# Value-at-Risk- and Expectile-based Systemic Risk Measures and Second-order Asymptotics: With Applications to Diversification

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#### Abstract

The systemic risk measure plays a crucial role in analyzing individual losses conditioned on extreme system-wide disasters. In this paper, we provide a unified asymptotic treatment for systemic risk measures. First, we classify them into two families of Value-at-Risk- (VaR-) and expectile-based systemic risk measures. While VaR has been extensively studied, in the latter family, we propose two new systemic risk measures named the Individual Conditional Expectile (ICE) and the Systemic Individual Conditional Expectile (SICE), as alternatives to Marginal Expected Shortfall (MES) and Systemic Expected Shortfall (SES). Second, to characterize general mutually dependent and heavy-tailed risks, we adopt a modeling framework where the system. represented by a vector of random loss variables, follows a multivariate Sarmanov distribution with a common marginal exhibiting second-order regular variation. Third, we provide second-order asymptotic results for both families of systemic risk measures. This analytical framework offers a more accurate estimate compared to traditional first-order asymptotics. Through numerical and analytical examples, we demonstrate the superiority of second-order asymptotics in accurately assessing systemic risk. Further, we conduct a comprehensive comparison between VaR-based and expectile-based systemic risk measures. We find that expectile-based measures output higher risk evaluation than VaR-based ones, emphasizing the former's potential advantages in reporting extreme events and tail risk. As a financial application, we use the asymptotic treatment to discuss the diversification benefits associated with systemic risk measures. The financial insight is that the expectile-based diversification benefits consistently deduce an underestimation and suggest a conservative approximation, while the VaR-based diversification benefits consistently deduce an overestimation and suggest behaving optimistically.

**Keywords**: Asymptotic approximation; Systemic risk; Expectile; Sarmanov distribution; Secondorder regular variation; Diversification benefit.

## 1 Introduction

Financial risks refer to the potential situation that can negatively affect the stability of an individual financial institution, a specific financial market, or even the global economy. Analyzing and preparing for extreme events that align with these adverse scenarios is always an important field of risk management. To study these extreme events, the extreme value theory (EVT) is a useful framework. The EVT offers a contemporary collection of statistical tools and techniques that can be used to

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address various questions related to risk assessment and management in finance. Financial risks are typically divided into different categories depending on their characteristics. Market risk, credit risk, and operational risk are the primary risk groups in banks that have been extensively studied using quantitative assessment methods and regulated by authorities. After the global financial crises in 2008-2009, the concept of systemic risk, which refers to the risk of multiple financial institutions failing together and causing widespread impact, has gained significant attention from regulators and researchers in the field. Various measures related to the systemic risk have been proposed in the literature, including Systemic Expected Shortfall (SES) and Marginal Expected Shortfall (MES) by Acharya et al. (2017) and Chen and Liu (2022), scenario-based risk measures by Wang and Ziegel (2021), conditional distortion risk measures by Dhaene et al. (2022), generalized risk measures by Fadina et al. (2024) and others.

Following recent studies of systemic risks in banking, finance and insurance, we quantify SES and MES in a general context of quantitative risk management. Let the aggregate risk  $S_n = \sum_{i=1}^n X_i$ , where the allocation of capital to each individual risk  $X_1, \ldots, X_n$ . For the sum  $S_n$ , for  $p \in (0, 1)$ , according to the Euler principle (see Dhaene et al. (2012) or Acharya et al. (2017)), the risk allocated to line  $m \in \{1, \ldots, n\}$  is defined by

$$\operatorname{MES}_{p,m}(S_n) := \mathbb{E}\left[X_m | S_n > \operatorname{VaR}_p(S_n)\right]$$

or

$$\operatorname{SES}_{p,m}(S_n) := \mathbb{E}\left[ \left( X_m - \operatorname{VaR}_p(X_m) \right)_+ | S_n > \operatorname{VaR}_p(S_n) \right],$$

where the Value-at-Risk (VaR) is defined as the loss distribution of X:

$$\operatorname{VaR}_p(X) := F_X^{\leftarrow}(p) = \inf\{t \in \mathbb{R} : F(t) \ge p\},\$$

where  $F_X^{\leftarrow}(p)$  is the inverse of the distribution function. Another popular risk measure is Expected Shortfall (ES):

$$\mathrm{ES}_p(X) := \frac{1}{1-p} \int_p^1 \mathrm{VaR}_t(X) \mathrm{d}t,$$

which is the average value on the tail above  $VaR_p$ . The ES (sometimes called Tail-Value-at-Risk (TVaR)) is a coherent risk measure in the sense of Artzner et al. (1999). If F is continuous, ES coincides with the Conditional Talied Expectation (CTE), which represents the conditional expected loss given that the loss exceeds its VaR:

$$\operatorname{ES}_p(X) = \operatorname{CTE}_p(X) := \mathbb{E}\left[X|X > \operatorname{VaR}_p(X)\right].$$

Though enjoying several merits, VaR and ES have some drawbacks. Specifically, VaR does not possess subadditivity, which excludes VaR from the good class of coherent risk measures. ES does not satisfy the *elicitability*, which is a property recently arousing interest in the field of risk management. Here, a risk measure is said to be elicitable if it can be defined as the minimizer of a suitable expected loss function. Elicitability is important in backtesting of a risk measure as it provides a natural methodology to perform backtesting. Meaningful point forecasts and forecast performance

comparisons then become possible for elicitable risk measures; see Ziegel (2016).

Following Newey and Powell (1987), the expectile  $e_p(X)$  of order  $p \in (0, 1)$  of the variable X can be defined as the minimizer of a piecewise quadratic loss function or, equivalently, as

$$\mathbf{e}_p(X) = \arg\min_{\theta \in \mathbb{R}} \left\{ p\mathbb{E}\left[ \left( (X-\theta)_+ \right)^2 \right] + (1-p)\mathbb{E}\left[ \left( (X-\theta)_- \right)^2 \right] \right\},\$$

where  $x_+ := \max(x, 0)$  and  $x_- := \min(x, 0)$ . The presence of terms  $X_+^2$  and  $X_-^2$  makes this problem well defined indeed as soon as  $X \in L^2$  (i.e.  $\mathbb{E}|X|^2 < \infty$ ). The related first-order necessary condition optimality can be written in several ways, one of them being

$$e_p(X) - \mathbb{E}[X] = \frac{2p-1}{1-p} \mathbb{E}\left[(X - e_p(X))_+\right].$$
 (1.1)

This equation has a unique solution for all  $X \in L^1$ . Thenceforth expectiles of a distribution function F with a finite absolute first-order moment are well defined, and we assume that  $\mathbb{E}|X| < \infty$ throughout. Expectiles summarize the distribution function in much the same way as the quantiles; see Gneiting (2011). The expectile and VaR are elicitable while ES is not; see Gneiting (2011). Actually, the expectile  $e_p$  with  $p \geq \frac{1}{2}$  is the only risk measure which is elicitable, law-invariant and coherent; see Bellini and Bignozzi (2015). As a result, the expectile is suggested (see Emmer et al. (2015)) as a potential alternative to both VaR and ES. The study on expectiles also becomes increasingly popular in the econometric literature; see, for example, De Rossi and Harvey (2009), Kuan et al. (2009).

In this paper, we provide a unified asymptotic treatment to the systemic risk measures. The treatment has three steps. The first step is that the systemic risk measures are classified into two representative families: Value-at-Risk- (VaR-) and expectile-based measures. From the definitions, VaR<sub>p</sub> and  $e_p$  exhibit distinct mathematical properties. As we can see below, both of them lead to many systemic risk measures; e.g.,  $CTE_p$  (or  $ES_p$ ),  $MES_{p,m}$  and  $SES_{p,m}$  are categorized within the family of VaR-based risk measures. As a result,  $VaR_p$  and  $e_p$  can serve as building blocks in assessing systemic risk and form essential foundations for further study. In particular, Taylor (2008) introduced an expectile-based alternative of ES, known as the Conditional Expectile ( $CE_p$ ):

$$\operatorname{CE}_p(X) := \mathbb{E}\left[X|X > \operatorname{e}_p(X)\right],$$

where  $CE_p$  represents the expectation of exceedances beyond the *p*-th expectile  $e_p$  of the distribution of X.

To evaluate the allocation of each individual agent to the systemic risk, we further propose two new expectile-based systemic risk measures on the sum variable. We call them the *Individual Conditional Expectile* (ICE) and the *Systemic Individual Conditional Expectile* (SICE):

$$\operatorname{ICE}_{p,m}(S_n) := \mathbb{E}\left[X_m | S_n > e_p(S_n)\right],$$

and

$$\operatorname{SICE}_{p,m}(S_n) := \mathbb{E}\left[ \left( X_m - \operatorname{e}_p(X_m) \right)_+ | S_n > \operatorname{e}_p(S_n) \right].$$

ICE and SICE stand from a view of expectile to capture an individual agent's risk profile conditional on a system-wide catastrophe. For comparison, the VaR estimation knows only whether an observation is below or above the predictor. It would be inaccurate to measure an extreme risk based on only the frequency of tail losses and not on their values. The expectile makes more efficient use of the available data since it optimizes the discrepancy between the observations and the predictor. Particularly, ICE represents the potential losses an individual would suffer conditional on the tail of the system's loss distribution. SICE is an improved version of ICE and reveals the individual's excess loss to his/her expectile  $e_p(X_m)$  conditional on the systemic catastrophe.

In the second step, to characterize a general dependence and heavy-tailed risks, we assume that the risks  $X_1, \ldots, X_n$  are dependent on each other through a multivariate Sarmanov distribution. This characterizes a more general dependence structure than the commonly used Farlie-Gumbel-Morgenstern (FGM) copula; see Yang and Hashorva (2013). Meanwhile, the FGM copula has some drawbacks in terms of correlation coefficients; see Section 2.2. In particular, the Sarmanov distribution is flexible in combining different types of marginals, making it suitable for modeling various risks. The advantage of using the Sarmanov distribution lies in its ability to capture dependencies and helps the evaluation of joint probabilities.

Systemic risk measure	First-order asymptotic	Second-order asymptotic
$\operatorname{VaR}_p(S_n)$	Bingham et al. (1989); Barbe	Degen et al. (2010); Mao and Yang
	et al. (2006); Embrechts et al.	(2015); Theorem 3.1 of our paper
	(2009b) and so on.	
$\operatorname{CTE}_p(S_n)$	Alink et al. (2005); Chen et al.	Mao and Hu (2013); Lv et al. (2013);
	(2012); Kley et al. $(2020)$ and so	Theorem $3.1$ of our paper
	on.	
$\operatorname{MES}_{p,m}(S_n)$	Asimit et al. (2011); Joe and Li	Hua and Joe (2011); Theorem 3.2 of
	(2011); Jaunė and Šiaulys $(2022)$	our paper
	and so on.	
$SES_{p,m}(S_n)$	Chen and Liu (2022)	Theorem 3.2 of our paper
$e_p(S_n)$	Bellini et al. (2014); Bellini and	Mao et al. (2015); Mao and Yang
	Di Bernardino (2017)	(2015); Theorem 4.1 of our paper
$\operatorname{CE}_p(S_n)$	Dhaene et al. (2022)	Theorem 4.1 of our paper
$ICE_{p,m}(S_n)$	Emmer et al. (2015); Tadese and	Theorem 4.2 of our paper
	Drapeau (2020)	
$\operatorname{SICE}_{p,m}(S_n)$	Theorem 4.2 of our paper	Theorem 4.2 of our paper

Table 1: Contribution of our paper compared to the literature. Here VaR-based systemic risk measures include VaR, CTE, MES and SES, while expectile-based systemic risk measures include e, CE, ICE and SICE.

In the third step, we obtain the second-order asymptotics of the two families of systemic risk measures. Here we make our most theoretical contributions on asymptotic approximations. First, we investigate the second-order expansions of the tail probability of  $S_n$  under multivariate Sarmanov distribution (Proposition 3.1 below), which generalizes the results of Mao and Hu (2013) and Theorem 4.4 of Mao and Yang (2015). Second, we study second-order asymptotic formulas of VaR<sub>p</sub>( $S_n$ ), CTE<sub>p</sub>( $S_n$ ) (Theorem 3.1 below) and MES<sub>p,m</sub>( $S_n$ ), SES<sub>p,m</sub>( $S_n$ ) (Theorem 3.2 below). Third, we use different methods to obtain the second-order asymptotic estimation of expectile, which extends theorem 3.1 of Mao and Yang (2015) (Proposition 4.1 below). Fourth, we consider the secondorder asymptotic formulas of  $e_p(S_n)$ ,  $CE_p(S_n)$  (Theorem 4.1 below) and  $ICE_{p,m}(S_n)$ ,  $SICE_{p,m}(S_n)$ (Theorem 4.2). Lastly, we apply two examples to explain different risk measures based on VaR and expectile, where we use the Monte Carlo method to conduct the numerical simulation. Numerical and analytical examples illustrate that our second-order asymptotics provide an accurate estimate and behave much better than the first-order asymptotics. Further, we conduct a comprehensive comparison between these two families of systemic risk measures. We find that expectile-based systemic risk measures produce a larger risk evaluation than that of VaR-based systemic risk measures. Hence, the former has a potential advantage in reporting extreme events and amplifying the tail risk. Besides, this finding appeals for a lower confidence level when using expectile-based measures.

As a financial application, we explore economic insights on diversification with our asymptotic treatment. The idea of portfolio diversification dates back to the celebrated Markowitz mean-variance model, revealing the importance of mitigating risks in the investment. Diversification hence becomes a crucial topic in banking and insurance for risk management, integral to regulatory frameworks like Basel II and Solvency II. A lot of works propose quantitative ways to quantify the advantages of benefits. E.g., Chen et al. (2022) delved into comparing diversification advantages under the worstcase VaR and ES in the context of dependence uncertainty; see Cui et al. (2021) for more results. Among them, the *diversification benefit*, proposed by Bürgi et al. (2008), signifies the preserved capital achieved through collectively considering all risks in a portfolio versus addressing each risk in isolation; see Section 6 for a detailed definition. Based on our asymptotic treatment, we obtain the second-order asymptotics for the diversification benefits based on different risk measures, including VaR, CTE, e and MES. We find that CE, an expectile-based systemic risk measure, provides the most accurate approximation with an error range of less than 5 %. Further, the expectilebased diversification benefits consistently deduce an underestimation and suggest a conservative approximation, while the VaR-based diversification benefits consistently deduce an overestimation and suggest an optimistical approximation.

Finally, we discuss our results compared with the literature. In the field of risk management and capital allocation, it is necessary to determine how to allocate the acquired economic capital among different risks. In this case, the solvency capital has already been calculated using risk aggregation techniques; see Blanchet et al. (2020) for a recent treatment of risk aggregation. In recent years, the focus is on selection of appropriate models for multivariate risk factors, such as the choice of dependence structure model and the distributions of the marginals. Some recent contributions include the use of the FGM distribution (Yang and Hashorva (2013), Chen and Yang (2014), etc) and the Sarmanov distribution (Qu and Chen (2013), Yang and Wang (2013), etc), and multivariate regular variation (MRV) (Embrechts et al. (2009a), Asimit et al. (2011), etc). They usually aim to obtain the first-order asymptotics of some risk measures. On the contrary, our results provide a series of second-order asymptotics, which is much more accurate than the first-order asymptotics; this will be shown in the tables and figures later. This fact is also studied in Degen et al. (2010), Mao et al. (2012) and Mao and Hu (2013), but they obtained the second-order regularly varying tails. This independence assumption is too restrictive for practical problems. Mao and Yang (2015) studied

the case that  $X_1, \ldots, X_n$  are dependent on each other through a multivariate FGM distribution, and derived second-order approximations of the risk concentrations of Value-at-Risk and expectile. However, our study contributes to the advancement of systemic risk measurement by introducing novel measures, developing a modeling framework, and providing enhanced asymptotic tools for risk assessment. Technically, the proposed two wide families of systemic risk measures can include the risk measures in Mao and Yang (2015). In particular, we provide rigorous and necessary lemmas for asymptotic treatment (Proposition 3.1, Lemmas 8.1-8.3) and offer a simpler proof for the key results.

The rest of the paper is organized as follows. In Section 2, we introduce the definitions of  $\mathcal{RV}$  and  $2\mathcal{RV}$  and discuss the *n*-dimensional Sarmanov distribution. In Sections 3 and 4, we obtain several second-order asymptotics of VaR- and expectile-based systemic risk measures and present examples to explain the main results. In Section 5, we give concrete examples to numerically illustrate these risk measures. Further, we apply the above asymptotic treatment to discuss financial diversification in Section 6. Section 7 concludes the paper. The Appendix provides details for the proofs.

## 2 Preliminaries

In this section, we first review the definitions and basic properties of regular variation  $(\mathcal{RV})$  and the second-order regular variation  $(2\mathcal{RV})$ .  $2\mathcal{RV}$  is a concept that is generalization of regular variation  $(\mathcal{RV})$ , which has various applications in areas such as applied probability, statistics, risk management, telecommunication networks and so on. The idea of  $2\mathcal{RV}$  was initially proposed to investigate the rate of convergence of the extreme order statistics in EVT; see De Haan and Stadtmüller (1996) and De Haan and Ferreira (2006). Next, we introduce the *n*-dimensional Sarmanov distribution. Its applications in many insurance contexts show its flexible structure when modeling the dependence between multivariate risks given the marginal distributions; see Qu and Chen (2013), Abdallah et al. (2016), Ratovomirija (2016) and so on.

### 2.1 Regular Variation

### **Definition 2.1** (Regular variation)

A measurable function  $f : \mathbb{R}_+ \to \mathbb{R}_+$  is said to be regular varying at  $t_0 \in [-\infty, \infty]$  with index  $\alpha \in \mathbb{R}$ , denoted by  $f \in \mathcal{RV}^{t_0}_{\alpha}$ , if for all x > 0

$$\lim_{t \to t_0} \frac{f(tx)}{f(t)} = x^{\alpha}, \quad \text{for all} \quad x > 0.$$

When  $t_0 = \infty$ , we write  $\mathcal{RV}_{\alpha}^{t_0} = \mathcal{RV}_{\alpha}$ . In addition, if  $\alpha = 0$ , then f is said to be slowly varying at infinity.

### **Definition 2.2** (Second-order regular variation)

A measurable function  $f : \mathbb{R}_+ \to \mathbb{R}_+$  is said to be second-order regular varying with the first-order index  $\alpha \in \mathbb{R}$  and second-order index  $\beta \leq 0$ , denoted by  $f \in 2\mathcal{RV}^{t_0}_{\alpha,\beta}$ , if there exists some eventually positive or negative measurable function  $A(\cdot)$  with  $A(t) \to 0$  as  $t \to t_0$  such that

$$\lim_{t \to t_0} \frac{\frac{f(tx)}{f(t)} - x^{\alpha}}{A(t)} = x^{\alpha} \frac{x^{\beta} - 1}{\beta} := H_{\alpha,\beta}(x), \quad \text{for all} \quad x > 0.$$

Here  $H_{\alpha,\beta}(x)$  is  $x^{\alpha} \log x$  if  $\beta = 0$ , and  $A(\cdot)$  that is called an auxiliary function of f. It is worth noting that the auxiliary function  $A(\cdot) \in \mathcal{RV}_{\beta}$ ; see for example Theorem 2.3.3 of De Haan and Ferreira (2006). If  $t_0 = \infty$ , we write  $2\mathcal{RV}_{\alpha,\beta}^{t_0} = 2\mathcal{RV}_{\alpha,\beta}$ .

Let  $X_1, X_2, ..., X_n$  denote the financial losses, which are identically distributed random variables with a distribution function F. Here  $\overline{F}(\cdot)$  means the survival function  $\overline{F}(x) = 1 - F(x)$  and  $F^{\leftarrow}(\cdot)$ means the generalized inverse function  $F^{\leftarrow}(y) = \inf\{x \in \mathbb{R} : F(x) \ge y\}$ . The tail quantile function associated with the distribution function F is denoted by  $U_F(\cdot) = (1/\overline{F})^{\leftarrow}(\cdot) = F^{\leftarrow}(1-1/\cdot)$ . Note that  $\overline{F}(\cdot) \in \mathcal{RV}_{-\alpha}$  for all  $\alpha \in \mathbb{R}$  is equivalent to  $U_F(\cdot) \in \mathcal{RV}_{1/\alpha}$  (see Corollary 1.2.10 of De Haan and Ferreira (2006)). Furthermore, if  $\overline{F}(\cdot) \in 2\mathcal{RV}_{-\alpha,\beta}$  with  $\alpha > 0, \beta \le 0$  and auxiliary function  $A(\cdot)$ , by Theorem 2.3.9 of De Haan and Ferreira (2006), one can easily check that  $U_F(\cdot) \in 2\mathcal{RV}_{1/\alpha,\beta/\alpha}$  with auxiliary function  $\alpha^{-2}A \circ U_F(\cdot)$ . Generally, the equality  $F(F^{\leftarrow}(p)) = p$  does not hold true. It can be shown that if  $\overline{F}(\cdot) \in \mathcal{RV}_{-\alpha}$  with  $\alpha > 0$ , then  $F(F^{\leftarrow}(p)) \sim p$  as  $p \uparrow 1$ . If further assume that  $\overline{F}(\cdot) \in 2\mathcal{RV}_{-\alpha,\beta}$  with auxiliary function  $A(\cdot)$ , then

$$U_F(1/\overline{F}(t)) = t (1 + o(A(t))) \text{ as } t \to \infty, \quad F(F^{\leftarrow}(p)) = p (1 + o(F^{\leftarrow}(p))) \text{ as } p \uparrow 1;$$
(2.1)

see Mao and Yang (2015) and Exercise 2.11 of De Haan and Ferreira (2006).

### 2.2 Multivariate Sarmanov distribution

The Sarmanov distribution is widely studied in different fields. It was originally introduced by Sarmanov (1966) in the bivariate case. It was then extended by Ting Lee (1996) and Kotz et al. (2004) in the multivariate case:

$$\mathbb{P}\left(X_1 \in \mathrm{d}x_1, \dots, X_n \in \mathrm{d}x_n\right) = \left(1 + \sum_{1 \le i < j \le n} a_{ij}\phi_i(x_i)\phi_j(x_j)\right) \prod_{k=1}^n \mathrm{d}F(x_k),$$
(2.2)

where F is the corresponding marginal distribution of X. Particularly, the parameters  $a_{ij}$  are real numbers and the kernels  $\phi_i$  are functions satisfying

$$\mathbb{E}[\phi_i(X_i)] = 0, \quad i = 1, \dots, n,$$

and

$$1 + \sum_{1 \le i < j \le n} a_{ij} \phi_i(x_i) \phi_j(x_j) \ge 0, \quad \text{for all} \quad x_i \in D_{X_i}, i = 1, \dots, n,$$

where  $D_{X_i} = \{x \in \mathbb{R} : \mathbb{P}(X_i \in (x - \delta, x + \delta)) > 0 \text{ for all } \delta > 0\}, i = 1, \dots, n.$ 

Similarly to those pointed out in Yang and Wang (2013), two common choices for the kernels  $\phi_i, i = 1, ..., n$  are listed below:

(i)  $\phi_i(x) = 1 - 2F(x)$  for all  $x \in D_{X_i}$ , leading to the well-known standard FGM distribution;

(ii)  $\phi_i(x) = x^p - \mathbb{E}[X_i^p]$  for all  $x \in D_{X_i}$  and exist  $p \in \mathbb{R}$  such that  $\mathbb{E}[X_i^p] < \infty$ ; (iii)  $\phi_i(x) = e^{-x} - g_i$  with  $g_i = \mathbb{E}[e^{-X_i}]$  for all  $x \in D_{X_i}$ .

We further discuss the dependence structure of two rvs  $(X_1, X_2)$  following a Sarmanov distribution with different kernel functions. To model the dependence between the two rvs  $X_1$  and  $X_2$ , we shall use Pearson's correlation coefficient, which is defined as

$$\rho_{12} = \frac{\mathbb{E}[X_1 X_2] - \mathbb{E}[X_1] \mathbb{E}[X_2]}{\sqrt{Var(X_1)} \sqrt{Var(X_2)}}$$

In the case of the Sarmanov's distribution,  $\rho_{12}$  can be rewritten as

$$\rho_{12} = \frac{a_{12}\mathbb{E}[X_1\phi_1(X_1)]\mathbb{E}[X_2\phi_2(X_2)]}{\sqrt{Var(X_1)}\sqrt{Var(X_2)}}.$$
(2.3)

Based on (2.3), we hereafter present the Pearson's correlation coefficient for different kernel functions along with its maximal and minimal value.

**Case 1:** Set the kernel function  $\phi_i(x) = 1 - 2F(x)$ , which corresponds to the FGM distribution. It is well known that Pearson correlation coefficients  $\rho_{12}$  of the FGM distribution lie between  $-\frac{1}{3}$  and  $\frac{1}{3}$ (see Schucany et al. (1978)), which is an important drawback of the FGM distribution. Huang and Kotz (1984) show that, by considering the iterated generalization of FGM distribution proposed by Johnson and Kotz (1977), the range of correlation coefficients can be enlarged.

**Case 2:** Set the kernel function  $\phi_i(x) = x^p - \mathbb{E}[X_i^p]$ . A usual choice is p = 1, which leads to the Pearson correlation coefficients  $\rho_{12} = a_{12}\sigma_1\sigma_2$ . In this situation, according to Ting Lee (1996), if the correlation coefficient of  $X_1$  and  $X_2$  exists, however, if we denote by  $T_i$ , i = 1, 2 (the corresponding upper truncation points) and consider the marginal pdfs are defined only for non-negative values,  $a_{12}$  satisfies the condition that

$$\max\left\{\frac{-1}{\mu_1\mu_2}, \frac{-1}{(T_1-\mu_1)(T_2-\mu_2)}\right\} \le a_{12} \le \min\left\{\frac{1}{\mu_1(T_2-\mu_2)}, \frac{1}{\mu_2(T_1-\mu_1)}\right\},$$

where  $\mu_i = \mathbb{E}[X_i]$ . Then, the maximal and the minimal values of the correlation coefficient are, respectively, given by

$$\rho_{12}^{\max} = \frac{\sigma_1 \sigma_2}{\max(\mu_1(T_2 - \mu_2), \mu_2(T_1 - \mu_1))}, \ \rho_{12}^{\min} = \frac{-\sigma_1 \sigma_2}{\max\{\mu_1 \mu_2, (T_1 - \mu_1)(T_2 - \mu_2)\}}.$$

**Case 3:** Set the kernel function  $\phi_i(x) = e^{-x} - g_i$  with  $g_i = \mathbb{E}[e^{-X_i}]$  for all  $x \in D_{X_i}$ . In this case, if the correlation coefficient of  $X_1$  and  $X_2$  exists, the range of  $a_{12}$  is (see Ting Lee (1996))

$$\frac{-1}{\max(\mathcal{L}_1(1)\mathcal{L}_2(1), (1-\mathcal{L}_1(1))(1-\mathcal{L}_2(1)))} \le a_{12} \le \frac{1}{\max(\mathcal{L}_1(1)(1-\mathcal{L}_2(1)), \mathcal{L}_2(1)(1-\mathcal{L}_1(1)))}$$

where  $\mathcal{L}_i(t) = \int_0^\infty \exp(-tx_i) dF_i(x_i)$ . Then, according to (2.3), the maximal value of Pearson's correlation coefficient  $\rho_{12}$  can be written as follows

$$\rho_{12}^{\max} = \frac{[-\mathcal{L}_1'(1) - \mathcal{L}_1(1)\mu_1][-\mathcal{L}_2'(1) - \mathcal{L}_2(1)\mu_2]}{\max(\mathcal{L}_1(1)(1 - \mathcal{L}_2(1)), \mathcal{L}_2(1)(1 - \mathcal{L}_1(1)))\sigma_1\sigma_2}$$

and the minimal value can be expressed as

$$\rho_{12}^{\min} = -\frac{\left[-\mathcal{L}_{1}^{'}(1) - \mathcal{L}_{1}(1)\mu_{1}\right]\left[-\mathcal{L}_{2}^{'}(1) - \mathcal{L}_{2}(1)\mu_{2}\right]}{\max(\mathcal{L}_{1}(1)\mathcal{L}_{2}(1), (1 - \mathcal{L}_{1}(1))(1 - \mathcal{L}_{2}(1)))\sigma_{1}\sigma_{2}}$$

where  $\mu_i = \mathbb{E}[X_i]$ .

### 3 Second-order asymptotics of VaR-based systemic risk measures

In this section, we study the second-order asymptotics of VaR-based systemic risk measures with multivariate Sarmanov distributions. Denote the distribution function of the aggregate risk  $S_n = \sum_{i=1}^n X_i$  by  $G(t) := P(S_n \leq t)$ . Before stating some results, we use the following notation: (i)  $\eta_{\alpha} := \alpha \int_0^{1/2} ((1-x)^{-\alpha} - 1) x^{-\alpha-1} dx + 2^{2\alpha-1} - 2^{\alpha}$ ; (ii)  $\mu := \mathbb{E}[X]$ ; (iii)  $\mu(t) := \int_0^t x dF(x)$ ; (iv)  $\mu_i(t) := \int_0^t x \phi_i(x) dF(x), i = 1, \dots, n$ ;

First, we establish the second-order asymptotics of the random sum under multivariate Sarmanov distributions.

**Proposition 3.1** Let  $X_1, \ldots, X_n$  be nonnegative random variables with common marginal distribution F satisfying that  $\overline{F}(\cdot) \in 2\mathcal{RV}_{-\alpha,\beta}$  with  $\alpha > 0, \beta \leq 0$  and an auxiliary function  $A(\cdot)$ . Suppose that  $(X_1, \ldots, X_n)$  follows an n-dimensional Sarmanov distribution given by (2.2) and  $\lim_{t\to\infty} \phi_i(t) =$  $d_i \in \mathbb{R}, \phi_i(\cdot) - d_i \in \mathcal{RV}_{\rho_i}$  with  $\rho_i \leq 0$  for each  $i = 1, \ldots, n$ . Then as  $t \to \infty$ , we get that

$$\frac{\overline{G}(t)}{\overline{F}(t)} = n\left(1 + \widetilde{A_n}(t)\left(1 + o(1)\right)\right),\,$$

where

$$\widetilde{A_n}(t) = \begin{cases} \alpha t^{-1} \mu_n^*(t) + o\left(|A(t)| + \sum_{i=1}^n |\phi_i(t) - d_i|\right), & \alpha \ge 1, \\ \eta_\alpha \kappa_n \overline{F}(t) + o\left(|A(t)| + \sum_{i=1}^n |\phi_i(t) - d_i|\right), & 0 < \alpha < 1, \end{cases}$$
(3.1)

and

$$\mu_n^*(t) := (n-1)\mu(t) + \sum_{1 \le i < j \le n} \frac{a_{ij} (d_i \mu_j(t) + d_j \mu_i(t))}{n},$$
$$\kappa_n := n - 1 + \sum_{1 \le i < j \le n} \frac{2a_{ij} d_i d_j}{n}.$$

**Proof.** We require Lemma 8.1 in Appendix for this proof. For t > 0, denote the region  $\Omega_t = \{(x_1, \ldots, x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i > t\}$ . In addition, let  $X_1^*, \ldots, X_n^*$  be iid with a distribution function F. According to Proposition 1.1 of Yang and Wang (2013), there exist n constants  $c_i > 1, i = 1, \ldots, n$ , such that  $|\phi_i(x_i)| \leq c_i - 1$  for all  $x_i \in D_{X_i}$ . Let  $\widetilde{X}_1^*, \ldots, \widetilde{X}_n^*$  be mutually independent rvs with

marginal distributions  $\widetilde{F_1}, \ldots, \widetilde{F_n}$ , which are also independent of  $X_1^*, \ldots, X_n^*$ . Particularly,  $\widetilde{F_1}, \ldots, \widetilde{F_n}$  are defined by

$$\mathrm{d}\widetilde{F}_i(x_i) := \left(1 - \frac{\phi_i(x_i)}{c_i}\right) \mathrm{d}F(x_i), \quad i = 1, \dots, n.$$

By Lemma 8.1 in Appendix, we have  $\phi(t) = 1 - \frac{\phi_i(x_i)}{c_i}$ . Thus,  $\lim_{t \to \infty} \phi(t) = 1 - \frac{d_i}{c_i}$  and  $\phi_i(\cdot) - 1 + \frac{d_i}{c_i} \in \mathcal{RV}_{\rho_i}$ . It follows that  $\overline{\widetilde{F_i}}(\cdot) \in 2\mathcal{RV}_{-\alpha,\gamma_i}$  with  $\gamma_i = \max\{\beta, \rho_i\}$  and auxiliary function  $\widetilde{A^i}(\cdot) = A(\cdot) + \frac{\rho_i \alpha}{(c_i - d_i)(\alpha - \rho_i)} (\phi_i(\cdot) - d_i)$ . In addition, as  $t \to \infty$ , we obtain

$$\frac{\widetilde{F}_i(t)}{\overline{F}(t)} = \left(1 - \frac{d_i}{c_i}\right) \left(1 - \frac{\alpha}{(c_i - d_i)(\alpha - \rho_i)}(\phi_i(t) - d_i)(1 + o(1))\right),$$

for all i = 1, ..., n. Write  $\mu_i(t) := \int_0^t x \phi_i(x) dF(x), i = 1, ..., n$ . We have that

$$\int_0^t x \mathrm{d}\widetilde{F}_i(x) = \int_0^t x \left(1 - \frac{\phi_i(x)}{c_i}\right) \mathrm{d}F(x) = \mu(t) - \frac{\mu_i(t)}{c_i}$$

Next, we can split  $\overline{G}(t)$  as

$$\overline{G}(t) = \int_{\Omega_t} \left( 1 + \sum_{1 \le i < j \le n} a_{ij} \phi_i(x_i) \phi_j(x_j) \right) \prod_{k=1}^n \mathrm{d}F(x_k) \\
= \int_{\Omega_t} \left( 1 + \sum_{1 \le i < j \le n} a_{ij} c_i c_j \left( 1 - \left( 1 - \frac{\phi_i(x_i)}{c_i} \right) - \left( 1 - \frac{\phi_j(x_j)}{c_j} \right) \right) \\
+ \left( 1 - \frac{\phi_i(x_i)}{c_i} \right) \left( 1 - \frac{\phi_j(x_j)}{c_j} \right) \right) \right) \prod_{k=1}^n \mathrm{d}F(x_k) \\
= \int_{\Omega_t} \prod_{k=1}^n \mathrm{d}F(x_k) + \sum_{1 \le i < j \le n} a_{ij} c_i c_j \left( \int_{\Omega_t} \prod_{k=1}^n \mathrm{d}F(x_k) - \int_{\Omega_t} \prod_{k=1, k \ne i}^n \mathrm{d}F(x_k) \mathrm{d}\widetilde{F}_i(x_i) \right) \\
- \int_{\Omega_t} \prod_{k=1, k \ne j}^n \mathrm{d}F(x_k) \mathrm{d}\widetilde{F}_j(x_j) + \int_{\Omega_t} \prod_{k=1, k \ne i, j}^n \mathrm{d}F(x_k) \mathrm{d}\widetilde{F}_i(x_i) \mathrm{d}\widetilde{F}_j(x_j) \right) \\
:= I(t) + \sum_{1 \le i < j \le n} a_{ij} c_i c_j \left( I(t) - I_i(t) - I_j(t) + I_{i,j}(t) \right).$$
(3.2)

To deal with I(t), according to Propositions 3.6, 3.7 and Remark 3.1 of Mao and Ng (2015) with common distribution  $\overline{F}(\cdot) \in 2\mathcal{RV}_{-\alpha,\beta}$ , it follows that

$$\frac{I(t)}{\overline{F}(t)} = \begin{cases} n\left(1 + (n-1)\alpha t^{-1}\mu(t)\left(1 + o(1)\right)\right) + o\left(|A(t)|\right), & \alpha \ge 1, \\ n\left(1 + (n-1)\eta_{\alpha}\overline{F}(t)\left(1 + o(1)\right)\right) + o\left(|A(t)|\right), & 0 < \alpha < 1 \end{cases}$$

By the similar analysis,  $\overline{\widetilde{F}_i}(\cdot) \in 2\mathcal{RV}_{-\alpha,\gamma_i}$ . According to (8.1) in Appendix, we have

$$\frac{I_{i}(t)}{\overline{F}(t)} = \begin{cases} \left(n - \frac{d_{i}}{c_{i}}\right) \left(1 + \alpha(n-1)t^{-1}\mu(t)\left(1 + o(1)\right)\right) - \left(\frac{\alpha(n-1)\mu_{i}(t)}{c_{i}t} + \frac{\alpha(\phi_{i}(t) - d_{i})}{c_{i}(\alpha - \rho_{i})}\right) (1 + o(1)) \\ + o\left(|A(t)|\right), & \alpha \ge 1, \\ \left(n - \frac{d_{i}}{c_{i}}\right) + \eta_{\alpha}\left(n - 1\right) \left(n - \frac{2d_{i}}{c_{i}}\right) \overline{F}(t)(1 + o(1)) - \frac{\alpha(\phi_{i}(t) - d_{i})}{c_{i}(\alpha - \rho_{i})}(1 + o(1)) + o\left(|A(t)|\right), & 0 < \alpha < 1 \end{cases}$$

and

$$\frac{I_{i,j}(t)}{\overline{F}(t)} = \begin{cases} \left(n - \frac{d_i}{c_i} - \frac{d_j}{c_j}\right) \left(1 + \alpha t^{-1}(n-1)\mu(t)\left(1 + o(1)\right)\right) - \alpha \left(\left(n - 1 - \frac{d_j}{c_j}\right) \frac{\mu_i(t)}{c_i t} + \left(n - 1 - \frac{d_i}{c_i}\right) \frac{\mu_j(t)}{c_j t} + \frac{(\phi_i(t) - d_i)}{c_i(\alpha - \rho_i)} + \frac{(\phi_j(t) - d_j)}{c_j(\alpha - \rho_j)}\right) (1 + o(1)) + o\left(|A(t)|\right), & \alpha \ge 1, \\ \left(n - \frac{d_i}{c_i} - \frac{d_j}{c_j}\right) + \eta_\alpha \left((n-1)\left(n - 2\left(\frac{d_i}{c_i} + \frac{d_j}{c_j}\right)\right) + \frac{2d_i d_j}{c_i c_j}\right) \overline{F}(t) \left(1 + o(1)\right) - \left(\frac{\alpha(\phi_i(t) - d_i)}{c_i(\alpha - \rho_i)} + \frac{\alpha(\phi_j(t) - d_j)}{c_j(\alpha - \rho_j)}\right) (1 + o(1)) + o\left(|A(t)|\right), & 0 < \alpha < 1. \end{cases}$$

Pulling all the asymptotics for I(t),  $I_i(t)$ , and  $I_{i,j}(t)$  into (3.2) yields that

$$\frac{\overline{G}(t)}{n\overline{F}(t)} = \begin{cases} 1 + \alpha t^{-1} \left( (n-1)\mu(t) + \sum_{1 \le i < j \le n} \frac{a_{ij}(d_i\mu_j(t) + d_j\mu_i(t))}{n} \right) (1 + o(1)) \\ + o \left( |A(t)| + \sum_{i=1}^n |\phi_i(t) - d_i| \right), & \alpha \ge 1, \\ 1 + \eta_\alpha \left( n - 1 + \sum_{1 \le i < j \le n} \frac{2a_{ij}d_id_j}{n} \right) \overline{F}(t) (1 + o(1)) \\ + o \left( |A(t)| + \sum_{i=1}^n |\phi_i(t) - d_i| \right), & 0 < \alpha < 1 \end{cases}$$

Thus, this ends the proof.  $\blacksquare$ 

**Remark 3.1** If  $\phi_i(x) = 1 - 2F(x)$ , then  $d_i = -1$  and  $\rho_i = -\alpha$ . If  $\phi_i(x) = x^p - \mathbb{E}[X_i^p]$ , then  $d_i = -\mathbb{E}[X_i^p]$  and  $\rho_i = p$  for all  $p \le 0$ . If  $\phi_i(x) = e^{-x} - g_i$  then  $d_i = -g_i$  and  $\rho_i = -\infty$ . In addition, let  $\rho_0 = \max_{1 \le i \le n} \rho_i$  with  $\phi_0(\cdot)$  and  $d_0 \in \mathbb{R}$ .

In view of Proposition 3.1, we can easily obtain the  $2\mathcal{RV}$  property of  $\overline{G}(\cdot)$ .

**Corollary 3.1** Under the conditions of Proposition 3.1, we have  $\overline{G}(\cdot) \in 2\mathcal{RV}_{-\alpha,\lambda}$  with  $\lambda = \max\{-1, -\alpha, \beta\}$ and an auxiliary function  $A_G^{(n)}(\cdot)$  given by

$$A_G^{(n)}(t) = \begin{cases} A(t) - \alpha t^{-1} \mu_n^*(t), & \alpha \ge 1, \\ A(t) - \alpha \eta_\alpha \kappa_n \overline{F}(t), & 0 < \alpha < 1. \end{cases}$$
(3.3)

**Proof.** According to the definition of  $\widetilde{A}_n(t)$ , it is easy to check that  $\widetilde{A}_n(t) \in \mathcal{RV}_{\widetilde{\lambda}}$ , where  $\widetilde{\lambda} = \max\{-1, -\alpha\}$ . Note that  $\overline{F}(t) \in 2\mathcal{RV}_{-\alpha,\beta}$  with an auxiliary function A(t), for any x > 0, as  $t \to \infty$ ,

we have that

$$\begin{split} \overline{G}(tx) &= \overline{\overline{G}(tx)} \overline{\overline{F}(tx)} \overline{\overline{F}(tx)} \overline{\overline{F}(t)} \overline{\overline{G}(t)} \\ &= \frac{n\left(1 + \widetilde{A_n}(tx)(1+o(1))\right)}{n\left(1 + \widetilde{A_n}(t)(1+o(1))\right)} \left(x^{-\alpha} + H_{-\alpha,\beta}(x)A(t)\left(1+o(1)\right)\right) \\ &= x^{-\alpha} + H_{-\alpha,\beta}(x)A(t)\left(1+o(1)\right) + x^{-\alpha}(x^{\widetilde{\lambda}}-1)\widetilde{A_n}(t)\left(1+o(1)\right) \\ &= \begin{cases} x^{-\alpha} + \left(H_{-\alpha,\beta}(x)A(t) - H_{-\alpha,-1}(x)\alpha t^{-1}\mu_n^*(t)\right)\left(1+o(1)\right) + o\left(\sum_{i=1}^n |\phi_i(t) - d_i|\right), & \alpha \ge 1, \\ x^{-\alpha} + \left(H_{-\alpha,\beta}(x)A(t) - H_{-\alpha,-\alpha}(x)\alpha\eta_\alpha\kappa_n\overline{F}(t)\right)\left(1+o(1)\right) + o\left(\sum_{i=1}^n |\phi_i(t) - d_i|\right), & 0 < \alpha < 1 \end{split}$$

Thus, we complete this proof.  $\blacksquare$ 

**Remark 3.2** (1) When  $\phi_i(x) = 1 - 2F(x)$  for all  $x \in D_{X_i}$ , i = 1, ..., n, Proposition 3.1 and Corollary 3.1 reduce to Theorem 4.4 and Corollary 4.5 of Mao and Yang (2015).

(2) If  $\alpha \geq 1$  and  $\beta < -1$ , then  $A(t) = o(\mu_n^*(t))$ . If  $\alpha < 1$  and  $\beta < -\alpha$ , then  $A(t) = o(\overline{F}(t))$ . If  $\beta > -(1 \wedge \alpha)$ , then  $\mu_n^*(t) = o(A(t))$  and  $\overline{F}(t) = o(A(t))$ .

Second, we are ready to show the second-order asymptotics of  $\operatorname{VaR}_p(S_n)$  and  $\operatorname{CTE}_p(S_n)$ .

**Theorem 3.1** Under the conditions of Proposition 3.1, we have, as  $p \uparrow 1$ ,

$$\frac{\operatorname{VaR}_{p}(S_{n})}{F^{\leftarrow}(p)} = \begin{cases} n^{1/\alpha} \left( 1 + \left( \frac{\mu_{n}^{*}(F^{\leftarrow}(p))}{n^{1/\alpha}F^{\leftarrow}(p)} + \frac{n^{\beta/\alpha}-1}{\alpha\beta}A\left(F^{\leftarrow}(p)\right)\right)\left(1+o(1)\right) \right) \\ + o\left(\sum_{i=1}^{n} |\phi_{i}(t) - d_{i}|\right), & \alpha \ge 1, \\ n^{1/\alpha} \left( 1 + \left( \frac{\eta_{\alpha}\kappa_{n}}{\alpha n}\overline{F}\left(F^{\leftarrow}(p)\right) + \frac{n^{\beta/\alpha}-1}{\alpha\beta}A\left(F^{\leftarrow}(p)\right)\right)\left(1+o(1)\right) \right) \\ + o\left(\sum_{i=1}^{n} |\phi_{i}(t) - d_{i}|\right), & 0 < \alpha < 1. \end{cases}$$

For  $\alpha > 1$ , as  $p \uparrow 1$ ,

$$\frac{\text{CTE}_p(S_n)}{F^{\leftarrow}(p)} = \frac{\alpha n^{1/\alpha}}{\alpha - 1} \left( 1 + \zeta_{\alpha,\beta}^n A\left(F^{\leftarrow}(p)\right)\left(1 + o(1)\right) \right) + \frac{\mu_n^*\left(F^{\leftarrow}(p)\right)}{F^{\leftarrow}(p)} \left(1 + o(1)\right) + o\left(\sum_{i=1}^n |\phi_i(t) - d_i|\right),$$

and

$$\zeta_{\alpha,\beta}^{n} = \frac{1}{\alpha\beta} \left( \frac{n^{\beta/\alpha}(\alpha-1)}{\alpha-\beta-1} - 1 \right).$$
(3.4)

Here, by convention,  $\frac{n^{\beta/\alpha}-1}{\alpha\beta} := \alpha^{-2}\log n$  and  $\zeta_{\alpha,\beta}^n := \alpha^{-2}\log n$  if  $\beta = 0$ . Clearly, the first-order asymptotics of  $\operatorname{VaR}_p(S_n)$  and  $\operatorname{CTE}_p(S_n)$  are  $n^{1/\alpha}F^{\leftarrow}(p)$  and  $\frac{\alpha n^{1/\alpha}}{\alpha-1}F^{\leftarrow}(p)$ .

**Proof.** Since  $\overline{F}(\cdot) \in 2\mathcal{RV}_{-\alpha,\beta}$  with an auxiliary function  $A(\cdot)$  and Theorem 2.3.9 of De Haan and Ferreira (2006), one can easily check that  $U_F(\cdot) \in 2\mathcal{RV}_{1/\alpha,\beta/\alpha}$  with an auxiliary function  $\alpha^{-2}A \circ U_F(\cdot)$ .

Let  $t = G^{\leftarrow}(p)$ . If  $p \uparrow 1$ , then  $t \to \infty$ . By the relation (2.1) and Proposition 3.1, we get that

$$\begin{split} \frac{\operatorname{VaR}_p(S_n)}{F^{\leftarrow}(p)} &= \frac{G^{\leftarrow}(p)}{F^{\leftarrow}(p)} = \frac{U_F\left(1/\overline{F}(t)\right)}{U_F\left(1/\overline{G}(t)\right)\left(1 + o(A(t))\right)} \\ &= \left(\frac{\overline{G}(t)}{\overline{F}(t)}\right)^{1/\alpha} \left(1 + \frac{\left(\frac{\overline{G}(t)}{\overline{F}(t)}\right)^{\beta/\alpha} - 1}{\beta/\alpha} \alpha^{-2} A \circ U_F\left(\frac{1}{\overline{G}(t)}\right)\right) \\ &= \left(n\left(1 + \widetilde{A_n}(t)(1 + o(1))\right)\right)^{1/\alpha} \left(1 + \frac{n^{\beta/\alpha} - 1}{\alpha\beta} A \circ U_F\left(\frac{1}{\overline{G}(t)}\right)(1 + o(1))\right), \end{split}$$

where we use the transformation  $t \mapsto G^{\leftarrow}(p)$  and the first-order Taylor expansion. Note that  $\widetilde{A}_n(G^{\leftarrow}(p)) \sim \widetilde{A}_n(n^{1/\alpha}F^{\leftarrow}(p)) \sim n^{\widetilde{\lambda}/\alpha}\widetilde{A}_n(F^{\leftarrow}(p))$  with  $\widetilde{\lambda} = \max\{-1, -\alpha\}$ . As  $p \uparrow 1$ , we have that

$$\begin{split} \frac{\operatorname{VaR}_{p}(S_{n})}{F^{\leftarrow}(p)} &= n^{1/\alpha} \left( 1 + \frac{1}{\alpha} \widetilde{A_{n}} \left( G^{\leftarrow}(p) \right) \left( 1 + o(1) \right) + \frac{n^{\beta/\alpha} - 1}{\alpha\beta} A \left( F^{\leftarrow}(p) \right) \left( 1 + o(1) \right) \right) \\ &= n^{1/\alpha} \left( 1 + \frac{1}{\alpha} n^{\widetilde{\lambda}/\alpha} \widetilde{A_{n}} \left( F^{\leftarrow}(p) \right) \left( 1 + o(1) \right) + \frac{n^{\beta/\alpha} - 1}{\alpha\beta} A \left( F^{\leftarrow}(p) \right) \left( 1 + o(1) \right) \right) \\ &= \begin{cases} n^{1/\alpha} \left( 1 + \left( \frac{\mu_{n}^{*}(F^{\leftarrow}(p))}{n^{1/\alpha}F^{\leftarrow}(p)} + \frac{n^{\beta/\alpha} - 1}{\alpha\beta} A \left( F^{\leftarrow}(p) \right) \right) \left( 1 + o(1) \right) \right) \\ &+ o \left( \sum_{i=1}^{n} |\phi_{i}(t) - d_{i}| \right), & \alpha \ge 1 \\ n^{1/\alpha} \left( 1 + \left( \frac{\eta_{\alpha} \kappa_{n}}{\alpha n} \overline{F} \left( F^{\leftarrow}(p) \right) + \frac{n^{\beta/\alpha} - 1}{\alpha\beta} A \left( F^{\leftarrow}(p) \right) \right) \left( 1 + o(1) \right) \right) \\ &+ o \left( \sum_{i=1}^{n} |\phi_{i}(t) - d_{i}| \right). & 0 < \alpha < 1. \end{split}$$

By Proposition 3.1 and the definition of  $CTE_p(S_n)$ , we have

$$\begin{split} \frac{\operatorname{CTE}_{p}(S_{n})}{F^{\leftarrow}(p)} &= \frac{\mathbb{E}\left[S_{n} \middle| S_{n} > \operatorname{VaR}_{p}(S_{n})\right]}{F^{\leftarrow}(p)} \\ &= \frac{\alpha}{\alpha - 1} \left(1 + \frac{1}{\alpha(\alpha - \lambda - 1)} A_{G}^{(n)}\left(G^{\leftarrow}(p)\right)\right)(1 + o(1))\right) \frac{\operatorname{VaR}_{p}(S_{n})}{F^{\leftarrow}(p)} \\ &= \frac{\alpha n^{1/\alpha}}{\alpha - 1} \left(1 + \frac{n^{\lambda/\alpha}}{\alpha(\alpha - \lambda - 1)} A_{G}^{(n)}\left(F^{\leftarrow}(p)\right)\left(1 + o(1)\right)\right) \\ &\quad \cdot \left(1 + \left(\frac{\mu_{n}^{*}\left(F^{\leftarrow}(p)\right)}{n^{1/\alpha}F^{\leftarrow}(p)} + \frac{n^{\beta/\alpha} - 1}{\alpha\beta}A\left(F^{\leftarrow}(p)\right)\right)(1 + o(1))\right) \\ &= \frac{\alpha n^{1/\alpha}}{\alpha - 1} \left(1 + \frac{1}{\alpha\beta} \left(\frac{n^{\beta/\alpha}(\alpha - 1)}{\alpha - \beta - 1} - 1\right)A\left(F^{\leftarrow}(p)\right)(1 + o(1))\right) + \frac{\mu_{n}^{*}\left(F^{\leftarrow}(p)\right)}{F^{\leftarrow}(p)}\left(1 + o(1)\right) \\ &\quad + o\left(\sum_{i=1}^{n} \left|\phi_{i}(t) - d_{i}\right|\right). \end{split}$$

This ends the proof.  $\blacksquare$ 

**Remark 3.3** According to  $\operatorname{VaR}_p(S_n)$  in Theorem 3.1, we include Theorem 4.6 of Mao and Yang (2015) and our result provides a simpler proof. If  $a_{ij} = 0$ , for all  $1 \le i \ne j \le n$ , the n-dimensional Sarmanov distribution reduces to the independent rvs. In this case,  $\operatorname{CTE}_p(S_n)$  of Theorem 3.1 is consistent with Theorem 3.1 of Mao et al. (2012).

The following example is used to illustrate Theorem 3.1 under the Pareto distribution with different parameters  $\alpha$ .

**Example 3.1** (The Pareto distribution) A Pareto distribution function F satisfies that

$$F(x) = 1 - \left(\frac{k}{x+k}\right)^{\alpha}, \quad x, k, \alpha > 0.$$

It can be described that  $\overline{F}(\cdot) \in 2\mathcal{RV}_{-\alpha,-1}$  with an auxiliary function  $A(t) = \alpha k/t$  and  $F^{\leftarrow}(p) = k\left((1-p)^{-1/\alpha}-1\right)$ . If  $\alpha > 1$ , we have  $\mu = k/(\alpha-1)$  and  $\mu(t) = k/(\alpha-1) - (\alpha t+k)/(\alpha-1)\overline{F}(t)$ . Let  $X_1$  and  $X_2$  have an identical Pareto distribution F. Suppose that the random vector  $(X_1, X_2)$  follows an Sarmanov distribution in (2.2) with  $\phi_i(\cdot) = 1 - 2F(\cdot), i = 1, 2$ . Clearly,  $d_i = -1, \rho_i = -\alpha$  and  $\mu_i(t) = k/(2\alpha-1) - (2\alpha t+k)/(2\alpha-1)(\overline{F}(t))^2 - \mu(t), i = 1, 2$ . Then

$$\operatorname{VaR}_{p}(S_{2}) = \begin{cases} 2^{1/\alpha} F^{\leftarrow}(p) + \mu \left(F^{\leftarrow}(p)\right) - a_{12}\mu_{1} \left(F^{\leftarrow}(p)\right) + k \left(2^{1/\alpha} - 1\right), & \alpha \geq 1, \\ 2^{1/\alpha} F^{\leftarrow}(p) \left(1 + \frac{\eta_{\alpha}(1 + a_{12})}{2\alpha}(1 - p)\right) + k \left(2^{1/\alpha} - 1\right), & 0 < \alpha < 1. \end{cases}$$

and for  $\alpha > 1$ ,

$$CTE_p(S_2) = \frac{2^{1/\alpha}\alpha}{\alpha - 1} \left( F^{\leftarrow}(p) + k \right) - k + \mu \left( F^{\leftarrow}(p) \right) - a_{12}\mu_1 \left( F^{\leftarrow}(p) \right).$$

Table 2: Simulated values(MC) versus first-order(1st) and second-order(2nd) asymptotics values  $\operatorname{VaR}_p(S_2)$  and  $\operatorname{CTE}_p(S_2)$  with various values of  $\alpha$ . We use the Pareto distribution with k = 1 and p = 0.99,  $a_{12} = 0.5$ .

α	$\operatorname{VaR}_p(S_2)_{MC}$	$_{\mathbb{C}} \operatorname{VaR}_p(S_2)_{1st}$	$\operatorname{VaR}_p(S_2)_{2nd}$	$_{l}\mathrm{CTE}_{p}(S_{2})_{MC}$	$\operatorname{CTE}_p(S_2)_{1st}$	$CTE_p(S_2)_{2nd}$	$\frac{\operatorname{VaR}_p(S_2)_{1st}}{\operatorname{VaR}_p(S_2)_{MC}}$	$\frac{\mathrm{VaR}_p(S_2)_{2nd}}{\mathrm{VaR}_p(S_2)_{MC}}$	$\frac{\text{CTE}_p(S_2)_{1st}}{\text{CTE}_p(S_2)_{MC}}$	$\frac{\text{CTE}_p(S_2)_{2nd}}{\text{CTE}_p(S_2)_{MC}}$
1.1	126.8065	121.5690	126.1943	1561.6015	1337.2588	31360.6628	0.9587	0.9952	0.8563	0.8713
1.5	$5\ 35.3132$	32.6000	34.9843	102.9805	97.8000	103.3591	0.9232	0.9907	0.9497	1.0037
2.0	) 14.4215	12.7262	14.1893	28.5141	25.4524	28.3298	0.8824	0.9839	0.8926	0.9935
2.5	5 8.2435	7.0060	8.0581	13.8310	11.6767	13.6085	0.8499	0.9775	0.8442	0.9839
3.(	) 5.5535	4.5847	5.4053	8.5145	6.8771	8.3277	0.8255	0.9733	0.8077	0.9781
4.(	) 3.2479	2.5708	3.1404	4.5244	3.4277	4.3938	0.7915	0.9669	0.7576	0.9711
5.0	) 2.2591	1.7378	2.1739	2.9994	2.1722	2.8956	0.7692	0.9623	0.7242	0.9654

In Table 2 we find that the second-order asymptotics of VaR and CTE are closer to the simulation values than the first-order asymptotics. Specifically, the asymptotic values of CTE are not as accurate as those of VaR, because CTE may not exist if p is close to 1. In addition, the values of VaR and CTE decrease as  $\alpha$  rises.

The following theorem obtains second-order asymptotics of MES and SES under an n-dimensional Sarmanov distribution. These results are important in a wide range of systemic risk.

**Theorem 3.2** Let  $X_1, \ldots, X_n$  be nonnegative random variables with a common marginal distribution F satisfying that  $\overline{F}(\cdot) \in 2\mathcal{RV}_{-\alpha,\beta}$  with  $\alpha > 1$ ,  $\beta \leq 0$  and auxiliary function  $A(\cdot)$ . Suppose that  $(X_1, \ldots, X_n)$  follows an n-dimensional Sarmanov distribution given by (2.2) and  $\lim_{x_i \to \infty} \phi_i(x_i) = d_i \in \mathbb{R}, \phi_i(\cdot) - d_i \in \mathcal{RV}_{\rho_i}$  with  $\rho_i \leq 0$  for each  $i = 1, \ldots, n$ . Then as  $t \to \infty$ , we get that

$$\frac{\operatorname{MES}_{p,m}(S_n)}{F^{\leftarrow}(p)} = \frac{\alpha n^{1/\alpha}}{(\alpha-1)n} \left(1 + \zeta_{\alpha,\beta}^n A\left(F^{\leftarrow}(p)\right)\left(1 + o(1)\right)\right) + o\left(t^{-1} + o\left(\sum_{i=1}^n (\phi_i(t) - d_i)\right)\right),$$

where  $\zeta_{\alpha,\beta}^n$  is defined in (3.4) and

$$\frac{\operatorname{SES}_{p,m}(S_n)}{F^{\leftarrow}(p)} = \frac{\operatorname{MES}_{p,m}(S_n)}{F^{\leftarrow}(p)} - \frac{1}{n}$$

Obviously, the first-order asymptotics of  $\operatorname{MES}_p(S_n)$  and  $\operatorname{SES}_p(S_n)$  are  $\frac{\alpha n^{1/\alpha} F^{\leftarrow}(p)}{(\alpha-1)n}$  and  $\frac{\alpha n^{1/\alpha} F^{\leftarrow}(p)}{(\alpha-1)n} - \frac{F^{\leftarrow}(p)}{n}$ .

**Proof.** Here we require Lemma 8.4 in Appendix and Theorem 3.1. Define  $\widetilde{B}(t) = \frac{1}{\alpha(\alpha-\beta-1)}A(t) - \frac{\mu_n^*(t)}{t}$ . We have that  $\widetilde{B} \in \mathcal{RV}_{\rho}$  with  $\rho = \max\{-1, \beta\}$ . It follows that, as  $p \uparrow 1$ ,

$$\frac{\operatorname{MES}_{p,m}(S_n)}{F^{\leftarrow}(p)} = \frac{\mathbb{E}\left[X_m \middle| S_n > \operatorname{VaR}_p(S_n)\right]}{F^{\leftarrow}(p)} \\
= \frac{\alpha}{(\alpha - 1)n} \left(1 + \widetilde{B}\left(\operatorname{VaR}_p(S_n)\right)(1 + o(1))\right) \frac{\operatorname{VaR}_p(S_n)}{F^{\leftarrow}(p)} + o\left(\sum_{i=1}^n \left|\phi_i(t) - d_i\right|\right) \\
= \frac{\alpha n^{1/\alpha}}{(\alpha - 1)n} \left(1 + n^{\rho/\alpha} \widetilde{B}\left(F^{\leftarrow}(p)\right)(1 + o(1))\right) \\
\cdot \left(1 + \left(\frac{\mu_n^*\left(F^{\leftarrow}(p)\right)}{n^{1/\alpha}F^{\leftarrow}(p)} + \frac{n^{\beta/\alpha} - 1}{\alpha\beta}A\left(F^{\leftarrow}(p)\right)\right)(1 + o(1))\right) + o\left(\sum_{i=1}^n \left|\phi_i(t) - d_i\right|\right) \\
= \frac{\alpha n^{1/\alpha}}{(\alpha - 1)n} \left(1 + \frac{1}{\alpha\beta}\left(\frac{n^{\beta/\alpha}(\alpha - 1)}{\alpha - \beta - 1} - 1\right)A\left(F^{\leftarrow}(p)\right)\right)(1 + o(1)) + o\left(t^{-1} + \sum_{i=1}^n \left|\phi_i(t) - d_i\right|\right)$$

Using Lemma 8.5 in Appendix and Theorem 3.1, we conclude that

$$\frac{\operatorname{SES}_{p,m}(S_n)}{F^{\leftarrow}(p)} = \frac{\mathbb{E}\left[\left(X_m - \operatorname{VaR}_p(X_m)\right)_+ \middle| S_n > \operatorname{VaR}_p(S_n)\right]}{F^{\leftarrow}(p)}$$
$$= \frac{\operatorname{MES}_{p,m}(S_n)}{F^{\leftarrow}(p)} - \frac{1}{n}.$$

This completes the proof for  $MES_{p,m}(S_n)$  and  $SES_{p,m}(S_n)$ .

Lastly, Example 3.2 is used to explain Theorem 3.2 with Burr distribution. We use B(u, v) to represent Beta distribution.

**Example 3.2** (Burr distribution) A Burr distribution function F satisfies that

$$F(x) = 1 - (1 + x^{-\beta})^{\alpha/\beta}, \quad x, \alpha > 1, \beta < 0.$$

It is easy to check that  $\overline{F}(\cdot) \in 2\mathcal{RV}_{-\alpha,\beta}$  with auxiliary function  $A(t) = \alpha t^{\beta}$ ,  $\mu = -\alpha/\beta B\left(\frac{1-\alpha}{\beta}, \frac{\beta-1}{\beta}\right)$ and  $F^{\leftarrow}(p) = \left(\left(1-p\right)^{\beta/\alpha}-1\right)^{-1/\beta}$ . Let  $X_1$  and  $X_2$  have an identical Burr distribution F. Suppose that the random vector  $(X_1, X_2)$  follows an Sarmanov distribution in (2.2) with  $\phi_i(\cdot) = 1 - 2F(\cdot)$ . Clearly,  $d_i = -1, i = 1, 2$ . Then,

$$\operatorname{MES}_{p}(S_{2}) = \frac{2^{1/\alpha} \alpha F^{\leftarrow}(p)}{2(\alpha - 1)} \left( 1 + \frac{1}{\beta} \left( \frac{2^{\beta/\alpha} (\alpha - 1)}{\alpha - \beta - 1} - 1 \right) (F^{\leftarrow}(p))^{\beta} \right),$$

and

$$SES_p(S_2) = \frac{2^{1/\alpha} \alpha F^{\leftarrow}(p)}{2(\alpha - 1)} \left( 1 + \frac{1}{\beta} \left( \frac{2^{\beta/\alpha} (\alpha - 1)}{\alpha - \beta - 1} - 1 \right) (F^{\leftarrow}(p))^{\beta} \right) - \frac{F^{\leftarrow}(p)}{2}$$

It is easy to see that the first-order asymptotics of  $\operatorname{MES}_p(S_2)$  and  $\operatorname{SES}_p(S_2)$  are  $\frac{2^{1/\alpha}\alpha F^{\leftarrow}(p)}{2(\alpha-1)}$  and  $\frac{2^{1/\alpha}\alpha F^{\leftarrow}(p)}{2(\alpha-1)} - \frac{F^{\leftarrow}(p)}{2}$ . By Figure 1, the second-order asymptotics of  $\operatorname{MES}_p(S_2)$  and  $\operatorname{SES}_p(S_2)$  are much closer to the simulation value than the first-order asymptotics for  $p \in [0.95, 1)$ .



Figure 1: Simulated values(MC) versus the first-order and second-order asymptotics values of  $\text{MES}_p(S_2)$  for the left panel and  $\text{SES}_p(S_2)$  for the right panel. We use the Burr distribution with  $\alpha = 2, \beta = -0.5$  and  $a_{12} = 0.5$ .

## 4 Second-order asymptotics of expectile-based systemic risk measures

In this section, we consider the second-order asymptotics of expectile-based systemic risk measures under multivariate Sarmanov distributions. Firstly, we establish the second-order asymptotics of the expectile under a different method with the literature. **Proposition 4.1** Let  $\overline{F}(\cdot) \in 2\mathcal{RV}_{-\alpha,\beta}$  with  $\alpha > 1, \beta \leq 0$  and auxiliary function  $A(\cdot)$ . Then we have that as  $p \uparrow 1$ ,

$$\frac{e_p(X)}{F^{\leftarrow}(p)} = (\alpha - 1)^{-1/\alpha} \left(1 + \xi_{\alpha,\beta} A(F^{\leftarrow}(p)) \left(1 + o(1)\right)\right) + \frac{\mu}{\alpha F^{\leftarrow}(p)} \left(1 + o(1)\right) + \frac{\mu}{\alpha F^{\leftarrow}$$

where

$$\xi_{\alpha,\beta} = \frac{1}{\alpha\beta} \left( \frac{(\alpha-1)^{1-\beta/\alpha}}{\alpha-\beta-1} - 1 \right).$$
(4.1)

Here  $\xi_{\alpha,\beta} := -\alpha^{-2} \log(\alpha - 1)$  if  $\beta = 0$ .

### **Proof.** see Appendix.

Proposition 4.1 is derived by a different method with Corollary 1 of Daouia et al. (2018). In addition, due to  $1 - p \sim \overline{F}(F^{\leftarrow}(p))$  as  $p \uparrow 1$  and  $\alpha > 1$ , we derive that  $1 - p = o(1/F^{\leftarrow}(p))$ . Thus, Proposition 4.1 is consistent with Proposition 3.1 of Mao et al. (2015) or Theorem 3.1 of Mao and Yang (2015).

Secondly, we obtain the second-order asymptotics of  $e_p(S_n)$  and  $CE_p(S_n)$  with an *n*-dimensional Sarmanov distribution.

**Theorem 4.1** Let  $X_1, \ldots, X_n$  be nonnegative random variables with a common marginal distribution F satisfying that  $\overline{F}(\cdot) \in 2\mathcal{RV}_{-\alpha,\beta}$  with  $\alpha > 1$ ,  $\beta \leq 0$  and auxiliary function  $A(\cdot)$ . Suppose that  $(X_1, \ldots, X_n)$  follows an n-dimensional Sarmanov distribution given by (2.2) and  $\lim_{x_i \to \infty} \phi_i(x_i) = d_i \in \mathbb{R}, \phi_i(\cdot) - d_i \in \mathcal{RV}_{\rho_i}$  with  $\rho_i \leq 0$  for each  $i = 1, \ldots, n$ . Then as  $p \uparrow 1$ , we get that

$$\frac{\mathbf{e}_p(S_n)}{F^{\leftarrow}(p)} = \left(\frac{n}{\alpha-1}\right)^{1/\alpha} \left(1 + \frac{1}{\alpha\beta} \left(\frac{n^{\beta/\alpha}(\alpha-1)^{1-\beta/\alpha}}{\alpha-\beta-1} - 1\right) A\left(F^{\leftarrow}(p)\right)\left(1 + o(1)\right)\right) \\ + \frac{(\alpha-1)\mu_n^*\left(F^{\leftarrow}(p)\right) + n\mu}{\alpha F^{\leftarrow}(p)}\left(1 + o(1)\right) + o\left(\sum_{i=1}^n |\phi_i(t) - d_i|\right),$$

and

$$\frac{\operatorname{CE}_{p}(S_{n})}{F^{\leftarrow}(p)} = \frac{\alpha n^{1/\alpha}}{(\alpha - 1)^{1/\alpha + 1}} \left( 1 + \chi_{\alpha,\beta}^{n} A\left(F^{\leftarrow}(p)\right)\left(1 + o(1)\right) \right) + \frac{(\alpha - 2)\mu_{n}^{*}\left(F^{\leftarrow}(p)\right) + n\mu}{(\alpha - 1)F^{\leftarrow}(p)} \left(1 + o(1)\right) + o\left(\sum_{i=1}^{n} |\phi_{i}(t) - d_{i}|\right),$$

where

$$\chi_{\alpha,\beta}^{n} = \frac{1}{\alpha\beta} \left( \left( \frac{n}{\alpha - 1} \right)^{\beta/\alpha} \frac{\alpha + \beta - 1}{\alpha - \beta - 1} - 1 \right).$$
(4.2)

Here  $\chi_{\alpha,\beta}^n := \alpha^{-2} (\log n - \log(\alpha - 1))$  if  $\beta = 0$ . In addition, the first-order asymptotics of  $e_p(S_n)$  and  $\operatorname{CE}_p(S_n)$  are  $\frac{n^{1/\alpha}F^{\leftarrow}(p)}{(\alpha-1)^{1/\alpha}}$  and  $\frac{\alpha n^{1/\alpha}F^{\leftarrow}(p)}{(\alpha-1)^{1/\alpha+1}}$ .

**Proof.** Firstly, to deal with  $\mathbb{E}[S_n]$ , owing to  $(X_1, \ldots, X_n)$  follows an *n*-dimensional Sarmanov distribution in (2.2), we have

$$\mathbb{E}[S_n] = \int_{[0,\infty]^n} \sum_{i=1}^n x_i \left( 1 + \sum_{1 \le i < j \le n} a_{ij} \phi_i(x_i) \phi_j(x_j) \right) \prod_{k=1}^n \mathrm{d}F(x_k)$$
$$= \sum_{i=1}^n \int_{[0,\infty]^n} x_i \left( 1 + \sum_{1 \le j \le n, j \ne i} a_{ij} \phi_i(x_i) \phi_j(x_j) \right) \prod_{k=1}^n \mathrm{d}F(x_k)$$
$$= n\mu.$$

According to Proposition 4.1 and Corollary 3.1, we have

$$\begin{split} \mathbf{e}_{p}(S_{n}) &= (\alpha - 1)^{-1/\alpha} \, G^{\leftarrow}(p) \left( 1 + \xi_{\alpha,\lambda} A_{G}^{(n)} \left( G^{\leftarrow}(p) \right) (1 + o(1)) \right) + \frac{\mathbb{E}(S_{n})}{\alpha} \left( 1 + o(1) \right) \\ &= (\alpha - 1)^{-1/\alpha} \, G^{\leftarrow}(p) \left( 1 + n^{\lambda/\alpha} \xi_{\alpha,\lambda} A_{G}^{(n)} \left( F^{\leftarrow}(p) \right) (1 + o(1)) \right) + \frac{n\mu}{\alpha} \left( 1 + o(1) \right) \\ &= \left( \frac{n}{\alpha - 1} \right)^{1/\alpha} F^{\leftarrow}(p) \left( 1 + \left( \frac{\mu_{n}^{*} \left( F^{\leftarrow}(p) \right)}{n^{1/\alpha} F^{\leftarrow}(p)} + \frac{n^{\beta/\alpha} - 1}{\alpha\beta} A \left( F^{\leftarrow}(p) \right) \right) (1 + o(1)) \right) \\ &\quad \cdot \left( 1 + n^{\lambda/\alpha} \xi_{\alpha,\lambda} A_{G}^{(n)} \left( F^{\leftarrow}(p) \right) (1 + o(1)) \right) + \frac{n\mu}{\alpha} \left( 1 + o(1) \right) \\ &= \left( \frac{n}{\alpha - 1} \right)^{1/\alpha} F^{\leftarrow}(p) \left( 1 + \frac{1}{\alpha\beta} \left( \frac{n^{\beta/\alpha} (\alpha - 1)^{1 - \beta/\alpha}}{\alpha - \beta - 1} - 1 \right) A \left( F^{\leftarrow}(p) \right) (1 + o(1)) \right) \\ &\quad + \alpha^{-1} \left( (\alpha - 1) \mu_{n}^{*} \left( F^{\leftarrow}(p) \right) + n\mu \right) (1 + o(1)) + o \left( \sum_{i=1}^{n} |\phi_{i}(t) - d_{i}| \right). \end{split}$$

According to the definition of  ${\rm CE}_p(S_n)$  and Theorem 4.1, we obtain, as  $p\uparrow 1,$ 

$$\begin{split} \frac{\operatorname{CE}_{p}(S_{n})}{F^{\leftarrow}(p)} &= \frac{\mathbb{E}\left[S_{n} \middle| S_{n} > \operatorname{e}_{p}(S_{n})\right]}{F^{\leftarrow}(p)} \\ &= \frac{\alpha}{\alpha - 1} \left(1 + \frac{1}{\alpha(\alpha - \lambda - 1)} A_{G}^{(n)}(\operatorname{e}_{p}(S_{n}))(1 + o(1))\right) \frac{\operatorname{e}_{p}(S_{n})}{F^{\leftarrow}(p)} \\ &= \frac{\alpha}{\alpha - 1} \left(1 + \frac{1}{\alpha(\alpha - \lambda - 1)} \left(\frac{n}{\alpha - 1}\right)^{\lambda/\alpha} A_{G}^{(n)}\left(F^{\leftarrow}(p)\right)\right) \\ &\quad \cdot \left(\left(\frac{n}{\alpha - 1}\right)^{1/\alpha} \left(1 + \frac{1}{\alpha\beta} \left(\frac{n^{\beta/\alpha}(\alpha - 1)^{1 - \beta/\alpha}}{\alpha - \beta - 1} - 1\right) A\left(F^{\leftarrow}(p)\right)(1 + o(1))\right) \right. \\ &\quad + \frac{(\alpha - 1)\mu_{n}^{*}\left(F^{\leftarrow}(p)\right) + n\mu}{\alpha F^{\leftarrow}(p)} \left(1 + o(1)\right)\right) \\ &= \frac{\alpha n^{1/\alpha}}{(\alpha - 1)^{1/\alpha + 1}} \left(1 + \frac{1}{\alpha\beta} \left(\left(\frac{n}{\alpha - 1}\right)^{\beta/\alpha} \frac{\alpha + \beta - 1}{\alpha - \beta - 1} - 1\right) A\left(F^{\leftarrow}(p)\right)(1 + o(1))\right) \\ &\quad + \frac{(\alpha - 2)\mu_{n}^{*}\left(F^{\leftarrow}(p)\right) + n\mu}{(\alpha - 1)F^{\leftarrow}(p)} \left(1 + o(1)\right) + o\left(\sum_{i=1}^{n} |\phi_{i}(t) - d_{i}|\right). \end{split}$$

Thus, this completes the proof of the theorem.  $\blacksquare$ 

The following example is applied to interpret in interpretation Theorem 4.1 with absolute Student  $t_{\alpha}$  distribution.

**Example 4.1** (Absolute Student  $t_{\alpha}$  Distribution) A standard Student  $t_{\alpha}$  distribution with density function satisfies that

$$f(x) = \frac{\Gamma((\alpha+1)/2)}{\sqrt{\alpha\pi}\Gamma(\alpha/2)} \left(1 + \frac{x^2}{\alpha}\right)^{-(\alpha+1)/2}, \quad x \in \mathbb{R}, \quad \alpha > 1.$$

Denote by F the distribution function of |X|. According to Example 4.2 of Mao et al. (2012) and Example 4.4 of Hua and Joe (2011), we know that  $\overline{F} \in 2\mathcal{RV}_{-\alpha,-2}$  with auxiliary function  $A(t) = \frac{\alpha^2}{\alpha+2}t^{-2}, \ \mu = E|X| = \alpha/(\alpha-1)$  and  $F^{\leftarrow}(p) = t_{\alpha}^{\leftarrow}((1+p)/2)$ . Let  $X_1$  and  $X_2$  have an identical Student  $t_{\alpha}$  distribution F. Suppose that the random vector  $(X_1, X_2)$  follows an Sarmanov distribution in (2.2) with  $\phi_i(\cdot) = 1 - 2F(\cdot), i = 1, 2$ . Clearly,  $d_i = -1$ . Then

$$e_p(S_2) = \frac{2^{1/\alpha} F^{\leftarrow}(p)}{(\alpha - 1)^{1/\alpha}} + \frac{\alpha - 1}{\alpha} \left( \mu \left( F^{\leftarrow}(p) \right) - a_{12} \mu_1 \left( F^{\leftarrow}(p) \right) \right) + \frac{2}{\alpha - 1}$$

and

$$CE_p(S_2) = \frac{2^{1/\alpha} \alpha F^{\leftarrow}(p)}{(\alpha - 1)^{1/\alpha + 1}} + \frac{\alpha - 2}{\alpha - 1} \left( \mu \left( F^{\leftarrow}(p) \right) - a_{12} \mu_1 \left( F^{\leftarrow}(p) \right) \right) + \frac{2\alpha}{(\alpha - 1)^2}$$

Table 3: Simulated values(MC) versus first-order(1st) and second-order(2nd) asymptotic values  $e_p(S_2)$  and  $CE_p(S_2)$  with different p. We use the Student  $t_{\alpha}$  distribution with  $\alpha = 2.5$  and  $a_{12} = -0.5$ .

p	$e_p(S_2)_{MC}$	$e_p(S_2)_{1st}$	$e_p(S_2)_{2nd}$	$\operatorname{CE}_p(S_2)_{MC}$	$\operatorname{CE}_p(S_2)_{1st}$	$\operatorname{CE}_p(S_2)_{2nd}$	$\frac{\mathbf{e}_p(S_2)_{1st}}{\mathbf{e}_p(S_2)_{MC}}$	$\frac{\mathbf{e}_p(S_2)_{2nd}}{\mathbf{e}_p(S_2)_{MC}}$	$\frac{\operatorname{CE}_p(S_2)_{1st}}{\operatorname{CE}_p(S_2)_{MC}}$	$\frac{\operatorname{CE}_p(S_2)_{2nd}}{\operatorname{CE}_p(S_2)_{MC}}$
0.9500	5.7991	4.0097	5.5194	9.2238	6.6828	9.0031	0.6914	0.9518	0.7245	0.9761
0.9600	6.2368	4.4366	5.9522	9.9000	7.3943	9.7178	0.7114	0.9544	0.7469	0.9816
0.9700	6.8510	5.0387	6.5609	10.8592	8.3978	10.7250	0.7355	0.9577	0.7733	0.9876
0.9800	7.8270	6.0016	7.5313	12.4033	10.0027	12.3340	0.7668	0.9622	0.8065	0.9944
0.9900	9.8652	8.0309	9.5699	15.6877	13.3849	15.7213	0.8141	0.9701	0.8532	1.0021
0.9950	12.5120	10.6764	12.2216	20.0218	17.7940	20.1339	0.8533	0.9768	0.8887	1.0056
0.9990	22.2665	20.4956	22.0482	36.1840	34.1594	36.5034	0.9205	0.9902	0.9440	1.0088
0.9999	53.1965	51.3901	52.9461	88.2526	85.6502	87.9961	0.9660	0.9953	0.9705	0.9971

According to Table 3, the second-order asymptotics of  $e_p(S_2)$  and  $CE_p(S_2)$  are much closer to the simulation value as  $p \in [0.95, 1)$ .

Next, we get the second-order asymptotics of  $ICE_{p,m}(S_n)$  and  $SICE_{p,m}(S_n)$  with an *n*-dimensional Sarmanov distribution.

**Theorem 4.2** Under the conditions of Theorem 4.1, as  $p \uparrow 1$ , we get that

$$\frac{\text{ICE}_{p,m}(S_n)}{F^{\leftarrow}(p)} = \frac{\alpha}{(\alpha-1)n} \left( \left(\frac{n}{\alpha-1}\right)^{1/\alpha} \left(1 + \chi_{\alpha,\beta}^n A\left(F^{\leftarrow}(p)\right)\left(1 + o(1)\right)\right) + \frac{n\mu - \mu_n^*\left(F^{\leftarrow}(p)\right)}{\alpha F^{\leftarrow}(p)}(1 + o(1)) \right) + o\left(\sum_{i=1}^n |\phi_i(t) - d_i|\right),$$

with  $\chi^n_{\alpha,\beta}$  is defined in (4.2) and

$$\frac{\operatorname{SICE}_{p,m}(S_n)}{F^{\leftarrow}(p)} = \frac{\operatorname{ICE}_{p,m}(S_n)}{F^{\leftarrow}(p)} - \frac{\operatorname{e}_p(X_m)}{nF^{\leftarrow}(p)}$$

Moreover, the first-order asymptotics of  $\text{ICE}_{p,m}(S_n)$  and  $\text{SICE}_{p,m}(S_n)$  are  $\frac{n^{1/\alpha}\alpha F^{\leftarrow}(p)}{n(\alpha-1)^{1+1/\alpha}}$  and  $\frac{n^{1/\alpha}\alpha F^{\leftarrow}(p)}{n(\alpha-1)^{1+1/\alpha}} - \frac{F^{\leftarrow}(p)}{n(\alpha-1)^{1/\alpha}}$ .

**Proof.** We require Lemmas 8.4-8.5 in Appendix for this proof. By the definition of  $\text{ICE}_{p,m}(S_n)$ , Theorem 4.1 and Lemma 8.4, as  $p \uparrow 1$ , we have that

$$\begin{split} \frac{\operatorname{ICE}_{p,m}(S_n)}{F^{\leftarrow}(p)} &= \frac{\mathbb{E}\left[X_m | S_n > \operatorname{e}_p(S_n)\right]}{F^{\leftarrow}(p)} \\ &= \frac{\alpha}{(\alpha - 1)n} \left(1 + \widetilde{B}\left(\operatorname{e}(S_n)\right)\left(1 + o(1)\right)\right) \frac{\operatorname{e}(S_n)}{F^{\leftarrow}(p)} \\ &= \frac{\alpha}{(\alpha - 1)n} \left(\left(\frac{n}{\alpha - 1}\right)^{1/\alpha} \left(1 + \frac{1}{\alpha\beta} \left(\frac{n^{\beta/\alpha}(\alpha - 1)^{1 - \beta/\alpha}}{\alpha - \beta - 1} - 1\right) A\left(F^{\leftarrow}(p)\right)\left(1 + o(1)\right)\right) \\ &+ \frac{(\alpha - 1)\mu_n^*\left(F^{\leftarrow}(p)\right) + n\mu}{\alpha F^{\leftarrow}(p)} \left(1 + o(1)\right)\right) \left(1 + \left(\frac{n}{\alpha - 1}\right)^{\rho/\alpha} \widetilde{B}\left(F^{\leftarrow}(p)\right)\left(1 + o(1)\right)\right) \\ &= \frac{\alpha}{(\alpha - 1)n} \left(\left(\frac{n}{\alpha - 1}\right)^{1/\alpha} \left(1 + \chi_{\alpha,\beta}^n A\left(F^{\leftarrow}(p)\right)\left(1 + o(1)\right)\right) \\ &+ \frac{n\mu - \mu_n^*\left(F^{\leftarrow}(p)\right)}{\alpha F^{\leftarrow}(p)} \left(1 + o(1)\right)\right) + o\left(\sum_{i=1}^n |\phi_i(t) - d_i|\right). \end{split}$$

Applying Theorem 4.1 and Lemma 8.5, we conclude that

$$\frac{\operatorname{SICE}_{p,m}(S_n)}{F^{\leftarrow}(p)} = \frac{\mathbb{E}\left[\left(X_m - \operatorname{e}_p(X_m)\right)_+ |S_n > \operatorname{e}_p(S_n)\right]}{F^{\leftarrow}(p)}$$
$$= \frac{\operatorname{ICE}_{p,m}(S_n)}{F^{\leftarrow}(p)} - \frac{\operatorname{e}_p(X_m)}{nF^{\leftarrow}(p)}.$$

This completes the proof of  $ICE_{p,m}(S_n)$  and  $SICE_{p,m}(S_n)$ .

Lastly, the following example is applied to this result of Theorem 4.2 under the Fréchet distribution.

**Example 4.2** (Fréchet distribution) A Fréchet distribution function F satisfies that

$$F(x) = 1 - \exp(-x^{-\alpha}), \quad \alpha > 1.$$

p	$ICE_{p,1}(S_2)_{MC}$	$ICE_{p,1}(S_2)_{1st}$	$ICE_{p,1}(S_2)_{2nd}$	$\frac{\mathrm{ICE}_{p,1}(S_2)_{1st}}{\mathrm{ICE}_{p,1}(S_2)_{MC}}$	$\frac{\mathrm{ICE}_{p,1}(S_2)_{2nd}}{\mathrm{ICE}_{p,1}(S_2)_{MC}}$
0.9500	8.0518	6.2481	7.2566	0.7760	0.9012
0.9600	8.8178	7.0051	7.9763	0.7944	0.9046
0.9700	9.9471	8.1096	9.0379	0.8153	0.9086
0.9800	11.8197	9.9749	10.8508	0.8439	0.9180
0.9900	16.0617	14.1623	14.9689	0.8817	0.9320
0.9990	47.6943	45.0164	45.7053	0.9439	0.9583
0.9999	146.0483	144.7467	145.3974	0.9911	0.9955

Table 4: Simulated values  $ICE_p(S_2)_{MC}$  versus the first-order asymptotic values  $ICE_p(S_2)_{1st}$  and the second-order asymptotic values  $ICE_p(S_2)_{2nd}$  with  $\alpha = 2, a_{12} = 0.5$ .

Table 5: Simulated values  $\text{SICE}_p(S_2)_{MC}$  versus the first-order asymptotic values  $\text{SICE}_p(S_2)_{1st}$  and the second-order asymptotic values  $\text{SICE}_p(S_2)_{2nd}$  with  $\alpha = 2, a_{12} = 0.5$ .

p	$\mathrm{SICE}_{p,1}(S_2)_{MC}$	$\mathrm{SICE}_{p,1}(S_2)_{1st}$	$\mathrm{SICE}_{p,1}(S_2)_{2nd}$	$\frac{\operatorname{SICE}_{p,1}(S_2)_{1st}}{\operatorname{SICE}_{p,1}(S_2)_{MC}}$	$\frac{\operatorname{SICE}_{p,1}(S_2)_{2nd}}{\operatorname{SICE}_{p,1}(S_2)_{MC}}$
0.9500	4.4381	4.0390	4.6044	0.9101	1.0375
0.9600	4.9085	4.5284	5.0565	0.9226	1.0301
0.9700	5.6135	5.2424	5.7276	0.9339	1.0203
0.9800	6.7875	6.4482	6.8810	0.9500	1.0138
0.9900	9.4881	9.1552	9.5187	0.9649	1.0032
0.9990	29.4512	29.1007	29.4465	0.9881	0.9998
0.9999	93.8148	93.5710	93.7786	0.9974	0.9996

Obviously,  $\overline{F}(\cdot) \in 2\mathcal{RV}_{-\alpha,-\alpha}$  with auxiliary function  $A(t) = \alpha t^{-\alpha}/2$ . Let  $X_1$  and  $X_2$  have an identical Fréchet distribution F. Suppose that the random vector  $(X_1, X_2)$  follows a Sarmanov distribution in (2.2) with  $\phi_i(\cdot) = (\cdot)^{-1} - E[X_i^{-1}]$ . Then,  $d_i = -E[X_i^{-1}]$  and  $\rho_i = -1, i = 1, 2$ . Thus,

$$ICE_{p,1}(S_2) = \frac{2^{1/\alpha - 1} \alpha F^{\leftarrow}(p)}{(\alpha - 1)^{1/\alpha + 1}} + \frac{2\mu - \mu \left(F^{\leftarrow}(p)\right) - a_{12} d_1 \mu_1 \left(F^{\leftarrow}(p)\right)}{2(\alpha - 1)}$$

and

$$\operatorname{SICE}_{p,1}(S_2) = \operatorname{ICE}_{p,1}(S_2) - \left(\frac{F^{\leftarrow}(p)}{2(\alpha-1)^{1/\alpha}} + \frac{\mu}{2\alpha}\right)$$

Again, Tables 4-5 reveal that the second-order asymptotics of  $ICE_p(S_2)$  and  $CE_p(S_2)$  are close to simulation values for  $p \in [0.95, 1)$  and provide much better estimates than the first-order asymptotics.

### 5 Numerical illustration

In this section, we numerically illustrate our asymptotic results with a comprehensive comparison among different families and types of systemic risk measures.

**Example 5.1** Under Example 3.1 with  $\alpha = 2$ , k = 1 and  $a_{12} = -1, 0, 1$ , by Theorems 3.1, 3.2, 4.1 and 4.2, we have that the first-order asymptotics (as  $p \uparrow 1$ ) of VaR, Expectile, MES and ICE are



Figure 2: Comparison of the simulation values(MC), first-order and second-order asymptotic values of  $\operatorname{VaR}_p(S_2)/\operatorname{VaR}_p(X)$ and  $e_p(S_2)/\operatorname{VaR}_p(X)$ ,  $\operatorname{CTE}_p(S_2)/\operatorname{VaR}_p(X)$  and  $\operatorname{CE}_p(S_2)/\operatorname{VaR}_p(X)$ ,  $\operatorname{MES}_p(S_2)/\operatorname{VaR}_p(X)$  and  $\operatorname{ICE}_p(S_2)/\operatorname{VaR}_p(X)$ , and  $\operatorname{SES}_p(S_2)/\operatorname{VaR}_p(X)$  and  $\operatorname{SICE}_p(S_2)/\operatorname{VaR}_p(X)$ . We use the Pareto distribution and the Sarmanov distribution with  $a_{12} = -1$  for the left panel,  $a_{12} = 0$  for the middle panel and  $a_{12} = 1$  for the right panel.



Figure 3: Comparison of the simulation values(MC), first-order and second-order asymptotic values of  $\operatorname{VaR}_p(S_2)/\operatorname{VaR}_p(X)$  and  $\operatorname{e}_p(S_2)/\operatorname{VaR}_p(X)$ . We use the Burr distribution with  $\beta = -0.5$  for the left panel and  $\beta = -2$  for the right panel.

equivalent (the ratio to  $F^{\leftarrow}(p)$  is  $2^{1/2}$ ). Those of CTE and CE are equivalent (the ratio to  $F^{\leftarrow}(p)$  is  $2^{3/2}$ ). Those of SES and SICE are equivalent (the ratio to  $F^{\leftarrow}(p)$  is  $2^{1/2} - 1/2$ ). In Figure 2, we have the following observations:

- The second-order asymptotics of systemic risk measures are closer to simulation values than the first-order asymptotics as p ↑ 1. The expectile-based systemic risk measures are larger than VaR-based ones. One should consider a lower confidence level for expectile-based risk measures.
- The second-order asymptotics of VaR, CTE and MES are closer to the simulation values as  $a_{12}$  (i.e. dependence coefficient) decreases. The second-order asymptotics of expectile and CE are closer to the simulation values as  $a_{12}$  increases.
- The second-order asymptotics of ICE is closer to the empirical value as  $a_{12}$  decreases and and that of ICE is closer to the simulation value than that of MES.
- The second-order asymptotic of SICE is closer to the simulation values as  $a_{12}$  decreases and that of SES is closer to the simulation values than that of SICE.

**Example 5.2** Under the conditions of Example 3.2 with  $\alpha = 1.5$ ,  $\beta = -0.5$  or -2 under an Sarmanov distribution in (2.2) with  $\phi_i(\cdot) = 1 - 2F(\cdot)$ , i = 1, 2 and  $a_{12} = -0.5$ . Then by Theorem 3.1, 3.2, 4.1 and 4.2, we have that, as  $p \uparrow 1$ ,

- Figures 3-6 show the second-order asymptotics and simulation values of Theorems 3.1, 3.2, 4.1 and 4.2. Again, the second-order asymptotics can approximate the simulation values as p ↑ 1 better than the first-order asymptotics.
- Figures 3 and 5 reveal that the second-order asymptotics of VaR and MES with  $\beta = -0.5$  are better than those of  $\beta = -2$ , which the expectile and ICE have an opposite result.
- Figure 4 and 6 represent that the second-order asymptotics of CTE, CE, SES and SICE with  $\beta = -2$  are better than those of  $\beta = -0.5$ .

## 6 Application

Introduced by Bürgi et al. (2008), the concept of diversification benefit represents the retained capital gained by collectively managing all risks within a portfolio, in contrast to addressing each risk individually. For a fixed threshold of 0 , the*diversification benefit*is defined by

$$D_{p}^{\rho}(S_{n}) = 1 - \frac{\rho_{p}(S_{n}) - \mathbb{E}[S_{n}]}{\sum_{i=1}^{n} (\rho_{p}(X_{i}) - \mathbb{E}[X_{i}])},$$
(6.1)

where  $\rho_p$  represents a risk measure at a specific confidence level p (e.g., VaR<sub>p</sub>, e<sub>p</sub>, CTE<sub>p</sub>, etc).

Constructed as such, for a fixed systemic risk measure  $\rho$ ,  $D_p^{\rho} > 0$  indicates that diversification is advantageous, potentially reducing an insurer's risk by market engagement. Conversely,  $D_p^{\rho} \leq 0$ 



Figure 4: Comparison of the simulation values (MC), first-order and second-order asymptotic values of  $\operatorname{CTE}_p(S_2)/\operatorname{VaR}_p(X)$  and  $\operatorname{CE}_p(S_2)/\operatorname{VaR}_p(X)$ . We use the Burr distribution with  $\beta = -0.5$  for the left panel and  $\beta = -2$  for the right panel.



Figure 5: Comparison of the simulation values (MC), first-order and second-order asymptotic values of  $MES_p(S_2)/VaR_p(X)$  and  $ICE_p(S_2)/VaR_p(X)$ . We use the Burr distribution with  $\beta = -0.5$  for the left panel and  $\beta = -2$  for the right panel.



Figure 6: Comparison of the simulation values(MC), first-order and second-order asymptotic values of  $SES_p(S_2)/VaR_p(X)$  and  $SICE_p(S_2)/VaR_p(X)$ , Burr distribution with  $\beta = -0.5$  for the left panel and  $\beta = -2$  for the right panel.

suggests diversification is not advantageous for a single insurer. We are particularly interested in cases of heavy-tailed risks whether diversification may be beneficial. Further, this aspect of whether  $D_p^{\rho} > 0$  is technically linked to the risk measure  $\rho$ 's subadditivity. It further relates to the so-called coherence; e.g., see Artzner et al. (1999).

The diversification benefit aids in portfolio selection. By maximizing diversification benefits, the investor can mitigate the risk and boost the performance of a portfolio. It is worthwhile to mention that the usage of  $D_p^{\rho}(S_n)$  is not always applicable; its value depends on the number of risks involved and the specific risk measures employed. Recent findings from Dacorogna et al. (2018) and Chen et al. (2022) highlighted that diversification benefits vary notably based on the type of dependence and the risk measures.

Experts emphasize caution against careless diversification practices, especially when confronted with risks with heavy tails. By adopting the above results of risk measures and deriving formulas for diversification benefits, we can evaluate the performance of a portfolio  $S_n$  in contrast to individual risks operating independently. In the following, we first derive the second-order asymptotic of  $D_p^{\rho}(S_n)$ with  $\rho$  based on VaR, expectile, CTE and CE.

**Theorem 6.1** Under the conditions of Proposition 3.1 with  $\alpha > 1$ , we have, as  $p \uparrow 1$ ,

$$\begin{split} D_p^{\text{VaR}}(S_n) &= 1 - n^{1/\alpha - 1} \left( 1 + \left( \frac{\mu_n^* \left( F^{\leftarrow}(p) \right)}{n^{1/\alpha} F^{\leftarrow}(p)} + \frac{n^{\beta/\alpha} - 1}{\alpha \beta} A \left( F^{\leftarrow}(p) \right) \right) (1 + o(1)) \right) \\ &+ \frac{\left( 1 - n^{1/\alpha - 1} \right) \mu}{F^{\leftarrow}(p)} \left( 1 + o(1) \right) + o\left( \sum_{i=1}^n |\phi_i(t) - d_i| \right), \end{split}$$

$$D_p^{\mathbf{e}}(S_n) = 1 - n^{1/\alpha - 1} \left( 1 + \frac{\left(n^{\beta/\alpha} - 1\right) (\alpha - 1)^{1 - \beta/\alpha}}{\alpha\beta(\alpha - \beta - 1)} A\left(F^{\leftarrow}(p)\right) (1 + o(1)) \right) - \frac{(\alpha - 1)^{1/\alpha + 1} \mu_n^* \left(F^{\leftarrow}(p)\right)}{\alpha n F^{\leftarrow}(p)} \left(1 + o(1)\right) + o\left(\sum_{i=1}^n |\phi_i(t) - d_i|\right),$$

$$D_{p}^{\text{CTE}}(S_{n}) = 1 - n^{1/\alpha - 1} \left( 1 + \frac{1}{\alpha\beta} \left( \frac{n^{\beta/\alpha}(\alpha - 1) - \beta}{\alpha - \beta - 1} - 1 \right) A\left(F^{\leftarrow}(p)\right) (1 + o(1)) \right) - \frac{(\alpha - 1)\left(\mu_{n}^{*}\left(F^{\leftarrow}(p)\right) - \left(n - n^{1/\alpha}\right)\mu\right)}{n\alpha F^{\leftarrow}(p)} \left(1 + o(1)\right) + o\left(\sum_{i=1}^{n} |\phi_{i}(t) - d_{i}|\right),$$

and

$$D_{p}^{CE}(S_{n}) = 1 - n^{1/\alpha - 1} \left( 1 + \frac{\left(n^{\beta/\alpha} - 1\right)(\alpha - 1)^{-\beta/\alpha}(\alpha + \beta - 1)}{\alpha\beta(\alpha - \beta - 1)} A\left(F^{\leftarrow}(p)\right)(1 + o(1)) \right) - \frac{(\alpha - 1)^{1/\alpha}(\alpha - 2)\mu_{n}^{*}\left(F^{\leftarrow}(p)\right)}{\alpha n F^{\leftarrow}(p)} \left(1 + o(1)\right) + o\left(\sum_{i=1}^{n} |\phi_{i}(t) - d_{i}|\right).$$

 $Obviously, \ the \ first-order \ asymptotics \ of \ D_p^{\rm VaR}(S_n), \ D_p^{\rm e}(S_n), \ D_p^{\rm CTE}(S_n) \ and \ D_p^{\rm CE}(S_n) \ are \ 1-n^{1/\alpha-1}.$ 

### **Proof.** see Appendix.

For numerical illustration, we give an example of the Weiss distribution.

**Example 6.1** (Weiss distribution) A Burr distribution function F satisfies that

$$F(x) = 1 - x^{\alpha} \left( 1 + x^{-\beta} \right), \quad x, \alpha > 1, \beta < 0.$$

It is easy to check that  $\overline{F}(\cdot) \in 2\mathcal{RV}_{-\alpha,\beta}$  with auxiliary function  $A(t) = \beta t^{\beta}$ . Let  $X_1$  and  $X_2$  have an identical Weiss distribution F. Suppose that the random vector  $(X_1, X_2)$  follows an Sarmanov distribution in (2.2) with  $\phi_i(\cdot) = 1 - 2F(\cdot)$ . Clearly,  $d_i = -1, i = 1, 2$ .



Figure 7: Comparison of the simulation values (MC), first-order and second-order asymptotic values of  $D_{\text{VaR}}(S_2)$ ,  $D_{\text{e}}(X)$ ,  $D_{\text{CTE}}(S_2)$  and  $D_{\text{CE}}(S_2)$ . We use the Weiss distribution with  $\alpha = 2.5$ ,  $\beta = -1$  and  $a_{12} = 0.5$ .

In the context of Example 6.1, we aim to present the asymptotic performance for diversification benefits  $D_p^{\rho}$  across four risk measures (i.e. VaR, expectile, CTE, and CE) based on the results obtained in Theorem 6.1. Here, we denote by  $\hat{D}_p^{\rho}$  the second-order asymptotic and employ the ratio  $\hat{D}_p^{\rho}/D_p^{\rho}$  to assess the asymptotic performance of diversification benefits. A value of  $\hat{D}_p^{\rho}/D_p^{\rho}$  closer to 1 indicates a more accurate asymptotic result, while deviations from 1 imply poorer outcomes. Besides, according to Bürgi et al. (2008),  $\hat{D}_p^{\rho}/D_p^{\rho} > 1$  signifies the overestimation of the diversification benefit, while  $\hat{D}_p^{\rho}/D_p^{\rho} < 1$  corresponds to the underestimation.

The numerical experiment is shown in Figure 7. For the comparing purpose, we use blue lines to represent the outcomes derived from VaR-based systemic risk measures (i.e. VaR and CTE), while red lines represent the outcomes obtained from expectile-based systemic risk measures (i.e. expectile and CE).

We can observe that, in both plots, the values of  $\hat{D}_p^{\rho}/D_p^{\rho}$  obtained from alternative expectile-based systemic risk measures (i.e. expectile and CE) generally exhibit higher accuracy compared to those

obtained from VaR-based systemic risk measures (i.e. VaR and CTE). The particularly noteworthy finding is the performance of CE, the ratio of  $\hat{D}_p^{\rho}/D_p^{\rho}$  consistently greater than 0.95 when  $p \in [0.95, 1]$ , indicating an error range of less than 5%. It suggests that the expectile inherently incorporates more information than VaR when estimated from the empirical dataset. This insight shows the potential of expectile-based systemic risk measures in the quantification of diversification benefits.

Significantly, the second insight is that when  $\rho$  is an expectile-based systemic risk measure,  $\hat{D}_p^{\rho}/D_p^{\rho} < 1$  consistently with  $p \in [0.95, 1]$ , implying an underestimation of the diversification benefit and suggesting a conservative approximation. Conversely, when  $\rho$  is a VaR-based systemic risk measure,  $\hat{D}_p^{\rho}/D_p^{\rho} > 1$  consistently with  $p \in [0.95, 1]$ , which overestimates the diversification benefit and reflects an optimistic view of diversification. These observations reveal different features of expectile-based and VaR-based systemic risk measures in diversification benefits. Financial practitioners and regulators can sophisticatedly choose one from them according to their distinct purposes and attitudes (conservative or optimistic).

There are more new ways to quantify diversification. E.g., the diversification quotient, was proposed in Han et al. (2022, 2023). Our asymptotic treatment provides a unified framework to investigate these new quotients, which will be studied in the future.

## 7 Conclusion

In this paper, we study systemic risk measures with multivariate Sarmanov distribution. We first classify them into two families of VaR- and expectile-based systemic risk measures. We have the second-order asymptotics of VaR, CTE, MES and SES in the first family. Furthermore, we obtain the second-order asymptotics of expectile, CE, ICE and SICE in the second family. In addition, we give concrete analytical and numerical examples to illustrate the main results. We emphasize that the second-order asymptotics can provide a much better approximation as  $p \uparrow 1$  than the first-order asymptotics. Moreover, we provide a comprehensive comparison among VaR- and expectile-based systemic risk measures. We find that expectile-based measures deduce a larger risk evaluation than VaR-based measures, suggesting a lower confidence level when the expectile is adopted. Finally, we apply the asymptotic treatment to financial diversification and provide instructive insights for risk management. We believe that our results consolidate future research in risk management and our findings have implications for financial practitioners and regulators striving to better understand and mitigate systemic risks in complex financial systems.

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## 8 Appendix

**Lemma 8.1** Assume that  $\overline{F}(\cdot) \in 2\mathcal{RV}_{-\alpha,\beta}$  with  $\alpha > 0, \beta \leq 0$  and auxiliary function  $A(\cdot)$ . Define  $W(t) = \int_0^t \phi(x) dF(x)$ , where  $\phi : \mathbb{R} \to \mathbb{R}$  is a function with  $\mathbb{E}(\phi(X)) = 0$ ,  $\lim_{t\to\infty} \phi(t) = b$  and  $\phi(\cdot) - b \in \mathcal{RV}_{\rho}$  with  $b \in \mathbb{R}$  and  $\rho \leq 0$ . We have  $\overline{W}(\cdot) \in 2\mathcal{RV}_{-\alpha,\gamma}$  with  $\gamma = \max\{\beta, \rho\}$  and  $\widetilde{A}(\cdot) = A(\cdot) + \frac{\rho\alpha}{b(\alpha-\rho)}(\phi(\cdot) - b)$ . In addition, as  $t \to \infty$ , we have

$$\frac{W(t)}{\overline{F}(t)} = b + \frac{\alpha}{\alpha - \rho} \left(\phi(t) - b\right) \left(1 + o(1)\right).$$
(8.1)

Moreover,  $\widetilde{A}(\cdot) = \rho \phi(\cdot)$  if b = 0.

**Proof.** Firstly, we need to prove that  $\overline{W}(\cdot) \in \mathcal{RV}_{-\alpha}$ . According to  $\lim_{t\to\infty} \phi(t) = b$ , for any  $\epsilon > 0$ , there exists  $t_0 > 0$  such that

$$b - \epsilon \le \phi(t) \le b + \epsilon, \quad \forall t > t_0.$$

Fix any x > 0. we have

$$\lim_{t \to \infty} \frac{\overline{W}(tx)}{\overline{W}(t)} = \lim_{t \to \infty} \frac{\int_{tx}^{\infty} (\phi(y) - b) \mathrm{d}F(y) + b\overline{F}(tx)}{\int_{t}^{\infty} (\phi(y) - b) \mathrm{d}F(y) + b\overline{F}(t)} = x^{-\alpha}.$$

Secondly, according to  $\overline{F} \in RV_{-\alpha}$ ,  $\phi(\cdot) - b \in \mathcal{RV}_{\rho}$  and Potter's inequality (Proposition B.1.9 (5) of De Haan and Ferreira (2006)), for any  $\delta > 0$ , there exists  $t_1 > t_0$  such that for  $t, ty > t_1$ ,

$$\left|\frac{\overline{F}(ty)}{\overline{F}(t)} - y^{-\alpha}\right| \le \epsilon \max\{y^{-\alpha+\delta}, y^{-\alpha-\delta}\},\$$

and

$$\left|\frac{\phi(ty)-b}{\phi(t)-b} - x^{\rho}\right| \le \epsilon \max\{y^{\rho+\delta}, y^{\rho-\delta}\}.$$

For any  $t > t_1$ , by the dominated convergence theorem, we have

$$\begin{split} \overline{\overline{F}(t)} &= b + \int_t^\infty \frac{\phi(x) - b}{\overline{F}(t)} \mathrm{d}F(x) \\ &= b - \frac{\phi(t) - b}{\overline{F}(t)} \int_1^\infty \left(\frac{\phi(tx) - b}{\phi(t) - b} - x^\rho + x^\rho\right) \mathrm{d}\overline{F}(tx) \\ &= b - \frac{\phi(t) - b}{\overline{F}(t)} \int_1^\infty x^\rho \mathrm{d}\overline{F}(tx) \left(1 + o(1)\right) \\ &= b + \left(1 + \int_1^\infty \left(\frac{\overline{F}(tx)}{\overline{F}(t)} - x^{-\alpha} + x^{-\alpha}\right) \mathrm{d}x^\rho\right) (\phi(t) - b) \left(1 + o(1)\right) \\ &= b + \left(1 + \int_1^\infty x^{-\alpha} \mathrm{d}x^\rho\right) (\phi(t) - b) \left(1 + o(1)\right) \\ &= b + \frac{\alpha}{\alpha - \rho} \left(\phi(t) - b\right) \left(1 + o(1)\right) \end{split}$$

By  $\overline{F}(\cdot) \in 2\mathcal{RV}_{-\alpha,\beta}$  and Drees-type inequality in Mao (2013), there exists  $t_2 > t_1$  such that for all y > 0 and  $t > \max\{t_2, \frac{t_2}{y}\}$ ,

$$\left|\frac{1}{A(t)}\left(\frac{\overline{F}(ty)}{\overline{F}(t)} - y^{-\alpha}\right) - H_{-\alpha,\beta}(y)\right| \le \epsilon y^{-\alpha+\rho} \max\{y^{\delta}, y^{-\delta}\}.$$

For any  $t > \max\{t_2, \frac{t_2}{x}\}$ , according to the dominated convergence theorem, it follows that

$$\begin{split} \overline{W}(tx) &- x^{-\alpha} = \frac{\int_{tx}^{\infty} \phi(y) \mathrm{d}F(y)}{\int_{t}^{\infty} \phi(y) \mathrm{d}F(y)} - x^{-\alpha} \\ &= \frac{b\overline{F}(tx) + \int_{tx}^{\infty} (\phi(y) - b) \mathrm{d}F(y)}{b\overline{F}(t) + \int_{t}^{\infty} (\phi(y) - b) \mathrm{d}F(y)} - x^{-\alpha} \\ &= \frac{b\overline{F}(tx) - \int_{1}^{\infty} (\phi(txy) - b) \mathrm{d}\overline{F}(txy)}{b\overline{F}(t) - \int_{1}^{\infty} (\phi(ty) - b) \mathrm{d}\overline{F}(ty)} - x^{-\alpha} \\ &= \frac{b\overline{F}(tx) - \int_{1}^{\infty} (\frac{\phi(ty) - b}{\phi(t) - b} - (xy)^{\rho} + (xy)^{\rho}) \mathrm{d}\overline{F}(txy) (\phi(t) - b)}{b\overline{F}(t) - \int_{1}^{\infty} (\frac{\phi(ty) - b}{\phi(t) - b} - y^{\rho} + y^{\rho}) \mathrm{d}\overline{F}(ty) (\phi(t) - b)} - x^{-\alpha} \\ &= \frac{b\overline{F}(tx) - \int_{1}^{\infty} (xy)^{\rho} \mathrm{d}\overline{F}(txy) (\phi(t) - b) (1 + o(1))}{b\overline{F}(t) - \int_{1}^{\infty} y^{\rho} \mathrm{d}\overline{F}(ty) (\phi(t) - b) (1 + o(1))} - x^{-\alpha} \\ &= \frac{b\overline{F}(tx) - \int_{1}^{\infty} y^{\rho} \mathrm{d}\overline{F}(ty) (\phi(t) - b) (1 + o(1))}{b\overline{F}(t) - \int_{1}^{\infty} y^{\rho} \mathrm{d}\overline{F}(ty) (\phi(t) - b) (1 + o(1))} - x^{-\alpha} \\ &= \frac{b\overline{F}(tx) - \int_{1}^{\infty} y^{\rho} \mathrm{d}\overline{F}(ty) (\phi(t) - b) (1 + o(1))}{b + (1 + \int_{1}^{\infty} (\frac{\overline{F}(txy)}{F(t)} - (xy)^{-\alpha} + (xy)^{-\alpha}) \mathrm{d}y^{\rho}) (\phi(t) - b) (1 + o(1))} - x^{-\alpha} \\ &= \frac{bx^{-\alpha} \left(1 + \frac{x^{\beta} - 1}{\beta} A(t) (1 + o(1))\right) + x^{-\alpha + \rho} \left(1 + \int_{1}^{\infty} y^{-\alpha} \mathrm{d}y^{\rho} \right) (\phi(t) - b) (1 + o(1))}{b - (1 + \int_{1}^{\infty} y^{-\alpha} \mathrm{d}y^{\rho}) (\phi(t) - b) (1 + o(1))} - x^{-\alpha} \\ &= x^{-\alpha} \frac{x^{\beta} - 1}{\beta} A(t) (1 + o(1)) + x^{-\alpha} \frac{x^{\rho} - 1}{\rho} \frac{\rho\alpha}{b(\alpha - \rho)} (\phi(t) - b) (1 + o(1)). \end{split}$$

Thus, this ends the proof of Lemma 8.1.  $\blacksquare$ 

**Proof of Proposition 4.1.** Because of equation (1.1), for large enough  $p \uparrow 1$  satisfying  $e_p(X) > 0$ , we have

$$1 - \frac{\mathbb{E}(X)}{\mathbf{e}_p(X)} = \frac{2p-1}{1-p} \mathbb{E}\left(\left[\frac{X}{\mathbf{e}_p(X)} - 1\right] \mathbf{1}_{\{X/\mathbf{e}_p(X) \ge 1\}}\right).$$

Applying the integration by parts, we have

$$\begin{split} \mathbb{E}\left(\left[\frac{X}{\mathbf{e}_{p}(X)}-1\right]\mathbf{1}_{\{X/\mathbf{e}_{p}(X)\geq1\}}\right) &= \int_{\mathbf{e}_{p}(X)}^{\infty} \left(\frac{x}{\mathbf{e}_{p}(X)}-1\right) \mathrm{d}F(x) \\ &= -\int_{\mathbf{e}_{p}(X)}^{\infty} \left(\frac{x}{\mathbf{e}_{p}(X)}-1\right) \mathrm{d}\overline{F}(x) \\ &= -\left(\frac{x}{\mathbf{e}_{p}(X)}-1\right) \overline{F}(x)\Big|_{\mathbf{e}_{p}(X)}^{\infty} + \frac{1}{\mathbf{e}_{p}(X)}\int_{\mathbf{e}_{p}(X)}^{\infty} \overline{F}(x)\mathrm{d}x \\ &= \int_{1}^{\infty} \overline{F}\left(x\mathbf{e}_{p}(X)\right) \mathrm{d}x \\ &= \overline{F}\left(\mathbf{e}_{p}(X)\right) \left(\int_{1}^{\infty} x^{-\alpha}\mathrm{d}x + \int_{1}^{\infty} \frac{\overline{F}\left(x\mathbf{e}_{p}(X)\right)}{\overline{F}\left(\mathbf{e}_{p}(X)\right)} - x^{-\alpha}\mathrm{d}x\right) \\ &= \overline{F}\left(\mathbf{e}_{p}(X)\right) \left(\frac{1}{\alpha-1} + \int_{1}^{\infty} H_{-\alpha,\beta}(x)A(\mathbf{e}_{p}(X))\left(1+o(1)\right)\mathrm{d}x\right) \\ &= \frac{\overline{F}\left(\mathbf{e}_{p}(X)\right)}{\alpha-1} \left(1 + \frac{1}{\alpha-\beta-1}A(\mathbf{e}_{p}(X))\left(1+o(1)\right)\right), \end{split}$$

where the third step is due to the dominated convergence theorem ensured by Theorem 2.3.9 of De Haan and Ferreira (2006). In particular, Bellini et al. (2014) shows that

$$e_p(X) \sim (\alpha - 1)^{-1/\alpha} F^{\leftarrow}(p), \qquad p \uparrow 1.$$

Since  $e_p(X) \to \infty$ ,  $1 - p \downarrow 0$  and  $A(e_p(X)) \downarrow 0$  as  $p \uparrow 1$ , by the first-order Taylor expansion, we have that

$$\begin{split} \frac{1-p}{\overline{F}\left(\mathbf{e}_{p}(X)\right)} &= \frac{1}{\alpha-1} (1-2(1-p)) \left(1 - \frac{\mu}{\mathbf{e}_{p}(X)}\right)^{-1} \left(1 + \frac{1}{\alpha-\beta-1} A(\mathbf{e}_{p}(X)) \left(1 + o(1)\right)\right) \\ &= \frac{1}{\alpha-1} (1-2(1-p)) \left(1 + \frac{\mu}{\mathbf{e}_{p}(X)} \left(1 + o(1)\right)\right) \left(1 + \frac{1}{\alpha-\beta-1} A(\mathbf{e}_{p}(X)) \left(1 + o(1)\right)\right) \\ &= \frac{1}{\alpha-1} \left(1 + \frac{1}{\alpha-\beta-1} A(\mathbf{e}_{p}(X)) \left(1 + o(1)\right) + \frac{\mu}{\mathbf{e}_{p}(X)} \left(1 + o(1)\right) - 2(1-p) \left(1 + o(1)\right)\right) \\ &= \frac{1}{\alpha-1} \left(1 + \frac{1}{\alpha-\beta-1} A(\mathbf{e}_{p}(X)) \left(1 + o(1)\right) + \frac{\mu}{\mathbf{e}_{p}(X)} \left(1 + o(1)\right) - 2\overline{F}(F^{\leftarrow}(p)) \left(1 + o(1)\right)\right) \\ &= \frac{1}{\alpha-1} \left(1 + \frac{(\alpha-1)^{-\beta/\alpha}}{\alpha-\beta-1} A(F^{\leftarrow}(p)) \left(1 + o(1)\right) + \frac{(\alpha-1)^{1/\alpha}\mu}{F^{\leftarrow}(p)} \left(1 + o(1)\right)\right), \end{split}$$

where in the third step we use  $1 - p \sim \overline{F}(F^{\leftarrow}(p))$  as  $p \uparrow 1$ . Notably, in the second last step, we use  $\lim_{p\uparrow 1} \overline{F}(F^{\leftarrow}(p))F^{\leftarrow}(p) = 0$ , and thus  $\overline{F}(F^{\leftarrow}(p)) = o(1/F^{\leftarrow}(p))$ . In addition, due to the fact that

 $U_F(\cdot) \in \mathcal{RV}_{1/\alpha,\beta/\alpha}$  with auxiliary function  $\alpha^{-2}A \circ U_F(\cdot)$ , it follows that

$$\begin{split} \frac{\mathbf{e}_{p}(X)}{F^{\leftarrow}(p)} &= \frac{U_{F}\left(1/\overline{F}\left(\mathbf{e}_{p}(X)\right)\right)}{U_{F}(1/(1-p))} \\ &= \left(\frac{1-p}{\overline{F}\left(\mathbf{e}_{p}(X)\right)}\right)^{1/\alpha} \left(1 + \frac{\left(\frac{1-p}{\overline{F}\left(\mathbf{e}_{p}(X)\right)}\right)^{\beta/\alpha} - 1}{\beta/\alpha} \alpha^{-2}A \circ U_{F}\left(1/(1-p)\right)\right) \\ &= \left(\frac{1}{\alpha-1} \left(1 + \frac{(\alpha-1)^{-\beta/\alpha}}{\alpha-\beta-1}A(F^{\leftarrow}(p))\left(1+o(1)\right) + \frac{(\alpha-1)^{1/\alpha}\mu}{F^{\leftarrow}(p)}\left(1+o(1)\right)\right)\right)^{1/\alpha} \\ &\times \left(1 + \frac{(\alpha-1)^{-\beta/\alpha} - 1}{\alpha\beta}A(F^{\leftarrow}(p))\left(1+o(1)\right)\right) \\ &= (\alpha-1)^{-1/\alpha} \left(1 + \frac{(\alpha-1)^{-\beta/\alpha}}{\alpha(\alpha-\beta-1)}A(F^{\leftarrow}(p))\left(1+o(1)\right) + \frac{(\alpha-1)^{1/\alpha}\mu}{\alpha F^{\leftarrow}(p)}\left(1+o(1)\right)\right) \\ &\times \left(1 + \frac{(\alpha-1)^{-\beta/\alpha} - 1}{\alpha\beta}A(F^{\leftarrow}(p))\left(1+o(1)\right)\right) \\ &= (\alpha-1)^{-1/\alpha} \left(1 + \left(\frac{1}{\alpha\beta} \left(\frac{(\alpha-1)^{1-\beta/\alpha}}{\alpha-\beta-1} - 1\right)A(F^{\leftarrow}(p)) + \frac{(\alpha-1)^{1/\alpha}\mu}{\alpha F^{\leftarrow}(p)}\right)\left(1+o(1)\right)\right) \end{split}$$

Thus, we obtain the desired results.

**Lemma 8.2** Let Y be the nonnegative rv with a distribution H satisfying that  $\overline{H}(\cdot) \in 2\mathcal{RV}_{-\alpha,\rho}$  with  $\alpha > 1, \rho < 0$  and auxiliary function  $A_H(\cdot)$ . As  $t \to \infty$ , we have

$$\mathbb{E}\left[Y\big|Y>t\right] = \frac{\alpha t}{\alpha - 1} \left(1 + \frac{1}{\alpha(\alpha - \rho - 1)}A_H(t)(1 + o(1))\right).$$

**Proof.** By the dominated convergence theorem ensured by Theorem 2.3.9 of De Haan and Ferreira (2006), it follows that, as  $t \to \infty$ ,

$$\mathbb{E}\left[Y|Y>t\right] = \int_0^\infty \frac{\mathbb{P}\left(Y>z, Y>t\right)}{\mathbb{P}(Y>t)} dz$$
  
$$= \int_0^t \frac{\mathbb{P}\left(Y>t\right)}{\mathbb{P}(Y>t)} dz + t \int_1^\infty \frac{\mathbb{P}\left(Y>zt\right)}{\mathbb{P}(Y>t)} dz$$
  
$$= t \left(1 + \int_1^\infty z^{-\alpha} \left(1 + \frac{z^{\rho} - 1}{\rho} A_H(t)(1+o(1))\right) dz\right)$$
  
$$= t \left(\frac{\alpha}{\alpha - 1} + \frac{1}{(\alpha - \rho - 1)(\alpha - 1)} A_H(t)(1+o(1))\right)$$
  
$$= \frac{\alpha t}{\alpha - 1} \left(1 + \frac{1}{\alpha(\alpha - \rho - 1)} A_H(t)(1+o(1))\right).$$

Thus, we prove this lemma.  $\blacksquare$ 

The next lemma extends Lemma 2.4 of Mao and Hu (2013).

**Lemma 8.3** Let F be the distribution function of a nonnegative random variable satisfying  $\overline{F}(\cdot) \in$ 

 $\mathcal{RV}_{-\alpha}$  with  $\alpha > 1$ . For any fixed  $z \in (0,1)$  and  $\beta > 0$ , define

$$V_{\beta}(zt) = \int_0^{zt} \left( \left(1 - \frac{y}{t}\right)^{-\beta} - 1 \right) \mathrm{d}F(y), \quad t > 0.$$

Then, as  $t \to \infty$ , we have

$$V_{\beta}(zt) \sim \beta t^{-1} \mu(t).$$

**Proof.** Since  $\alpha > 1$  and  $\mu(t) \to \mu$  as  $t \to \infty$ , we have that  $\mu(t) < \infty$  and  $\frac{\mu(t)}{t} \in \mathcal{RV}_{-1}$ . We have

$$\mu(t) = \int_0^t x \mathrm{d}F(x) = -\int_0^t x \mathrm{d}\overline{F}(x) = -t\overline{F}(t) + \int_0^t \overline{F}(x) \mathrm{d}x.$$

According to Karamata's theorem, it is easy to check that

$$\mu(t) \sim \int_0^t \overline{F}(x) dx \quad \text{as} \quad t \to \infty, \quad \text{and} \quad \lim_{t \to \infty} \frac{t\overline{F}(t)}{\int_0^t \overline{F}(y) dy} = 0.$$
(8.2)

By the integration by parts, it follows that

$$V_{\alpha}(zt) = -\int_{0}^{zt} \left(1 - \frac{y}{t}\right)^{-\beta} - 1d\overline{F}(y)$$
$$= -(1 - z)^{-\beta}\overline{F}(zt) + \frac{\beta}{t}\int_{0}^{zt}\overline{F}(y)\left(1 - \frac{y}{t}\right)^{-\beta - 1}dy.$$

For any fixed  $z \in (0, 1)$  and (8.2),

$$\lim_{t \to \infty} \frac{t\overline{F}(zt)}{\int_0^{zt} \overline{F}(y) \left(1 - \frac{y}{t}\right)^{-\alpha - 1} \mathrm{d}y} \le \lim_{t \to \infty} \frac{t\overline{F}(zt)}{\int_0^{zt} \overline{F}(y) \mathrm{d}y} = 0.$$

Since (8.2) holds for all  $\alpha > 1$  and  $1 + (\beta + 1)x \le (1 - x)^{-\beta - 1} \le 1 + (\beta + 1)(1 - z)^{-\beta - 2}x$  for  $x \in (0, z)$ , we obtain

$$\begin{split} \lim_{t \to \infty} \frac{V_{\beta}(zt)}{\mu(zt)} &= \beta t^{-1} \lim_{t \to \infty} \frac{\int_{0}^{zt} \overline{F}(y) \left(1 - \frac{y}{t}\right)^{-\beta - 1} \mathrm{d}y}{\int_{0}^{zt} \overline{F}(y) \mathrm{d}y} \\ &\geq \beta t^{-1} \lim_{t \to \infty} \frac{\int_{0}^{zt} \overline{F}(y) \mathrm{d}y + (\beta + 1) \int_{0}^{zt} \overline{F}(y) y/t \mathrm{d}y}{\int_{0}^{zt} \overline{F}(y) \mathrm{d}y} \\ &= \beta t^{-1} \left(1 + (\beta + 1) \lim_{t \to \infty} \frac{\int_{0}^{zt} \overline{F}(y) \mathrm{d}y}{t \int_{0}^{zt} \overline{F}(y) \mathrm{d}y}\right) \\ &= \beta t^{-1} \left(1 + (\beta + 1) \lim_{t \to \infty} \frac{z t \overline{F}(zt)}{\int_{0}^{zt} \overline{F}(y) \mathrm{d}y + t \overline{F}(zt)}\right) \\ &= \beta t^{-1}, \end{split}$$

and

$$\lim_{t \to \infty} \frac{V_{\alpha}(zt)}{\mu(zt)} \le \beta t^{-1} \left( 1 + (\beta + 1)(1 - z)^{-\beta - 2} \lim_{t \to \infty} \frac{\int_0^{zt} \overline{F}(y) y \mathrm{d}y}{t \int_0^{zt} \overline{F}(y) \mathrm{d}y} \right) = \beta t^{-1}.$$

By  $\frac{\mu(t)}{t} \in \mathcal{RV}_{-1}$ , it follows that

$$V_{\beta}(zt) \sim \beta t^{-1} \mu(zt) \sim \beta t^{-1} \mu(t), \text{ as } t \to \infty.$$

This ends the proof.  $\blacksquare$ 

**Lemma 8.4** Under the conditions of Theorem 3.2, as  $t \to \infty$ , we have that

$$\mathbb{E}\left[X_m \middle| S_n > t\right] = \frac{\alpha t}{(\alpha - 1)n} \left(1 + \widetilde{B}(t)(1 + o(1))\right) + o\left(\sum_{i=1}^n |\phi_i(t) - d_i|\right),$$

where

$$\widetilde{B}(t) = \frac{1}{\alpha(\alpha - \beta - 1)} A(t) - \frac{\mu_n^*(t)}{t}.$$
(8.3)

**Proof.** It follows that, as  $t \to \infty$ ,

$$\begin{split} \mathbb{E}\left[X_m \middle| S_n > t\right] &= \int_0^\infty \frac{\mathbb{P}\left(S_n > t, X_m > z\right)}{\mathbb{P}(S_n > t)} \mathrm{d}z \\ &= t \int_0^1 \frac{\mathbb{P}\left(S_n > t, X_m > zt\right)}{\mathbb{P}(S_n > t)} \mathrm{d}z + t \int_1^\infty \frac{\mathbb{P}(X_m > zt)}{\mathbb{P}(S_n > t)} \mathrm{d}z \\ &= t \left(1 - \int_0^1 \frac{\mathbb{P}\left(S_n > t, X_m \le zt\right)}{\mathbb{P}(S_n > t)} \mathrm{d}z + \int_1^\infty \frac{\mathbb{P}(X_m > zt)}{\mathbb{P}(S_n > t)} \mathrm{d}z\right) \\ &= t \left(1 + \frac{\mathbb{P}(X > t)}{\mathbb{P}(S_n > t)} \left(\int_1^\infty \frac{\mathbb{P}(X_m > zt)}{\mathbb{P}(X > t)} \mathrm{d}z - \int_0^1 \frac{\mathbb{P}\left(S_n > t, X_m \le zt\right)}{\mathbb{P}(X > t)} \mathrm{d}z\right)\right) \\ &:= t \left(1 + \frac{\mathbb{P}(X > t)}{\mathbb{P}(S_n > t)} \left(Q_1(t) - Q_2(t)\right)\right). \end{split}$$

For  $Q_1(t)$ , using the fact that  $\overline{F}(\cdot) \in 2\mathcal{RV}_{-\alpha,\beta}$  with  $\alpha > 1$   $\beta < 0$  and the auxiliary function A(t), as  $t \to \infty$ , we have

$$Q_1(t) = \int_1^\infty z^{-\alpha} \left( 1 + \frac{z^\beta - 1}{\beta} A(t)(1 + o(1)) \right) dz$$
  
=  $\frac{1}{\alpha - 1} \left( 1 + \frac{1}{\alpha - \beta - 1} A(t)(1 + o(1)) \right).$ 

For t > 0 and  $z \in (0,1)$ , write  $\Omega_{t,z} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i > t, x_m \leq zt\}$ . For  $Q_2(t)$ , the

key idea is to connect  $\mathbb{P}(S_n > t, X_m \leq zt)$ . Similar to the proof of Proposition 3.1, we have that

$$\mathbb{P}\left(S_{n} > t, X_{m} \leq zt\right) \\
= \int_{\Omega_{t,z}} \left(1 + \sum_{1 \leq i < j \leq n} a_{ij}\phi_{i}(x_{i})\phi_{j}(x_{j})\right) \prod_{k=1}^{n} \mathrm{d}F(x_{k}) \\
= \int_{\Omega_{t,z}} \prod_{k=1}^{n} \mathrm{d}F(x_{k}) + \sum_{1 \leq i < j \leq n} a_{ij}c_{i}c_{j} \left(\int_{\Omega_{t,z}} \prod_{k=1}^{n} \mathrm{d}F(x_{k}) - \int_{\Omega_{t,z}} \prod_{k=1, k \neq i}^{n} \mathrm{d}F(x_{k})\mathrm{d}\widetilde{F}_{i}(x_{i}) \\
- \int_{\Omega_{t,z}} \prod_{k=1, k \neq j}^{n} \mathrm{d}F(x_{k})\mathrm{d}\widetilde{F}_{j}(x_{j}) + \int_{\Omega_{t,z}} \prod_{k=1, k \neq i, j}^{n} \mathrm{d}F(x_{k})\mathrm{d}\widetilde{F}_{i}(x_{j})\mathrm{d}\widetilde{F}_{j}(x_{j})\right) \\
:= J(t, z) + \sum_{1 \leq i < j \leq n} a_{ij}c_{i}c_{j} \left(J(t, z) - K_{i}(t, z) - K_{j}(t, z) + K_{i,j}(t, z)\right).$$
(8.4)

For simplicity, denote  $S_n^{(m)} = \sum_{i \neq m}^n X_i^*$  which has the distribution  $G_m$ . By Theorem 3.5 of Mao and Hu (2013) with  $\alpha > 1$  and the induction assumption, it is easy to check that

$$\overline{\frac{G_m(t)}{\overline{F}(t)}} = (n-1)\left(1 + (n-2)\alpha t^{-1}\mu(t)\left(1 + o(1)\right)\right) + o(A(t)).$$

Obviously,  $\overline{G_m}(\cdot) \in 2\mathcal{RV}_{-\alpha,\lambda}$  with auxiliary function B(t), where  $\lambda = -\min\{1, -\beta\}$  and B(t) is given by

$$B(t) = A(t) - (n-2)\alpha t^{-1}\mu(t).$$

For J(t, z), by the dominated convergence theorem, it follows that

$$\begin{split} J(t,z) &= \int_0^{zt} \overline{G_m}(t-y) \mathrm{d}F(y) \\ &= \overline{G_m}(t) \int_0^{zt} \frac{\overline{G_m}(t-y)}{\overline{G_m}(t)} \mathrm{d}F(y) \\ &= \overline{G_m}(t) \int_0^{zt} \left(1 - \frac{y}{t}\right)^{-\alpha} + H_{-\alpha,\lambda} \left(1 - \frac{y}{t}\right) B(t) \left(1 + o(1)\right) \mathrm{d}F(y) \\ &:= \overline{G_m}(t) \left(J_1(t,z) + J_2(t,z)\right). \end{split}$$

For  $J_1(t, z)$ , by Lemma 8.3 and  $\overline{F}(t) = o(\frac{\mu(t)}{t})$ , we have that

$$J_{1}(t,z) = \int_{0}^{zt} \left( \left(1 - \frac{y}{t}\right)^{-\alpha} - 1 \right) dF(y) + F(zt)$$
  
=  $\alpha t^{-1} \mu(t) + 1 - \overline{F}(zt)$   
=  $1 + \alpha t^{-1} \mu(t) (1 + o(1)).$ 

For  $J_2(t,z)$ , since  $H_{\alpha,\lambda}\left(1-\frac{y}{t}\right) \leq \frac{(1-z)^{-\alpha}}{|\lambda|}\left(\left(1-\frac{y}{t}\right)^{\lambda}-1\right)$  for any  $y \in (0,zt)$  and  $z \in (0,1)$  and

Lemma 5.6 in Barbe and McCormick (2005), we have

$$\int_0^{zt} H_{\alpha,\lambda}\left(1-\frac{y}{t}\right) \mathrm{d}F(y) \le \int_0^{zt} \frac{(1-z)^{-\alpha}}{|\lambda|} \left(\left(1-\frac{y}{t}\right)^{\lambda}-1\right) \mathrm{d}F(y) = 0.$$

Then,

$$J_2(t,z) = o\left(B(t)\right).$$

Thus, according to  $\overline{G_m}(t)$ ,  $J_1(t,z)$  and  $J_2(t,z)$ , it follows that

$$\frac{J(t,z)}{\overline{F}(t)} = \frac{\overline{G_m}(t)}{\overline{F}(t)} \left( 1 + \alpha t^{-1} \mu(t) \left( 1 + o(1) \right) + o\left( B(t) \right) \right)$$
$$= (n-1) \left( 1 + (n-1)\alpha t^{-1} \mu(t) \left( 1 + o(1) \right) + o\left( |A(t)| \right) \right).$$

Similarly, it is easy to see that

$$\frac{K_i(t,z)}{\overline{F}(t)} = \left(n - 1 - \frac{d_i}{c_i}\right) \left(1 + \alpha t^{-1}(n-1)\mu(t)\left(1 + o(1)\right)\right) + o\left(|A(t)|\right) \\ - \left(\frac{\alpha(n-1)\mu_i(t)}{c_i t} + \frac{\alpha(\phi_i(t) - d_i)}{c_i(\alpha - \rho_i)}\right) (1 + o(1)),$$

and

$$\frac{K_{i,m}(t,z)}{\overline{F}(t)} = \left(n - 1 - \frac{d_i}{c_i} - \frac{d_m}{c_m}\right) \left(1 + \alpha t^{-1}(n-1)\mu(t)\left(1 + o(1)\right)\right) + o\left(|A(t)|\right) \\ - \left(\frac{\alpha(n-1)\mu_i(t)}{c_it} + \frac{\alpha(\phi_i(t) - d_i)}{c_i(\alpha - \rho_i)} + \frac{\alpha(n-1)\mu_m(t)}{c_mt} + \frac{\alpha(\phi_m(t) - d_m)}{c_m(\alpha - \rho_m)}\right) (1 + o(1)).$$

Pulling J(t, z),  $K_i(t, z)$ ,  $K_m(t, z)$  and  $K_{i,m}(t, z)$  into (8.4) yields that

$$\frac{\mathbb{P}(S_n > t, X_m \le zt)}{\overline{F}(t)} = (n-1)\left(1 + \alpha t^{-1}\mu_n^*(t)\left(1 + o(1)\right)\right) + o\left(|A(t)| + \sum_{i=1}^n |\phi_i(t) - d_i|\right).$$

Thus, we have that

$$Q_2(t) = \int_0^1 \frac{\mathbb{P}(S_n > t, X_m \le zt)}{\overline{F}(t)} dz = (n-1) \left(1 + \alpha t^{-1} \mu_n^*(t)\right) + o\left(|A(t)| + \sum_{i=1}^n |\phi_i(t) - d_i|\right).$$

According to the dominated convergence theorem, it follows that as  $t \to \infty$ ,

$$\mathbb{E}\left[X_{m}|S_{n}>t\right] = t\left(1 + \left(n\left(1 + \widetilde{A}_{n}(t)(1+o(1))\right)\right)^{-1}\left(Q_{1}(t) - Q_{2}(t)\right)\right)$$
$$= t\left(1 + \frac{1}{n}\left(1 - \alpha t^{-1}\mu_{n}^{*}(t)(1+o(1)) + +o\left(A(t) + \sum_{i=1}^{n}(\phi_{i}(t) - d_{i})\right)\right)\right)$$
$$\cdot \left(\frac{1}{\alpha - 1}\left(1 + \frac{1}{\alpha - \beta - 1}A(t)(1+o(1))\right) - (n-1)\left(1 + \alpha t^{-1}\mu_{n}^{*}(t) + o(|A(t)|)\right)\right)\right)$$
$$= \frac{\alpha t}{(\alpha - 1)n}\left(1 + \left(\frac{1}{\alpha(\alpha - \beta - 1)}A(t) - \frac{\mu_{n}^{*}(t)}{t}\right)(1+o(1))\right) + o\left(\sum_{i=1}^{n}|\phi_{i}(t) - d_{i}|\right).$$

This completes the proof of the lemma.  $\blacksquare$ 

**Lemma 8.5** Under the conditions of Theorem 3.2, there exist some large enough  $t_1$  such that for any  $t \ge t_1$ ,

$$\mathbb{E}\left[ (X_m - t_1)_+ \left| S_n > t \right] = \frac{\alpha t}{(\alpha - 1)n} \left( 1 + \widetilde{B}(t)(1 + o(1)) \right) - \frac{t_1}{n} + o\left( \sum_{i=1}^n |\phi_i(t) - d_i| \right),$$

where  $\widetilde{B}(t)$  is defined in (8.3).

**Proof.** This proof proceeds along similar lines as in the proof of Proposition 3.1. Applying the integration by part, we conclude that

$$\begin{split} \mathbb{E}\left[ (X_m - t_1)_+ \left| S_n > t \right] &= \int_{t_1}^{\infty} (x - t_1) \mathrm{d}F_{X_m \mid S > t}(x) \\ &= -\int_{t_1}^{\infty} (x - t_1) \mathrm{d}\overline{F}_{X_m \mid S > t}(x) \\ &= -(x - t_1) \overline{F}_{X_m \mid S > t}(x) \Big|_{t_1}^{\infty} + \int_{t_1}^{\infty} \overline{F}_{X_m \mid S > t}(x) \mathrm{d}x \\ &= \int_{t_1}^{\infty} \mathbb{P}\left(X_m > x \mid S > t\right) \mathrm{d}x \\ &= t \int_{t_1}^{t_1} \frac{\mathbb{P}\left(X_m > zt, S > t\right)}{\mathbb{P}\left(S > t\right)} \mathrm{d}z \\ &= t \left(\int_{t_1}^{t_1} \frac{\mathbb{P}\left(S_n > t, X_m > zt\right)}{\mathbb{P}(X > t)} \mathrm{d}z + \int_{1}^{\infty} \frac{\mathbb{P}(X_m > zt)}{\mathbb{P}(X > t)} \mathrm{d}z \right) \\ &= t \left(1 - \frac{t_1}{t} + \frac{\mathbb{P}(X > t)}{\mathbb{P}(S_n > t)} \left(\int_{1}^{\infty} \frac{\mathbb{P}(X_m > zt)}{\mathbb{P}(X > t)} \mathrm{d}z - \int_{t_1}^{t_1} \frac{\mathbb{P}\left(S_n > t, X_m \le zt\right)}{\mathbb{P}(X > t)} \mathrm{d}z \right) \right). \end{split}$$

Following a similar analysis of  $Q_2(t)$  of Proposition 3.1, we have that

$$\int_{\frac{t_1}{t}}^{1} \frac{\mathbb{P}(S_n > t, X_m \le zt)}{\mathbb{P}(X > t)} dz = (1 - t_1/t) (n - 1) \left(1 + \alpha t^{-1} \mu_n^*(t)\right) + o\left(A(t) + \sum_{i=1}^n (\phi_i(t) - d_i)\right).$$

Thus, by first-order Taylor expansion, as  $t \to \infty,$  we have

$$\mathbb{E}\left[ (X_m - t_1)_+ \left| S_n > t \right] \right]$$
  
=  $t \left( 1 - \frac{t_1}{t} + \left( n \left( 1 + \widetilde{A}_n(t)(1 + o(1)) \right) \right)^{-1} \left( \frac{1}{\alpha - 1} \left( 1 + \frac{1}{\alpha - \beta - 1} A(t) \right) - (1 - t_1/t) (n - 1) \left( 1 + \alpha t^{-1} \mu_n^*(t) + o \left( A(t) + \sum_{i=1}^n (\phi_i(t) - d_i) \right) \right) \right) \right)$   
=  $\frac{\alpha t}{(\alpha - 1)n} \left( 1 + \left( \frac{1}{\alpha(\alpha - \beta - 1)} A(t) - \frac{\mu_n^*(t)}{t} \right) (1 + o(1)) \right) - \frac{t_1}{n} + o \left( \sum_{i=1}^n (\phi_i(t) - d_i) \right) \right).$ 

We complete this proof of Lemma.  $\blacksquare$ 

**Proof of Theorem 6.1.** By Theorem 3.1 and (6.1), as  $p \uparrow 1$ , based on VaR, we obtain

$$\begin{split} D_p^{\text{VaR}}(S_n) &= 1 - \frac{\text{VaR}_p(S_n) - n\mu}{n(F^{\leftarrow}(p) - \mu)} \\ &= 1 - \frac{\text{VaR}_p(S_n) - n\mu}{nF^{\leftarrow}(p)} \left(1 - \frac{\mu}{F^{\leftarrow}(p)}\right)^{-1} \\ &= 1 - \frac{\text{VaR}_p(S_n) - n\mu}{nF^{\leftarrow}(p)} \left(1 + \frac{\mu}{F^{\leftarrow}(p)}(1 + o(1))\right) \\ &= 1 - \left(n^{1/\alpha - 1} \left(1 + \left(\frac{\mu_n^* \left(F^{\leftarrow}(p)\right)}{n^{1/\alpha}F^{\leftarrow}(p)} + \frac{n^{\beta/\alpha} - 1}{\alpha\beta}A\left(F^{\leftarrow}(p)\right)\right)(1 + o(1))\right) \\ &- \frac{\mu}{F^{\leftarrow}(p)}\right) \left(1 + \frac{\mu}{F^{\leftarrow}(p)}(1 + o(1))\right) + o\left(\sum_{i=1}^n (\phi_i(t) - d_i)\right) \\ &= 1 - n^{1/\alpha - 1} \left(1 + \left(\frac{\mu_n^* \left(F^{\leftarrow}(p)\right)}{n^{1/\alpha}F^{\leftarrow}(p)} + \frac{n^{\beta/\alpha} - 1}{\alpha\beta}A\left(F^{\leftarrow}(p)\right)\right)(1 + o(1))\right) \\ &+ \frac{\left(1 - n^{1/\alpha - 1}\right)\mu}{F^{\leftarrow}(p)} \left(1 + o(1)\right) + o\left(\sum_{i=1}^n (\phi_i(t) - d_i)\right). \end{split}$$

According to Proposition 4.1 and Theorem 4.1, as  $p\uparrow 1,$  we have

$$D_p^{e}(S_n) = 1 - \frac{e_p(S_n) - n\mu}{n (e_p(X) - \mu)}$$
  
=  $1 - \frac{e_p(S_n)/n - \mu}{e_p(X) - \mu}$   
=  $1 - n^{1/\alpha - 1} \left( 1 + \frac{(n^{\beta/\alpha} - 1) (\alpha - 1)^{1 - \beta/\alpha}}{\alpha\beta(\alpha - \beta - 1)} A (F^{\leftarrow}(p)) (1 + o(1)) \right)$   
 $- \frac{(\alpha - 1)^{1/\alpha + 1} \mu_n^* (F^{\leftarrow}(p))}{\alpha n F^{\leftarrow}(p)} (1 + o(1)) + o \left( \sum_{i=1}^n (\phi_i(t) - d_i) \right).$ 

Based on CTE and Lemma 8.2, as  $p \uparrow 1$ , we have

$$\begin{split} D_p^{\text{CTE}}(S_n) &= 1 - \frac{\text{CTE}_p(S_n) - n\mu}{n \left( \text{CTE}_p(X) - \mu \right)} \\ &= 1 - \frac{\text{CTE}_p(S_n) - n\mu}{\frac{n\alpha F^{\leftarrow}(p)}{\alpha - 1} \left( 1 + \frac{1}{\alpha(\alpha - \beta - 1)} A \left( F^{\leftarrow}(p) \right) - \frac{(\alpha - 1)\mu}{\alpha F^{\leftarrow}(p)} \right)}{1 + \alpha(1)} \\ &= 1 - n^{1/\alpha - 1} \left( 1 + \left( \zeta_{\alpha,\beta}^n - \frac{1}{\alpha(\alpha - \beta - 1)} \right) A \left( F^{\leftarrow}(p) \right) (1 + o(1)) \right) + o \left( \sum_{i=1}^n (\phi_i(t) - d_i) \right) \\ &- \left( \frac{(\alpha - 1)\mu_n^* \left( F^{\leftarrow}(p) - n\mu \right)}{n\alpha F^{\leftarrow}(p)} + \frac{(\alpha - 1)n^{1/\alpha - 1}\mu}{\alpha F^{\leftarrow}(p)} \right) (1 + o(1)) \\ &= 1 - n^{1/\alpha - 1} \left( 1 + \frac{1}{\alpha\beta} \left( \frac{n^{\beta/\alpha}(\alpha - 1) - \beta}{\alpha - \beta - 1} - 1 \right) A \left( F^{\leftarrow}(p) \right) (1 + o(1)) \right) \\ &- \frac{(\alpha - 1) \left( \mu_n^* \left( F^{\leftarrow}(p) \right) - (n - n^{1/\alpha}) \mu \right)}{n\alpha F^{\leftarrow}(p)} \left( 1 + o(1) \right) + o \left( \sum_{i=1}^n (\phi_i(t) - d_i) \right). \end{split}$$

In addition, applying Lemma 8.2, it follows that

$$\begin{split} D_{p}^{\text{CE}}(S_{n}) &= 1 - \frac{\text{CE}_{p}(S_{n}) - n\mu}{n\left(\text{CE}_{p}(X) - \mu\right)} \\ &= 1 - \frac{\text{CE}_{p}(S_{n}) - n\mu}{\frac{n\alpha e_{p}(X)}{\alpha - 1}\left(1 + \frac{1}{\alpha(\alpha - \beta - 1)}A\left(e_{p}(X)\right)\right) - n\mu} \\ &= 1 - \frac{\text{CE}_{p}(S_{n})/n - \mu}{\frac{\alpha e_{p}(X)}{\alpha - 1}\left(1 + \frac{(\alpha - 1)^{-\beta/\alpha}}{\alpha(\alpha - \beta - 1)}A\left(F^{\leftarrow}(p)\right)\left(1 + o(1)\right)\right) - \mu} \\ &= 1 - n^{1/\alpha - 1}\left(1 + \frac{\left(n^{\beta/\alpha} - 1\right)\left(\alpha - 1\right)^{-\beta/\alpha}\left(\alpha + \beta - 1\right)}{\alpha\beta(\alpha - \beta - 1)}A\left(F^{\leftarrow}(p)\right)\left(1 + o(1)\right)\right) \\ &- \frac{(\alpha - 1)^{1/\alpha}(\alpha - 2)\mu_{n}^{*}\left(F^{\leftarrow}(p)\right)}{\alpha nF^{\leftarrow}(p)}\left(1 + o(1)\right) + o\left(\sum_{i=1}^{n}(\phi_{i}(t) - d_{i})\right). \end{split}$$

## References

- Abdallah, A., Boucher, J.-P., Cossette, H., and Trufin, J. (2016). Sarmanov family of bivariate distributions for multivariate loss reserving analysis. North American Actuarial Journal, 20(2):184–200.
- Acharya, V. V., Pedersen, L. H., Philippon, T., and Richardson, M. (2017). Measuring systemic risk. Review of Financial Studies, 30(1):2–47.
- Alink, S., Löwe, M., and Wüthrich, M. V. (2005). Analysis of the expected shortfall of aggregate dependent risks. ASTIN Bulletin, 35(1):25–43.
- Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent measures of risk. Mathematical Finance, 9(3):203–228.
- Asimit, A. V., Furman, E., Tang, Q., and Vernic, R. (2011). Asymptotics for risk capital allocations based on conditional tail expectation. *Insurance: Mathematics and Economics*, 49(3):310–324.
- Barbe, P., Fougeres, A.-L., and Genest, C. (2006). On the tail behavior of sums of dependent risks. ASTIN Bulletin, 36(2):361–373.
- Barbe, P. and McCormick, W. P. (2005). Asymptotic expansions of convolutions of regularly varying distributions. Journal of the Australian Mathematical Society, 78(3):339–371.
- Bellini, F. and Bignozzi, V. (2015). On elicitable risk measures. Quantitative Finance, 15(5):725–733.
- Bellini, F. and Di Bernardino, E. (2017). Risk management with expectiles. *European Journal of Finance*, 23(6):487–506.
- Bellini, F., Klar, B., Müller, A., and Gianin, E. R. (2014). Generalized quantiles as risk measures. *Insurance: Mathematics and Economics*, 54:41–48.
- Bingham, N. H., Goldie, C. M., and Teugels, J. L. (1989). Regular Variation. Cambridge university press.
- Blanchet, J., Lam, H., Liu, Y., and Wang, R. (2020). Convolution bounds on quantile aggregation. arXiv preprint arXiv:2007.09320.
- Bürgi, R., Dacorogna, M. M., and Iles, R. (2008). Risk aggregation, dependence structure and diversification benefit. Stress testing for financial institutions.
- Chen, D., Mao, T., Pan, X., and Hu, T. (2012). Extreme value behavior of aggregate dependent risks. *Insurance: Mathematics and Economics*, 50(1):99–108.
- Chen, Y. and Liu, J. (2022). An asymptotic study of systemic expected shortfall and marginal expected shortfall. *Insurance: Mathematics and Economics*, 105:238–251.
- Chen, Y., Liu, P., Liu, Y., and Wang, R. (2022). Ordering and inequalities of mixtures on risk aggregation. Mathematical Finance, 32(1):421–451.
- Chen, Y. and Yang, Y. (2014). Ruin probabilities with insurance and financial risks having an FGM dependence structure. *Science China Mathematics*, 57:1071–1082.
- Cui, H., Tan, K. S., and Yang, F. (2021). Diversification in catastrophe insurance markets. *ASTIN Bulletin*, 51(3):753–778.
- Dacorogna, M., Elbahtouri, L., and Kratz, M. (2018). Validation of aggregated risks models. Annals of Actuarial Science, 12(2):433–454.
- Daouia, A., Girard, S., and Stupfler, G. (2018). Estimation of tail risk based on extreme expectiles. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 80(2):263–292.

De Haan, L. and Ferreira, A. (2006). Extreme Value Theory: An Introduction. Springer.

- De Haan, L. and Stadtmüller, U. (1996). Generalized regular variation of second order. Journal of the Australian Mathematical Society, 61(3):381–395.
- De Rossi, G. and Harvey, A. (2009). Quantiles, expectiles and splines. Journal of Econometrics, 152(2):179– 185.
- Degen, M., Lambrigger, D. D., and Segers, J. (2010). Risk concentration and diversification: Second-order properties. *Insurance: Mathematics and Economics*, 46(3):541–546.
- Dhaene, J., Laeven, R. J. A., and Zhang, Y. (2022). Systemic risk: Conditional distortion risk measures. Insurance: Mathematics and Economics, 102:126–145.
- Dhaene, J., Tsanakas, A., Valdez, E. A., and Vanduffel, S. (2012). Optimal capital allocation principles. Journal of Risk and Insurance, 79(1):1–28.
- Embrechts, P., Lambrigger, D. D., and Wüthrich, M. V. (2009a). Multivariate extremes and the aggregation of dependent risks: examples and counter-examples. *Extremes*, 12:107–127.
- Embrechts, P., Nešlehová, J., and Wüthrich, M. V. (2009b). Additivity properties for Value-at-Risk under Archimedean dependence and heavy-tailedness. *Insurance: Mathematics and Economics*, 44(2):164–169.
- Emmer, S., Kratz, M., and Tasche, D. (2015). What is the best risk measure in practice? A comparison of standard measures. *Journal of Risk*, 18(2):31–60.
- Fadina, T., Liu, Y., and Wang, R. (2024). A framework for measures of risk under uncertainty. *Finance and Stochastics*. Forthcoming: https://doi.org/10.1007/s00780-024-00528-2.
- Gneiting, T. (2011). Making and evaluating point forecasts. Journal of the American Statistical Association, 106(494):746–762.
- Han, X., Lin, L., and Wang, R. (2022). Diversification quotients: Quantifying diversification via risk measures. SSRN: 4149069.
- Han, X., Lin, L., and Wang, R. (2023). Diversification quotients based on VaR and ES. Insurance: Mathematics and Economics, 113:185–197.
- Hua, L. and Joe, H. (2011). Second order regular variation and conditional tail expectation of multiple risks. Insurance: Mathematics and Economics, 49(3):537–546.
- Huang, J. and Kotz, S. (1984). Correlation structure in iterated farlie-gumbel-morgenstern distributions. Biometrika, 71(3):633–636.
- Jaunė, E. and Šiaulys, J. (2022). Asymptotic risk decomposition for regularly varying distributions with tail dependence. *Applied Mathematics and Computation*, 427:127164.
- Joe, H. and Li, H. (2011). Tail risk of multivariate regular variation. Methodology and Computing in Applied Probability, 13:671–693.
- Johnson, N. L. and Kotz, S. (1977). On some generalized farlie-gumbel-morgenstern distributions-ii regression, correlation and further generalizations. *Communications in Statistics-Theory and Methods*, 6(6):485–496.
- Kley, O., Klüppelberg, C., and Paterlini, S. (2020). Modelling extremal dependence for operational risk by a bipartite graph. Journal of Banking & Finance, 117:105855.
- Kotz, S., Balakrishnan, N., and Johnson, N. L. (2004). Continuous Multivariate Distributions, Volume 1: Models and Applications. John Wiley & Sons.

- Kuan, C.-M., Yeh, J.-H., and Hsu, Y.-C. (2009). Assessing value at risk with CARE, the conditional autoregressive expectile models. *Journal of Econometrics*, 150(2):261–270.
- Lv, W., Pan, X., and Hu, T. (2013). Asymptotics of the risk concentration based on the tail distortion risk measure. *Statistics & Probability Letters*, 83(12):2703–2710.
- Mao, T. (2013). Second-order conditions of regular variation and drees-type inequalities. In Stochastic Orders in Reliability and Risk: In Honor of Professor Moshe Shaked, pages 313–330. Springer.
- Mao, T. and Hu, T. (2013). Second-order properties of risk concentrations without the condition of asymptotic smoothness. *Extremes*, 16(4):383–405.
- Mao, T., Lv, W., and Hu, T. (2012). Second-order expansions of the risk concentration based on CTE. Insurance: Mathematics and Economics, 51(2):449–456.
- Mao, T. and Ng, K. W. (2015). Second-order properties of tail probabilities of sums and randomly weighted sums. *Extremes*, 18(3):403–435.
- Mao, T., Ng, K. W., and Hu, T. (2015). Asymptotic expansions of generalized quantiles and expectiles for extreme risks. Probability in the Engineering and Informational Sciences, 29(3):309–327.
- Mao, T. and Yang, F. (2015). Risk concentration based on expectiles for extreme risks under FGM copula. Insurance: Mathematics and Economics, 64:429–439.
- Newey, W. K. and Powell, J. L. (1987). Asymmetric least squares estimation and testing. *Econometrica*, 55(4):819–847.
- Qu, Z. and Chen, Y. (2013). Approximations of the tail probability of the product of dependent extremal random variables and applications. *Insurance: Mathematics and Economics*, 53(1):169–178.
- Ratovomirija, G. (2016). On mixed erlang reinsurance risk: aggregation, capital allocation and default risk. European Actuarial Journal, 6(1):149–175.
- Sarmanov, O. V. (1966). Generalized normal correlation and two-dimensional Fréchet classes. 168(1):32–35.
- Schucany, W. R., Parr, W. C., and Boyer, J. E. (1978). Correlation structure in farlie-gumbel-morgenstern distributions. *Biometrika*, 65(3):650–653.
- Tadese, M. and Drapeau, S. (2020). Relative bound and asymptotic comparison of expectile with respect to expected shortfall. *Insurance: Mathematics and Economics*, 93:387–399.
- Taylor, J. W. (2008). Estimating value at risk and expected shortfall using expectiles. Journal of Financial Econometrics, 6(2):231–252.
- Ting Lee, M.-L. (1996). Properties and applications of the Sarmanov family of bivariate distributions. Communications in Statistics-Theory and Methods, 25(6):1207–1222.
- Wang, R. and Ziegel, J. F. (2021). Scenario-based risk evaluation. *Finance and Stochastics*, 25:725–756.
- Yang, Y. and Hashorva, E. (2013). Extremes and products of multivariate AC-product risks. Insurance: Mathematics and Economics, 52(2):312–319.
- Yang, Y. and Wang, Y. (2013). Tail behavior of the product of two dependent random variables with applications to risk theory. *Extremes*, 16(1):55–74.
- Ziegel, J. F. (2016). Coherence and elicitability. *Mathematical Finance*, 26(4):901–918.