# Value-at-Risk- and Expectile-based Systemic Risk Measures and Second-order Asymptotics: With Applications to Diversification 

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#### Abstract

The systemic risk measure plays a crucial role in analyzing individual losses conditioned on extreme system-wide disasters. In this paper, we provide a unified asymptotic treatment for systemic risk measures. First, we classify them into two families of Value-at-Risk- (VaR-) and expectile-based systemic risk measures. While VaR has been extensively studied, in the latter family, we propose two new systemic risk measures named the Individual Conditional Expectile (ICE) and the Systemic Individual Conditional Expectile (SICE), as alternatives to Marginal Expected Shortfall (MES) and Systemic Expected Shortfall (SES). Second, to characterize general mutually dependent and heavy-tailed risks, we adopt a modeling framework where the system, represented by a vector of random loss variables, follows a multivariate Sarmanov distribution with a common marginal exhibiting second-order regular variation. Third, we provide second-order asymptotic results for both families of systemic risk measures. This analytical framework offers a more accurate estimate compared to traditional first-order asymptotics. Through numerical and analytical examples, we demonstrate the superiority of second-order asymptotics in accurately assessing systemic risk. Further, we conduct a comprehensive comparison between VaR-based and expectile-based systemic risk measures. We find that expectile-based measures output higher risk evaluation than VaR-based ones, emphasizing the former's potential advantages in reporting extreme events and tail risk. As a financial application, we use the asymptotic treatment to discuss the diversification benefits associated with systemic risk measures. The financial insight is that the expectile-based diversification benefits consistently deduce an underestimation and suggest a conservative approximation, while the VaR-based diversification benefits consistently deduce an overestimation and suggest behaving optimistically.


Keywords: Asymptotic approximation; Systemic risk; Expectile; Sarmanov distribution; Secondorder regular variation; Diversification benefit.

## 1 Introduction

Financial risks refer to the potential situation that can negatively affect the stability of an individual financial institution, a specific financial market, or even the global economy. Analyzing and preparing for extreme events that align with these adverse scenarios is always an important field of risk management. To study these extreme events, the extreme value theory (EVT) is a useful framework. The EVT offers a contemporary collection of statistical tools and techniques that can be used to

[^0]address various questions related to risk assessment and management in finance. Financial risks are typically divided into different categories depending on their characteristics. Market risk, credit risk, and operational risk are the primary risk groups in banks that have been extensively studied using quantitative assessment methods and regulated by authorities. After the global financial crises in 2008-2009, the concept of systemic risk, which refers to the risk of multiple financial institutions failing together and causing widespread impact, has gained significant attention from regulators and researchers in the field. Various measures related to the systemic risk have been proposed in the literature, including Systemic Expected Shortfall (SES) and Marginal Expected Shortfall (MES) by Acharya et al. (2017) and Chen and Liu (2022), scenario-based risk measures by Wang and Ziegel (2021), conditional distortion risk measures by Dhaene et al. (2022), generalized risk measures by Fadina et al. (2024) and others.

Following recent studies of systemic risks in banking, finance and insurance, we quantify SES and MES in a general context of quantitative risk management. Let the aggregate risk $S_{n}=\sum_{i=1}^{n} X_{i}$, where the allocation of capital to each individual risk $X_{1}, \ldots, X_{n}$. For the sum $S_{n}$, for $p \in(0,1)$, according to the Euler principle (see Dhaene et al. (2012) or Acharya et al. (2017)), the risk allocated to line $m \in\{1, \ldots, n\}$ is defined by

$$
\operatorname{MES}_{p, m}\left(S_{n}\right):=\mathbb{E}\left[X_{m} \mid S_{n}>\operatorname{VaR}_{p}\left(S_{n}\right)\right],
$$

or

$$
\operatorname{SES}_{p, m}\left(S_{n}\right):=\mathbb{E}\left[\left(X_{m}-\operatorname{VaR}_{p}\left(X_{m}\right)\right)_{+} \mid S_{n}>\operatorname{VaR}_{p}\left(S_{n}\right)\right],
$$

where the Value-at-Risk (VaR) is defined as the loss distribution of $X$ :

$$
\operatorname{VaR}_{p}(X):=F_{X}^{\overleftarrow{X}}(p)=\inf \{t \in \mathbb{R}: F(t) \geq p\}
$$

where $F_{X}^{\overleftarrow{X}}(p)$ is the inverse of the distribution function. Another popular risk measure is Expected Shortfall (ES):

$$
\operatorname{ES}_{p}(X):=\frac{1}{1-p} \int_{p}^{1} \operatorname{VaR}_{t}(X) \mathrm{d} t
$$

which is the average value on the tail above $\operatorname{VaR}_{p}$. The ES (sometimes called Tail-Value-at-Risk (TVaR)) is a coherent risk measure in the sense of Artzner et al. (1999). If $F$ is continuous, ES coincides with the Conditional Talied Expectation (CTE), which represents the conditional expected loss given that the loss exceeds its VaR:

$$
\operatorname{ES}_{p}(X)=\operatorname{CTE}_{p}(X):=\mathbb{E}\left[X \mid X>\operatorname{VaR}_{p}(X)\right] .
$$

Though enjoying several merits, VaR and ES have some drawbacks. Specifically, VaR does not possess subadditivity, which excludes VaR from the good class of coherent risk measures. ES does not satisfy the elicitability, which is a property recently arousing interest in the field of risk management. Here, a risk measure is said to be elicitable if it can be defined as the minimizer of a suitable expected loss function. Elicitability is important in backtesting of a risk measure as it provides a natural methodology to perform backtesting. Meaningful point forecasts and forecast performance
comparisons then become possible for elicitable risk measures; see Ziegel (2016).
Following Newey and Powell (1987), the expectile $\mathrm{e}_{p}(X)$ of order $p \in(0,1)$ of the variable $X$ can be defined as the minimizer of a piecewise quadratic loss function or, equivalently, as

$$
\mathrm{e}_{p}(X)=\arg \min _{\theta \in \mathbb{R}}\left\{p \mathbb{E}\left[\left((X-\theta)_{+}\right)^{2}\right]+(1-p) \mathbb{E}\left[\left((X-\theta)_{-}\right)^{2}\right]\right\},
$$

where $x_{+}:=\max (x, 0)$ and $x_{-}:=\min (x, 0)$. The presence of terms $X_{+}^{2}$ and $X_{-}^{2}$ makes this problem well defined indeed as soon as $X \in \mathrm{~L}^{2}$ (i.e. $\mathbb{E}|X|^{2}<\infty$ ). The related first-order necessary condition optimality can be written in several ways, one of them being

$$
\begin{equation*}
\mathrm{e}_{p}(X)-\mathbb{E}[X]=\frac{2 p-1}{1-p} \mathbb{E}\left[\left(X-\mathrm{e}_{p}(X)\right)_{+}\right] \tag{1.1}
\end{equation*}
$$

This equation has a unique solution for all $X \in \mathrm{~L}^{1}$. Thenceforth expectiles of a distribution function $F$ with a finite absolute first-order moment are well defined, and we assume that $\mathbb{E}|X|<\infty$ throughout. Expectiles summarize the distribution function in much the same way as the quantiles; see Gneiting (2011). The expectile and VaR are elicitable while ES is not; see Gneiting (2011). Actually, the expectile $\mathrm{e}_{p}$ with $p \geq \frac{1}{2}$ is the only risk measure which is elicitable, law-invariant and coherent; see Bellini and Bignozzi (2015). As a result, the expectile is suggested (see Emmer et al. (2015)) as a potential alternative to both VaR and ES. The study on expectiles also becomes increasingly popular in the econometric literature; see, for example, De Rossi and Harvey (2009), Kuan et al. (2009).

In this paper, we provide a unified asymptotic treatment to the systemic risk measures. The treatment has three steps. The first step is that the systemic risk measures are classified into two representative families: Value-at-Risk- (VaR-) and expectile-based measures. From the definitions, $\mathrm{VaR}_{p}$ and $e_{p}$ exhibit distinct mathematical properties. As we can see below, both of them lead to many systemic risk measures; e.g., $\mathrm{CTE}_{p}$ (or $\mathrm{ES}_{p}$ ), $\mathrm{MES}_{p, m}$ and $\mathrm{SES}_{p, m}$ are categorized within the family of VaR-based risk measures. As a result, $\mathrm{VaR}_{p}$ and $e_{p}$ can serve as building blocks in assessing systemic risk and form essential foundations for further study. In particular, Taylor (2008) introduced an expectile-based alternative of ES, known as the Conditional Expectile ( $\mathrm{CE}_{p}$ ):

$$
\mathrm{CE}_{p}(X):=\mathbb{E}\left[X \mid X>\mathrm{e}_{p}(X)\right],
$$

where $\mathrm{CE}_{p}$ represents the expectation of exceedances beyond the $p$-th expectile $\mathrm{e}_{p}$ of the distribution of $X$.

To evaluate the allocation of each individual agent to the systemic risk, we further propose two new expectile-based systemic risk measures on the sum variable. We call them the Individual Conditional Expectile (ICE) and the Systemic Individual Conditional Expectile (SICE):

$$
\operatorname{ICE}_{p, m}\left(S_{n}\right):=\mathbb{E}\left[X_{m} \mid S_{n}>\mathrm{e}_{p}\left(S_{n}\right)\right]
$$

and

$$
\operatorname{SICE}_{p, m}\left(S_{n}\right):=\mathbb{E}\left[\left(X_{m}-\mathrm{e}_{p}\left(X_{m}\right)\right)_{+} \mid S_{n}>\mathrm{e}_{p}\left(S_{n}\right)\right] .
$$

ICE and SICE stand from a view of expectile to capture an individual agent's risk profile conditional on a system-wide catastrophe. For comparison, the VaR estimation knows only whether an observation is below or above the predictor. It would be inaccurate to measure an extreme risk based on only the frequency of tail losses and not on their values. The expectile makes more efficient use of the available data since it optimizes the discrepancy between the observations and the predictor. Particularly, ICE represents the potential losses an individual would suffer conditional on the tail of the system's loss distribution. SICE is an improved version of ICE and reveals the individual's excess loss to his/her expectile $\mathrm{e}_{p}\left(X_{m}\right)$ conditional on the systemic catastrophe.

In the second step, to characterize a general dependence and heavy-tailed risks, we assume that the risks $X_{1}, \ldots, X_{n}$ are dependent on each other through a multivariate Sarmanov distribution. This characterizes a more general dependence structure than the commonly used Farlie-GumbelMorgenstern (FGM) copula; see Yang and Hashorva (2013). Meanwhile, the FGM copula has some drawbacks in terms of correlation coefficients; see Section 2.2. In particular, the Sarmanov distribution is flexible in combining different types of marginals, making it suitable for modeling various risks. The advantage of using the Sarmanov distribution lies in its ability to capture dependencies and helps the evaluation of joint probabilities.

| Systemic risk measure | First-order asymptotic | Second-order asymptotic |
| :---: | :--- | :--- |
| $\operatorname{VaR}_{p}\left(S_{n}\right)$ | Bingham et al. (1989); Barbe <br> et al. (2006); Embrechts et al. <br> (2009b) and so on. | Degen et al. (2010); Mao and Yang <br> (2015); Theorem 3.1 of our paper |
| $\operatorname{CTE}_{p}\left(S_{n}\right)$ | Alink et al. (2005); Chen et al. <br> (2012); Kley et al. (2020) and so <br> on. | Mao and Hu (2013); Lv et al. (2013); <br> Theorem 3.1 of our paper |
| $\operatorname{MES}_{p, m}\left(S_{n}\right)$ | Asimit et al. (2011); Joe and Li <br> (2011); Jaune and Siaulys (2022) <br> and so on. | Hua and Joe (2011); Theorem 3.2 of <br> our paper |
| $\operatorname{SES}_{p, m}\left(S_{n}\right)$ | Chen and Liu (2022) | Theorem 3.2 of our paper |
| $\mathrm{e}_{p}\left(S_{n}\right)$ | Bellini et al. (2014); Bellini and <br> Di Bernardino (2017) | Mao et al. (2015); Mao and Yang <br> $(2015) ;$ Theorem 4.1 of our paper |
| $\operatorname{CE}_{p}\left(S_{n}\right)$ | Dhaene et al. (2022) | Theorem 4.1 of our paper |
| $\operatorname{ICE}_{p, m}\left(S_{n}\right)$ | Emmer et al. (2015); Tadese and <br> Drapeau (2020) | Theorem 4.2 of our paper |
| $\operatorname{SICE}_{p, m}\left(S_{n}\right)$ | Theorem 4.2 of our paper | Theorem 4.2 of our paper |

Table 1: Contribution of our paper compared to the literature. Here VaR-based systemic risk measures include VaR, CTE, MES and SES, while expectile-based systemic risk measures include e, CE, ICE and SICE.

In the third step, we obtain the second-order asymptotics of the two families of systemic risk measures. Here we make our most theoretical contributions on asymptotic approximations. First, we investigate the second-order expansions of the tail probability of $S_{n}$ under multivariate Sarmanov distribution (Proposition 3.1 below), which generalizes the results of Mao and Hu (2013) and Theorem 4.4 of Mao and Yang (2015). Second, we study second-order asymptotic formulas of $\operatorname{VaR}_{p}\left(S_{n}\right), \operatorname{CTE}_{p}\left(S_{n}\right)$ (Theorem 3.1 below) and $\operatorname{MES}_{p, m}\left(S_{n}\right), \operatorname{SES}_{p, m}\left(S_{n}\right)$ (Theorem 3.2 below). Third, we use different methods to obtain the second-order asymptotic estimation of expectile, which
extends theorem 3.1 of Mao and Yang (2015) (Proposition 4.1 below). Fourth, we consider the secondorder asymptotic formulas of $\mathrm{e}_{p}\left(S_{n}\right), \mathrm{CE}_{p}\left(S_{n}\right)$ (Theorem 4.1 below) and $\operatorname{ICE}_{p, m}\left(S_{n}\right), \operatorname{SICE}_{p, m}\left(S_{n}\right)$ (Theorem 4.2). Lastly, we apply two examples to explain different risk measures based on VaR and expectile, where we use the Monte Carlo method to conduct the numerical simulation. Numerical and analytical examples illustrate that our second-order asymptotics provide an accurate estimate and behave much better than the first-order asymptotics. Further, we conduct a comprehensive comparison between these two families of systemic risk measures. We find that expectile-based systemic risk measures produce a larger risk evaluation than that of VaR-based systemic risk measures. Hence, the former has a potential advantage in reporting extreme events and amplifying the tail risk. Besides, this finding appeals for a lower confidence level when using expectile-based measures.

As a financial application, we explore economic insights on diversification with our asymptotic treatment. The idea of portfolio diversification dates back to the celebrated Markowitz mean-variance model, revealing the importance of mitigating risks in the investment. Diversification hence becomes a crucial topic in banking and insurance for risk management, integral to regulatory frameworks like Basel II and Solvency II. A lot of works propose quantitative ways to quantify the advantages of benefits. E.g., Chen et al. (2022) delved into comparing diversification advantages under the worstcase VaR and ES in the context of dependence uncertainty; see Cui et al. (2021) for more results. Among them, the diversification benefit, proposed by Bürgi et al. (2008), signifies the preserved capital achieved through collectively considering all risks in a portfolio versus addressing each risk in isolation; see Section 6 for a detailed definition. Based on our asymptotic treatment, we obtain the second-order asymptotics for the diversification benefits based on different risk measures, including VaR, CTE, e and MES. We find that CE, an expectile-based systemic risk measure, provides the most accurate approximation with an error range of less than $5 \%$. Further, the expectilebased diversification benefits consistently deduce an underestimation and suggest a conservative approximation, while the VaR-based diversification benefits consistently deduce an overestimation and suggest an optimistical approximation.

Finally, we discuss our results compared with the literature. In the field of risk management and capital allocation, it is necessary to determine how to allocate the acquired economic capital among different risks. In this case, the solvency capital has already been calculated using risk aggregation techniques; see Blanchet et al. (2020) for a recent treatment of risk aggregation. In recent years, the focus is on selection of appropriate models for multivariate risk factors, such as the choice of dependence structure model and the distributions of the marginals. Some recent contributions include the use of the FGM distribution (Yang and Hashorva (2013), Chen and Yang (2014), etc) and the Sarmanov distribution (Qu and Chen (2013), Yang and Wang (2013), etc), and multivariate regular variation (MRV) (Embrechts et al. (2009a), Asimit et al. (2011), etc). They usually aim to obtain the first-order asymptotics of some risk measures. On the contrary, our results provide a series of second-order asymptotics, which is much more accurate than the first-order asymptotics; this will be shown in the tables and figures later. This fact is also studied in Degen et al. (2010), Mao et al. (2012) and Mao and Hu (2013), but they obtained the second-order asymptotics for independent and identically distributed (iid) random variables (rvs) with second-order regularly varying tails. This independence assumption is too restrictive for practical problems. Mao and Yang (2015) studied
the case that $X_{1}, \ldots, X_{n}$ are dependent on each other through a multivariate FGM distribution, and derived second-order approximations of the risk concentrations of Value-at-Risk and expectile. However, our study contributes to the advancement of systemic risk measurement by introducing novel measures, developing a modeling framework, and providing enhanced asymptotic tools for risk assessment. Technically, the proposed two wide families of systemic risk measures can include the risk measures in Mao and Yang (2015). In particular, we provide rigorous and necessary lemmas for asymptotic treatment (Proposition 3.1, Lemmas 8.1-8.3) and offer a simpler proof for the key results.

The rest of the paper is organized as follows. In Section 2, we introduce the definitions of $\mathcal{R} \mathcal{V}$ and $2 \mathcal{R} \mathcal{V}$ and discuss the $n$-dimensional Sarmanov distribution. In Sections 3 and 4 , we obtain several second-order asymptotics of VaR- and expectile-based systemic risk measures and present examples to explain the main results. In Section 5, we give concrete examples to numerically illustrate these risk measures. Further, we apply the above asymptotic treatment to discuss financial diversification in Section 6. Section 7 concludes the paper. The Appendix provides details for the proofs.

## 2 Preliminaries

In this section, we first review the definitions and basic properties of regular variation $(\mathcal{R} \mathcal{V})$ and the second-order regular variation $(2 \mathcal{R} \mathcal{V}) .2 \mathcal{R} \mathcal{V}$ is a concept that is generalization of regular variation $(\mathcal{R} \mathcal{V})$, which has various applications in areas such as applied probability, statistics, risk management, telecommunication networks and so on. The idea of $2 \mathcal{R} \mathcal{V}$ was initially proposed to investigate the rate of convergence of the extreme order statistics in EVT; see De Haan and Stadtmüller (1996) and De Haan and Ferreira (2006). Next, we introduce the $n$-dimensional Sarmanov distribution. Its applications in many insurance contexts show its flexible structure when modeling the dependence between multivariate risks given the marginal distributions; see Qu and Chen (2013), Abdallah et al. (2016), Ratovomirija (2016) and so on.

### 2.1 Regular Variation

Definition 2.1 (Regular variation)
A measurable function $f: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$is said to be regular varying at $t_{0} \in[-\infty, \infty]$ with index $\alpha \in \mathbb{R}$, denoted by $f \in \mathcal{R} \mathcal{V}_{\alpha}^{t_{0}}$, if for all $x>0$

$$
\lim _{t \rightarrow t_{0}} \frac{f(t x)}{f(t)}=x^{\alpha}, \quad \text { for all } \quad x>0
$$

When $t_{0}=\infty$, we write $\mathcal{R} \mathcal{V}_{\alpha}^{t_{0}}=\mathcal{R} \mathcal{V}_{\alpha}$. In addition, if $\alpha=0$, then $f$ is said to be slowly varying at infinity.

Definition 2.2 (Second-order regular variation)
A measurable function $f: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$is said to be second-order regular varying with the first-order index $\alpha \in \mathbb{R}$ and second-order index $\beta \leq 0$, denoted by $f \in 2 \mathcal{R} \mathcal{V}_{\alpha, \beta}^{t_{0}}$, if there exists some eventually
positive or negative measurable function $A(\cdot)$ with $A(t) \rightarrow 0$ as $t \rightarrow t_{0}$ such that

$$
\lim _{t \rightarrow t_{0}} \frac{\frac{f(t x)}{f(t)}-x^{\alpha}}{A(t)}=x^{\alpha} \frac{x^{\beta}-1}{\beta}:=H_{\alpha, \beta}(x), \quad \text { for all } \quad x>0
$$

Here $H_{\alpha, \beta}(x)$ is $x^{\alpha} \log x$ if $\beta=0$, and $A(\cdot)$ that is called an auxiliary function of $f$. It is worth noting that the auxiliary function $A(\cdot) \in \mathcal{R} \mathcal{V}_{\beta}$; see for example Theorem 2.3.3 of De Haan and Ferreira (2006). If $t_{0}=\infty$, we write $2 \mathcal{R} \mathcal{V}_{\alpha, \beta}^{t_{0}}=2 \mathcal{R} \mathcal{V}_{\alpha, \beta}$.

Let $X_{1}, X_{2}, \ldots, X_{n}$ denote the financial losses, which are identically distributed random variables with a distribution function $F$. Here $\bar{F}(\cdot)$ means the survival function $\bar{F}(x)=1-F(x)$ and $F^{\leftarrow}(\cdot)$ means the generalized inverse function $F^{\leftarrow}(y)=\inf \{x \in \mathbb{R}: F(x) \geq y\}$. The tail quantile function associated with the distribution function $F$ is denoted by $U_{F}(\cdot)=(1 / \bar{F})^{\leftarrow}(\cdot)=F^{\leftarrow}(1-1 / \cdot)$. Note that $\bar{F}(\cdot) \in \mathcal{R} \mathcal{V}_{-\alpha}$ for all $\alpha \in \mathbb{R}$ is equivalent to $U_{F}(\cdot) \in \mathcal{R} \mathcal{V}_{1 / \alpha}$ (see Corollary 1.2.10 of De Haan and Ferreira (2006)). Furthermore, if $\bar{F}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \beta}$ with $\alpha>0, \beta \leq 0$ and auxiliary function $A(\cdot)$, by Theorem 2.3.9 of De Haan and Ferreira (2006), one can easily check that $U_{F}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{1 / \alpha, \beta / \alpha}$ with auxiliary function $\alpha^{-2} A \circ U_{F}(\cdot)$. Generally, the equality $F\left(F^{\leftarrow}(p)\right)=p$ does not hold true. It can be shown that if $\bar{F}(\cdot) \in \mathcal{R} \mathcal{V}_{-\alpha}$ with $\alpha>0$, then $F\left(F^{\leftarrow}(p)\right) \sim p$ as $p \uparrow 1$. If further assume that $\bar{F}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \beta}$ with auxiliary function $A(\cdot)$, then

$$
\begin{equation*}
U_{F}(1 / \bar{F}(t))=t(1+o(A(t))) \text { as } t \rightarrow \infty, \quad F\left(F^{\leftarrow}(p)\right)=p\left(1+o\left(F^{\leftarrow}(p)\right)\right) \text { as } p \uparrow 1 \tag{2.1}
\end{equation*}
$$

see Mao and Yang (2015) and Exercise 2.11 of De Haan and Ferreira (2006).

### 2.2 Multivariate Sarmanov distribution

The Sarmanov distribution is widely studied in different fields. It was originally introduced by Sarmanov (1966) in the bivariate case. It was then extended by Ting Lee (1996) and Kotz et al. (2004) in the multivariate case:

$$
\begin{equation*}
\mathbb{P}\left(X_{1} \in \mathrm{~d} x_{1}, \ldots, X_{n} \in \mathrm{~d} x_{n}\right)=\left(1+\sum_{1 \leq i<j \leq n} a_{i j} \phi_{i}\left(x_{i}\right) \phi_{j}\left(x_{j}\right)\right) \prod_{k=1}^{n} \mathrm{~d} F\left(x_{k}\right) \tag{2.2}
\end{equation*}
$$

where $F$ is the corresponding marginal distribution of $X$. Particularly, the parameters $a_{i j}$ are real numbers and the kernels $\phi_{i}$ are functions satisfying

$$
\mathbb{E}\left[\phi_{i}\left(X_{i}\right)\right]=0, \quad i=1, \ldots, n
$$

and

$$
1+\sum_{1 \leq i<j \leq n} a_{i j} \phi_{i}\left(x_{i}\right) \phi_{j}\left(x_{j}\right) \geq 0, \quad \text { for all } x_{i} \in D_{X_{i}}, i=1, \ldots, n
$$

where $D_{X_{i}}=\left\{x \in \mathbb{R}: \mathbb{P}\left(X_{i} \in(x-\delta, x+\delta)\right)>0\right.$ for all $\left.\delta>0\right\}, i=1, \ldots, n$.
Similarly to those pointed out in Yang and Wang (2013), two common choices for the kernels $\phi_{i}, i=1, \ldots, n$ are listed below:
(i) $\phi_{i}(x)=1-2 F(x)$ for all $x \in D_{X_{i}}$, leading to the well-known standard FGM distribution;
(ii) $\phi_{i}(x)=x^{p}-\mathbb{E}\left[X_{i}^{p}\right]$ for all $x \in D_{X_{i}}$ and exist $p \in \mathbb{R}$ such that $\mathbb{E}\left[X_{i}^{p}\right]<\infty$;
(iii) $\phi_{i}(x)=e^{-x}-g_{i}$ with $g_{i}=\mathbb{E}\left[e^{-X_{i}}\right]$ for all $x \in D_{X_{i}}$.

We further discuss the dependence structure of two rvs ( $X_{1}, X_{2}$ ) following a Sarmanov distribution with different kernel functions. To model the dependence between the two rvs $X_{1}$ and $X_{2}$, we shall use Pearson's correlation coefficient, which is defined as

$$
\rho_{12}=\frac{\mathbb{E}\left[X_{1} X_{2}\right]-\mathbb{E}\left[X_{1}\right] \mathbb{E}\left[X_{2}\right]}{\sqrt{\operatorname{Var}\left(X_{1}\right)} \sqrt{\operatorname{Var}\left(X_{2}\right)}} .
$$

In the case of the Sarmanov's distribution, $\rho_{12}$ can be rewritten as

$$
\begin{equation*}
\rho_{12}=\frac{a_{12} \mathbb{E}\left[X_{1} \phi_{1}\left(X_{1}\right)\right] \mathbb{E}\left[X_{2} \phi_{2}\left(X_{2}\right)\right]}{\sqrt{\operatorname{Var}\left(X_{1}\right)} \sqrt{\operatorname{Var}\left(X_{2}\right)}} . \tag{2.3}
\end{equation*}
$$

Based on (2.3), we hereafter present the Pearson's correlation coefficient for different kernel functions along with its maximal and minimal value.
Case 1: Set the kernel function $\phi_{i}(x)=1-2 F(x)$, which corresponds to the FGM distribution. It is well known that Pearson correlation coefficients $\rho_{12}$ of the FGM distribution lie between $-\frac{1}{3}$ and $\frac{1}{3}$ (see Schucany et al. (1978)), which is an important drawback of the FGM distribution. Huang and Kotz (1984) show that, by considering the iterated generalization of FGM distribution proposed by Johnson and Kotz (1977), the range of correlation coefficients can be enlarged.
Case 2: Set the kernel function $\phi_{i}(x)=x^{p}-\mathbb{E}\left[X_{i}^{p}\right]$. A usual choice is $p=1$, which leads to the Pearson correlation coefficients $\rho_{12}=a_{12} \sigma_{1} \sigma_{2}$. In this situation, according to Ting Lee (1996), if the correlation coefficient of $X_{1}$ and $X_{2}$ exists, however, if we denote by $T_{i}, i=1,2$ (the corresponding upper truncation points) and consider the marginal pdfs are defined only for non-negative values, $a_{12}$ satisfies the condition that

$$
\max \left\{\frac{-1}{\mu_{1} \mu_{2}}, \frac{-1}{\left(T_{1}-\mu_{1}\right)\left(T_{2}-\mu_{2}\right)}\right\} \leq a_{12} \leq \min \left\{\frac{1}{\mu_{1}\left(T_{2}-\mu_{2}\right)}, \frac{1}{\mu_{2}\left(T_{1}-\mu_{1}\right)}\right\}
$$

where $\mu_{i}=\mathbb{E}\left[X_{i}\right]$. Then, the maximal and the minimal values of the correlation coefficient are, respectively, given by

$$
\rho_{12}^{\max }=\frac{\sigma_{1} \sigma_{2}}{\max \left(\mu_{1}\left(T_{2}-\mu_{2}\right), \mu_{2}\left(T_{1}-\mu_{1}\right)\right)}, \rho_{12}^{\min }=\frac{-\sigma_{1} \sigma_{2}}{\max \left\{\mu_{1} \mu_{2},\left(T_{1}-\mu_{1}\right)\left(T_{2}-\mu_{2}\right)\right\}} .
$$

Case 3: Set the kernel function $\phi_{i}(x)=e^{-x}-g_{i}$ with $g_{i}=\mathbb{E}\left[e^{-X_{i}}\right]$ for all $x \in D_{X_{i}}$. In this case, if the correlation coefficient of $X_{1}$ and $X_{2}$ exists, the range of $a_{12}$ is (see Ting Lee (1996))

$$
\frac{-1}{\max \left(\mathcal{L}_{1}(1) \mathcal{L}_{2}(1),\left(1-\mathcal{L}_{1}(1)\right)\left(1-\mathcal{L}_{2}(1)\right)\right)} \leq a_{12} \leq \frac{1}{\max \left(\mathcal{L}_{1}(1)\left(1-\mathcal{L}_{2}(1)\right), \mathcal{L}_{2}(1)\left(1-\mathcal{L}_{1}(1)\right)\right)}
$$

where $\mathcal{L}_{i}(t)=\int_{0}^{\infty} \exp \left(-t x_{i}\right) \mathrm{d} F_{i}\left(x_{i}\right)$. Then, according to (2.3), the maximal value of Pearson's correlation coefficient $\rho_{12}$ can be written as follows

$$
\rho_{12}^{\max }=\frac{\left[-\mathcal{L}_{1}^{\prime}(1)-\mathcal{L}_{1}(1) \mu_{1}\right]\left[-\mathcal{L}_{2}^{\prime}(1)-\mathcal{L}_{2}(1) \mu_{2}\right]}{\max \left(\mathcal{L}_{1}(1)\left(1-\mathcal{L}_{2}(1)\right), \mathcal{L}_{2}(1)\left(1-\mathcal{L}_{1}(1)\right)\right) \sigma_{1} \sigma_{2}}
$$

and the minimal value can be expressed as

$$
\rho_{12}^{\min }=-\frac{\left[-\mathcal{L}_{1}^{\prime}(1)-\mathcal{L}_{1}(1) \mu_{1}\right]\left[-\mathcal{L}_{2}^{\prime}(1)-\mathcal{L}_{2}(1) \mu_{2}\right]}{\max \left(\mathcal{L}_{1}(1) \mathcal{L}_{2}(1),\left(1-\mathcal{L}_{1}(1)\right)\left(1-\mathcal{L}_{2}(1)\right)\right) \sigma_{1} \sigma_{2}},
$$

where $\mu_{i}=\mathbb{E}\left[X_{i}\right]$.

## 3 Second-order asymptotics of VaR-based systemic risk measures

In this section, we study the second-order asymptotics of VaR-based systemic risk measures with multivariate Sarmanov distributions. Denote the distribution function of the aggregate risk $S_{n}=$ $\sum_{i=1}^{n} X_{i}$ by $G(t):=P\left(S_{n} \leq t\right)$. Before stating some results, we use the following notation:
(i) $\eta_{\alpha}:=\alpha \int_{0}^{1 / 2}\left((1-x)^{-\alpha}-1\right) x^{-\alpha-1} \mathrm{~d} x+2^{2 \alpha-1}-2^{\alpha}$;
(ii) $\mu:=\mathbb{E}[X]$;
(iii) $\mu(t):=\int_{0}^{t} x \mathrm{~d} F(x)$;
(iv) $\mu_{i}(t):=\int_{0}^{t} x \phi_{i}(x) \mathrm{d} F(x), i=1, \ldots, n$;

First, we establish the second-order asymptotics of the random sum under multivariate Sarmanov distributions.

Proposition 3.1 Let $X_{1}, \ldots, X_{n}$ be nonnegative random variables with common marginal distribution $F$ satisfying that $\bar{F}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \beta}$ with $\alpha>0, \beta \leq 0$ and an auxiliary function $A(\cdot)$. Suppose that $\left(X_{1}, \ldots, X_{n}\right)$ follows an $n$-dimensional Sarmanov distribution given by (2.2) and $\lim _{t \rightarrow \infty} \phi_{i}(t)=$ $d_{i} \in \mathbb{R}, \phi_{i}(\cdot)-d_{i} \in \mathcal{R} \mathcal{V}_{\rho_{i}}$ with $\rho_{i} \leq 0$ for each $i=1, \ldots, n$. Then as $t \rightarrow \infty$, we get that

$$
\frac{\bar{G}(t)}{\bar{F}(t)}=n\left(1+\widetilde{A_{n}}(t)(1+o(1))\right)
$$

where

$$
\widetilde{A_{n}}(t)= \begin{cases}\alpha t^{-1} \mu_{n}^{*}(t)+o\left(|A(t)|+\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right), & \alpha \geq 1  \tag{3.1}\\ \eta_{\alpha} \kappa_{n} \bar{F}(t)+o\left(|A(t)|+\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right), & 0<\alpha<1\end{cases}
$$

and

$$
\begin{gathered}
\mu_{n}^{*}(t):=(n-1) \mu(t)+\sum_{1 \leq i<j \leq n} \frac{a_{i j}\left(d_{i} \mu_{j}(t)+d_{j} \mu_{i}(t)\right)}{n}, \\
\kappa_{n}:=n-1+\sum_{1 \leq i<j \leq n} \frac{2 a_{i j} d_{i} d_{j}}{n} .
\end{gathered}
$$

Proof. We require Lemma 8.1 in Appendix for this proof. For $t>0$, denote the region $\Omega_{t}=$ $\left\{\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{R}_{+}^{n}: \sum_{i=1}^{n} x_{i}>t\right\}$. In addition, let $X_{1}^{*}, \ldots, X_{n}^{*}$ be iid with a distribution function $F$. According to Proposition 1.1 of Yang and Wang (2013), there exist $n$ constants $c_{i}>1, i=1, \ldots, n$, such that $\left|\phi_{i}\left(x_{i}\right)\right| \leq c_{i}-1$ for all $x_{i} \in D_{X_{i}}$. Let $\widetilde{X_{1}^{*}}, \ldots, \widetilde{X_{n}^{*}}$ be mutually independent rvs with
marginal distributions $\widetilde{F_{1}}, \ldots, \widetilde{F_{n}}$, which are also independent of $X_{1}^{*}, \ldots, X_{n}^{*}$. Particularly, $\widetilde{F_{1}}, \ldots, \widetilde{F_{n}}$ are defined by

$$
\mathrm{d} \widetilde{F}_{i}\left(x_{i}\right):=\left(1-\frac{\phi_{i}\left(x_{i}\right)}{c_{i}}\right) \mathrm{d} F\left(x_{i}\right), \quad i=1, \ldots, n
$$

By Lemma 8.1 in Appendix, we have $\phi(t)=1-\frac{\phi_{i}\left(x_{i}\right)}{c_{i}}$. Thus, $\lim _{t \rightarrow \infty} \phi(t)=1-d_{i} / c_{i}$ and $\phi_{i}(\cdot)-$ $1+d_{i} / c_{i} \in \mathcal{R} \mathcal{V}_{\rho_{i}}$. It follows that $\widetilde{\widetilde{F}}_{i}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \gamma_{i}}$ with $\gamma_{i}=\max \left\{\beta, \rho_{i}\right\}$ and auxiliary function $\widetilde{A^{i}}(\cdot)=A(\cdot)+\frac{\rho_{i} \alpha}{\left(c_{i}-d_{i}\right)\left(\alpha-\rho_{i}\right)}\left(\phi_{i}(\cdot)-d_{i}\right)$. In addition, as $t \rightarrow \infty$, we obtain

$$
\frac{\overline{\widetilde{F}_{i}}(t)}{\bar{F}(t)}=\left(1-\frac{d_{i}}{c_{i}}\right)\left(1-\frac{\alpha}{\left(c_{i}-d_{i}\right)\left(\alpha-\rho_{i}\right)}\left(\phi_{i}(t)-d_{i}\right)(1+o(1))\right)
$$

for all $i=1, \ldots, n$. Write $\mu_{i}(t):=\int_{0}^{t} x \phi_{i}(x) \mathrm{d} F(x), i=1, \ldots, n$. We have that

$$
\int_{0}^{t} x \mathrm{~d} \widetilde{F}_{i}(x)=\int_{0}^{t} x\left(1-\frac{\phi_{i}(x)}{c_{i}}\right) \mathrm{d} F(x)=\mu(t)-\frac{\mu_{i}(t)}{c_{i}}
$$

Next, we can split $\bar{G}(t)$ as

$$
\begin{align*}
\bar{G}(t)= & \int_{\Omega_{t}}\left(1+\sum_{1 \leq i<j \leq n} a_{i j} \phi_{i}\left(x_{i}\right) \phi_{j}\left(x_{j}\right)\right) \prod_{k=1}^{n} \mathrm{~d} F\left(x_{k}\right) \\
= & \int_{\Omega_{t}}\left(1+\sum_{1 \leq i<j \leq n} a_{i j} c_{i} c_{j}\left(1-\left(1-\frac{\phi_{i}\left(x_{i}\right)}{c_{i}}\right)-\left(1-\frac{\phi_{j}\left(x_{j}\right)}{c_{j}}\right)\right.\right. \\
& \left.\left.+\left(1-\frac{\phi_{i}\left(x_{i}\right)}{c_{i}}\right)\left(1-\frac{\phi_{j}\left(x_{j}\right)}{c_{j}}\right)\right)\right) \prod_{k=1}^{n} \mathrm{~d} F\left(x_{k}\right) \\
= & \int_{\Omega_{t}} \prod_{k=1}^{n} \mathrm{~d} F\left(x_{k}\right)+\sum_{1 \leq i<j \leq n} a_{i j} c_{i} c_{j}\left(\int_{\Omega_{t}} \prod_{k=1}^{n} \mathrm{~d} F\left(x_{k}\right)-\int_{\Omega_{t}} \prod_{k=1, k \neq i}^{n} \mathrm{~d} F\left(x_{k}\right) \mathrm{d} \widetilde{F}_{i}\left(x_{i}\right)\right. \\
& \left.-\int_{\Omega_{t}} \prod_{k=1, k \neq j}^{n} \mathrm{~d} F\left(x_{k}\right) \mathrm{d} \widetilde{F}_{j}\left(x_{j}\right)+\int_{\Omega_{t}} \prod_{k=1, k \neq i, j}^{n} \mathrm{~d} F\left(x_{k}\right) \mathrm{d} \widetilde{F}_{i}\left(x_{i}\right) \mathrm{d} \widetilde{F}_{j}\left(x_{j}\right)\right) \\
:= & I(t)+\sum_{1 \leq i<j \leq n} a_{i j} c_{i} c_{j}\left(I(t)-I_{i}(t)-I_{j}(t)+I_{i, j}(t)\right) . \tag{3.2}
\end{align*}
$$

To deal with $I(t)$, according to Propositions 3.6, 3.7 and Remark 3.1 of Mao and Ng (2015) with common distribution $\bar{F}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \beta}$, it follows that

$$
\frac{I(t)}{\bar{F}(t)}= \begin{cases}n\left(1+(n-1) \alpha t^{-1} \mu(t)(1+o(1))\right)+o(|A(t)|), & \alpha \geq 1 \\ n\left(1+(n-1) \eta_{\alpha} \bar{F}(t)(1+o(1))\right)+o(|A(t)|), & 0<\alpha<1\end{cases}
$$

By the similar analysis, $\widetilde{F}_{i}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \gamma_{i}}$. According to (8.1) in Appendix, we have

$$
\frac{I_{i}(t)}{\bar{F}(t)}= \begin{cases}\left(n-\frac{d_{i}}{c_{i}}\right)\left(1+\alpha(n-1) t^{-1} \mu(t)(1+o(1))\right)-\left(\frac{\alpha(n-1) \mu_{i}(t)}{c_{i} t}+\frac{\alpha\left(\phi_{i}(t)-d_{i}\right)}{c_{i}\left(\alpha-\rho_{i}\right)}\right)(1+o(1)) & \\ \quad+o(|A(t)|), & \alpha \geq 1 \\ \left(n-\frac{d_{i}}{c_{i}}\right)+\eta_{\alpha}(n-1)\left(n-\frac{2 d_{i}}{c_{i}}\right) \bar{F}(t)(1+o(1))-\frac{\alpha\left(\phi_{i}(t)-d_{i}\right)}{c_{i}\left(\alpha-\rho_{i}\right)}(1+o(1))+o(|A(t)|), & 0<\alpha<1\end{cases}
$$

and

$$
\frac{I_{i, j}(t)}{\bar{F}(t)}=\left\{\begin{array}{rlr}
\left(n-\frac{d_{i}}{c_{i}}-\frac{d_{j}}{c_{j}}\right)\left(1+\alpha t^{-1}(n-1) \mu(t)(1+o(1))\right)-\alpha\left(\left(n-1-\frac{d_{j}}{c_{j}}\right) \frac{\mu_{i}(t)}{c_{i} t}\right. & \\
+\left(n-1-\frac{d_{i}}{c_{i}}\right) \frac{\mu_{j}(t)}{c_{j} t}+\frac{\left(\phi_{i}(t)-d_{i}\right)}{c_{i}\left(\alpha-\rho_{i}\right)}+\frac{\left(\frac{\left(\phi_{j}(t)-d_{j}\right)}{c_{j}\left(\alpha-\rho_{j}\right)}\right)(1+o(1))+o(|A(t)|),}{} & \alpha \geq 1, \\
\left(n-\frac{d_{i}}{c_{i}}-\frac{d_{j}}{c_{j}}\right)+\eta_{\alpha}\left((n-1)\left(n-2\left(\frac{d_{i}}{c_{i}}+\frac{d_{j}}{c_{j}}\right)\right)+\frac{2 d_{i} d_{j}}{c_{i} c_{j}}\right) \bar{F}(t)(1+o(1)) & \\
-\left(\frac{\alpha\left(\phi_{i}(t)-d_{i}\right)}{c_{i}\left(\alpha-\rho_{i}\right)}+\frac{\alpha\left(\phi_{j}(t)-d_{j}\right)}{c_{j}\left(\alpha-\rho_{j}\right)}\right)(1+o(1))+o(|A(t)|), & 0<\alpha<1 .
\end{array}\right.
$$

Pulling all the asymptotics for $I(t), I_{i}(t)$, and $I_{i, j}(t)$ into (3.2) yields that

$$
\frac{\bar{G}(t)}{n \bar{F}(t)}= \begin{cases}1+\alpha t^{-1}\left((n-1) \mu(t)+\sum_{1 \leq i<j \leq n} \frac{a_{i j}\left(d_{i} \mu_{j}(t)+d_{j} \mu_{i}(t)\right)}{n}\right)(1+o(1)) & \\ +o\left(|A(t)|+\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right), & \alpha \geq 1, \\ 1+\eta_{\alpha}\left(n-1+\sum_{1 \leq i<j \leq n} \frac{2 a_{i j} d_{i} d_{j}}{n}\right) \bar{F}(t)(1+o(1)) & \\ +o\left(|A(t)|+\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right), & 0<\alpha<1,\end{cases}
$$

Thus, this ends the proof.
Remark 3.1 If $\phi_{i}(x)=1-2 F(x)$, then $d_{i}=-1$ and $\rho_{i}=-\alpha$. If $\phi_{i}(x)=x^{p}-\mathbb{E}\left[X_{i}^{p}\right]$, then $d_{i}=-\mathbb{E}\left[X_{i}^{p}\right]$ and $\rho_{i}=p$ for all $p \leq 0$. If $\phi_{i}(x)=e^{-x}-g_{i}$ then $d_{i}=-g_{i}$ and $\rho_{i}=-\infty$. In addition, let $\rho_{0}=\max _{1 \leq i \leq n} \rho_{i}$ with $\phi_{0}(\cdot)$ and $d_{0} \in \mathbb{R}$.

In view of Proposition 3.1, we can easily obtain the $2 \mathcal{R} \mathcal{V}$ property of $\bar{G}(\cdot)$.
Corollary 3.1 Under the conditions of Proposition 3.1, we have $\bar{G}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \lambda}$ with $\lambda=\max \{-1,-\alpha, \beta\}$ and an auxiliary function $A_{G}^{(n)}(\cdot)$ given by

$$
A_{G}^{(n)}(t)= \begin{cases}A(t)-\alpha t^{-1} \mu_{n}^{*}(t), & \alpha \geq 1,  \tag{3.3}\\ A(t)-\alpha \eta_{\alpha} \kappa_{n} \bar{F}(t), & 0<\alpha<1\end{cases}
$$

Proof. According to the definition of $\widetilde{A_{n}}(t)$, it is easy to check that $\widetilde{A_{n}}(t) \in \mathcal{R} \mathcal{V}_{\tilde{\lambda}}$, where $\widetilde{\lambda}=$ $\max \{-1,-\alpha\}$. Note that $\bar{F}(t) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \beta}$ with an auxiliary function $A(t)$, for any $x>0$, as $t \rightarrow \infty$,
we have that

$$
\begin{aligned}
\frac{\bar{G}(t x)}{\bar{G}(t)} & =\frac{\bar{G}(t x)}{\bar{F}(t x)} \frac{\bar{F}(t x)}{\bar{F}(t)} \overline{\bar{F}(t)} \overline{\bar{G}(t)} \\
& =\frac{n\left(1+\widetilde{A_{n}}(t x)(1+o(1))\right)}{n\left(1+\widetilde{A_{n}}(t)(1+o(1))\right)}\left(x^{-\alpha}+H_{-\alpha, \beta}(x) A(t)(1+o(1))\right) \\
& =x^{-\alpha}+H_{-\alpha, \beta}(x) A(t)(1+o(1))+x^{-\alpha}\left(x^{\widetilde{\lambda}}-1\right) \widetilde{A_{n}}(t)(1+o(1)) \\
& = \begin{cases}x^{-\alpha}+\left(H_{-\alpha, \beta}(x) A(t)-H_{-\alpha,-1}(x) \alpha t^{-1} \mu_{n}^{*}(t)\right)(1+o(1))+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right), & \alpha \geq 1, \\
x^{-\alpha}+\left(H_{-\alpha, \beta}(x) A(t)-H_{-\alpha,-\alpha}(x) \alpha \eta_{\alpha} \kappa_{n} \bar{F}(t)\right)(1+o(1))+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right), & 0<\alpha<1 .\end{cases}
\end{aligned}
$$

Thus, we complete this proof.
Remark 3.2 (1) When $\phi_{i}(x)=1-2 F(x)$ for all $x \in D_{X_{i}}, i=1, \ldots, n$, Proposition 3.1 and Corollary 3.1 reduce to Theorem 4.4 and Corollary 4.5 of Mao and Yang (2015).
(2) If $\alpha \geq 1$ and $\beta<-1$, then $A(t)=o\left(\mu_{n}^{*}(t)\right)$. If $\alpha<1$ and $\beta<-\alpha$, then $A(t)=o(\bar{F}(t))$. If $\beta>-(1 \wedge \alpha)$, then $\mu_{n}^{*}(t)=o(A(t))$ and $\bar{F}(t)=o(A(t))$.

Second, we are ready to show the second-order asymptotics of $\operatorname{VaR}_{p}\left(S_{n}\right)$ and $\operatorname{CTE}_{p}\left(S_{n}\right)$.
Theorem 3.1 Under the conditions of Proposition 3.1, we have, as $p \uparrow 1$,

$$
\frac{\operatorname{VaR}_{p}\left(S_{n}\right)}{F^{\leftarrow}(p)}= \begin{cases}n^{1 / \alpha}\left(1+\left(\frac{\mu_{n}^{*}\left(F^{\leftarrow}(p)\right)}{n^{1 / \alpha} F^{\leftarrow}(p)}+\frac{n^{\beta / \alpha}-1}{\alpha \beta} A\left(F^{\leftarrow}(p)\right)\right)(1+o(1))\right) & \\ \quad+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right), & \alpha \geq 1, \\ n^{1 / \alpha}\left(1+\left(\frac{\eta_{\alpha} \kappa_{n}}{\alpha n} \bar{F}\left(F^{\leftarrow}(p)\right)+\frac{n^{\beta / \alpha}-1}{\alpha \beta} A\left(F^{\leftarrow}(p)\right)\right)(1+o(1))\right) & \\ \quad+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right), & 0<\alpha<1 .\end{cases}
$$

For $\alpha>1$, as $p \uparrow 1$,

$$
\frac{\operatorname{CTE}_{p}\left(S_{n}\right)}{F^{\leftarrow}(p)}=\frac{\alpha n^{1 / \alpha}}{\alpha-1}\left(1+\zeta_{\alpha, \beta}^{n} A\left(F^{\leftarrow}(p)\right)(1+o(1))\right)+\frac{\mu_{n}^{*}\left(F^{\leftarrow}(p)\right)}{F^{\leftarrow}(p)}(1+o(1))+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right),
$$

and

$$
\begin{equation*}
\zeta_{\alpha, \beta}^{n}=\frac{1}{\alpha \beta}\left(\frac{n^{\beta / \alpha}(\alpha-1)}{\alpha-\beta-1}-1\right) . \tag{3.4}
\end{equation*}
$$

Here, by convention, $\frac{n^{\beta / \alpha}-1}{\alpha \beta}:=\alpha^{-2} \log n$ and $\zeta_{\alpha, \beta}^{n}:=\alpha^{-2} \log n$ if $\beta=0$. Clearly, the first-order asymptotics of $\operatorname{VaR}_{p}\left(S_{n}\right)$ and $\operatorname{CTE}_{p}\left(S_{n}\right)$ are $n^{1 / \alpha} F^{\leftarrow}(p)$ and $\frac{\alpha n^{1 / \alpha}}{\alpha-1} F^{\leftarrow}(p)$.
Proof. Since $\bar{F}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \beta}$ with an auxiliary function $A(\cdot)$ and Theorem 2.3.9 of De Haan and Ferreira (2006), one can easily check that $U_{F}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{1 / \alpha, \beta / \alpha}$ with an auxiliary function $\alpha^{-2} A \circ U_{F}(\cdot)$.

Let $t=G^{\leftarrow}(p)$. If $p \uparrow 1$, then $t \rightarrow \infty$. By the relation (2.1) and Proposition 3.1, we get that

$$
\begin{aligned}
\frac{\operatorname{VaR}_{p}\left(S_{n}\right)}{F^{\leftarrow}(p)} & =\frac{G \leftarrow(p)}{F \leftarrow(p)}=\frac{U_{F}(1 / \bar{F}(t))}{U_{F}(1 / \bar{G}(t))(1+o(A(t)))} \\
& =\left(\frac{\bar{G}(t)}{\bar{F}(t)}\right)^{1 / \alpha}\left(1+\frac{\left(\frac{\bar{G}(t)}{\bar{F}(t)}\right)^{\beta / \alpha}-1}{\beta / \alpha} \alpha^{-2} A \circ U_{F}\left(\frac{1}{\bar{G}(t)}\right)\right) \\
& =\left(n\left(1+\widetilde{A_{n}}(t)(1+o(1))\right)\right)^{1 / \alpha}\left(1+\frac{n^{\beta / \alpha}-1}{\alpha \beta} A \circ U_{F}\left(\frac{1}{\bar{G}(t)}\right)(1+o(1))\right),
\end{aligned}
$$

where we use the transformation $t \underset{\sim}{\mapsto} G^{\leftarrow}(p)$ and the first-order Taylor expansion. Note that $\widetilde{A_{n}}\left(G^{\leftarrow}(p)\right) \sim \widetilde{A_{n}}\left(n^{1 / \alpha} F^{\leftarrow}(p)\right) \sim n^{\widetilde{\lambda} / \alpha} \widetilde{A_{n}}\left(F^{\leftarrow}(p)\right)$ with $\widetilde{\lambda}=\max \{-1,-\alpha\}$. As $p \uparrow 1$, we have that

$$
\begin{aligned}
\frac{\operatorname{VaR}_{p}\left(S_{n}\right)}{F^{\leftarrow}(p)}= & n^{1 / \alpha}\left(1+\frac{1}{\alpha} \widetilde{A_{n}}\left(G^{\leftarrow}(p)\right)(1+o(1))+\frac{n^{\beta / \alpha}-1}{\alpha \beta} A\left(F^{\leftarrow}(p)\right)(1+o(1))\right) \\
= & n^{1 / \alpha}\left(1+\frac{1}{\alpha} n^{\widetilde{\lambda} / \alpha} \widetilde{A_{n}}\left(F^{\leftarrow}(p)\right)(1+o(1))+\frac{n^{\beta / \alpha}-1}{\alpha \beta} A\left(F^{\leftarrow}(p)\right)(1+o(1))\right) \\
= & \left\{\begin{array}{rr}
n^{1 / \alpha}\left(1+\left(\frac{\mu_{n}^{*}\left(F^{\leftarrow}(p)\right)}{n^{1 / \alpha} F^{\leftarrow(p)}}+\frac{n^{\beta / \alpha}-1}{\alpha \beta} A\left(F^{\leftarrow}(p)\right)\right)(1+o(1))\right) \\
+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right), & \alpha \geq 1 \\
n^{1 / \alpha}\left(1+\left(\frac{\eta_{\alpha} \kappa_{n}}{\alpha n} \bar{F}\left(F^{\leftarrow}(p)\right)+\frac{n^{\beta / \alpha}-1}{\alpha \beta} A\left(F^{\leftarrow}(p)\right)\right)(1+o(1))\right) \\
& +o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right) .
\end{array}\right.
\end{aligned}
$$

By Proposition 3.1 and the definition of $\operatorname{CTE}_{p}\left(S_{n}\right)$, we have

$$
\begin{aligned}
\frac{\operatorname{CTE}_{p}\left(S_{n}\right)}{F^{\leftarrow}(p)}= & \frac{\mathbb{E}\left[S_{n} \mid S_{n}>\operatorname{VaR}_{p}\left(S_{n}\right)\right]}{F^{\leftarrow}(p)} \\
= & \left.\frac{\alpha}{\alpha-1}\left(1+\frac{1}{\alpha(\alpha-\lambda-1)} A_{G}^{(n)}\left(G^{\leftarrow}(p)\right)\right)(1+o(1))\right) \frac{\operatorname{VaR}_{p}\left(S_{n}\right)}{F^{\leftarrow(p)}} \\
= & \frac{\alpha n^{1 / \alpha}}{\alpha-1}\left(1+\frac{n^{\lambda / \alpha}}{\alpha(\alpha-\lambda-1)} A_{G}^{(n)}\left(F^{\leftarrow}(p)\right)(1+o(1))\right) \\
& \cdot\left(1+\left(\frac{\mu_{n}^{*}\left(F^{\leftarrow(p))}\right.}{\left.\left.n^{1 / \alpha} F^{\leftarrow(p)}+\frac{n^{\beta / \alpha}-1}{\alpha \beta} A\left(F^{\leftarrow}(p)\right)\right)(1+o(1))\right)}\right.\right. \\
= & \frac{\alpha n^{1 / \alpha}}{\alpha-1}\left(1+\frac{1}{\alpha \beta}\left(\frac{n^{\beta / \alpha}(\alpha-1)}{\alpha-\beta-1}-1\right) A\left(F^{\leftarrow}(p)\right)(1+o(1))\right)+\frac{\mu_{n}^{*}\left(F^{\leftarrow}(p)\right)}{F^{\leftarrow}(p)}(1+o(1)) \\
& +o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right) .
\end{aligned}
$$

This ends the proof.

Remark 3.3 According to $\operatorname{VaR}_{p}\left(S_{n}\right)$ in Theorem 3.1, we include Theorem 4.6 of Mao and Yang (2015) and our result provides a simpler proof. If $a_{i j}=0$, for all $1 \leq i \neq j \leq n$, the $n$-dimensional Sarmanov distribution reduces to the independent rvs. In this case, $\mathrm{CTE}_{p}\left(S_{n}\right)$ of Theorem 3.1 is consistent with Theorem 3.1 of Mao et al. (2012).

The following example is used to illustrate Theorem 3.1 under the Pareto distribution with different parameters $\alpha$.

Example 3.1 (The Pareto distribution) A Pareto distribution function F satisfies that

$$
F(x)=1-\left(\frac{k}{x+k}\right)^{\alpha}, \quad x, k, \alpha>0 .
$$

It can be described that $\bar{F}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha,-1}$ with an auxiliary function $A(t)=\alpha k / t$ and $F^{\leftarrow}(p)=$ $k\left((1-p)^{-1 / \alpha}-1\right)$. If $\alpha>1$, we have $\mu=k /(\alpha-1)$ and $\mu(t)=k /(\alpha-1)-(\alpha t+k) /(\alpha-1) \bar{F}(t)$. Let $X_{1}$ and $X_{2}$ have an identical Pareto distribution $F$. Suppose that the random vector ( $X_{1}, X_{2}$ ) follows an Sarmanov distribution in (2.2) with $\phi_{i}(\cdot)=1-2 F(\cdot), i=1,2$. Clearly, $d_{i}=-1, \rho_{i}=-\alpha$ and $\mu_{i}(t)=k /(2 \alpha-1)-(2 \alpha t+k) /(2 \alpha-1)(\bar{F}(t))^{2}-\mu(t), i=1,2$. Then

$$
\operatorname{VaR}_{p}\left(S_{2}\right)= \begin{cases}2^{1 / \alpha} F^{\leftarrow}(p)+\mu\left(F^{\leftarrow}(p)\right)-a_{12} \mu_{1}\left(F^{\leftarrow}(p)\right)+k\left(2^{1 / \alpha}-1\right), & \alpha \geq 1, \\ 2^{1 / \alpha} F^{\leftarrow}(p)\left(1+\frac{\eta_{\alpha}\left(1+a_{12}\right)}{2 \alpha}(1-p)\right)+k\left(2^{1 / \alpha}-1\right), & 0<\alpha<1 .\end{cases}
$$

and for $\alpha>1$,

$$
\operatorname{CTE}_{p}\left(S_{2}\right)=\frac{2^{1 / \alpha} \alpha}{\alpha-1}\left(F^{\leftarrow}(p)+k\right)-k+\mu\left(F^{\leftarrow}(p)\right)-a_{12} \mu_{1}\left(F^{\leftarrow}(p)\right) .
$$

Table 2: Simulated values(MC) versus first-order(1st) and second-order(2nd) asymptotics values $\operatorname{VaR}_{p}\left(S_{2}\right)$ and $\operatorname{CTE}_{p}\left(S_{2}\right)$ with various values of $\alpha$. We use the Pareto dstribution with $k=1$ and $p=0.99, a_{12}=0.5$.

|  |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1.1126 .8065121 .5690126 .19431561 .60151337 .25881360 .6628 |  |  |  |  |  |  | 0.9587 | 0.9952 | 0.8563 | 0.8713 |
| 1.5 | 35.3132 | 32.6000 | 34.9843 | 102.9805 | 97.8000 | 103.3591 | 0.9232 | 0.9907 | 0.9497 | 1.0037 |
| 2.0 | 14.4215 | 12.7262 | 14.1893 | 28.5141 | 25.4524 | 28.3298 | 0.8824 | 0.9839 | 0.8926 | 0.9935 |
| 2.5 | 8.2435 | 7.0060 | 8.0581 | 13.8310 | 11.6767 | 13.6085 | 0.8499 | 0.9775 | 0.8442 | 0.9839 |
| 3.0 | 5.5535 | 4.5847 | 5.4053 | 8.5145 | 6.8771 | 8.3277 | 0.8255 | 0.9733 | 0.8077 | 0.9781 |
| 4.0 | 3.2479 | 2.5708 | 3.1404 | 4.5244 | 3.4277 | 4.3938 | 0.7915 | 0.9669 | 0.7576 | 0.9711 |
|  | 2.2591 | 1.7378 | 2.1739 | 2.9994 | 2.1722 | 2.8956 | 0.7692 | 0.9623 | 0.7242 | 0.9654 |

In Table 2 we find that the second-order asymptotics of VaR and CTE are closer to the simulation values than the first-order asymptotics. Specifically, the asymptotic values of CTE are not as accurate as those of VaR, because CTE may not exist if $p$ is close to 1 . In addition, the values of VaR and CTE decrease as $\alpha$ rises.

The following theorem obtains second-order asymptotics of MES and SES under an $n$-dimensional Sarmanov distribution. These results are important in a wide range of systemic risk.

Theorem 3.2 Let $X_{1}, \ldots, X_{n}$ be nonnegative random variables with a common marginal distribution $F$ satisfying that $\bar{F}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \beta}$ with $\alpha>1, \beta \leq 0$ and auxiliary function $A(\cdot)$. Suppose that $\left(X_{1}, \ldots, X_{n}\right)$ follows an $n$-dimensional Sarmanov distribution given by (2.2) and $\lim _{x_{i} \rightarrow \infty} \phi_{i}\left(x_{i}\right)=d_{i} \in$ $\mathbb{R}, \phi_{i}(\cdot)-d_{i} \in \mathcal{R} \mathcal{V}_{\rho_{i}}$ with $\rho_{i} \leq 0$ for each $i=1, \ldots, n$. Then as $t \rightarrow \infty$, we get that

$$
\frac{\operatorname{MES}_{p, m}\left(S_{n}\right)}{F^{\leftarrow}(p)}=\frac{\alpha n^{1 / \alpha}}{(\alpha-1) n}\left(1+\zeta_{\alpha, \beta}^{n} A\left(F^{\leftarrow}(p)\right)(1+o(1))\right)+o\left(t^{-1}+o\left(\sum_{i=1}^{n}\left(\phi_{i}(t)-d_{i}\right)\right)\right),
$$

where $\zeta_{\alpha, \beta}^{n}$ is defined in (3.4) and

$$
\frac{\operatorname{SES}_{p, m}\left(S_{n}\right)}{F \leftarrow(p)}=\frac{\operatorname{MES}_{p, m}\left(S_{n}\right)}{F \leftarrow(p)}-\frac{1}{n} .
$$

Obviously, the first-order asymptotics of $\operatorname{MES}_{p}\left(S_{n}\right)$ and $\operatorname{SES}_{p}\left(S_{n}\right)$ are $\frac{\alpha n^{1 / \alpha} F^{\leftarrow}(p)}{(\alpha-1) n}$ and $\frac{\alpha n^{1 / \alpha} F^{\leftarrow}(p)}{(\alpha-1) n}-$ $\frac{F^{\leftarrow}(p)}{n}$.

Proof. Here we require Lemma 8.4 in Appendix and Theorem 3.1. Define $\widetilde{B}(t)=\frac{1}{\alpha(\alpha-\beta-1)} A(t)-$ $\frac{\mu_{n}^{*}(t)}{t}$. We have that $\widetilde{B} \in \mathcal{R} \mathcal{V}_{\rho}$ with $\rho=\max \{-1, \beta\}$. It follows that, as $p \uparrow 1$,

$$
\begin{aligned}
\frac{\operatorname{MES}_{p, m}\left(S_{n}\right)}{F^{\leftarrow}(p)}= & \frac{\mathbb{E}\left[X_{m} \mid S_{n}>\operatorname{VaR}_{p}\left(S_{n}\right)\right]}{F^{\leftarrow(p)}} \\
= & \frac{\alpha}{(\alpha-1) n}\left(1+\widetilde{B}\left(\operatorname{VaR}_{p}\left(S_{n}\right)\right)(1+o(1))\right) \frac{\operatorname{VaR}_{p}\left(S_{n}\right)}{F^{\leftarrow}(p)}+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right) \\
= & \frac{\alpha n^{1 / \alpha}}{(\alpha-1) n}\left(1+n^{\rho / \alpha} \widetilde{B}\left(F^{\leftarrow}(p)\right)(1+o(1))\right) \\
& \cdot\left(1+\left(\frac{\mu_{n}^{*}\left(F^{\leftarrow}(p)\right)}{n^{1 / \alpha} F^{\leftarrow}(p)}+\frac{n^{\beta / \alpha}-1}{\alpha \beta} A\left(F^{\leftarrow}(p)\right)\right)(1+o(1))\right)+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right) \\
= & \frac{\alpha n^{1 / \alpha}}{(\alpha-1) n}\left(1+\frac{1}{\alpha \beta}\left(\frac{n^{\beta / \alpha}(\alpha-1)}{\alpha-\beta-1}-1\right) A\left(F^{\leftarrow}(p)\right)\right)(1+o(1))+o\left(t^{-1}+\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right) .
\end{aligned}
$$

Using Lemma 8.5 in Appendix and Theorem 3.1, we conclude that

$$
\begin{aligned}
\frac{\operatorname{SES}_{p, m}\left(S_{n}\right)}{F \leftarrow(p)} & =\frac{\mathbb{E}\left[\left(X_{m}-\operatorname{VaR}_{p}\left(X_{m}\right)\right)_{+} \mid S_{n}>\operatorname{VaR}_{p}\left(S_{n}\right)\right]}{F \leftarrow(p)} \\
& =\frac{\operatorname{MES}_{p, m}\left(S_{n}\right)}{F^{\leftarrow}(p)}-\frac{1}{n} .
\end{aligned}
$$

This completes the proof for $\operatorname{MES}_{p, m}\left(S_{n}\right)$ and $\operatorname{SES}_{p, m}\left(S_{n}\right)$.
Lastly, Example 3.2 is used to explain Theorem 3.2 with Burr distribution. We use $B(u, v)$ to represent Beta distribution.

Example 3.2 (Burr distribution) A Burr distribution function $F$ satisfies that

$$
F(x)=1-\left(1+x^{-\beta}\right)^{\alpha / \beta}, \quad x, \alpha>1, \beta<0 .
$$

It is easy to check that $\bar{F}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \beta}$ with auxiliary function $A(t)=\alpha t^{\beta}, \mu=-\alpha / \beta B\left(\frac{1-\alpha}{\beta}, \frac{\beta-1}{\beta}\right)$ and $F^{\leftarrow}(p)=\left((1-p)^{\beta / \alpha}-1\right)^{-1 / \beta}$. Let $X_{1}$ and $X_{2}$ have an identical Burr distribution $F$. Suppose that the random vector ( $X_{1}, X_{2}$ ) follows an Sarmanov distribution in (2.2) with $\phi_{i}(\cdot)=1-2 F(\cdot)$. Clearly, $d_{i}=-1, i=1,2$. Then,

$$
\operatorname{MES}_{p}\left(S_{2}\right)=\frac{2^{1 / \alpha} \alpha F^{\leftarrow}(p)}{2(\alpha-1)}\left(1+\frac{1}{\beta}\left(\frac{2^{\beta / \alpha}(\alpha-1)}{\alpha-\beta-1}-1\right)\left(F^{\leftarrow}(p)\right)^{\beta}\right),
$$

and

$$
\operatorname{SES}_{p}\left(S_{2}\right)=\frac{2^{1 / \alpha} \alpha F^{\leftarrow}(p)}{2(\alpha-1)}\left(1+\frac{1}{\beta}\left(\frac{2^{\beta / \alpha}(\alpha-1)}{\alpha-\beta-1}-1\right)\left(F^{\leftarrow}(p)\right)^{\beta}\right)-\frac{F^{\leftarrow}(p)}{2}
$$

It is easy to see that the first-order asymptotics of $\operatorname{MES}_{p}\left(S_{2}\right)$ and $\operatorname{SES}_{p}\left(S_{2}\right)$ are $\frac{2^{1 / \alpha} \alpha F^{\leftarrow}(p)}{2(\alpha-1)}$ and $\frac{2^{1 / \alpha} \alpha F^{\leftarrow}(p)}{2(\alpha-1)}-\frac{F^{\leftarrow}(p)}{2}$. By Figure 1, the second-order asymptotics of $\operatorname{MES}_{p}\left(S_{2}\right)$ and $\operatorname{SES}_{p}\left(S_{2}\right)$ are much closer to the simulation value than the first-order asymptotics for $p \in[0.95,1)$.


Figure 1: Simulated values(MC) versus the first-order and second-order asymptotics values of $\mathrm{MES}_{p}\left(S_{2}\right)$ for the left panel and $\operatorname{SES}_{p}\left(S_{2}\right)$ for the right panel. We use the Burr distribution with $\alpha=2, \beta=-0.5$ and $a_{12}=0.5$.

## 4 Second-order asymptotics of expectile-based systemic risk measures

In this section, we consider the second-order asymptotics of expectile-based systemic risk measures under multivariate Sarmanov distributions. Firstly, we establish the second-order asymptotics of the expectile under a different method with the literature.

Proposition 4.1 Let $\bar{F}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \beta}$ with $\alpha>1, \beta \leq 0$ and auxiliary function $A(\cdot)$. Then we have that as $p \uparrow 1$,

$$
\frac{\mathrm{e}_{p}(X)}{F^{\leftarrow(p)}}=(\alpha-1)^{-1 / \alpha}\left(1+\xi_{\alpha, \beta} A\left(F^{\leftarrow}(p)\right)(1+o(1))\right)+\frac{\mu}{\alpha F^{\leftarrow}(p)}(1+o(1)),
$$

where

$$
\begin{equation*}
\xi_{\alpha, \beta}=\frac{1}{\alpha \beta}\left(\frac{(\alpha-1)^{1-\beta / \alpha}}{\alpha-\beta-1}-1\right) \tag{4.1}
\end{equation*}
$$

Here $\xi_{\alpha, \beta}:=-\alpha^{-2} \log (\alpha-1)$ if $\beta=0$.
Proof. see Appendix.
Proposition 4.1 is derived by a different method with Corollary 1 of Daouia et al. (2018). In addition, due to $1-p \sim \bar{F}\left(F^{\leftarrow}(p)\right)$ as $p \uparrow 1$ and $\alpha>1$, we derive that $1-p=o\left(1 / F^{\leftarrow}(p)\right)$. Thus, Proposition 4.1 is consistent with Proposition 3.1 of Mao et al. (2015) or Theorem 3.1 of Mao and Yang (2015).

Secondly, we obtain the second-order asymptotics of $\mathrm{e}_{p}\left(S_{n}\right)$ and $\mathrm{CE}_{p}\left(S_{n}\right)$ with an $n$-dimensional Sarmanov distribution.

Theorem 4.1 Let $X_{1}, \ldots, X_{n}$ be nonnegative random variables with a common marginal distribution $F$ satisfying that $\bar{F}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \beta}$ with $\alpha>1, \beta \leq 0$ and auxiliary function $A(\cdot)$. Suppose that $\left(X_{1}, \ldots, X_{n}\right)$ follows an $n$-dimensional Sarmanov distribution given by (2.2) and $\lim _{x_{i} \rightarrow \infty} \phi_{i}\left(x_{i}\right)=d_{i} \in$ $\mathbb{R}, \phi_{i}(\cdot)-d_{i} \in \mathcal{R} \mathcal{V}_{\rho_{i}}$ with $\rho_{i} \leq 0$ for each $i=1, \ldots, n$. Then as $p \uparrow 1$, we get that

$$
\begin{aligned}
\frac{\mathrm{e}_{p}\left(S_{n}\right)}{F^{\leftarrow}(p)}= & \left(\frac{n}{\alpha-1}\right)^{1 / \alpha}\left(1+\frac{1}{\alpha \beta}\left(\frac{n^{\beta / \alpha}(\alpha-1)^{1-\beta / \alpha}}{\alpha-\beta-1}-1\right) A\left(F^{\leftarrow}(p)\right)(1+o(1))\right) \\
& +\frac{(\alpha-1) \mu_{n}^{*}\left(F^{\leftarrow}(p)\right)+n \mu}{\alpha F^{\leftarrow}(p)}(1+o(1))+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mathrm{CE}_{p}\left(S_{n}\right)}{F^{\leftarrow}(p)}= & \frac{\alpha n^{1 / \alpha}}{(\alpha-1)^{1 / \alpha+1}}\left(1+\chi_{\alpha, \beta}^{n} A\left(F^{\leftarrow}(p)\right)(1+o(1))\right) \\
& +\frac{(\alpha-2) \mu_{n}^{*}\left(F^{\leftarrow}(p)\right)+n \mu}{(\alpha-1) F^{\leftarrow}(p)}(1+o(1))+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right)
\end{aligned}
$$

where

$$
\begin{equation*}
\chi_{\alpha, \beta}^{n}=\frac{1}{\alpha \beta}\left(\left(\frac{n}{\alpha-1}\right)^{\beta / \alpha} \frac{\alpha+\beta-1}{\alpha-\beta-1}-1\right) \tag{4.2}
\end{equation*}
$$

Here $\chi_{\alpha, \beta}^{n}:=\alpha^{-2}(\log n-\log (\alpha-1))$ if $\beta=0$. In addition, the first-order asymptotics of $\mathrm{e}_{p}\left(S_{n}\right)$ and $\mathrm{CE}_{p}\left(S_{n}\right)$ are $\frac{n^{1 / \alpha} F^{\leftarrow}(p)}{(\alpha-1)^{1 / \alpha}}$ and $\frac{\alpha n^{1 / \alpha} F^{\leftarrow}(p)}{(\alpha-1)^{1 / \alpha+1}}$.

Proof. Firstly, to deal with $\mathbb{E}\left[S_{n}\right]$, owing to $\left(X_{1}, \ldots, X_{n}\right)$ follows an $n$-dimensional Sarmanov distribution in (2.2), we have

$$
\begin{aligned}
\mathbb{E}\left[S_{n}\right] & =\int_{[0, \infty]^{n}} \sum_{i=1}^{n} x_{i}\left(1+\sum_{1 \leq i<j \leq n} a_{i j} \phi_{i}\left(x_{i}\right) \phi_{j}\left(x_{j}\right)\right) \prod_{k=1}^{n} \mathrm{~d} F\left(x_{k}\right) \\
& =\sum_{i=1}^{n} \int_{[0, \infty]^{n}} x_{i}\left(1+\sum_{1 \leq j \leq n, j \neq i} a_{i j} \phi_{i}\left(x_{i}\right) \phi_{j}\left(x_{j}\right)\right) \prod_{k=1}^{n} \mathrm{~d} F\left(x_{k}\right) \\
& =n \mu .
\end{aligned}
$$

According to Proposition 4.1 and Corollary 3.1, we have

$$
\begin{aligned}
\mathrm{e}_{p}\left(S_{n}\right)= & (\alpha-1)^{-1 / \alpha} G^{\leftarrow}(p)\left(1+\xi_{\alpha, \lambda} A_{G}^{(n)}\left(G^{\leftarrow}(p)\right)(1+o(1))\right)+\frac{\mathbb{E}\left(S_{n}\right)}{\alpha}(1+o(1)) \\
= & (\alpha-1)^{-1 / \alpha} G^{\leftarrow}(p)\left(1+n^{\lambda / \alpha} \xi_{\alpha, \lambda} A_{G}^{(n)}\left(F^{\leftarrow}(p)\right)(1+o(1))\right)+\frac{n \mu}{\alpha}(1+o(1)) \\
= & \left(\frac{n}{\alpha-1}\right)^{1 / \alpha} F^{\leftarrow}(p)\left(1+\left(\frac{\mu_{n}^{*}\left(F^{\leftarrow}(p)\right)}{n^{1 / \alpha} F^{\leftarrow}(p)}+\frac{n^{\beta / \alpha}-1}{\alpha \beta} A\left(F^{\leftarrow}(p)\right)\right)(1+o(1))\right) \\
& \cdot\left(1+n^{\lambda / \alpha} \xi_{\alpha, \lambda} A_{G}^{(n)}\left(F^{\leftarrow}(p)\right)(1+o(1))\right)+\frac{n \mu}{\alpha}(1+o(1)) \\
= & \left(\frac{n}{\alpha-1}\right)^{1 / \alpha} F^{\leftarrow}(p)\left(1+\frac{1}{\alpha \beta}\left(\frac{n^{\beta / \alpha}(\alpha-1)^{1-\beta / \alpha}}{\alpha-\beta-1}-1\right) A\left(F^{\leftarrow}(p)\right)(1+o(1))\right) \\
& +\alpha^{-1}\left((\alpha-1) \mu_{n}^{*}\left(F^{\leftarrow}(p)\right)+n \mu\right)(1+o(1))+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right) .
\end{aligned}
$$

According to the definition of $\mathrm{CE}_{p}\left(S_{n}\right)$ and Theorem 4.1, we obtain, as $p \uparrow 1$,

$$
\begin{aligned}
\frac{\mathrm{CE}_{p}\left(S_{n}\right)}{F^{\leftarrow}(p)}= & \frac{\mathbb{E}\left[S_{n} \mid S_{n}>\mathrm{e}_{p}\left(S_{n}\right)\right]}{F^{\leftarrow(p)}}= \\
= & \frac{\alpha}{\alpha-1}\left(1+\frac{1}{\alpha(\alpha-\lambda-1)} A_{G}^{(n)}\left(\mathrm{e}_{p}\left(S_{n}\right)\right)(1+o(1))\right) \frac{\mathrm{e}_{p}\left(S_{n}\right)}{F^{\leftarrow(p)}} \\
= & \frac{\alpha}{\alpha-1}\left(1+\frac{1}{\alpha(\alpha-\lambda-1)}\left(\frac{n}{\alpha-1}\right)^{\lambda / \alpha} A_{G}^{(n)}\left(F^{\leftarrow}(p)\right)\right) \\
& \cdot\left(\left(\frac{n}{\alpha-1}\right)^{1 / \alpha}\left(1+\frac{1}{\alpha \beta}\left(\frac{n^{\beta / \alpha}(\alpha-1)^{1-\beta / \alpha}}{\alpha-\beta-1}-1\right) A\left(F^{\leftarrow}(p)\right)(1+o(1))\right)\right. \\
& \left.+\frac{(\alpha-1) \mu_{n}^{*}\left(F^{\leftarrow}(p)\right)+n \mu}{\alpha F^{\leftarrow}(p)}(1+o(1))\right) \\
= & \frac{\alpha n^{1 / \alpha}}{(\alpha-1)^{1 / \alpha+1}}\left(1+\frac{1}{\alpha \beta}\left(\left(\frac{n}{\alpha-1}\right)^{\beta / \alpha} \frac{\alpha+\beta-1}{\alpha-\beta-1}-1\right) A\left(F^{\leftarrow}(p)\right)(1+o(1))\right) \\
& +\frac{(\alpha-2) \mu_{n}^{*}\left(F^{\leftarrow}(p)\right)+n \mu}{(\alpha-1) F^{\leftarrow}(p)}(1+o(1))+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right) .
\end{aligned}
$$

Thus, this completes the proof of the theorem.

The following example is applied to interpret in interpretation Theorem 4.1 with absolute Student $t_{\alpha}$ distribution.

Example 4.1 (Absolute Student $t_{\alpha}$ Distribution) A standard Student $t_{\alpha}$ distribution with density function satisfies that

$$
f(x)=\frac{\Gamma((\alpha+1) / 2)}{\sqrt{\alpha \pi \Gamma(\alpha / 2)}}\left(1+\frac{x^{2}}{\alpha}\right)^{-(\alpha+1) / 2}, \quad x \in \mathbb{R}, \quad \alpha>1
$$

Denote by $F$ the distribution function of $|X|$. According to Example 4.2 of Mao et al. (2012) and Example 4.4 of Hua and Joe (2011), we know that $\bar{F} \in 2 \mathcal{R} \mathcal{V}_{-\alpha,-2}$ with auxiliary function $A(t)=\frac{\alpha^{2}}{\alpha+2} t^{-2}, \mu=E|X|=\alpha /(\alpha-1)$ and $F^{\leftarrow}(p)=t_{\alpha}^{\leftarrow}((1+p) / 2)$. Let $X_{1}$ and $X_{2}$ have an identical Student $t_{\alpha}$ distribution F. Suppose that the random vector ( $X_{1}, X_{2}$ ) follows an Sarmanov distribution in $(2.2)$ with $\phi_{i}(\cdot)=1-2 F(\cdot), i=1,2$. Clearly, $d_{i}=-1$. Then

$$
\mathrm{e}_{p}\left(S_{2}\right)=\frac{2^{1 / \alpha} F^{\leftarrow}(p)}{(\alpha-1)^{1 / \alpha}}+\frac{\alpha-1}{\alpha}\left(\mu\left(F^{\leftarrow}(p)\right)-a_{12} \mu_{1}\left(F^{\leftarrow}(p)\right)\right)+\frac{2}{\alpha-1},
$$

and

$$
\mathrm{CE}_{p}\left(S_{2}\right)=\frac{2^{1 / \alpha} \alpha F^{\leftarrow}(p)}{(\alpha-1)^{1 / \alpha+1}}+\frac{\alpha-2}{\alpha-1}\left(\mu\left(F^{\leftarrow}(p)\right)-a_{12} \mu_{1}\left(F^{\leftarrow}(p)\right)\right)+\frac{2 \alpha}{(\alpha-1)^{2}} .
$$

Table 3: Simulated values(MC) versus first-order(1st) and second-order(2nd) asymptotic values $\mathrm{e}_{p}\left(S_{2}\right)$ and $\mathrm{CE}_{p}\left(S_{2}\right)$ with different $p$. We use the Student $t_{\alpha}$ distribution with $\alpha=2.5$ and $a_{12}=-0.5$.


According to Table 3, the second-order asymptotics of $\mathrm{e}_{p}\left(S_{2}\right)$ and $\mathrm{CE}_{p}\left(S_{2}\right)$ are much closer to the simulation value as $p \in[0.95,1)$.

Next, we get the second-order asymptotics of $\operatorname{ICE}_{p, m}\left(S_{n}\right)$ and $\operatorname{SICE}_{p, m}\left(S_{n}\right)$ with an $n$-dimensional Sarmanov distribution.

Theorem 4.2 Under the conditions of Theorem 4.1, as $p \uparrow 1$, we get that

$$
\begin{aligned}
\frac{\operatorname{ICE}_{p, m}\left(S_{n}\right)}{F^{\leftarrow}(p)}= & \frac{\alpha}{(\alpha-1) n}\left(\left(\frac{n}{\alpha-1}\right)^{1 / \alpha}\left(1+\chi_{\alpha, \beta}^{n} A\left(F^{\leftarrow}(p)\right)(1+o(1))\right)\right. \\
& \left.+\frac{n \mu-\mu_{n}^{*}\left(F^{\leftarrow}(p)\right)}{\alpha F^{\leftarrow}(p)}(1+o(1))\right)+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right)
\end{aligned}
$$

with $\chi_{\alpha, \beta}^{n}$ is defined in (4.2) and

$$
\frac{\operatorname{SICE}_{p, m}\left(S_{n}\right)}{F_{\leftarrow} \leftarrow(p)}=\frac{\operatorname{ICE}_{p, m}\left(S_{n}\right)}{F^{\leftarrow}(p)}-\frac{\mathrm{e}_{p}\left(X_{m}\right)}{n F^{\leftarrow}(p)} .
$$

Moreover, the first-order asymptotics of $\operatorname{ICE}_{p, m}\left(S_{n}\right)$ and $\operatorname{SICE}_{p, m}\left(S_{n}\right)$ are $\frac{n^{1 / \alpha} \alpha F \leftarrow(p)}{n(\alpha-1)^{1+1 / \alpha}}$ and $\frac{n^{1 / \alpha} \alpha F \leftarrow(p)}{n(\alpha-1)^{1+1 / \alpha}}-$ $\frac{F^{\leftarrow}(p)}{n(\alpha-1)^{1 / \alpha}}$.

Proof. We require Lemmas 8.4-8.5 in Appendix for this proof. By the definition of $\operatorname{ICE}_{p, m}\left(S_{n}\right)$, Theorem 4.1 and Lemma 8.4, as $p \uparrow 1$, we have that

$$
\begin{aligned}
\frac{\operatorname{ICE}_{p, m}\left(S_{n}\right)}{F^{\leftarrow}(p)}= & \frac{\mathbb{E}\left[X_{m} \mid S_{n}>\mathrm{e}_{p}\left(S_{n}\right)\right]}{F^{\leftarrow}(p)} \\
= & \frac{\alpha}{(\alpha-1) n}\left(1+\widetilde{B}\left(\mathrm{e}\left(S_{n}\right)\right)(1+o(1))\right) \frac{\mathrm{e}\left(S_{n}\right)}{F^{\leftarrow(p)}} \\
= & \frac{\alpha}{(\alpha-1) n}\left(\left(\frac{n}{\alpha-1}\right)^{1 / \alpha}\left(1+\frac{1}{\alpha \beta}\left(\frac{n^{\beta / \alpha}(\alpha-1)^{1-\beta / \alpha}}{\alpha-\beta-1}-1\right) A\left(F^{\leftarrow}(p)\right)(1+o(1))\right)\right. \\
& +\frac{(\alpha-1) \mu_{n}^{*}\left(F^{\leftarrow}(p)\right)+n \mu}{\left.\alpha F^{\leftarrow(p)}(1+o(1))\right)\left(1+\left(\frac{n}{\alpha-1}\right)^{\rho / \alpha} \widetilde{B}\left(F^{\leftarrow}(p)\right)(1+o(1))\right)} \\
= & \frac{\alpha}{(\alpha-1) n}\left(\left(\frac{n}{\alpha-1}\right)^{1 / \alpha}\left(1+\chi_{\alpha, \beta}^{n} A\left(F^{\leftarrow}(p)\right)(1+o(1))\right)\right. \\
& \left.+\frac{n \mu-\mu_{n}^{*}\left(F^{\leftarrow}(p)\right)}{\alpha F^{\leftarrow}(p)}(1+o(1))\right)+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right) .
\end{aligned}
$$

Applying Theorem 4.1 and Lemma 8.5, we conclude that

$$
\begin{aligned}
\frac{\operatorname{SICE}_{p, m}\left(S_{n}\right)}{F^{\leftarrow}(p)} & =\frac{\mathbb{E}\left[\left(X_{m}-\mathrm{e}_{p}\left(X_{m}\right)\right)_{+} \mid S_{n}>\mathrm{e}_{p}\left(S_{n}\right)\right]}{F_{\leftarrow}(p)} \\
& =\frac{\operatorname{ICE}_{p, m}\left(S_{n}\right)}{F^{\leftarrow}(p)}-\frac{\mathrm{e}_{p}\left(X_{m}\right)}{n F^{\leftarrow}(p)} .
\end{aligned}
$$

This completes the proof of $\operatorname{ICE}_{p, m}\left(S_{n}\right)$ and $\operatorname{SICE}_{p, m}\left(S_{n}\right)$.
Lastly, the following example is applied to this result of Theorem 4.2 under the Fréchet distribution.

Example 4.2 (Fréchet distribution) A Fréchet distribution function $F$ satisfies that

$$
F(x)=1-\exp \left(-x^{-\alpha}\right), \quad \alpha>1
$$

Table 4: Simulated values $\operatorname{ICE}_{p}\left(S_{2}\right)_{M C}$ versus the first-order asymptotic values $\operatorname{ICE}_{p}\left(S_{2}\right)_{1 s t}$ and the secondorder asymptotic values $\operatorname{ICE}_{p}\left(S_{2}\right)_{2 n d}$ with $\alpha=2, a_{12}=0.5$.

| $p$ | $\mathrm{ICE}_{p, 1}\left(S_{2}\right)_{M C}$ | $\mathrm{ICE}_{p, 1}\left(S_{2}\right)_{1 s t}$ | $\mathrm{ICE}_{p, 1}\left(S_{2}\right)_{2 n d}$ | $\operatorname{ICE}_{p, 1}\left(S_{2}\right)_{1 s t}$ | $\mathrm{ICE}_{p, 1}\left(S_{2}\right)_{2 n d}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.9500 | 8.0518 | 6.2481 | 7.2566 | 0.7760 | 0.9012 |
| 0.9600 | 8.8178 | 7.0051 | 7.9763 | 0.7944 | 0.9046 |
| 0.9700 | 9.9471 | 8.1096 | 9.0379 | 0.8153 | 0.9086 |
| 0.9800 | 11.8197 | 9.9749 | 10.8508 | 0.8439 | 0.9180 |
| 0.9900 | 16.0617 | 14.1623 | 14.9689 | 0.8817 | 0.9320 |
| 0.9990 | 47.6943 | 45.0164 | 45.7053 | 0.9439 | 0.9583 |
| 0.9999 | 146.0483 | 144.7467 | 145.3974 | 0.9911 | 0.9955 |

Table 5: Simulated values $\operatorname{SICE}_{p}\left(S_{2}\right)_{M C}$ versus the first-order asymptotic values $\operatorname{SICE}_{p}\left(S_{2}\right)_{1 s t}$ and the secondorder asymptotic values $\operatorname{SICE}_{p}\left(S_{2}\right)_{2 n d}$ with $\alpha=2, a_{12}=0.5$.

| $p$ | SICE $_{p, 1}\left(S_{2}\right)_{M C}$ | $\operatorname{SICE}_{p, 1}\left(S_{2}\right)_{1 s t}$ | SICE $_{p, 1}\left(S_{2}\right)_{2 n d}$ | $\frac{\text { SICE }_{p, 1}\left(S_{2}\right)_{1 s t}}{\operatorname{SICE}_{p, 1}\left(S_{2}\right)_{M C}}$ | $\frac{\operatorname{SICE}_{p, 1}\left(S_{2}\right)_{2 n d}}{\operatorname{SICE}_{p, 1}\left(S_{2}\right)_{M C}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.9500 | 4.4381 | 4.0390 | 4.6044 | 0.9101 | 1.0375 |
| 0.9600 | 4.9085 | 4.5284 | 5.0565 | 0.9226 | 1.0301 |
| 0.9700 | 5.6135 | 5.2424 | 5.7276 | 0.9339 | 1.0203 |
| 0.9800 | 6.7875 | 6.4482 | 6.8810 | 0.9500 | 1.0138 |
| 0.9900 | 9.4881 | 9.1552 | 9.5187 | 0.9649 | 1.0032 |
| 0.9990 | 29.4512 | 29.1007 | 29.4465 | 0.9881 | 0.9998 |
| 0.9999 | 93.8148 | 93.5710 | 93.7786 | 0.9974 | 0.9996 |

Obviously, $\bar{F}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha,-\alpha}$ with auxiliary function $A(t)=\alpha t^{-\alpha} / 2$. Let $X_{1}$ and $X_{2}$ have an identical Fréchet distribution $F$. Suppose that the random vector $\left(X_{1}, X_{2}\right)$ follows a Sarmanov distribution in (2.2) with $\phi_{i}(\cdot)=(\cdot)^{-1}-E\left[X_{i}^{-1}\right]$. Then, $d_{i}=-E\left[X_{i}^{-1}\right]$ and $\rho_{i}=-1, i=1,2$. Thus,

$$
\operatorname{ICE}_{p, 1}\left(S_{2}\right)=\frac{2^{1 / \alpha-1} \alpha F^{\leftarrow}(p)}{(\alpha-1)^{1 / \alpha+1}}+\frac{2 \mu-\mu\left(F^{\leftarrow}(p)\right)-a_{12} d_{1} \mu_{1}\left(F^{\leftarrow}(p)\right)}{2(\alpha-1)}
$$

and

$$
\operatorname{SICE}_{p, 1}\left(S_{2}\right)=\operatorname{ICE}_{p, 1}\left(S_{2}\right)-\left(\frac{F^{\leftarrow}(p)}{2(\alpha-1)^{1 / \alpha}}+\frac{\mu}{2 \alpha}\right)
$$

Again, Tables 4-5 reveal that the second-order asymptotics of $\operatorname{ICE}_{p}\left(S_{2}\right)$ and $\mathrm{CE}_{p}\left(S_{2}\right)$ are close to simulation values for $p \in[0.95,1)$ and provide much better estimates than the first-order asymptotics.

## 5 Numerical illustration

In this section, we numerically illustrate our asymptotic results with a comprehensive comparison among different families and types of systemic risk measures.

Example 5.1 Under Example 3.1 with $\alpha=2, k=1$ and $a_{12}=-1,0,1$, by Theorems 3.1, 3.2, 4.1 and 4.2, we have that the first-order asymptotics (as $p \uparrow 1$ ) of VaR, Expectile, MES and ICE are


Figure 2: Comparison of the simulation values(MC), first-order and second-order asymptotic values of $\operatorname{VaR}_{p}\left(S_{2}\right) / \operatorname{VaR}_{p}(X)$ and $\mathrm{e}_{p}\left(S_{2}\right) / \operatorname{VaR}_{p}(X), \operatorname{CTE}_{p}\left(S_{2}\right) / \operatorname{VaR}_{p}(X)$ and $\mathrm{CE}_{p}\left(S_{2}\right) / \operatorname{VaR}_{p}(X), \operatorname{MES}_{p}\left(S_{2}\right) / \operatorname{VaR}_{p}(X)$ and $\operatorname{ICE}_{p}\left(S_{2}\right) / \operatorname{VaR}_{p}(X)$, and $\operatorname{SES}_{p}\left(S_{2}\right) / \operatorname{VaR}_{p}(X)$ and $\operatorname{SICE}_{p}\left(S_{2}\right) / \operatorname{VaR}_{p}(X)$. We use the Pareto distribution and the Sarmanov distribution with $a_{12}=-1$ for the left panel, $a_{12}=0$ for the middle panel and $a_{12}=1$ for the right panel.


Figure 3: Comparison of the simulation values(MC), first-order and second-order asymptotic values of $\operatorname{VaR}_{p}\left(S_{2}\right) / \operatorname{VaR}_{p}(X)$ and $\mathrm{e}_{p}\left(S_{2}\right) / \operatorname{VaR}_{p}(X)$. We use the Burr distribution with $\beta=-0.5$ for the left panel and $\beta=-2$ for the right panel.
equivalent (the ratio to $F^{\leftarrow}(p)$ is $2^{1 / 2}$ ). Those of CTE and CE are equivalent (the ratio to $F^{\leftarrow}(p)$ is $2^{3 / 2}$ ). Those of SES and SICE are equivalent (the ratio to $F^{\leftarrow}(p)$ is $2^{1 / 2}-1 / 2$ ). In Figure 2, we have the following observations:

- The second-order asymptotics of systemic risk measures are closer to simulation values than the first-order asymptotics as $p \uparrow 1$. The expectile-based systemic risk measures are larger than VaR-based ones. One should consider a lower confidence level for expectile-based risk measures.
- The second-order asymptotics of VaR, CTE and MES are closer to the simulation values as $a_{12}$ (i.e. dependence coefficient) decreases. The second-order asymptotics of expectile and CE are closer to the simulation values as $a_{12}$ increases.
- The second-order asymptotics of ICE is closer to the empirical value as $a_{12}$ decreases and and that of ICE is closer to the simulation value than that of MES.
- The second-order asymptotic of SICE is closer to the simulation values as $a_{12}$ decreases and that of SES is closer to the simulation values than that of SICE.

Example 5.2 Under the conditions of Example 3.2 with $\alpha=1.5$, $\beta=-0.5$ or -2 under an Sarmanov distribution in (2.2) with $\phi_{i}(\cdot)=1-2 F(\cdot), i=1,2$ and $a_{12}=-0.5$. Then by Theorem 3.1, 3.2, 4.1 and 4.2, we have that, as $p \uparrow 1$,

- Figures 3-6 show the second-order asymptotics and simulation values of Theorems 3.1, 3.2, 4.1 and 4.2. Again, the second-order asymptotics can approximate the simulation values as $p \uparrow 1$ better than the first-order asymptotics.
- Figures 3 and 5 reveal that the second-order asymptotics of VaR and MES with $\beta=-0.5$ are better than those of $\beta=-2$, which the expectile and ICE have an opposite result.
- Figure 4 and 6 represent that the second-order asymptotics of CTE, CE, SES and SICE with $\beta=-2$ are better than those of $\beta=-0.5$.


## 6 Application

Introduced by Bürgi et al. (2008), the concept of diversification benefit represents the retained capital gained by collectively managing all risks within a portfolio, in contrast to addressing each risk individually. For a fixed threshold of $0<p<1$, the diversification benefit is defined by

$$
\begin{equation*}
D_{p}^{\rho}\left(S_{n}\right)=1-\frac{\rho_{p}\left(S_{n}\right)-\mathbb{E}\left[S_{n}\right]}{\sum_{i=1}^{n}\left(\rho_{p}\left(X_{i}\right)-\mathbb{E}\left[X_{i}\right]\right)} \tag{6.1}
\end{equation*}
$$

where $\rho_{p}$ represents a risk measure at a specific confidence level $p$ (e.g., $\mathrm{VaR}_{p}, \mathrm{e}_{p}, \mathrm{CTE}_{p}$, etc).
Constructed as such, for a fixed systemic risk measure $\rho, D_{p}^{\rho}>0$ indicates that diversification is advantageous, potentially reducing an insurer's risk by market engagement. Conversely, $D_{p}^{\rho} \leq 0$


Figure 4: Comparison of the simulation values(MC), first-order and second-order asymptotic values of $\operatorname{CTE}_{p}\left(S_{2}\right) / \operatorname{VaR}_{p}(X)$ and $\mathrm{CE}_{p}\left(S_{2}\right) / \operatorname{VaR}_{p}(X)$. We use the Burr distribution with $\beta=-0.5$ for the left panel and $\beta=-2$ for the right panel.


Figure 5: Comparison of the simulation values(MC), first-order and second-order asymptotic values of $\operatorname{MES}_{p}\left(S_{2}\right) / \operatorname{VaR}_{p}(X)$ and $\operatorname{ICE}_{p}\left(S_{2}\right) / \operatorname{VaR}_{p}(X)$. We use the Burr distribution with $\beta=-0.5$ for the left panel and $\beta=-2$ for the right panel.


Figure 6: Comparison of the simulation values(MC), first-order and second-order asymptotic values of $\operatorname{SES}_{p}\left(S_{2}\right) / \operatorname{VaR}_{p}(X)$ and $\operatorname{SICE}_{p}\left(S_{2}\right) / \operatorname{VaR}_{p}(X)$, Burr distribution with $\beta=-0.5$ for the left panel and $\beta=-2$ for the right panel.
suggests diversification is not advantageous for a single insurer. We are particularly interested in cases of heavy-tailed risks whether diversification may be beneficial. Further, this aspect of whether $D_{p}^{\rho}>0$ is technically linked to the risk measure $\rho$ 's subadditivity. It further relates to the so-called coherence; e.g., see Artzner et al. (1999).

The diversification benefit aids in portfolio selection. By maximizing diversification benefits, the investor can mitigate the risk and boost the performance of a portfolio. It is worthwhile to mention that the usage of $D_{p}^{\rho}\left(S_{n}\right)$ is not always applicable; its value depends on the number of risks involved and the specific risk measures employed. Recent findings from Dacorogna et al. (2018) and Chen et al. (2022) highlighted that diversification benefits vary notably based on the type of dependence and the risk measures.

Experts emphasize caution against careless diversification practices, especially when confronted with risks with heavy tails. By adopting the above results of risk measures and deriving formulas for diversification benefits, we can evaluate the performance of a portfolio $S_{n}$ in contrast to individual risks operating independently. In the following, we first derive the second-order asymptotic of $D_{p}^{\rho}\left(S_{n}\right)$ with $\rho$ based on VaR, expectile, CTE and CE.

Theorem 6.1 Under the conditions of Proposition 3.1 with $\alpha>1$, we have, as $p \uparrow 1$,

$$
\begin{aligned}
D_{p}^{\mathrm{VaR}}\left(S_{n}\right)= & 1-n^{1 / \alpha-1}\left(1+\left(\frac{\mu_{n}^{*}\left(F^{\leftarrow}(p)\right)}{n^{1 / \alpha} F^{\leftarrow}(p)}+\frac{n^{\beta / \alpha}-1}{\alpha \beta} A\left(F^{\leftarrow}(p)\right)\right)(1+o(1))\right) \\
& +\frac{\left(1-n^{1 / \alpha-1}\right) \mu}{F^{\leftarrow}(p)}(1+o(1))+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right), \\
D_{p}^{\mathrm{e}}\left(S_{n}\right)= & 1-n^{1 / \alpha-1}\left(1+\frac{\left(n^{\beta / \alpha}-1\right)(\alpha-1)^{1-\beta / \alpha}}{\alpha \beta(\alpha-\beta-1)} A\left(F^{\leftarrow}(p)\right)(1+o(1))\right) \\
& -\frac{(\alpha-1)^{1 / \alpha+1} \mu_{n}^{*}\left(F^{\leftarrow(p))}\right.}{\alpha n F^{\leftarrow}(p)}(1+o(1))+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right), \\
D_{p}^{\mathrm{CTE}}\left(S_{n}\right)= & 1-n^{1 / \alpha-1}\left(1+\frac{1}{\alpha \beta}\left(\frac{n^{\beta / \alpha}(\alpha-1)-\beta}{\alpha-\beta-1}-1\right) A\left(F^{\leftarrow}(p)\right)(1+o(1))\right) \\
- & \frac{(\alpha-1)\left(\mu_{n}^{*}\left(F^{\leftarrow}(p)\right)-\left(n-n^{1 / \alpha}\right) \mu\right)}{n \alpha F^{\leftarrow}(p)}(1+o(1))+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right),
\end{aligned}
$$

and

$$
\begin{aligned}
D_{p}^{\mathrm{CE}}\left(S_{n}\right)= & 1-n^{1 / \alpha-1}\left(1+\frac{\left(n^{\beta / \alpha}-1\right)(\alpha-1)^{-\beta / \alpha}(\alpha+\beta-1)}{\alpha \beta(\alpha-\beta-1)} A\left(F^{\leftarrow}(p)\right)(1+o(1))\right) \\
& -\frac{(\alpha-1)^{1 / \alpha}(\alpha-2) \mu_{n}^{*}\left(F^{\leftarrow}(p)\right)}{\alpha n F^{\leftarrow}(p)}(1+o(1))+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right) .
\end{aligned}
$$

Obviously, the first-order asymptotics of $D_{p}^{\mathrm{VaR}}\left(S_{n}\right), D_{p}^{\mathrm{e}}\left(S_{n}\right), D_{p}^{\mathrm{CTE}}\left(S_{n}\right)$ and $D_{p}^{\mathrm{CE}}\left(S_{n}\right)$ are $1-n^{1 / \alpha-1}$.

Proof. see Appendix.
For numerical illustration, we give an example of the Weiss distribution.
Example 6.1 (Weiss distribution) A Burr distribution function $F$ satisfies that

$$
F(x)=1-x^{\alpha}\left(1+x^{-\beta}\right), \quad x, \alpha>1, \beta<0 .
$$

It is easy to check that $\bar{F}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \beta}$ with auxiliary function $A(t)=\beta t^{\beta}$. Let $X_{1}$ and $X_{2}$ have an identical Weiss distribution $F$. Suppose that the random vector $\left(X_{1}, X_{2}\right)$ follows an Sarmanov distribution in (2.2) with $\phi_{i}(\cdot)=1-2 F(\cdot)$. Clearly, $d_{i}=-1, i=1,2$.


Figure 7: Comparison of the simulation values(MC), first-order and second-order asymptotic values of $D_{\mathrm{VaR}}\left(S_{2}\right), D_{\mathrm{e}}(X), D_{\mathrm{CTE}}\left(S_{2}\right)$ and $D_{\mathrm{CE}}\left(S_{2}\right)$. We use the Weiss distribution with $\alpha=2.5, \beta=-1$ and $a_{12}=0.5$.

In the context of Example 6.1, we aim to present the asymptotic performance for diversification benefits $D_{p}^{\rho}$ across four risk measures (i.e. VaR, expectile, CTE, and CE) based on the results obtained in Theorem 6.1. Here, we denote by $\hat{D}_{p}^{\rho}$ the second-order asymptotic and employ the ratio $\hat{D_{p}^{\rho}} / D_{p}^{\rho}$ to assess the asymptotic performance of diversification benefits. A value of $\hat{D}_{p}^{\rho} / D_{p}^{\rho}$ closer to 1 indicates a more accurate asymptotic result, while deviations from 1 imply poorer outcomes. Besides, according to Bürgi et al. (2008), $\hat{D}_{p}^{\rho} / D_{p}^{\rho}>1$ signifies the overestimation of the diversification benefit, while $\hat{D}_{p}^{\rho} / D_{p}^{\rho}<1$ corresponds to the underestimation.

The numerical experiment is shown in Figure 7. For the comparing purpose, we use blue lines to represent the outcomes derived from VaR-based systemic risk measures (i.e. VaR and CTE), while red lines represent the outcomes obtained from expectile-based systemic risk measures (i.e. expectile and CE).

We can observe that, in both plots, the values of $\hat{D}_{p}^{\rho} / D_{p}^{\rho}$ obtained from alternative expectile-based systemic risk measures (i.e. expectile and CE) generally exhibit higher accuracy compared to those
obtained from VaR-based systemic risk measures (i.e. VaR and CTE). The particularly noteworthy finding is the performance of CE , the ratio of $\hat{D}_{p}^{\rho} / D_{p}^{\rho}$ consistently greater than 0.95 when $p \in[0.95,1]$, indicating an error range of less than $5 \%$. It suggests that the expectile inherently incorporates more information than VaR when estimated from the empirical dataset. This insight shows the potential of expectile-based systemic risk measures in the quantification of diversification benefits.

Significantly, the second insight is that when $\rho$ is an expectile-based systemic risk measure, $\hat{D_{p}^{\rho}} / D_{p}^{\rho}<1$ consistently with $p \in[0.95,1]$, implying an underestimation of the diversification benefit and suggesting a conservative approximation. Conversely, when $\rho$ is a VaR-based systemic risk measure, $\hat{D}_{p}^{\rho} / D_{p}^{\rho}>1$ consistently with $p \in[0.95,1]$, which overestimates the diversification benefit and reflects an optimistic view of diversification. These observations reveal different features of expectile-based and VaR-based systemic risk measures in diversification benefits. Financial practitioners and regulators can sophisticatedly choose one from them according to their distinct purposes and attitudes (conservative or optimistic).

There are more new ways to quantify diversification. E.g., the diversification quotient, was proposed in Han et al. (2022, 2023). Our asymptotic treatment provides a unified framework to investigate these new quotients, which will be studied in the future.

## 7 Conclusion

In this paper, we study systemic risk measures with multivariate Sarmanov distribution. We first classify them into two families of VaR- and expectile-based systemic risk measures. We have the second-order asymptotics of VaR, CTE, MES and SES in the first family. Furthermore, we obtain the second-order asymptotics of expectile, CE, ICE and SICE in the second family. In addition, we give concrete analytical and numerical examples to illustrate the main results. We emphasize that the second-order asymptotics can provide a much better approximation as $p \uparrow 1$ than the first-order asymptotics. Moreover, we provide a comprehensive comparison among VaR- and expectile-based systemic risk measures. We find that expectile-based measures deduce a larger risk evaluation than VaR-based measures, suggesting a lower confidence level when the expectile is adopted. Finally, we apply the asymptotic treatment to financial diversification and provide instructive insights for risk management. We believe that our results consolidate future research in risk management and our findings have implications for financial practitioners and regulators striving to better understand and mitigate systemic risks in complex financial systems.

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## 8 Appendix

Lemma 8.1 Assume that $\bar{F}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \beta}$ with $\alpha>0, \beta \leq 0$ and auxiliary function $A(\cdot)$. Define $W(t)=\int_{0}^{t} \phi(x) \mathrm{d} F(x)$, where $\phi: \mathbb{R} \rightarrow \mathbb{R}$ is a function with $\mathbb{E}(\phi(X))=0$, $\lim _{t \rightarrow \infty} \phi(t)=b$ and $\phi(\cdot)-b \in \mathcal{R} \mathcal{V}_{\rho}$ with $b \in \mathbb{R}$ and $\rho \leq 0$. We have $\bar{W}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \gamma}$ with $\gamma=\max \{\beta, \rho\}$ and $\widetilde{A}(\cdot)=A(\cdot)+\frac{\rho \alpha}{b(\alpha-\rho)}(\phi(\cdot)-b)$. In addition, as $t \rightarrow \infty$, we have

$$
\begin{equation*}
\frac{\bar{W}(t)}{\bar{F}(t)}=b+\frac{\alpha}{\alpha-\rho}(\phi(t)-b)(1+o(1)) \tag{8.1}
\end{equation*}
$$

Moreover, $\widetilde{A}(\cdot)=\rho \phi(\cdot)$ if $b=0$.
Proof. Firstly, we need to prove that $\bar{W}(\cdot) \in \mathcal{R} \mathcal{V}_{-\alpha}$. According to $\lim _{t \rightarrow \infty} \phi(t)=b$, for any $\epsilon>0$, there exists $t_{0}>0$ such that

$$
b-\epsilon \leq \phi(t) \leq b+\epsilon, \quad \forall t>t_{0}
$$

Fix any $x>0$. we have

$$
\lim _{t \rightarrow \infty} \frac{\bar{W}(t x)}{\bar{W}(t)}=\lim _{t \rightarrow \infty} \frac{\int_{t x}^{\infty}(\phi(y)-b) \mathrm{d} F(y)+b \bar{F}(t x)}{\int_{t}^{\infty}(\phi(y)-b) \mathrm{d} F(y)+b \bar{F}(t)}=x^{-\alpha}
$$

Secondly, according to $\bar{F} \in R V_{-\alpha}, \phi(\cdot)-b \in \mathcal{R} \mathcal{V}_{\rho}$ and Potter's inequality (Proposition B.1.9 (5) of De Haan and Ferreira (2006)), for any $\delta>0$, there exists $t_{1}>t_{0}$ such that for $t$, ty $>t_{1}$,

$$
\left|\frac{\bar{F}(t y)}{\bar{F}(t)}-y^{-\alpha}\right| \leq \epsilon \max \left\{y^{-\alpha+\delta}, y^{-\alpha-\delta}\right\}
$$

and

$$
\left|\frac{\phi(t y)-b}{\phi(t)-b}-x^{\rho}\right| \leq \epsilon \max \left\{y^{\rho+\delta}, y^{\rho-\delta}\right\}
$$

For any $t>t_{1}$, by the dominated convergence theorem, we have

$$
\begin{aligned}
\frac{\bar{W}(t)}{\bar{F}(t)} & =b+\int_{t}^{\infty} \frac{\phi(x)-b}{\bar{F}(t)} \mathrm{d} F(x) \\
& =b-\frac{\phi(t)-b}{\bar{F}(t)} \int_{1}^{\infty}\left(\frac{\phi(t x)-b}{\phi(t)-b}-x^{\rho}+x^{\rho}\right) \mathrm{d} \bar{F}(t x) \\
& =b-\frac{\phi(t)-b}{\bar{F}(t)} \int_{1}^{\infty} x^{\rho} \mathrm{d} \bar{F}(t x)(1+o(1)) \\
& =b+\left(1+\int_{1}^{\infty}\left(\frac{\bar{F}(t x)}{\bar{F}(t)}-x^{-\alpha}+x^{-\alpha}\right) \mathrm{d} x^{\rho}\right)(\phi(t)-b)(1+o(1)) \\
& =b+\left(1+\int_{1}^{\infty} x^{-\alpha} \mathrm{d} x^{\rho}\right)(\phi(t)-b)(1+o(1)) \\
& =b+\frac{\alpha}{\alpha-\rho}(\phi(t)-b)(1+o(1))
\end{aligned}
$$

By $\bar{F}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \beta}$ and Drees-type inequality in Mao (2013), there exists $t_{2}>t_{1}$ such that for all $y>0$ and $t>\max \left\{t_{2}, \frac{t_{2}}{y}\right\}$,

$$
\left|\frac{1}{A(t)}\left(\frac{\bar{F}(t y)}{\bar{F}(t)}-y^{-\alpha}\right)-H_{-\alpha, \beta}(y)\right| \leq \epsilon y^{-\alpha+\rho} \max \left\{y^{\delta}, y^{-\delta}\right\} .
$$

For any $t>\max \left\{t_{2}, \frac{t_{2}}{x}\right\}$, according to the dominated convergence theorem, it follows that

$$
\begin{aligned}
\frac{\bar{W}(t x)}{\bar{W}(t)}-x^{-\alpha} & =\frac{\int_{t x}^{\infty} \phi(y) \mathrm{d} F(y)}{\int_{t}^{\infty} \phi(y) \mathrm{d} F(y)}-x^{-\alpha} \\
& =\frac{b \bar{F}(t x)+\int_{t x}^{\infty}(\phi(y)-b) \mathrm{d} F(y)}{b \bar{F}(t)+\int_{t}^{\infty}(\phi(y)-b) \mathrm{d} F(y)}-x^{-\alpha} \\
& =\frac{b \bar{F}(t x)-\int_{1}^{\infty}(\phi(t x y)-b) \mathrm{d} \bar{F}(t x y)}{b \bar{F}(t)-\int_{1}^{\infty}(\phi(t y)-b) \mathrm{d} \bar{F}(t y)}-x^{-\alpha} \\
& =\frac{b \bar{F}(t x)-\int_{1}^{\infty}\left(\frac{\phi(t x y)-b}{\phi(t)-b}-(x y)^{\rho}+(x y)^{\rho}\right) \mathrm{d} \bar{F}(t x y)(\phi(t)-b)}{b \bar{F}(t)-\int_{1}^{\infty}\left(\frac{\phi(t y)-b}{\phi(t)-b}-y^{\rho}+y^{\rho}\right) \mathrm{d} \bar{F}(t y)(\phi(t)-b)}-x^{-\alpha} \\
& =\frac{b \bar{F}(t x)-\int_{1}^{\infty}(x y)^{\rho} \mathrm{d} \bar{F}(t x y)(\phi(t)-b)(1+o(1))}{b \bar{F}(t)-\int_{1}^{\infty} y^{\rho} \mathrm{d} \bar{F}(t y)(\phi(t)-b)(1+o(1))}-x^{-\alpha} \\
& =\frac{\frac{b \bar{F}(t x)}{\bar{F}(t)}+x^{\rho}\left(\frac{\bar{F}(t x)}{\bar{F}(t)}+\int_{1}^{\infty}\left(\frac{\bar{F}(t x y)}{\bar{F}(t)}-(x y)^{-\alpha}+(x y)^{-\alpha}\right) \mathrm{d} y^{\rho}\right)(\phi(t)-b)(1+o(1))}{b+\left(1+\int_{1}^{\infty}\left(\frac{\bar{F}(t y)}{\bar{F}(t)}-y^{-\alpha}+y^{-\alpha}\right) \mathrm{d} y^{\rho}\right)(\phi(t)-b)(1+o(1))}-x^{-\alpha} \\
& =\frac{b x^{-\alpha}\left(1+\frac{x^{\beta}-1}{\beta} A(t)(1+o(1))\right)+x^{-\alpha+\rho}\left(1+\int_{1}^{\infty} y^{-\alpha} \mathrm{d} y^{\rho}\right)(\phi(t)-b)(1+o(1))}{b-\left(1+\int_{1}^{\infty} y^{-\alpha} \mathrm{d} y^{\rho}\right)(\phi(t)-b)(1+o(1))}-x^{-\alpha} \\
& =x^{-\alpha} \frac{x^{\beta}-1}{\beta} A(t)(1+o(1))+x^{-\alpha} \frac{x^{\rho}-1}{\rho} \frac{\rho \alpha}{b(\alpha-\rho)}(\phi(t)-b)(1+o(1)) .
\end{aligned}
$$

Thus, this ends the proof of Lemma 8.1.
Proof of Proposition 4.1. Because of equation (1.1), for large enough $p \uparrow 1$ satisfying $\mathrm{e}_{p}(X)>0$, we have

$$
1-\frac{\mathbb{E}(X)}{\mathrm{e}_{p}(X)}=\frac{2 p-1}{1-p} \mathbb{E}\left(\left[\frac{X}{\mathrm{e}_{p}(X)}-1\right] \mathbf{1}_{\left\{X / \mathrm{e}_{p}(X) \geq 1\right\}}\right) .
$$

Applying the integration by parts, we have

$$
\begin{aligned}
\mathbb{E}\left(\left[\frac{X}{\mathrm{e}_{p}(X)}-1\right] \mathbf{1}_{\left\{X / \mathrm{e}_{p}(X) \geq 1\right\}}\right) & =\int_{\mathrm{e}_{p}(X)}^{\infty}\left(\frac{x}{\mathrm{e}_{p}(X)}-1\right) \mathrm{d} F(x) \\
& =-\int_{\mathrm{e}_{p}(X)}^{\infty}\left(\frac{x}{\mathrm{e}_{p}(X)}-1\right) \mathrm{d} \bar{F}(x) \\
& =-\left.\left(\frac{x}{\mathrm{e}_{p}(X)}-1\right) \bar{F}(x)\right|_{\mathrm{e}_{p}(X)} ^{\infty}+\frac{1}{\mathrm{e}_{p}(X)} \int_{\mathrm{e}_{p}(X)}^{\infty} \bar{F}(x) \mathrm{d} x \\
& =\int_{1}^{\infty} \bar{F}\left(x \mathrm{e}_{p}(X)\right) \mathrm{d} x \\
& =\bar{F}\left(\mathrm{e}_{p}(X)\right)\left(\int_{1}^{\infty} x^{-\alpha} \mathrm{d} x+\int_{1}^{\infty} \frac{\bar{F}\left(x \mathrm{e}_{p}(X)\right)}{\bar{F}\left(\mathrm{e}_{p}(X)\right)}-x^{-\alpha} \mathrm{d} x\right) \\
& =\bar{F}\left(\mathrm{e}_{p}(X)\right)\left(\frac{1}{\alpha-1}+\int_{1}^{\infty} H_{-\alpha, \beta}(x) A\left(\mathrm{e}_{p}(X)\right)(1+o(1)) \mathrm{d} x\right) \\
& =\frac{\bar{F}\left(\mathrm{e}_{p}(X)\right)}{\alpha-1}\left(1+\frac{1}{\alpha-\beta-1} A\left(\mathrm{e}_{p}(X)\right)(1+o(1))\right),
\end{aligned}
$$

where the third step is due to the dominated convergence theorem ensured by Theorem 2.3.9 of De Haan and Ferreira (2006). In particular, Bellini et al. (2014) shows that

$$
\mathrm{e}_{p}(X) \sim(\alpha-1)^{-1 / \alpha} F^{\leftarrow}(p), \quad p \uparrow 1
$$

Since $\mathrm{e}_{p}(X) \rightarrow \infty, 1-p \downarrow 0$ and $A\left(\mathrm{e}_{p}(X)\right) \downarrow 0$ as $p \uparrow 1$, by the first-order Taylor expansion, we have that

$$
\begin{aligned}
\frac{1-p}{\bar{F}\left(\mathrm{e}_{p}(X)\right)} & =\frac{1}{\alpha-1}(1-2(1-p))\left(1-\frac{\mu}{\mathrm{e}_{p}(X)}\right)^{-1}\left(1+\frac{1}{\alpha-\beta-1} A\left(\mathrm{e}_{p}(X)\right)(1+o(1))\right) \\
& =\frac{1}{\alpha-1}(1-2(1-p))\left(1+\frac{\mu}{\mathrm{e}_{p}(X)}(1+o(1))\right)\left(1+\frac{1}{\alpha-\beta-1} A\left(\mathrm{e}_{p}(X)\right)(1+o(1))\right) \\
& =\frac{1}{\alpha-1}\left(1+\frac{1}{\alpha-\beta-1} A\left(\mathrm{e}_{p}(X)\right)(1+o(1))+\frac{\mu}{\mathrm{e}_{p}(X)}(1+o(1))-2(1-p)(1+o(1))\right) \\
& =\frac{1}{\alpha-1}\left(1+\frac{1}{\alpha-\beta-1} A\left(\mathrm{e}_{p}(X)\right)(1+o(1))+\frac{\mu}{\mathrm{e}_{p}(X)}(1+o(1))-2 \bar{F}\left(F^{\leftarrow}(p)\right)(1+o(1))\right) \\
& =\frac{1}{\alpha-1}\left(1+\frac{(\alpha-1)^{-\beta / \alpha}}{\alpha-\beta-1} A\left(F^{\leftarrow}(p)\right)(1+o(1))+\frac{(\alpha-1)^{1 / \alpha} \mu}{F^{\leftarrow}(p)}(1+o(1))\right),
\end{aligned}
$$

where in the third step we use $1-p \sim \bar{F}\left(F^{\leftarrow}(p)\right)$ as $p \uparrow 1$. Notably, in the second last step, we use $\lim _{p \uparrow 1} \bar{F}\left(F^{\leftarrow}(p)\right) F^{\leftarrow}(p)=0$, and thus $\bar{F}\left(F^{\leftarrow}(p)\right)=o\left(1 / F^{\leftarrow}(p)\right)$. In addition, due to the fact that
$U_{F}(\cdot) \in \mathcal{R} \mathcal{V}_{1 / \alpha, \beta / \alpha}$ with auxiliary function $\alpha^{-2} A \circ U_{F}(\cdot)$, it follows that

$$
\begin{aligned}
\frac{\mathrm{e}_{p}(X)}{F^{\leftarrow}(p)}= & \frac{U_{F}\left(1 / \bar{F}\left(\mathrm{e}_{p}(X)\right)\right)}{U_{F}(1 /(1-p))} \\
= & \left(\frac{1-p}{F\left(\mathrm{e}_{p}(X)\right)}\right)^{1 / \alpha}\left(1+\frac{\left(\frac{1-p}{\bar{F}\left(e_{p}(X)\right)}\right)^{\beta / \alpha}-1}{\beta / \alpha} \alpha^{-2} A \circ U_{F}(1 /(1-p))\right) \\
= & \left(\frac{1}{\alpha-1}\left(1+\frac{(\alpha-1)^{-\beta / \alpha}}{\alpha-\beta-1} A\left(F^{\leftarrow}(p)\right)(1+o(1))+\frac{(\alpha-1)^{1 / \alpha} \mu}{F^{\leftarrow}(p)}(1+o(1))\right)\right)^{1 / \alpha} \\
& \times\left(1+\frac{(\alpha-1)^{-\beta / \alpha}-1}{\alpha \beta} A\left(F^{\leftarrow}(p)\right)(1+o(1))\right) \\
= & (\alpha-1)^{-1 / \alpha}\left(1+\frac{(\alpha-1)^{-\beta / \alpha}}{\alpha(\alpha-\beta-1)} A\left(F^{\leftarrow}(p)\right)(1+o(1))+\frac{(\alpha-1)^{1 / \alpha} \mu}{\alpha F^{\leftarrow}(p)}(1+o(1))\right) \\
& \times\left(1+\frac{(\alpha-1)^{-\beta / \alpha}-1}{\alpha \beta} A\left(F^{\leftarrow}(p)\right)(1+o(1))\right) \\
= & (\alpha-1)^{-1 / \alpha}\left(1+\left(\frac{1}{\alpha \beta}\left(\frac{(\alpha-1)^{1-\beta / \alpha}}{\alpha-\beta-1}-1\right) A\left(F^{\leftarrow}(p)\right)+\frac{(\alpha-1)^{1 / \alpha} \mu}{\alpha F^{\leftarrow}(p)}\right)(1+o(1))\right) .
\end{aligned}
$$

Thus, we obtain the desired results.
Lemma 8.2 Let $Y$ be the nonnegative rv with a distribution $H$ satisfying that $\bar{H}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \rho}$ with $\alpha>1, \rho<0$ and auxiliary function $A_{H}(\cdot)$. As $t \rightarrow \infty$, we have

$$
\mathbb{E}[Y \mid Y>t]=\frac{\alpha t}{\alpha-1}\left(1+\frac{1}{\alpha(\alpha-\rho-1)} A_{H}(t)(1+o(1))\right) .
$$

Proof. By the dominated convergence theorem ensured by Theorem 2.3.9 of De Haan and Ferreira (2006), it follows that, as $t \rightarrow \infty$,

$$
\begin{aligned}
\mathbb{E}[Y \mid Y>t] & =\int_{0}^{\infty} \frac{\mathbb{P}(Y>z, Y>t)}{\mathbb{P}(Y>t)} \mathrm{d} z \\
& =\int_{0}^{t} \frac{\mathbb{P}(Y>t)}{\mathbb{P}(Y>t)} \mathrm{d} z+t \int_{1}^{\infty} \frac{\mathbb{P}(Y>z t)}{\mathbb{P}(Y>t)} \mathrm{d} z \\
& =t\left(1+\int_{1}^{\infty} z^{-\alpha}\left(1+\frac{z^{\rho}-1}{\rho} A_{H}(t)(1+o(1))\right) \mathrm{d} z\right) \\
& =t\left(\frac{\alpha}{\alpha-1}+\frac{1}{(\alpha-\rho-1)(\alpha-1)} A_{H}(t)(1+o(1))\right) \\
& =\frac{\alpha t}{\alpha-1}\left(1+\frac{1}{\alpha(\alpha-\rho-1)} A_{H}(t)(1+o(1))\right) .
\end{aligned}
$$

Thus, we prove this lemma.
The next lemma extends Lemma 2.4 of Mao and Hu (2013).
Lemma 8.3 Let $F$ be the distribution function of a nonnegative random variable satisfying $\bar{F}(\cdot) \in$
$\mathcal{R} \mathcal{V}_{-\alpha}$ with $\alpha>1$. For any fixed $z \in(0,1)$ and $\beta>0$, define

$$
V_{\beta}(z t)=\int_{0}^{z t}\left(\left(1-\frac{y}{t}\right)^{-\beta}-1\right) \mathrm{d} F(y), \quad t>0 .
$$

Then, as $t \rightarrow \infty$, we have

$$
V_{\beta}(z t) \sim \beta t^{-1} \mu(t)
$$

Proof. Since $\alpha>1$ and $\mu(t) \rightarrow \mu$ as $t \rightarrow \infty$, we have that $\mu(t)<\infty$ and $\frac{\mu(t)}{t} \in \mathcal{R} \mathcal{V}_{-1}$. We have

$$
\mu(t)=\int_{0}^{t} x \mathrm{~d} F(x)=-\int_{0}^{t} x \mathrm{~d} \bar{F}(x)=-t \bar{F}(t)+\int_{0}^{t} \bar{F}(x) \mathrm{d} x .
$$

According to Karamata's theorem, it is easy to check that

$$
\begin{equation*}
\mu(t) \sim \int_{0}^{t} \bar{F}(x) \mathrm{d} x \quad \text { as } \quad t \rightarrow \infty, \quad \text { and } \quad \lim _{t \rightarrow \infty} \frac{t \bar{F}(t)}{\int_{0}^{t} \bar{F}(y) \mathrm{d} y}=0 . \tag{8.2}
\end{equation*}
$$

By the integration by parts, it follows that

$$
\begin{aligned}
V_{\alpha}(z t) & =-\int_{0}^{z t}\left(1-\frac{y}{t}\right)^{-\beta}-1 \mathrm{~d} \bar{F}(y) \\
& =-(1-z)^{-\beta} \bar{F}(z t)+\frac{\beta}{t} \int_{0}^{z t} \bar{F}(y)\left(1-\frac{y}{t}\right)^{-\beta-1} \mathrm{~d} y .
\end{aligned}
$$

For any fixed $z \in(0,1)$ and (8.2),

$$
\lim _{t \rightarrow \infty} \frac{t \bar{F}(z t)}{\int_{0}^{z t} \bar{F}(y)\left(1-\frac{y}{t}\right)^{-\alpha-1} \mathrm{~d} y} \leq \lim _{t \rightarrow \infty} \frac{t \bar{F}(z t)}{\int_{0}^{z t} \bar{F}(y) \mathrm{d} y}=0 .
$$

Since (8.2) holds for all $\alpha>1$ and $1+(\beta+1) x \leq(1-x)^{-\beta-1} \leq 1+(\beta+1)(1-z)^{-\beta-2} x$ for $x \in(0, z)$, we obtain

$$
\begin{aligned}
\lim _{t \rightarrow \infty} \frac{V_{\beta}(z t)}{\mu(z t)} & =\beta t^{-1} \lim _{t \rightarrow \infty} \frac{\int_{0}^{z t} \bar{F}(y)\left(1-\frac{y}{t}\right)^{-\beta-1} \mathrm{~d} y}{\int_{0}^{z t} \bar{F}(y) \mathrm{d} y} \\
& \geq \beta t^{-1} \lim _{t \rightarrow \infty} \frac{\int_{0}^{z t} \bar{F}(y) \mathrm{d} y+(\beta+1) \int_{0}^{z t} \bar{F}(y) y / t \mathrm{~d} y}{\int_{0}^{z t} \bar{F}(y) \mathrm{d} y} \\
& =\beta t^{-1}\left(1+(\beta+1) \lim _{t \rightarrow \infty} \frac{\int_{0}^{z t} \bar{F}(y) y \mathrm{~d} y}{t \int_{0}^{z t} \bar{F}(y) \mathrm{d} y}\right) \\
& =\beta t^{-1}\left(1+(\beta+1) \lim _{t \rightarrow \infty} \frac{z t \bar{F}(z t)}{\int_{0}^{z t} \bar{F}(y) \mathrm{d} y+t \bar{F}(z t)}\right) \\
& =\beta t^{-1},
\end{aligned}
$$

and

$$
\lim _{t \rightarrow \infty} \frac{V_{\alpha}(z t)}{\mu(z t)} \leq \beta t^{-1}\left(1+(\beta+1)(1-z)^{-\beta-2} \lim _{t \rightarrow \infty} \frac{\int_{0}^{z t} \bar{F}(y) y \mathrm{~d} y}{t \int_{0}^{z t} \bar{F}(y) \mathrm{d} y}\right)=\beta t^{-1} .
$$

By $\frac{\mu(t)}{t} \in \mathcal{R} \mathcal{V}_{-1}$, it follows that

$$
V_{\beta}(z t) \sim \beta t^{-1} \mu(z t) \sim \beta t^{-1} \mu(t), \quad \text { as } t \rightarrow \infty .
$$

This ends the proof.
Lemma 8.4 Under the conditions of Theorem 3.2, as $t \rightarrow \infty$, we have that

$$
\mathbb{E}\left[X_{m} \mid S_{n}>t\right]=\frac{\alpha t}{(\alpha-1) n}(1+\widetilde{B}(t)(1+o(1)))+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right),
$$

where

$$
\begin{equation*}
\widetilde{B}(t)=\frac{1}{\alpha(\alpha-\beta-1)} A(t)-\frac{\mu_{n}^{*}(t)}{t} . \tag{8.3}
\end{equation*}
$$

Proof. It follows that, as $t \rightarrow \infty$,

$$
\begin{aligned}
\mathbb{E}\left[X_{m} \mid S_{n}>t\right] & =\int_{0}^{\infty} \frac{\mathbb{P}\left(S_{n}>t, X_{m}>z\right)}{\mathbb{P}\left(S_{n}>t\right)} \mathrm{d} z \\
& =t \int_{0}^{1} \frac{\mathbb{P}\left(S_{n}>t, X_{m}>z t\right)}{\mathbb{P}\left(S_{n}>t\right)} \mathrm{d} z+t \int_{1}^{\infty} \frac{\mathbb{P}\left(X_{m}>z t\right)}{\mathbb{P}\left(S_{n}>t\right)} \mathrm{d} z \\
& =t\left(1-\int_{0}^{1} \frac{\mathbb{P}\left(S_{n}>t, X_{m} \leq z t\right)}{\mathbb{P}\left(S_{n}>t\right)} \mathrm{d} z+\int_{1}^{\infty} \frac{\mathbb{P}\left(X_{m}>z t\right)}{\mathbb{P}\left(S_{n}>t\right)} \mathrm{d} z\right) \\
& =t\left(1+\frac{\mathbb{P}(X>t)}{\mathbb{P}\left(S_{n}>t\right)}\left(\int_{1}^{\infty} \frac{\mathbb{P}\left(X_{m}>z t\right)}{\mathbb{P}(X>t)} \mathrm{d} z-\int_{0}^{1} \frac{\mathbb{P}\left(S_{n}>t, X_{m} \leq z t\right)}{\mathbb{P}(X>t)} \mathrm{d} z\right)\right) \\
& :=t\left(1+\frac{\mathbb{P}(X>t)}{\mathbb{P}\left(S_{n}>t\right)}\left(Q_{1}(t)-Q_{2}(t)\right)\right) .
\end{aligned}
$$

For $Q_{1}(t)$, using the fact that $\bar{F}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \beta}$ with $\alpha>1 \beta<0$ and the auxiliary function $A(t)$, as $t \rightarrow \infty$, we have

$$
\begin{aligned}
Q_{1}(t) & =\int_{1}^{\infty} z^{-\alpha}\left(1+\frac{z^{\beta}-1}{\beta} A(t)(1+o(1))\right) \mathrm{d} z \\
& =\frac{1}{\alpha-1}\left(1+\frac{1}{\alpha-\beta-1} A(t)(1+o(1))\right) .
\end{aligned}
$$

For $t>0$ and $z \in(0,1)$, write $\Omega_{t, z}:=\left\{\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{R}_{+}^{n}: \sum_{i=1}^{n} x_{i}>t, x_{m} \leq z t\right\}$. For $Q_{2}(t)$, the
key idea is to connect $\mathbb{P}\left(S_{n}>t, X_{m} \leq z t\right)$. Similar to the proof of Proposition 3.1, we have that

$$
\begin{align*}
& \mathbb{P}\left(S_{n}>t, X_{m} \leq z t\right) \\
= & \int_{\Omega_{t, z}}\left(1+\sum_{1 \leq i<j \leq n} a_{i j} \phi_{i}\left(x_{i}\right) \phi_{j}\left(x_{j}\right)\right) \prod_{k=1}^{n} \mathrm{~d} F\left(x_{k}\right) \\
= & \int_{\Omega_{t, z}} \prod_{k=1}^{n} \mathrm{~d} F\left(x_{k}\right)+\sum_{1 \leq i<j \leq n} a_{i j} c_{i} c_{j}\left(\int_{\Omega_{t, z}} \prod_{k=1}^{n} \mathrm{~d} F\left(x_{k}\right)-\int_{\Omega_{t, z}} \prod_{k=1, k \neq i}^{n} \mathrm{~d} F\left(x_{k}\right) \mathrm{d} \widetilde{F}_{i}\left(x_{i}\right)\right. \\
& \left.-\int_{\Omega_{t, z}} \prod_{k=1, k \neq j}^{n} \mathrm{~d} F\left(x_{k}\right) \mathrm{d} \widetilde{F}_{j}\left(x_{j}\right)+\int_{\Omega_{t, z}} \prod_{k=1, k \neq i, j}^{n} \mathrm{~d} F\left(x_{k}\right) \mathrm{d} \widetilde{F}_{i}\left(x_{i}\right) \mathrm{d} \widetilde{F}_{j}\left(x_{j}\right)\right) \\
:= & J(t, z)+\sum_{1 \leq i<j \leq n} a_{i j} c_{i} c_{j}\left(J(t, z)-K_{i}(t, z)-K_{j}(t, z)+K_{i, j}(t, z)\right) . \tag{8.4}
\end{align*}
$$

For simplicity, denote $S_{n}^{(m)}=\sum_{i \neq m}^{n} X_{i}^{*}$ which has the distribution $G_{m}$. By Theorem 3.5 of Mao and Hu (2013) with $\alpha>1$ and the induction assumption, it is easy to check that

$$
\frac{\overline{G_{m}}(t)}{\bar{F}(t)}=(n-1)\left(1+(n-2) \alpha t^{-1} \mu(t)(1+o(1))\right)+o(A(t)) .
$$

Obviously, $\overline{G_{m}}(\cdot) \in 2 \mathcal{R} \mathcal{V}_{-\alpha, \lambda}$ with auxiliary function $B(t)$, where $\lambda=-\min \{1,-\beta\}$ and $B(t)$ is given by

$$
B(t)=A(t)-(n-2) \alpha t^{-1} \mu(t) .
$$

For $J(t, z)$, by the dominated convergence theorem, it follows that

$$
\begin{aligned}
J(t, z) & =\int_{0}^{z t} \overline{G_{m}}(t-y) \mathrm{d} F(y) \\
& =\overline{G_{m}}(t) \int_{0}^{z t} \frac{\overline{G_{m}}(t-y)}{\overline{G_{m}}(t)} \mathrm{d} F(y) \\
& =\overline{G_{m}}(t) \int_{0}^{z t}\left(1-\frac{y}{t}\right)^{-\alpha}+H_{-\alpha, \lambda}\left(1-\frac{y}{t}\right) B(t)(1+o(1)) \mathrm{d} F(y) \\
& :=\overline{G_{m}}(t)\left(J_{1}(t, z)+J_{2}(t, z)\right) .
\end{aligned}
$$

For $J_{1}(t, z)$, by Lemma 8.3 and $\bar{F}(t)=o\left(\frac{\mu(t)}{t}\right)$, we have that

$$
\begin{aligned}
J_{1}(t, z) & =\int_{0}^{z t}\left(\left(1-\frac{y}{t}\right)^{-\alpha}-1\right) \mathrm{d} F(y)+F(z t) \\
& =\alpha t^{-1} \mu(t)+1-\bar{F}(z t) \\
& =1+\alpha t^{-1} \mu(t)(1+o(1)) .
\end{aligned}
$$

For $J_{2}(t, z)$, since $H_{\alpha, \lambda}\left(1-\frac{y}{t}\right) \leq \frac{(1-z)^{-\alpha}}{|\lambda|}\left(\left(1-\frac{y}{t}\right)^{\lambda}-1\right)$ for any $y \in(0, z t)$ and $z \in(0,1)$ and

Lemma 5.6 in Barbe and McCormick (2005), we have

$$
\int_{0}^{z t} H_{\alpha, \lambda}\left(1-\frac{y}{t}\right) \mathrm{d} F(y) \leq \int_{0}^{z t} \frac{(1-z)^{-\alpha}}{|\lambda|}\left(\left(1-\frac{y}{t}\right)^{\lambda}-1\right) \mathrm{d} F(y)=0 .
$$

Then,

$$
J_{2}(t, z)=o(B(t)) .
$$

Thus, according to $\overline{G_{m}}(t), J_{1}(t, z)$ and $J_{2}(t, z)$, it follows that

$$
\begin{aligned}
\frac{J(t, z)}{\bar{F}(t)} & =\frac{\overline{G_{m}}(t)}{\bar{F}(t)}\left(1+\alpha t^{-1} \mu(t)(1+o(1))+o(B(t))\right) \\
& =(n-1)\left(1+(n-1) \alpha t^{-1} \mu(t)(1+o(1))+o(|A(t)|)\right) .
\end{aligned}
$$

Similarly, it is easy to see that

$$
\begin{aligned}
\frac{K_{i}(t, z)}{\bar{F}(t)}= & \left(n-1-\frac{d_{i}}{c_{i}}\right)\left(1+\alpha t^{-1}(n-1) \mu(t)(1+o(1))\right)+o(|A(t)|) \\
& -\left(\frac{\alpha(n-1) \mu_{i}(t)}{c_{i} t}+\frac{\alpha\left(\phi_{i}(t)-d_{i}\right)}{c_{i}\left(\alpha-\rho_{i}\right)}\right)(1+o(1)),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{K_{i, m}(t, z)}{\bar{F}(t)}= & \left(n-1-\frac{d_{i}}{c_{i}}-\frac{d_{m}}{c_{m}}\right)\left(1+\alpha t^{-1}(n-1) \mu(t)(1+o(1))\right)+o(|A(t)|) \\
& -\left(\frac{\alpha(n-1) \mu_{i}(t)}{c_{i} t}+\frac{\alpha\left(\phi_{i}(t)-d_{i}\right)}{c_{i}\left(\alpha-\rho_{i}\right)}+\frac{\alpha(n-1) \mu_{m}(t)}{c_{m} t}+\frac{\alpha\left(\phi_{m}(t)-d_{m}\right)}{c_{m}\left(\alpha-\rho_{m}\right)}\right)(1+o(1)) .
\end{aligned}
$$

Pulling $J(t, z), K_{i}(t, z), K_{m}(t, z)$ and $K_{i, m}(t, z)$ into (8.4) yields that

$$
\frac{\mathbb{P}\left(S_{n}>t, X_{m} \leq z t\right)}{\bar{F}(t)}=(n-1)\left(1+\alpha t^{-1} \mu_{n}^{*}(t)(1+o(1))\right)+o\left(|A(t)|+\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right) .
$$

Thus, we have that

$$
Q_{2}(t)=\int_{0}^{1} \frac{\mathbb{P}\left(S_{n}>t, X_{m} \leq z t\right)}{\bar{F}(t)} \mathrm{d} z=(n-1)\left(1+\alpha t^{-1} \mu_{n}^{*}(t)\right)+o\left(|A(t)|+\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right)
$$

According to the dominated convergence theorem, it follows that as $t \rightarrow \infty$,

$$
\begin{aligned}
\mathbb{E}\left[X_{m} \mid S_{n}>t\right]= & t\left(1+\left(n\left(1+\widetilde{A}_{n}(t)(1+o(1))\right)\right)^{-1}\left(Q_{1}(t)-Q_{2}(t)\right)\right) \\
= & t\left(1+\frac{1}{n}\left(1-\alpha t^{-1} \mu_{n}^{*}(t)(1+o(1))++o\left(A(t)+\sum_{i=1}^{n}\left(\phi_{i}(t)-d_{i}\right)\right)\right)\right. \\
& \left.\cdot\left(\frac{1}{\alpha-1}\left(1+\frac{1}{\alpha-\beta-1} A(t)(1+o(1))\right)-(n-1)\left(1+\alpha t^{-1} \mu_{n}^{*}(t)+o(|A(t)|)\right)\right)\right) \\
= & \frac{\alpha t}{(\alpha-1) n}\left(1+\left(\frac{1}{\alpha(\alpha-\beta-1)} A(t)-\frac{\mu_{n}^{*}(t)}{t}\right)(1+o(1))\right)+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right) .
\end{aligned}
$$

This completes the proof of the lemma.
Lemma 8.5 Under the conditions of Theorem 3.2, there exist some large enough $t_{1}$ such that for any $t \geq t_{1}$,

$$
\mathbb{E}\left[\left(X_{m}-t_{1}\right)_{+} \mid S_{n}>t\right]=\frac{\alpha t}{(\alpha-1) n}(1+\widetilde{B}(t)(1+o(1)))-\frac{t_{1}}{n}+o\left(\sum_{i=1}^{n}\left|\phi_{i}(t)-d_{i}\right|\right),
$$

where $\widetilde{B}(t)$ is defined in (8.3).
Proof. This proof proceeds along similar lines as in the proof of Proposition 3.1. Applying the integration by part, we conclude that

$$
\begin{aligned}
\mathbb{E}\left[\left(X_{m}-t_{1}\right)_{+} \mid S_{n}>t\right]= & \int_{t_{1}}^{\infty}\left(x-t_{1}\right) \mathrm{d} F_{X_{m} \mid S>t}(x) \\
= & -\int_{t_{1}}^{\infty}\left(x-t_{1}\right) \mathrm{d} \bar{F}_{X_{m} \mid S>t}(x) \\
= & -\left.\left(x-t_{1}\right) \bar{F}_{X_{m} \mid S>t}(x)\right|_{t_{1}} ^{\infty}+\int_{t_{1}}^{\infty} \bar{F}_{X_{m} \mid S>t}(x) \mathrm{d} x \\
= & \int_{t_{1}}^{\infty} \mathbb{P}\left(X_{m}>x \mid S>t\right) \mathrm{d} x \\
= & t \int_{\frac{t_{1}}{t}}^{\infty} \frac{\mathbb{P}\left(X_{m}>z t, S>t\right)}{\mathbb{P}(S>t)} \mathrm{d} z \\
= & t\left(\int_{\frac{t_{1}}{t}}^{1} \frac{\mathbb{P}\left(S_{n}>t, X_{m}>z t\right)}{\mathbb{P}(X>t)} \mathrm{d} z+\int_{1}^{\infty} \frac{\mathbb{P}\left(X_{m}>z t\right)}{\mathbb{P}(X>t)} \mathrm{d} z\right) \\
= & t\left(1-\frac{t_{1}}{t}+\frac{\mathbb{P}(X>t)}{\mathbb{P}\left(S_{n}>t\right)}\left(\int_{1}^{\infty} \frac{\mathbb{P}\left(X_{m}>z t\right)}{\mathbb{P}(X>t)} \mathrm{d} z\right.\right. \\
& \left.\left.-\int_{\frac{t_{1}}{t}}^{1} \frac{\mathbb{P}\left(S_{n}>t, X_{m} \leq z t\right)}{\mathbb{P}(X>t)} \mathrm{d} z\right)\right) .
\end{aligned}
$$

Following a similar analysis of $Q_{2}(t)$ of Proposition 3.1, we have that

$$
\int_{\frac{t_{1}}{t}}^{1} \frac{\mathbb{P}\left(S_{n}>t, X_{m} \leq z t\right)}{\mathbb{P}(X>t)} \mathrm{d} z=\left(1-t_{1} / t\right)(n-1)\left(1+\alpha t^{-1} \mu_{n}^{*}(t)\right)+o\left(A(t)+\sum_{i=1}^{n}\left(\phi_{i}(t)-d_{i}\right)\right) .
$$

Thus, by first-order Taylor expansion, as $t \rightarrow \infty$, we have

$$
\begin{aligned}
& \mathbb{E}\left[\left(X_{m}-t_{1}\right)_{+} \mid S_{n}>t\right] \\
= & t\left(1-\frac{t_{1}}{t}+\left(n\left(1+\widetilde{A}_{n}(t)(1+o(1))\right)\right)^{-1}\left(\frac{1}{\alpha-1}\left(1+\frac{1}{\alpha-\beta-1} A(t)\right)\right.\right. \\
& \left.\left.-\left(1-t_{1} / t\right)(n-1)\left(1+\alpha t^{-1} \mu_{n}^{*}(t)+o\left(A(t)+\sum_{i=1}^{n}\left(\phi_{i}(t)-d_{i}\right)\right)\right)\right)\right) \\
= & \frac{\alpha t}{(\alpha-1) n}\left(1+\left(\frac{1}{\alpha(\alpha-\beta-1)} A(t)-\frac{\mu_{n}^{*}(t)}{t}\right)(1+o(1))\right)-\frac{t_{1}}{n}+o\left(\sum_{i=1}^{n}\left(\phi_{i}(t)-d_{i}\right)\right) .
\end{aligned}
$$

We complete this proof of Lemma.
Proof of Theorem 6.1. By Theorem 3.1 and (6.1), as $p \uparrow 1$, based on VaR, we obtain

$$
\begin{aligned}
D_{p}^{\mathrm{VaR}}\left(S_{n}\right)= & 1-\frac{\operatorname{VaR}_{p}\left(S_{n}\right)-n \mu}{n\left(F^{\leftarrow}(p)-\mu\right)} \\
= & 1-\frac{\operatorname{VaR}_{p}\left(S_{n}\right)-n \mu}{n F^{\leftarrow}(p)}\left(1-\frac{\mu}{F^{\leftarrow(p)}}\right)^{-1} \\
= & 1-\frac{\operatorname{VaR}_{p}\left(S_{n}\right)-n \mu}{n F^{\leftarrow}(p)}\left(1+\frac{\mu}{F^{\leftarrow(p)}}(1+o(1))\right) \\
= & 1-\left(n^{1 / \alpha-1}\left(1+\left(\frac{\mu_{n}^{*}\left(F^{\leftarrow}(p)\right)}{n^{1 / \alpha} F^{\leftarrow(p)}}+\frac{n^{\beta / \alpha}-1}{\alpha \beta} A\left(F^{\leftarrow}(p)\right)\right)(1+o(1))\right)\right. \\
& \left.-\frac{\mu}{F^{\leftarrow(p)}}\right)\left(1+\frac{\mu}{\left.F^{\leftarrow(p)}(1+o(1))\right)+o\left(\sum_{i=1}^{n}\left(\phi_{i}(t)-d_{i}\right)\right)}\right. \\
= & 1-n^{1 / \alpha-1}\left(1+\left(\frac{\mu_{n}^{*}\left(F^{\leftarrow}(p)\right)}{n^{1 / \alpha} F^{\leftarrow(p)}}+\frac{n^{\beta / \alpha}-1}{\alpha \beta} A\left(F^{\leftarrow}(p)\right)\right)(1+o(1))\right) \\
& +\frac{\left(1-n^{1 / \alpha-1}\right) \mu}{F_{\leftarrow}^{\leftarrow}(p)}(1+o(1))+o\left(\sum_{i=1}^{n}\left(\phi_{i}(t)-d_{i}\right)\right) .
\end{aligned}
$$

According to Proposition 4.1 and Theorem 4.1, as $p \uparrow 1$, we have

$$
\begin{aligned}
D_{p}^{\mathrm{e}}\left(S_{n}\right)= & 1-\frac{\mathrm{e}_{p}\left(S_{n}\right)-n \mu}{n\left(\mathrm{e}_{p}(X)-\mu\right)} \\
= & 1-\frac{\mathrm{e}_{p}\left(S_{n}\right) / n-\mu}{\mathrm{e}_{p}(X)-\mu} \\
= & 1-n^{1 / \alpha-1}\left(1+\frac{\left(n^{\beta / \alpha}-1\right)(\alpha-1)^{1-\beta / \alpha}}{\alpha \beta(\alpha-\beta-1)} A\left(F^{\leftarrow}(p)\right)(1+o(1))\right) \\
& -\frac{(\alpha-1)^{1 / \alpha+1} \mu_{n}^{*}\left(F^{\leftarrow}(p)\right)}{\alpha n F^{\leftarrow}(p)}(1+o(1))+o\left(\sum_{i=1}^{n}\left(\phi_{i}(t)-d_{i}\right)\right) .
\end{aligned}
$$

Based on CTE and Lemma 8.2, as $p \uparrow 1$, we have

$$
\begin{aligned}
D_{p}^{\mathrm{CTE}}\left(S_{n}\right)= & 1-\frac{\operatorname{CTE}_{p}\left(S_{n}\right)-n \mu}{n\left(\operatorname{CTE}_{p}(X)-\mu\right)} \\
= & 1-\frac{\mathrm{CTE}_{p}\left(S_{n}\right)-n \mu}{\frac{n \alpha F^{\leftarrow(p)}}{\alpha-1}\left(1+\frac{1}{\alpha(\alpha-\beta-1)} A\left(F^{\leftarrow}(p)\right)-\frac{(\alpha-1) \mu}{\alpha F^{\leftarrow(p)}}\right)} \\
= & 1-n^{1 / \alpha-1}\left(1+\left(\zeta_{\alpha, \beta}^{n}-\frac{1}{\alpha(\alpha-\beta-1)}\right) A\left(F^{\leftarrow}(p)\right)(1+o(1))\right)+o\left(\sum_{i=1}^{n}\left(\phi_{i}(t)-d_{i}\right)\right) \\
& -\left(\frac{(\alpha-1) \mu_{n}^{*}\left(F^{\leftarrow}(p)-n \mu\right)}{n \alpha F^{\leftarrow}(p)}+\frac{(\alpha-1) n^{1 / \alpha-1} \mu}{\alpha F^{\leftarrow}(p)}\right)(1+o(1)) \\
= & 1-n^{1 / \alpha-1}\left(1+\frac{1}{\alpha \beta}\left(\frac{n^{\beta / \alpha}(\alpha-1)-\beta}{\alpha-\beta-1}-1\right) A\left(F^{\leftarrow}(p)\right)(1+o(1))\right) \\
& -\frac{(\alpha-1)\left(\mu_{n}^{*}\left(F^{\leftarrow}(p)\right)-\left(n-n^{1 / \alpha}\right) \mu\right)}{n \alpha F^{\leftarrow}(p)}(1+o(1))+o\left(\sum_{i=1}^{n}\left(\phi_{i}(t)-d_{i}\right)\right)
\end{aligned}
$$

In addition, applying Lemma 8.2, it follows that

$$
\begin{aligned}
D_{p}^{\mathrm{CE}}\left(S_{n}\right)= & 1-\frac{\mathrm{CE}_{p}\left(S_{n}\right)-n \mu}{n\left(\mathrm{CE}_{p}(X)-\mu\right)} \\
= & 1-\frac{\mathrm{CE}_{p}\left(S_{n}\right)-n \mu}{\frac{n \alpha \mathrm{e}_{p}(X)}{\alpha-1}\left(1+\frac{1}{\alpha(\alpha-\beta-1)} A\left(\mathrm{e}_{p}(X)\right)\right)-n \mu} \\
= & 1-\frac{\mathrm{CE}_{p}\left(S_{n}\right) / n-\mu}{\frac{\alpha \mathrm{e}_{p}(X)}{\alpha-1}\left(1+\frac{(\alpha-1)^{-\beta / \alpha}}{\alpha(\alpha-\beta-1)} A\left(F^{\leftarrow}(p)\right)(1+o(1))\right)-\mu} \\
= & 1-n^{1 / \alpha-1}\left(1+\frac{\left(n^{\beta / \alpha}-1\right)(\alpha-1)^{-\beta / \alpha}(\alpha+\beta-1)}{\alpha \beta(\alpha-\beta-1)} A\left(F^{\leftarrow}(p)\right)(1+o(1))\right) \\
& -\frac{(\alpha-1)^{1 / \alpha}(\alpha-2) \mu_{n}^{*}\left(F^{\leftarrow}(p)\right)}{\alpha n F^{\leftarrow}(p)}(1+o(1))+o\left(\sum_{i=1}^{n}\left(\phi_{i}(t)-d_{i}\right)\right)
\end{aligned}
$$

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