LEGO-like Small-Model Constructions for Åqvist's Logics

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Abstract

Åqvist's logics (**E**, **F**, **F+(CM)**, and **G**) are among the best-known systems in the long tradition of preference-based approaches for modeling conditional obligation. While the general semantics of preference models align well with philosophical intuitions, more constructive characterizations are needed to assess computational complexity and facilitate automated deduction. Existing small model constructions from conditional logics (due to Friedman and Halpern) are applicable only to **F+(CM)** and **G**, while recently developed proof-theoretic characterizations leave unresolved the exact complexity of theoremhood in logic **F**. In this paper, we introduce alternative small model constructions, obtained uniformly for all four Åqvist's logics. Our constructions propose alternative semantical characterizations and imply co-NP-completeness of theoremhood. Furthermore, they can be naturally encoded in classical propositional logic for automated deduction.

Keywords: deontic logic, preference models, small model property

1 Introduction

The analysis of various normative scenarios and deontic paradoxes led to the formalization of obligations as conditionals, i.e. as dyadic modalities $\bigcirc(\gamma \mid \alpha)$ read "γ is obligatory if α holds". One standard way to formalize conditionals is to use relational models with a certain preference relation on worlds inducing a notion of maximality (or minimality), known as preference models. In the deontic context, a landmark framework implementing the preference-based approach is Åqvist's family [1], comprising logics **E**, **F**, and **G**. **G** is a deontic version of Lewis' counterfactual logic **VTA** [13] and it is too strong in many deontic contexts, necessitating weaker modifications **E** and **F**. In the latter two logics the preference relation is not required to be transitive and *smoothness* assumption (a.k.a. Lewis' "Limit assumption") is either completely discarded (in **E**) or is replaced with a more general *limitedness* property (in **F**). This makes logics **E** and **F** rather unique among conditional logics (the smoothness assumption is ubiquitous in other conditional formalisms, e.g. [3,11,13]). A

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later addition to Åqvist's family is **F+(CM)** [14] which extends **F** with the cautious monotonicity principle, well-known in non-monotonic reasoning [8].

In recent decades, significant progress has been made in exploring variations of preference-model characterization for Åqvist's logics and their corresponding axiomatizations, surveyed in [15]. Now, there's a growing focus on the computational properties of these logics, which is the main motivation for this paper too. In [15] the decidability of theoremhood for all four logics is proven through alternative semantics based on selection functions, and embedding of the weakest logic **E** into Higher-Order Logic (HOL) from [2] is suggested as a potential approach for automated deduction. These approaches however are not suitable for assessment of the exact complexity of logics, which requires more constructive characterizations. One such characterization came from the proof-theoretic side in the form of cut-free hypersequent calculi, developed recently for all four Åqvist's logics [4,5,6]. For **E**, **F+(CM)** and **G** the proof search in the calculi has optimal co-NP complexity, and polynomial-size preference countermodels can be reconstructed from failed derivations [4,9]. At the same time, the limitedness condition of **F** seems difficult to handle both modal-theoretically and proof-theoretically. The calculus for **F** [5], which is an even more complicated variation of the calculi for logic **GL** [16], gives only a co-NEXP upper bound for theoremhood (which is the best estimation so far) and no countermodel construction.

Another powerful approach for establishing computational complexity of conditional logics is *small-model constructions* proposed by Friedman and Halpern [7] for Burgess' logic **PCL** [3] and its extensions, which transforms any satisfying model into a satisfying model of bounded size. Their approach covers in particular extensions **PCA** and **VTA** (in terminology of [9]), which coincide with Åqvist's logics **F+(CM)** and **G** respectively, and establishes co-NP-completeness of theoremhood for them. However, this approach significantly relies on the smoothness of the preference relation and therefore is not applicable for weaker logics **E** and **F** (see Remark 3.24 for details).

In this paper, we propose alternative small model constructions to uniformly handle all four Åqvist's logics. We compose a model of polynomial size by assembling elementary building blocks (chains, antichains, and cliques of worlds selected from any given model) like LEGO. We provide sufficient conditions for such construction to be a countermodel and define a suitable construction for each Åqvist's logic. There are two main applications for our constructions, obtained uniformly for all logics.

Alternative semantical characterizations of theoremhood. Our results imply that theoremhood can be characterized by finite models. Moreover, for finite models the complicated properties of limitedness and smoothness (which are not frame properties) can be replaced by natural frame properties: acyclicity and transitivity of the preference relation, respectively.

Complexity and automated deduction. The polynomial size of models together with easily checkable frame properties immediately imply co-NP-completeness of theoremhood (including logic **F**, for which it was an open problem) and allow for natural encodings in classical propositional logics, which can be utilized for efficient automated deduction using SAT-solvers.

2 Preliminaries

The syntax of Åqvist's logics extends the usual propositional language with two modalities: unary $\Box$ for necessity and binary $\bigcirc(\cdot\,|\,\cdot)$ for conditional obligation. We define the formulas over the set *Var* of propositional variables.

$$\mathcal{F} ::= \ x \in \mathit{Var} \ \mid \ \neg\mathcal{F} \ \mid \ \mathcal{F} \wedge \mathcal{F} \ \mid \ \Box\mathcal{F} \ \mid \ \bigcirc(\mathcal{F}\,|\,\mathcal{F})$$

We will use small Greek letters to denote formulas. $|\varphi|$ will denote size of the formula (number of symbols), $\mathit{Sub}\mathcal{F}(\varphi)$ will denote the set of all subformulas of φ (including φ), and $\mathrm{Cond}(\varphi) = \{\alpha \mid \bigcirc(\gamma\,|\,\alpha) \in \mathit{Sub}\mathcal{F}(\varphi)\}$.

Definition 2.1 A *preference model* is a triple $\langle W, \succeq, V\rangle$ where W is a (non-empty) set of worlds, $\succeq$ is a binary relation on W, and $V \colon \mathit{Var} \to 2^W$ is a valuation function. We denote by $W(M)$ the set of worlds of a given model.

The semantics of obligation is based on the notion of "best" worlds in the preference model. There are different definitions of bestness appearing in the literature (see [10,14] for the comparison of different definitions), we will use the most common one — maximality: a world is a best world when there are no worlds that are *strictly* more preferable. As usual we denote by $\succ$ a strict version of $\succeq$ ($w_1 \succ w_2$ when $w_1 \succeq w_2$ and $w_2 \not\succeq w_1$). We will use the notation $\mathit{Bet}_{\succ}(v) = \{w \in W \mid w \succ v\}$ for a set of worlds strictly preferable (better) than the given one.

Definition 2.2 For a preference model $M = \langle W, \succeq, V\rangle$ and $U \subseteq W$ we define $\max(U) = \{v \in U \ \mid \ \nexists u \in U \colon u \succ v\}$.

Satisfaction of $\bigcirc(\gamma\,|\,\alpha)$ is defined using this notion of bestness: $\bigcirc(\gamma\,|\,\alpha)$ is true when γ is true in all maximal worlds satisfying α (we will call such worlds *α-maximal*). And $\Box\beta$ is true when β is true in all worlds (so we treat $\Box$ as the universal **S5** modality).

Definition 2.3 (Satisfaction) For a preference model $M = \langle W, \succeq, V\rangle$ the *truth set* $||\varphi||^M$ of a formula φ is defined inductively:

- $w \in ||x||^M$ for $x \in \mathit{Var}$ when $w \in V(x)$,
- $w \in ||\neg\psi||^M$ when $w \notin ||\psi||^M$,
- $w \in ||\psi_1 \wedge \psi_2||^M$ when $w \in ||\psi_1||^M$ and $w \in ||\psi_2||^M$,
- $w \in ||\Box\beta||^M$ when $||\beta||^M = W$,
- $w \in ||\bigcirc(\gamma\,|\,\alpha)||^M$ when $\max(||\alpha||^M) \subseteq ||\gamma||^M$.

We say that w satisfies φ in M (denoted $M, w \models \varphi$) when $w \in ||\varphi||^M$, and that M validates φ (denoted $M \models \varphi$) when $||\varphi||^M = W$.

Notice that the satisfaction of both $\Box\beta$ and $\bigcirc(\gamma \mid \alpha)$ does not depend on the world of evaluation.

Logic	Properties of the models
E	no properties
F	limited
F+(CM)	smooth and transitive
G	smooth, transitive and total

Fig. 1. Preference-semantical characterizations for Åqvist's logics from [15, Tab. 1 and 2] (with maximality as the notion of bestness).

Different Åqvist's logics are defined by different classes of preference models. Some of these classes are defined using the properties of preference relation $\succeq$ in the model, we will use two properties: transitivity ($\succeq$ is transitive when $w_1 \succeq w_2$ and $w_2 \succeq w_3$ imply $w_1 \succeq w_3$) and totalness ($\succeq$ is total when for any $w_1, w_2 \in W$ either $w_1 \succeq w_2$ or $w_2 \succeq w_1$). Another property used for the characterization of deontic logic is what Lewis called "limit assumption", which ensures the existence of best worlds. The are different formal definitions of these assumptions in the literature, we will use two versions from [15]: *limitedness* and *smoothness*.

Definition 2.4 (Limit conditions) Let $M = (W, \succeq, V) \in \mathcal{M}$. M is *limited* when for any formula α if $||\alpha||^M \neq \emptyset$ then $\max(||\alpha||^M) \neq \emptyset$. M is *smooth* when for any formula α and any world $w \in ||\alpha||^M$ there exists $u \in \max(||\alpha||^M)$ such that either $u = w$ or $u \succ w$.

We rely on the semantical characterizations of the four Åqvist logics in Tab. 1, which are presented (among various other characterizations) in [15].

Definition 2.5 Formula φ is a theorem of Åqvist's logic $\mathcal{L}$ iff $M \vDash \varphi$ for any preference model M that satisfies model conditions for logic $\mathcal{L}$ in Fig. 1.

We will call a preference model M a *countermodel for a formula* φ if $M \not\vDash \varphi$ and we will further call it an $\mathcal{L}$*-countermodel* if it belongs to a class of models corresponding to a logic $\mathcal{L}$ in Tab. 1.

3 Small Model Constructions

This section contains the main technical result of the paper: we will show how for any formula φ an arbitrary given $\mathcal{L}$-countermodel M can be transformed into a $\mathcal{L}$-countermodel with the number of worlds bounded polynomially w.r.t. $|\varphi|$ for every logic $\mathcal{L}$ from the Åqvist family. We will achieve this by selecting a finite number of worlds from M, possibly adding copies for some of them, and defining a new preference relation on these selected worlds without changing the valuation. We call such transformation a *rearrangement* of a model.

Definition 3.1 We say that a model $M' = \langle W', \succeq', V' \rangle$ *rearranges* the model $M = \langle W, \succeq, V \rangle$ when there exists a *prototype function* $prot \colon W' \to W$ such that $w' \in V'(x)$ is equivalent to $prot(w) \in V(x)$ for all $x \in Var$.

Our main goal for the rearranged model is to have each of its worlds satisfying exactly the same subformulas of φ as its prototype does (we do not care about the satisfaction of other formulas, since the evaluation of a formula in a world only involves subformulas). The definition of the satisfaction of a formula in a world concerns other worlds of the model only in the cases of $\Box$ and $\bigcirc(\cdot\,|\,\cdot)$ modalities. Therefore we only need to ensure that the rearranged model validates exactly the same modalities among subformulas of φ as the original model does, while the satisfaction (and non-satisfaction) for other subformulas will be preserved in the rearranged model automatically (due to preservation of valuation for variables).

We will examine the cases of validated and non-validated modalities separately. Let us denote by $Box^+(\varphi, M)$ (resp. $Ob^+(\varphi, M)$) the set of subformulas of φ of the form $\Box\beta$ (resp. $\bigcirc(\gamma \mid \alpha)$) that are validated by M, and by $Box^-(\varphi, M)$ and $Ob^-(\varphi, M)$ the sets of subformulas of φ of the corresponding form that are not validated by M. To falsify $\Box\beta \in Box^-(\varphi, M)$ and $\bigcirc(\gamma\,|\,\alpha) \in Ob^-(\varphi, M)$ we need to take in M' some worlds that were falsifying these modalities in M. While the evaluation of $\Box\beta$ modalities relies only on the presence of the worlds satisfying β in the model, special care is needed to ensure that the evaluation of $\bigcirc(\gamma \mid \alpha)$ is the same: if a world w was made not α-maximal in M by some world $u \in ||\alpha||^M$ such that $u \succ w$ we need to preserve this violation of maximality in M', and conversely we need to ensure we are not breaking α-maximality in M' for the world falsifying $\bigcirc(\gamma \mid \alpha)$. To ensure this we will introduce the condition that $u' \succ' v'$ in M' requires $prot(u') \succ prot(v')$ in M (we need this condition only for the worlds v' falsifying some $\bigcirc(\gamma \mid \alpha) \in Ob^-(\varphi, M)$), and a converse condition that if there exist some α-world $\succ$-better than $prot(w')$ in M then there should exist some α-world $\succ'$-better than w' in M (for all worlds $w' \in W(M')$ and all conditions α such that $\bigcirc(\gamma \mid \alpha) \in Box^+(\varphi, M)$). All that is left to make M' a countermodel for φ is to take in M' any world falsifying φ in the original model. This reasoning leads to the following four conditions sufficient to ensure that a rearranged model M' is a countermodel for φ.

Theorem 3.2 *Suppose* $M = \langle W, \succeq, V\rangle$ *is a countermodel for* φ *and* $M' = \langle W', \succeq', V'\rangle$ *rearranges* M *with the prototype function* $prot\colon W' \to W$. *Then the following conditions are sufficient for* $M' \not\models \varphi$.

- (i) *There exists* $v' \in W'$ *such that* $M, prot(v') \not\models \varphi$.
- (ii) *For any* $\Box\beta \in Box^-(\varphi, M)$ *there exists* $v' \in W'$ *such that* $M, prot(v') \not\models \beta$.
- (iii) *For any* $\bigcirc(\gamma\,|\,\alpha) \in Ob^-(\varphi, M)$ *there exists* $v' \in W'$ *such that* $prot(v') \in \max(||\alpha||^M) \setminus ||\gamma||^M$ *and for all* $u' \succ' v'$ *holds* $prot(u') \succ prot(v')$.
- (iv) *For any* $w' \in W'$, *for all* $\bigcirc(\gamma\,|\,\alpha) \in Ob^+(\varphi, M)$ *if there exists* $u \succ prot(w')$ *such that* $M, u \models \alpha$ *then there exists* $s' \succ' w'$ *such that* $M, prot(s') \models \alpha$.

Proof. We will prove the goal statement described above: for any $w' \in W'$ and any $\psi \in Sub\mathcal{F}(\varphi)$ holds $M', w' \models \psi$ iff $M, prot(w') \models \psi$. Then $M' \not\models \varphi$ follows from condition (i). The proof is by induction on ψ with case analysis on ψ belonging to $Box^+(\varphi, M)$ or $Box^-(\varphi, M)$ for $\psi = \Box\beta$ and on ψ belonging to $Ob^+(\varphi, M)$ or $Ob^-(\varphi, M)$ for $\psi = \bigcirc(\gamma \mid \alpha)$. Conditions ($ii$), ($iii$), and ($iv$) directly cover case $\psi \in Box^-(\varphi, M)$, case $\psi \in Ob^-(\varphi, M)$, and case $\psi \in Ob^-(\varphi, M)$ respectively (see appendix A for details). $\square$

Ensuring conditions (i) and (ii) is simple: we need to take arbitrary worlds from $(W \setminus ||\varphi||^M)$ and from $(W \setminus ||\beta||^M)$ for each $\square\beta \in Box^-(\varphi, M)$. For this, we will use a *representative function* $rep\colon (2^W \setminus \emptyset) \to W$ that for any given non-empty subset S of W chooses an element $rep(S) \in S$ (thus, we use the axiom of choice explicitly in our construction). We will also need representatives of $(\max(||\alpha||^M) \setminus ||\gamma||^M)$ for every $\bigcirc(\gamma \mid \alpha) \in Ob^-(\varphi, M)$ for condition (iii). Let us denote the set of all such falsifying worlds $Fal(\varphi, M)$.

Definition 3.3 (Falsifying worlds) For a model $M = \langle W, \succeq, V\rangle$ such that $M \not\models \varphi$, $Fal(\varphi, M) = rep(W \setminus ||\varphi||^M) \cup Fal^{\square}(\varphi, M) \cup Fal^{\bigcirc}(\varphi, M)$, where $Fal^{\square}(\varphi, M) = \{rep(W \setminus ||\beta||^M) \mid \square\beta \in Box^-(\varphi, M)\}$, $Fal^{\bigcirc}(\varphi, M) = \{rep((\max(||\alpha||^M) \setminus ||\gamma||^M \mid \bigcirc(\gamma \mid \alpha) \in Ob^-(\varphi, M)\}$.

The rest of the rearranged models will be chosen to ensure the satisfaction of conditions (iii) and (iv). We will represent our small model constructions as *composite models*, assembled from *blocks*. A *block* B is a finite selection of worlds from M with some new ordering on them (in our cases it will be either an empty relation, a strict linear order, or a universal relation).

Definition 3.4 (Block) A *block* on M is a tuple $\langle W^B, \succeq^B\rangle$ where $W^B \subseteq W(M)$ and $\succeq^B$ is a binary relation on W^B. We will use $W(B)$ to refer to W^B.

If there are no $w_1, w_2 \in W^B$ such that $w_1 \succ^B w_2$, we call B *flat*.

A composite construction consists of the number of blocks with an additional preference relation on them. Each composite construction generates a model rearranging M, in which the new preference relation is given by combining the relation between blocks and the relations inside a block. To allow multiple occurrences of the same block in the construction we define it formally using labels.

Definition 3.5 (Composite construction) A *composite construction on* M is a tuple $\langle L, \succeq^L, cont\rangle$ where L is a set of labels, $\succeq^L$ is a binary relation on L, and $cont$ is a function mapping every label from L into a block on M. We will denote by $\mathcal{CC}(M)$ a set of all composite constructions on M. Each $C = \langle L, \succeq^L, cont\rangle \in \mathcal{CC}(M)$ *generates* a model $gen(C) = \langle W^C, \succeq^C, V^C\rangle$ where $W^C = \{(l, w) \mid l \in L, w \in W(cont(l))\}$, $(l_1, w_1) \succeq^C (l_2, w_2)$ iff either $l_1 \succeq^L l_2$ or both $l_1 = l_2$ and $w_1 \succeq^B w_2$ for the preference relation $\succeq^B$ in the block $cont(l_1)$, and $(l, w) \in V^C(x)$ iff $w \in V(x)$ for the valuation V in M.

We can now simplify conditions of Th. 3.2 for models generated by composite constructions. Specifically, we can ensure (iv) separately for each block by either ensuring it inside this block already or having another block $\succ^L$-preferred to it that has all worlds required in (iv).

Definition 3.6 We say that block B is (iv)-*safe* if condition (iv) is satisfied for the model generated just by this block. We say that block B' (iv)-*covers* block B if for any $\alpha \in \mathrm{Cond}(\varphi)$ and $v \in W(B)$ if there exists $u \in \mathit{Bet}_{\succ}(w)$ such that $M, u' \models \alpha$ then there exists $u' \in W(B')$ such that $M, u' \models \alpha$.

Other conditions can be ensured by having every falsifying world v in the composite model in some flat block with blocks covering it containing only worlds from $\mathit{Bet}_{\succ}(v)$.

Theorem 3.7 *Let $M \not\models \varphi$ and $C \in \mathcal{CC}(M)$. For $gen(C) \not\models \varphi$ it is sufficient that: (a) for each $v \in \mathit{Fal}(\varphi, M)$ there is a flat block B_v in C such that $v \in W(B_v)$ and for every $B' \succ^L B_v$ holds $W(B') \subseteq \mathit{Bet}_{\succ}(v)$; (b) each block B in C is either (iv)-safe or (iv)-covered by some other block B' such that $B' \succ^L B$.*

Proof. $gen(C)$ rearranges M (with $prot((l, w)) = w$), so we can apply Th. 3.2. (a) ensures conditions (i)-(iii) and (b) ensures condition (iv). $\square$

We now define composite constructions for each Åqvist's logic satisfying these conditions and the required model conditions for the logic from Tab. 1.

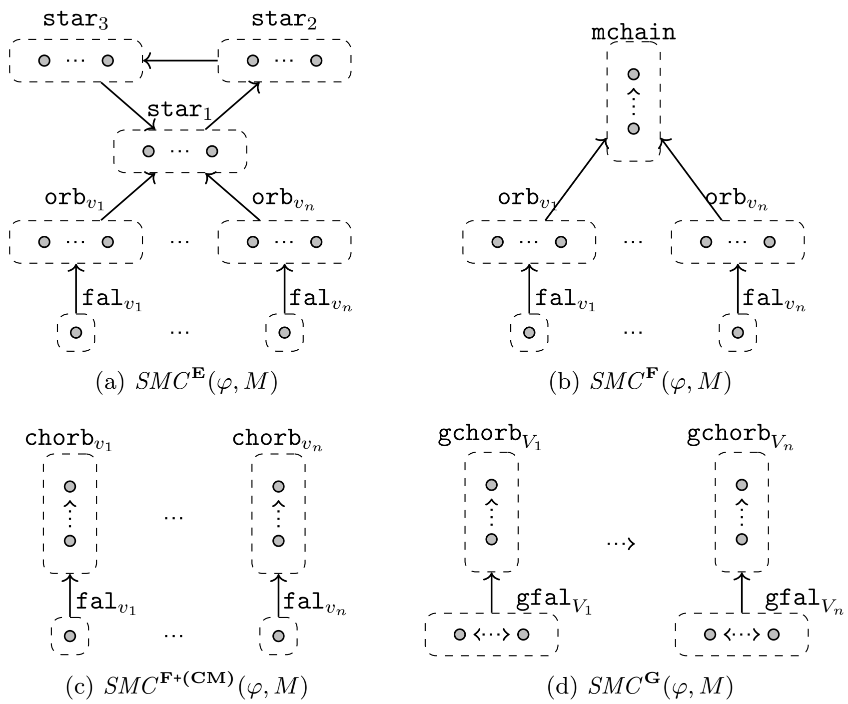

Fig. 2. Small model constructions for Åqvist's logics. Gray circles represent worlds, dashed rectangles represent blocks. Symbol '$\cdots$' in a block indicates an antichain, a dotted vertical arrow indicates a chain, and a dotted two-sided arrow indicates a clique. Solid arrows represent the preference relation $\succeq^L$ between blocks: an arrow from a block l_1 to a block l_2 means $l_2 \succeq^L l_1$. A dotted arrow between blocks in construction $\mathit{SMC}^{\mathbf{G}}(\varphi, M)$ means that there is a linear order on blocks. Note that the preference relation in constructions $\mathit{SMC}^{\mathbf{E}}(\varphi, M)$ and $\mathit{SMC}^{\mathbf{F}}(\varphi, M)$ is not transitive.

3.1 Small Model Construction for Logic E

In the case of logic **E** there are no model conditions we need to satisfy in our countermodel, so we can use a preference relation that is both non-transitive and contains cycles. In this simple case, all blocks of the countermodel construction will be *antichains*.

Definition 3.8 $\mathit{antichain}(U) = \langle U, \succeq_a \rangle$ where $\succeq_a$ is an empty relation.

We start our composite construction with a dedicated one-world block $\mathit{antichain}(\{v\})$ for each world $v \in \mathit{Fal}(\varphi, M)$, labeled $\mathtt{fal}_v$. The simplest way to (*iv*)-cover such block with linearly many (w.r.t. $|\varphi|$) worlds without violating condition (*iii*) is to go through formulas from $\mathrm{Cond}(\varphi)$ satisfied by some world in $\mathit{Bet}_{\succ}(v)$ and select one representative for each. Below such selection is defined more generally, for an arbitrary set of formulas $\mathcal{A}$ and an arbitrary set of worlds U to select from.

Definition 3.9 (Selection) For a set $U \subseteq W(M)$ and a set of formulas $\mathcal{A}$, $\mathit{Sel}_M(U, \mathcal{A}) = \{\mathit{rep}(||\alpha||^M \cap U) \mid \alpha \in \mathcal{A}, ||\alpha||^M \cap U \neq \emptyset\}$.

We can show that such a selection can (*iv*)-cover not only single-world blocks like $\mathtt{fal}_v$, but any block B as long as U contains all worlds $\succ$-preferable to any world in B.

Lemma 3.10 *If $\mathit{Bet}_{\succ}(w) \subseteq U$ for all $w \in W(B)$ in some block B then $\mathit{antichain}(\mathit{Sel}_M(U, \mathit{Cond}(\varphi)))$ (iv)-covers B.*

Proof. If $\alpha \in \mathrm{Cond}(\varphi)$ is satisfiable in $\mathit{Bet}_{\succ}(w)$ for some $w \in W(B)$ then there will be a representative satisfying α in $\mathit{Sel}_M(U, \mathrm{Cond}(\varphi))$. □

A block to (*iv*)-cover $\mathtt{fal}_v$, which we will call *orbit* and label $\mathtt{orb}_v$, can be defined as $\mathit{Orbit}(M, \varphi, v) = \mathit{antichain}(\mathit{Sel}_M(\mathit{Bet}_{\succ}(v), \mathrm{Cond}(\varphi)))$. To (*iv*)-cover orbits themselves, we can make another selection, this time from the whole $W(M)$, as we do not need to care about condition (*iii*) for these blocks. So the block $\mathit{Star}(M, \varphi) = \mathit{antichain}(\mathit{Sel}_M(W(M), \mathrm{Cond}(\varphi)))$, which we will label $\mathtt{star}_1$, can be used to (*iv*)-cover all orbits. Finally, to cover $\mathtt{star}_1$ we can add two more copies of $\mathit{Star}(M, \varphi)$ (labeled $\mathtt{star}_2$ and $\mathtt{star}_3$) and have a non-transitive loop on these three copies, which will (*iv*)-cover each other circularly. This leads to the following small model construction for **E**.

Definition 3.11 (Small Model Construction for E)
If M is an **E**-countermodel for φ, $\mathit{SMC}^{\mathbf{E}}(\varphi, M) = \langle L, \succeq^L, \mathit{comp} \rangle$ where $L = \{\mathtt{fal}_v, \mathtt{orb}_v \mid v \in \mathit{Fal}(\varphi, M)\} \cup \{\mathtt{star}_i \mid i \in \{1, 2, 3\}\}$, $\mathit{comp}(\mathtt{fal}_v) = \mathit{antichain}(\{v\})$, $\mathit{comp}(\mathtt{orb}_v) = \mathit{Orbit}(M, \varphi, v)$, $\mathit{comp}(\mathtt{star}_i) = \mathit{Star}(M, \varphi)$ and a preference relation $\succeq^L$ on blocks is demonstrated on Fig. 2a.

Theorem 3.12 *If M is a $\boldsymbol{E}$-countermodel for φ then $\mathit{gen}(\mathit{SMC}^{\boldsymbol{E}}(\varphi, M))$ is a $\boldsymbol{E}$-countermodel for φ and $|W(\mathit{gen}(\mathit{SMC}^{\boldsymbol{E}}(\varphi, M)))| = \mathcal{O}(|\varphi|^2)$.* [2]

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[2] As usual, the notation $f(\varphi, M) = \mathcal{O}(g(\varphi, M))$ for integer-valued functions f and g means that there exists a constant C such that $f(\varphi, M) \leq C \cdot g(\varphi, M)$ for all φ and M.

Proof. $SMC^{\mathbf{E}}(\varphi, M)$ is a countermodel for φ by Th. 3.7, because we have $W(\mathtt{orb}_v) \subseteq Bet_{\succ}(v)$ and all blocks are (iv)-covered by Lem. 3.10. $|W(gen(SMC^{\mathbf{E}}(\varphi, M)))| = \mathcal{O}(|\varphi|^2)$ since $SMC^{\mathbf{E}}(\varphi, M)$ contains $(2 \cdot |Fal(\varphi, M)| + 3)$ blocks with at most $|\mathrm{Cond}(\varphi)|$ worlds each. □

3.2 Small Model Construction for Logic F

For logic **F**, we will utilize the limitedness of the countermodel M to construct a small countermodel with an acyclic $\succ$, which will automatically make it limited (and thus an **F**-countermodel) too.

Lemma 3.13 *Model $\langle W, \succeq, V\rangle$ is limited if W is finite and $\succ$ is acyclic.*

Proof. If there is some $w_0 \in ||\alpha||^M$, consider (any) longest path $w_0 \prec w_1 \prec w_2 \prec \ldots$ with worlds from $||\alpha||^M$ staring from w_0. Since W is finite and there can be no repetitions on the path (due to acyclicity of $\succ$), the path is finite and there is the last world w_m for which there is no $u \in ||\alpha||^M$ such that $u \succ w_m$, and so $w_m \in \max(||\alpha||^M)$ by definition. □

For acyclicity, we will modify our construction $SMC^{\mathbf{E}}(\varphi, M)$ by replacing a non-transitive cycle on blocks $\mathtt{star}_1, \mathtt{star}_2, \mathtt{star}_3$ with one finite *chain* (i.e. block with linear ordering). To work with chains we will use lists (finite ordered sequences). We will use the notation $[a_1, \ldots, a_m]$ for a list containing given elements, the notation $[\,]$ for the empty list, and the notation $a :: S$ for the list in which element a is appended to the beginning of the list S.

Definition 3.14 (Chain) For a list $S = [w_1, \ldots, w_m]$ of worlds from $W(M)$, $chain(S) = \langle \{w_i\}_{i=1}^n, \succeq^{ch}\rangle$ where $w_i \succeq^{ch} w_j$ when $i \leq j$.

Our goal is to define a finite chain that is (iv)-safe and satisfies any $\alpha \in \mathrm{Cond}(\varphi)$ that is satisfiable in M (which will allow us to use the chain to (iv)-cover any block). We construct such a chain through an iterative process, that selects maximal worlds for disjunctions of conditions. At the beginning of the process, we have $\mathcal{A}_0 = \mathrm{Cond}(\alpha)$ as the set of conditions for which we need satisfying worlds. If at least one of conditions in $\mathcal{A}_0$ is satisfied by some world in M, then $||\bigvee_{\alpha\in\mathcal{A}_0} \alpha||^M \neq \emptyset$, then by limitedness there exists some $z_0 \in \max(||\bigvee_{\alpha\in\mathcal{A}_0} \alpha||^M)$. We can safely take z_0 as the first (i.e. $\succ^B$-greatest) world in the chain, since there are no worlds $u \succ z_0$ in M satisfying conditions from $\mathcal{A}_0$. z_0 satisfies some conditions from $\mathcal{A}_0$ (since $M, z_0 \models \bigvee_{\alpha\in\mathcal{A}_0} \alpha$), therefore we can move on to the next step with a strictly smaller set $\mathcal{A}_1$ of conditions for which we still need satisfying worlds. We can safely repeat this process by taking worlds from $z_i \in \max(||\bigvee_{\alpha\in\mathcal{A}_i} \alpha||^M)$ at every iteration: z_i has no $\succ$-prefferable α-worlds for all remaining conditions $\alpha \in \mathcal{A}_i$, while for all already removed formulas there is a satisfying world somewhere earlier in the chain (i.e. $\succ^B$-preferrable to z_i), thus condition (iv) will be satisfied for this world. After a linear number of iterations, the chain will contain satisfying worlds for all formulas from $\mathcal{A}$ satisfiable in M. Below is the formal definition of the described chain of maximal worlds. We give a generalized version that selects maximal worlds from any given subset of worlds U and any given set of formulas $\mathcal{A}_i$, the same way as we did for $Sel_M(U, \mathcal{A})$. We will need this generalized version for logics **F+(CM)** and **G**.

Definition 3.15 For any $U \subseteq W(M)$ and a finite set of formulas $\mathcal{A}_i$,

$$MaxSeq_M(U, \mathcal{A}_i) = \begin{cases} [\,], & \text{if } \mathcal{A}_i = \emptyset \\ [\,], & \text{if } \mathcal{D}(U, \mathcal{A}_i) = \emptyset \\ d(U, \mathcal{A}_i) :: MaxSeq_M(U, \mathcal{A}_{i+1}), & \text{otherwise} \end{cases}$$

where $\mathcal{D}(U, \mathcal{A}) = U \cap \max(||\bigvee_{\alpha \in \mathcal{A}_i} \alpha||^M)$, $d(U, \mathcal{A}) = rep(\mathcal{D}(U, \mathcal{A}))$ and $\mathcal{A}_{i+1} = \{\alpha \in \mathcal{A}_i \mid M, d(U, \mathcal{A}) \not\models \alpha\}$.

Notice that for a finite $\mathcal{A}_0$ this sequence is well-defined (representative $d(\mathcal{A}_i,)$ is always taken from a non-empty set and $|\mathcal{A}_i|$ decreases) and always has length at most $|\mathcal{A}_0|$. The reasoning above that shows (*iv*)-safeness of the chain built from this sequence works in the general case with arbitrary U and doesn't even require the limitedness of M.

Lemma 3.16 *$chain(MaxSeq_M(U, Cond(\varphi)))$ is (iv)-safe for any $U \subseteq W(M)$.*

Proof. Let $z_k \in W(chain(MaxSeq_M(U, \mathrm{Cond}(\varphi))))$ and $\bigcirc(\gamma \mid \alpha) \in Ob^+(\varphi, M)$. $z_k \in \max(||\bigvee_{\alpha \in \mathcal{A}_k} \alpha||^M)$ for some step k and set $\mathcal{A}_k$ of remaining conditions. If there is $u \succ z_k$ such that $M, u \models \alpha$, then $\alpha \notin \mathcal{A}_k$, i.e. α was removed at some previous step, therefore there is z_j with $j < k$ such that $M, z_j \models \alpha$. □

For logic **F**, we select worlds in the chain from the whole $W(M)$: for a limited model M we define block $MaxChain(M, \varphi) = chain(MaxSeq_M(\mathrm{Cond}(\varphi), W(M)))$, which we will label `mchain`. $MaxChain(M, \varphi)$ contains satisfying worlds for all conditions from $\mathrm{Cond}(\varphi)$ satisfiable in M so it (*iv*)-covers any block on M.

Lemma 3.17 *For a limited M, $MaxChain(M, \varphi)$ (iv)-covers any block.*

Proof. If a condition $\alpha \in \mathrm{Cond}(\varphi)$ is satisfiable in M then it can not be among the remaining conditions when the chain is built (otherwise $\mathcal{A}_m \neq \emptyset$ and $\mathcal{D}(W(M), \mathcal{A}_m) \neq \emptyset$ due to limitedness of M), therefore for some world z in the chain $M, z \models \alpha$. □

Replacement of non-transitive triangle in $SMC^E(\varphi, M)$ with $MaxChain(M, \varphi)$ gives us the small model construction $SMC^F(\varphi, M)$ with an acyclic strict version of preference relation.

Definition 3.18 (Small Model Construction for F)
If M is an **F**-countermodel for φ, $SMC^{\mathbf{F}}(\varphi, M) = \langle L, \succeq^L, comp \rangle$ where $L = \{\mathtt{fal}_v, \mathtt{orb}_v \mid v \in Fal(\varphi, M)\} \cup \{\mathtt{mchain}\}$, $comp(\mathtt{fal}_v) = antichain(\{v\})$, $comp(\mathtt{orb}_v) = Orbit(M, \varphi, Bet_\succ(v))$, $comp(\mathtt{mchain}) = MaxChain(M, \varphi)$ and a preference relation $\succeq^L$ on blocks is demonstrated on Fig. 2b.

Theorem 3.19 *If M is a **F**-countermodel for φ then $gen(SMC^{\mathbf{F}}(\varphi, M))$ is a **F**-countermodel for φ and $|W(gen(SMC^{\mathbf{F}}(\varphi, M)))| = \mathcal{O}(|\varphi|^2)$.*

Proof. $\mathit{gen}(\mathit{SMC}^{\mathbf{F}}(\varphi, M))$ is a countermodel for φ by Th. 3.7, because we have $W(\mathtt{orb}_v) \subseteq \mathit{Bet}_{\succ}(v)$, `mchain` is (*iv*)-safe by Lem. 3.16 and all other blocks are (*iv*)-covered by Lem. 3.17 and Lem. 3.10. $\mathit{gen}(\mathit{SMC}^{\mathbf{F}}(\varphi, M))$ is an **F**-countermodel by Lem. 3.13. $|W(\mathit{gen}(\mathit{SMC}^{\mathbf{F}}(\varphi, M)))| = \mathcal{O}(|\varphi|^2)$ since $\mathit{SMC}^{\mathbf{F}}(\varphi, M)$ contains $(2 \cdot |\mathit{Fal}(\varphi, M)| + 1)$ blocks with at most $|\mathrm{Cond}(\varphi)|$ worlds each. $\square$

3.3 Small Model Construction for Logic F+(CM)

For logic **F+(CM)** we need to ensure the transitivity of preference relation in $\mathit{gen}(\mathit{SMC}^{\mathbf{F+(CM)}}(\varphi, M))$. It is enough to obtain an **F+(CM)**-countermodel since for finite models transitivity implies smoothness.

Lemma 3.20 *Any model $\langle W, \succeq, V\rangle$ is smooth if W is finite and $\succeq$ is transitive.*

Proof. If there is some $w_0 \in ||\alpha||^M$, consider (any) longest path $w_0 \prec w_1 \prec w_2 \prec \ldots$ with worlds from $||\alpha||^M$ staring from w_0. Since W is finite and there can be no repetitions on the path due to transitivity of $\succ$, the path is finite and there is the last world w_m for which there is no $u \in ||\alpha||^M$ such that $u \succ w_m$, so $w_m \in \max(||\alpha||^M)$ and either $w_m = w_0$ or $w_m \succ w_0$. $\square$

In $\mathit{SMC}^{\mathbf{F}}(\varphi, M)$ non-transitivity was essential: imposing $\mathtt{mchain} \succ^L \mathtt{fal}_v$ can violate condition (*iii*) since we selected worlds in the maximal chain from the whole initial model. However, smoothness allows us to select a maximal chain only among worlds in $\mathit{Bet}_{\succ}(v)$. Specifiaclly, for every falsifying world v we introduce individual *chain-orbit* $\mathit{ChainOrbit}(M, \varphi, v) = \mathit{chain}(\mathit{MaxSeq}_M(\mathit{Bet}_{\succ}(v), \mathrm{Cond}(\varphi)))$. We already know that this block is (*iv*)-safe by Lem. 3.16, and we can show that for an **F+(CM)**-model M it covers block $\mathtt{fal}_v$.

Lemma 3.21 *For a transitive and smooth M, $\mathit{ChainOrbit}(M, \varphi, v)$ (iv)-covers $\mathit{antichain}(\{v\})$.*

Proof. Suppose that (1) there is some $u \succ v$ in M such that $M, u \models \alpha$ for some $\alpha \in \mathrm{Cond}(\varphi)$, we need to show that there is a world z_k in the chain such that $M, z_k \models \alpha$. Similarly to Lem. 3.17, we show it by proving that in this case α is removed from the set of conditions $\mathcal{A}_i$ at some point. And to show this, it is enough to prove that (*) $\alpha \in \mathcal{A}_i$ implies $\mathcal{D}(\mathit{Bet}_{\succ}(v), \mathcal{A}_i) \neq \emptyset$ (then the sequence of maximal worlds cannot end while α belongs to $\mathcal{A}_i$).

Let us prove (*). Suppose that $\alpha \in \mathcal{A}_i$. From this and (1) follows $u \in ||\bigvee_{\alpha \in \mathcal{A}_i} \alpha||^M$. By smoothness of M it implies that (2) there is $u' \in \max(||\bigvee_{\alpha \in \mathcal{A}_i} \alpha||^M)$ such that either $u' = u$ or $u' \succ u$. In either case $u' \succ v$ (since $u \succ v$ by (1) and $\succeq$ is transitive). So, we have (3) $u' \in \mathit{Bet}_{\succ}(v)$. (2) and (3) together imply $u' \in \mathcal{D}(\mathit{Bet}_{\succ}(v), \mathcal{A}_i)$, concluding the proof of (*). $\square$

So we can obtain a small model construction for **F+(CM)** by replacing each orbit $\mathtt{orb}_v$ with an individual maximal chain $\mathit{ChainOrbit}(M, \varphi, v)$ (which we will label $\mathtt{chorb}_v$). The common chain `mchain` from $\mathit{SMC}^{\mathbf{F}}(\varphi, M)$ is not needed anymore.

Definition 3.22 (Small Model Construction for F+(CM))
If M is an **F+(CM)**-countermodel for φ, $\mathit{SMC}^{\mathbf{F+(CM)}}(\varphi, M) = \langle L, \succeq^L, \mathit{comp}\rangle$ where $L = \{\mathtt{fal}_v, \mathtt{chorb}_v \mid v \in \mathit{Fal}(\varphi, M)\}$, $\mathit{comp}(\mathtt{fal}_v) = \mathit{antichain}(\{v\})$, $\mathit{comp}(\mathtt{chorb}_v) = \mathit{ChainOrbit}(M, \varphi, v)$ and a preference relation $\succeq^L$ on blocks is demonstrated on Fig. 2c.

Theorem 3.23 *If M is a **F+(CM)**-countermodel for φ then $\mathit{gen}(\mathit{SMC}^{\mathbf{F+(CM)}}(\varphi, M))$ is a **F+(CM)**-countermodel for φ and $|W(\mathit{gen}(\mathit{SMC}^{\mathbf{F+(CM)}}(\varphi, M)))| = \mathcal{O}(|\varphi|^2)$.*

Proof. $\mathit{SMC}^{\mathbf{F+(CM)}}(\varphi, M)$ is a countermodel for φ by Th. 3.7, because we have $W(\mathtt{chorb}_v) \subseteq \mathit{Bet}_\succ(v)$, each $\mathtt{chorb}_v$ is (iv)-safe by Lem. 3.16 and each $\mathtt{fal}_v$ is (iv)-covered by Lem. 3.21. $\mathit{SMC}^{\mathbf{F+(CM)}}(\varphi, M)$ is an **F+(CM)**-countermodel by Lem. 3.20. $|W(\mathit{gen}(\mathit{SMC}^{\mathbf{F+(CM)}}(\varphi, M)))| = \mathcal{O}(|\varphi|^2)$ since $\mathit{SMC}^{\mathbf{F+(CM)}}(\varphi, M)$ contains $(2 \cdot |\mathit{Fal}(\varphi, M)|)$ blocks with at most $|\mathrm{Cond}(\varphi)|$ worlds each. □

Remark 3.24 The form of countermodel we obtain — a union of incomparable finite chains — is the same as a Friedman-Halpern countermodel for logic **PCA** (i.e. **F+(CM)**) [7]. However, we have achieved it by using different methods: they use a finite-model property of **PCL** extensions (shown in [3]) and extend the preference relation to a linear order, then construct chains by selecting the greatest world w.r.t. extended order independently for each conditional in $Ob^+(\varphi, M)$, while we do it using an iterative procedure. The possibility of their selection fully relies on finitedness and transitivity, which due to Lem. 3.20 is only possible in smooth models, so it cannot be applied to the weaker logics **E** and **F**. Furthermore, the Horn fragment [3] of **PCA** was studied extensively in the area of non-monotonic reasoning, where it is known as the KLM logic **P** [11] of preferential reasoning. A small model construction for **P** has been introduced in [12] and consists of a single chain of polynomial size (by essentially the same method as Friedman-Halpern). Notice, that both Friedman-Halpern and our constructions turn into a single chain when restricted to Horn formulas.

3.4 Small Model Construction for Logic G

For logic **G** we also need to ensure the totalness of the transformed model by leveraging that the falsifying worlds in $\mathit{Fal}(\varphi, M)$ are ordered in the initial model by $\succeq$ which in a **G**-model is a total preorder.

Let us consider first a simple case where $\succeq$ in the given **G**-countermodel is asymmetric (and therefore a strict linear order). Then there exists an ordering $v_1 \prec \cdots \prec v_n$ of worlds from $\mathit{Fal}(\varphi, M)$. Then we can linearly order blocks of $\mathit{SMC}^{\mathbf{F+(CM)}}(\varphi, M)$ with the following order: $\mathtt{fal}_{v_1} \prec^L \mathtt{chorb}_{v_1} \prec^L \ldots \prec^L \mathtt{fal}_{v_n} \prec^L \mathtt{chorb}_{v_n}$. The (iii)-preservation of every block $\mathtt{fal}_{v_i}$ will be still satisfied with such ordering, because for every $j \geq i$ we have $W(\mathtt{chorb}_{v_j}) \subseteq \mathit{Bet}_\succ(v_j)$ and $\mathit{Bet}_\succ(v_j) \subseteq \mathit{Bet}_\succ(v_i)$ due to transitivity of $\succeq$.

[3] Conditional Horn formula is a formula of a form $\bigcirc(\gamma_1 \,|\, \alpha_1) \wedge \cdots \wedge \bigcirc(\gamma_n \,|\, \alpha_n) \to \bigcirc(\gamma_0 \,|\, \alpha_0)$

In general, $\succeq$ is not necessarily asymmetric, but we can generalize the same idea by grouping together $\succeq$-equivalent worlds as in the following definition.

Definition 3.25 (Stratification) For a finite set U and a binary relation $\succeq$ on it, a list $[U_1, \ldots, U_n]$ is called a *stratification of* U when U is the disjoint union of non-empty subsets $\{U_i\}$ and $u_i \succeq u_j$ iff $i \geq j$ for every $u_i \in U_i, u_j \in U_j$.

For total and transitive $\succeq$ (as in **G**-models) the unique stratification of any finite set is given by its factorization w.r.t. $\succeq$-equivalence (see appendix B for details). So we can take the stratification $[V_1, \ldots, V_n]$ of $Fal(\varphi, M)$ w.r.t. $\succeq$ and create a clique block for every group V_i.

Definition 3.26 (Clique) For $U \subseteq W(M)$, $clique(U) = \langle U, \succeq^{cl} \rangle$ where $u_1 \succeq_{cl} u_2$ for all $u_1, u_2 \in U$.

Notice that for $v_1, v_2 \in V_i$ both $v_1 \succeq v_2$ and $v_2 \succeq v_1$ so $Bet_{\succ}(v_1) = Bet_{\succ}(v_2)$ (due to transitivity). Therefore, to (iv)-cover block $clique(V_i)$ we can take orbit-chain $ChainOrbit(M, \varphi, rep(V_i))$ with arbitrary representative of V_i.

Lemma 3.27 *For a transitive and smooth $M = \langle W, \succeq, V \rangle$ and $V \subseteq W(M)$, if $v \succeq v'$ for all $v, v' \in V$ then $ChainOrbit(M, \varphi, rep(V))$ (iv)-covers $clique(V)$.*

Proof. For any $v \in V_i$ holds $Bet_{\succ}(v) = Bet_{\succ}(rep(V))$ (since $\succeq$ is transitive), so any formula from $\mathrm{Cond}(\varphi)$ satisfied in some world from $Bet_{\succ}(v)$ is also satisfied by some world in $ChainOrbit(M, \varphi, rep(V))$ by Lem. 3.21. □

Then we can take as the construction $SMC^{\mathbf{G}}(\varphi, M)$ a linearly-oredered sequence of blocks in which cliques $clique(V_i)$, labeled $\mathtt{gfal}_{V_i}$, are interleaved with chain-orbits $ChainOrbit(M, \varphi, rep(V_i))$, labeled $\mathtt{gchorb}_{V_i}$.

Definition 3.28 (Small Model Construction for G) For a **G**-countermodel $M = \langle W, \succeq, V \rangle$ for a formula φ, let $[V_1, \ldots, V_n]$ be the stratification of $Fal(\varphi, M)$ w.r.t. $\succeq$. Then $SMC^{\mathbf{G}}(\varphi, M) = \langle L, \succeq^L, comp \rangle$ where $L = \bigcup_{i=1}^{n} \{\mathtt{gfal}_{V_i}, \mathtt{gchorb}_{V_i}\}$, $comp(\mathtt{gfal}_{V_i}) = clique(V_i)$, $comp(\mathtt{gchorb}_{V_i}) = chain(ChainOrbit(M, \varphi, rep(V_i)))$ and the blocks are ordered linearly as follows: $\mathtt{gfal}_{V_1} \prec^L \mathtt{gchorb}_{V_1} \prec^L \ldots \prec^L \mathtt{gfal}_{V_n} \prec^L \mathtt{gchorb}_{V_n}$.

Theorem 3.29 *If M is an **G**-countermodel for φ then $gen(SMC^{\mathbf{G}}(\varphi, M))$ is a **G**-countermodel for φ and $|W(gen(SMC^{\mathbf{G}}(\varphi, M)))| = \mathcal{O}(|\varphi|^2)$.*

Proof. $SMC^{\mathbf{G}}(\varphi, M)$ is a countermodel for φ by Th. 3.7: $\mathtt{gfal}_{V_i}$ is flat and (iii)-preserving (since $\succeq$ in M is transitive), each $\mathtt{gchorb}_{V_i}$ is (iv)-safe by Lem. 3.16 and each $\mathtt{gfal}_{V_i}$ is (iv)-covered by Lem. 3.27. $SMC^{\mathbf{G}}(\varphi, M)$ is a **G**-countermodel since its preference relation is transitive and total, and also smooth by Lem. 3.20. $|W(gen(SMC^{\mathbf{G}}(\varphi, M)))| = \mathcal{O}(|\varphi|^2)$ since $SMC^{\mathbf{G}}(\varphi, M)$ contains at most $(2 \cdot |Fal(\varphi, M)|)$ blocks with at most $|\mathrm{Cond}(\varphi)|$ worlds each. □

Remark 3.30 Friedman and Halpern also provide a counter-model for logic **VTA** (i.e. **G**) [7]. They use an ad hoc approach, different from the one they use for the other extensions of **PCL**. For **VTA** for each conditional they simply take one world from the original model without changing the preference relation, resulting in a model of linear size. Although Th. 3.2 can be also used to establish the adequacy of their constriction, we provided a different construction (with a new explicitly defined preference relation and with potentially a quadratic number of worlds) for uniformity with the constructions for the other three logics.

4 Applications

In this section, we describe two applications of our small model constructions: alternative semantical characterizations and computational applications.

4.1 Alternative semantical characterizations

New semantical characterizations for theoremhood can be extracted from the specific form of our small model constructions. Namely, any model property satisfied by $SMC^{\mathcal{L}}(\varphi, M)$ (for all φ and M) that is stronger than some existing characterization for $\mathcal{L}$ (e.g. from Tab. 1) can be used as an alternative characterization: theorems are satisfied in all models with this property, while any non-theorem φ has a countermodel $SMC^{\mathcal{L}}(\varphi, M)$ with this property.

We can use this method to characterize theoremhood in Åqvist logics with *frame properties*, i.e. properties of the preference relation. Notice that the limit conditions (limitedness and smoothness) used for the characterization of **F**, **F+(CM)**, and **G** are not frame properties: they impose conditions only on truth sets of the model. This choice plays a vital role in establishing correspondence between semantics and known axiomatizations of Åqvist's logics, but it makes it hard to work with these models since you need to distinguish which subsets of worlds can be a truth set.

However, our constructions satisfy some stronger frame properties. In fact, we already used these properties to prove that $SMC^{\mathcal{L}}(\varphi, M)$ generates an $\mathcal{L}$-countermodel. If we consider only finite models (which can be seen as a frame condition itself) limit conditions can be replaced with much more natural conditions on relations that our constructions satisfy: acyclicity of $\succ$ instead of limitedness (due to Lem. 3.13) and transitivity instead of smoothness (due to Lem. 3.20). Moreover, $gen(SMC^{\mathbf{F}}(\varphi, M))$ almost satisfies the acyclicity of $\succeq$, not only acyclicity of $\succ$. The only cycles in $SMC^{\mathbf{F}}(\varphi, M)$ are loops on the worlds in the chains, which can be removed without affecting evaluation in the model. Thus we get the following convenient finite-model characterizations.

Corollary 4.1 *Formula φ is a theorem of Åqvist logic $\mathcal{L}$ iff $M \models \varphi$ for all:*

- *finite models M (for $\mathcal{L} =$ **E**);*
- *finite models M with acyclic preference relation (for $\mathcal{L} =$ **F**);*
- *finite models M with transitive preference relation (for $\mathcal{L} =$ **F+(CM)**);*
- *finite models M with transitive and total preference relation (for $\mathcal{L} =$ **G**).*

In addition to that our models for **E**, **F**, **F+(CM)** satisfy antisymmetry so it can be augmented to any existing characterization for these logics, but not to characterizations of **G** (see appendix C). Either reflexivity or irreflexivity can also be added since it is trivial to force them in any model without changing it satisfaction relation. Thus, **F+(CM)** is characterized by finite models where $\succeq$ is a partial order (strict or non-strict). At the same time, finite models where $\succeq$ is a linear order give some logic that is stronger than **G** (see the same appendix C). Even more specialized properties can be extracted from our construction, e.g. **F+(CM)** can be characterized by models that are unions of non-comparable finite chains.

4.2 Complexity and automated deduction

Our small model constructions show that for any non-valid formula there exists a countermodel with at most $N(\varphi)$ worlds, where $N(\varphi)$ is a certain upper bound polynomial w.r.t. $|\varphi|$. Plus, the stronger frame properties from Cor. 4.1 can be easily checked in polynomial time w.r.t. the model size. This immediately implies co-NP-completeness of theoremhood.

Corollary 4.2 *Theoremhood is co-NP-complete for any Åqvist's logic.*

Proof. Non-theoremhood can be checked non-deterministically in polynomial time by guessing a countermodel M of size at most $N(\varphi)$ (i.e. guessing preference relation and valuation for all variables occurring in φ) and then checking $M \not\models \varphi$ and the properties from Cor. 4.1 for the required logic. co-NP-hardness follows from co-NP-completeness of theoremhood in classical logic (since propositional formula is a classical theorem iff it is a theorem of an Åqvist logic). □

Moreover, with simpler finite-model characterization from Cor. 4.1 a countermodel definition can be rather straightforwardly encoded with a propositional formula of a polynomial size: having variables $p_{i,j}$ to encode the $w_i \succeq w_j$ and variables v_i^ψ to encode the fact $M, w_i \models \psi$ for $\psi \in Sub\mathcal{F}(\varphi)$, it is trivial to encode both evaluation and the frame properties (for acyclicity via additional variables for a transitive closure of the preference relation), and then require v_1^φ to be false. This propositional formula can be given to any SAT-solver for efficient theoremhood checking and countermodels can be trivially reconstructed from classical models found by the solver.

Concluding remark

In this paper, we provide small model constructions for Åqvist's logics, which can be used to understand theoretical properties of these logics (such as finite-model semantical characterizations and complexity) and to generate countermodels for non-valid formulas using SAT-solvers. Ideally, this should be complemented by analytic calculi which provide transparent derivations for valid formulas. We plan to further investigate the connection between our constructions and hypersequent calculi in hope that it can hint at simpler proof-theoretic characterizations, especially for the tricky logic **F**.

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Appendix

A Detailed proof of Th. 3.2

Theorem A.1 *Let φ be a formula and $M = \langle W, \succeq, V \rangle \in \mathcal{M}$ such that $M \not\models \varphi$. If a model $M' = \langle W', \succeq', V' \rangle \in \mathcal{M}$ rearranges M with the prototype function $prot\colon W' \to W$ then the following four conditions are sufficient for $M' \not\models \varphi$.*

(i) *There exists $v' \in W'$ such that $M, prot(v') \not\models \varphi$.*

(ii) *For any $\Box\beta \in Box^-(\varphi, M)$ there exists $v' \in W'$ such that $M, prot(v') \not\models \beta$.*

(iii) *For any $\bigcirc(\gamma \,|\, \alpha) \in Ob^-(\varphi, M)$ there exists $v' \in W'$ such that $prot(v') \in \max(||\alpha||^M) \setminus ||\gamma||^M$ and for all $u' \succ' v'$ holds $prot(u') \succ prot(v')$.*

(iv) *For any $w' \in W'$, for all $\bigcirc(\gamma \,|\, \alpha) \in Ob^+(\varphi, M)$ if there exists $u \succ prot(w')$ such that $M, u \models \alpha$ then there exists $u' \succ' w'$ such that $M, prot(u') \models \alpha$.*

Proof. We will prove that for any $w' \in W'$ and any $\psi \in Sub\mathcal{F}(\varphi)$ holds $M', w' \models \psi$ iff $M, prot(w') \models \psi$. Then $M' \not\models \varphi$ follows by the condition (i). The proof is by induction on ψ (we use the abbreviation IH(s) to refer to the inductive hypothesis(-es)).

- $\psi = x$. $w \in V(x)$ iff $prot(w) \in V'(x)$ by the definition of the prototype function.
- $\psi = \neg\psi'$. Directly from IH for ψ'.
- $\psi = \psi_1 \wedge \psi_2$. Directly from IHs ψ_1 and ψ_2.
- $\psi = \Box\beta$ and $\psi \in Box^+(\varphi, M)$. $M \models \Box\beta$, so for all $w' \in W'$ holds $M, prot(w') \models \beta$, so by IH for all $w' \in W'$ holds $M', w' \models \beta$, so $M' \models \Box\beta$.
- $\psi = \Box\beta$ and $\psi \in Box^-(\varphi, M)$. By ($ii$) there is $v' \in W'$ such that $M, prot(v') \not\models \beta$, so by IH holds $M', v' \not\models \beta$, so $M' \not\models \Box\beta$.
- $\psi = \bigcirc(\gamma \mid \alpha)$ and $\psi \in Ob^+(\varphi, M)$. Take any $w' \in W'$ such that $w' \in \max(||\alpha||^{M'})$. Then (1) $prot(w') \in \max(||\alpha||^M)$: $prot(w') \in ||\alpha||^M$ by IH, and there can be no $s \succ prot(w')$ such that $s \in ||\alpha||^M$ (otherwise there would be $u' \succ' w'$ such that $M', u' \models \alpha$ by (iv) and IH). Since $M \models \bigcirc(\gamma \,|\, \alpha)$, (1) implies $M, prot(w') \models \gamma$, which implies $M', w' \models \gamma$ by IH. Thus $M' \models \bigcirc(\gamma \,|\, \alpha)$.
- $\psi = \bigcirc(\gamma \,|\, \alpha)$ and $\psi \in Ob^-(\varphi, M)$. For the corresponding world $v' \in W'$ from (iii) we have $v' \notin ||\gamma||^{M'}$ (by IH and the choice of v') and $v' \in \max(||\alpha||^{M'})$ (since $v' \in ||\alpha||^{M'}$ by IH and the choice of v', and for all $u' \succ' v'$ we have $u' \notin ||\alpha||^{M'}$ by (iii) and IH), so $M' \not\models \bigcirc(\gamma \,|\, \alpha)$.

□

B Stratification

Definition B.1 (Stratification) For a finite set U and a binary relation $\succeq$ on it, list $[U_1, \ldots, U_n]$ is called *stratification of* U when U is the disjoint union of non-empty subsets $\{U_i\}$ and $u_i \succeq u_j$ iff $i \geq j$ for every $u_i \in U_i, u_j \in U_j$.

Lemma B.2 *For a finite set U and a transitive and total relation $\succeq$ on it, there exists a unique stratification of U w.r.t $\succeq$.*

Proof. Consider an equivalence relation $\approx$ on U where $u_1 \approx u_2$ means that both $u_1 \succeq u_2$ and $u_2 \succeq u_1$. Consider further a relation $\succeq_s$ on the set of equivalence classes of U w.r.t. $\approx$ where $U_i \succeq_s U_j$ when there exist $u_i \in U_i$ and $u_j \in U_j$ such that $u_i \succeq u_j$. Notice that for a transitive and total $\succeq$ the relation $\succeq_s$ is a linear order: it is antisymmetric due to definitions of $\approx$ and $\succeq_s$, and it is transitive and total (and hense reflexive) due to the transitivity and totalness of $\succeq$. This linear ordering gives a stratification by definition. Notice also that it is the only stratification: every element of a stratification should be an equivalence class w.r.t. $\approx$ and their order in the list should preserve $\succeq_s$ (i.e. $U_i \succeq_s U_j$ for $i \geq j$) by definition, so it has to be the sequence of the same elements ordered the same way as in the construction described above. □

C Logic of finite linearly-ordered models

We showed that for logics **E**, **F** and **F+(CM)** antisymmetry can be added to our finite-model characterizations. The same is not true for the logic **G**: we characterize **G** by finite, transitive and total models, if we add antisymmetry to this list of properties we will get the logic of finite models with preference relation being a linear order, and this logic is strictly stronger than **G**.

In particular, the following principle of conditional excluded middle from conditional logics is validated in all linearly-ordered models.

$$\text{(CEM)} \qquad \bigcirc(\gamma \,|\, \alpha) \vee \bigcirc(\neg\gamma \,|\, \alpha)$$

Indeed, in any linear model M there is at most one α-maximal world (if there are two different α-maximal worlds, one of them should be strictly preferable to another due to totality and antisymmetry, which contradicts the definition of maximality). If there are no α-maximal worlds in M then M satisfies both $\bigcirc(\gamma \mid \alpha)$ and $\bigcirc(\neg\gamma \mid \alpha)$ by definition. And if there is a unique α-maximal world v in M then either $M, v \models \gamma$ (then $M \models \bigcirc(\gamma \,|\, \alpha)$) or $M, v \models \neg\gamma$ (then $M \models \neg\gamma \bigcirc(\gamma \,|\, \alpha)$). In either case $M \models \bigcirc(\gamma \,|\, \alpha) \vee \bigcirc(\neg\gamma \,|\, \alpha)$.

At the same time, **G** does not validate (CEM): for example consider the formula $\chi = \bigcirc(y \,|\, x) \vee \bigcirc(\neg y \,|\, x)$ (where $x, y \in \mathit{Var}$), which is an instance of (CEM) and consider the preference model $M = \langle W, \succ^{cl}, V \rangle$ where $W = \{w_1, w_2\}$, $w \succ^{cl} w'$ for any $w, w' \in W$, and $V(x) = \{w_1, w_2\}$ and $V(y) = \{w_1\}$. M is a **G**-model (by Lem. 3.20), both w_1 and w_2 are α-maximal by definition and $M, w_2 \not\models y$ and $M, w_1 \not\models \neg y$, so neither $M \models \bigcirc(y \,|\, x)$ nor $M \models \bigcirc(\neg y \,|\, x)$, so $M \not\models \chi$, therefore χ is not a theorem of **G**.

Thus, the logic of finite linearly-ordered models is some conditional logic strictly stronger than **G**, which satisfies (CEM).