebraic curves*, Trans. Amer. Math. Soc. **365** (2013), 3373-3396.
- [LZ19] X. Lu, K. Zuo: *The Oort conjecture on Shimura curves in the Torelli locus of curves*, J. Math. Pures Appl. **123** (2019), 41-77.
- [MT87] B. Moishezon, M. Teicher: *Simply-connected algebraic surfaces of positive index*, Invent. Math. **89** (3) (1987), 601-643.
- [Mo98] A. Moriwaki: *Relative Bogomolov's inequality and the cone of positive divisors on the moduli space of stable curves*, J. Amer. Math. Soc. **11** (1998), 569-600.
- [OS79] F. Oort, J. Steenbrink: *The local Torelli problem for algebraic curves*, Journées de Géométrie Algébrique d'Angers (1979), 157-204.
- [Pe81] U. Persson: *Chern invariants of surfaces of general type*, Compositio Math. **43** (1) (1981), 3-58.
- [Pe87] U. Persson: *An introduction to the geography of surfaces of general type*, in Algebraic Geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math. 46, Amer. Math. Soc., Providence, RI, 1987, 195-218.
- [Po77] H. Popp: *Moduli theory and classification theory of algebraic varieties*, Lect. Notes in Math. 620, Springer-Verlag, 1977.

- [RU15] X. Roulleau, G. Urzúa: *Chern slopes of simply connected complex surfaces of general type are dense in [2,3]*, Ann. of Math. **182** (2015), 287-306.
- [So84] A. J. Sommese: *On the density of ratios of Chern numbers of algebraic surfaces*, Math. Ann. **268** (1984), 207-221.
- [Ta94] S.-L. Tan: *On the base changes of pencils of curves, I*, Manus. Math. **84** (1994), 225–244.
- [Ta96] S.-L. Tan: *On the base changes of pencils of curves, II*, Math. Z. **222** (1996), 655–676.
- [Ta10] S.-L. Tan: *Chern numbers of a singular fiber, modular invariants and isotrivial families of curves*, Acta Math. Viet. **35** (1) (2010), 159-172.
- [Xi85] G. Xiao: *Surface fibrées en courbes de genre deux*, Lect. Notes in Math. 1137, Springer-Verlag, 1985.
- [Xi87] G. Xiao: *Fibred algebraic surfaces with low slope*, Math. Ann. **276** (1987), 449-466.
- [Xi91] G. Xiao: π_1 of elliptic and hyperelliptic surfaces, Inter. J. Math. **5** (2) (1991), 599-615.
- [Xi92] G. Xiao: *The fibrations of algebraic surfaces*, Shanghai Scientific & Technical Publishers, 1992 (in Chinese).

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