

SLICING CRITERION FOR IND-SMOOTH RING MAPS

LONGKE TANG

ABSTRACT. We show that ind-smoothness of flat ring maps can be tested on constructible stratifications, even for maps of non-Noetherian rings. We prove this by generalizing ind-smoothness and ind-lci to a sequence of conditions on animated ring maps called ind- d -smoothness, and showing all at once that they can be tested on constructible stratifications.

1. INTRODUCTION

1.1. Results. The Artin approximation theorem [Art69, Theorem 1.10] is a very useful technical result in commutative algebra with many important consequences, including Artin's representability theorem and the proper base change theorem for étale cohomology. The most memorable generalization of the Artin approximation theorem, proved by Popescu, is as follows:

Theorem 1.1 ([Pop86, Theorem 1.8]; [Stacks, 07GC]). *A map $R \rightarrow A$ of Noetherian rings is ind-smooth if and only if it is regular.*

While the statement of this theorem is clean, its proof is extremely technical. An excellent exposition of the proof is [Stacks, 07BW], where one first reduces to the case where R is a field and then treats it by very concrete constructions.

Recent developments in p -adic geometry have led us to consider non-Noetherian rings such as perfectoid rings and general valuation rings. For example, Bouis [Bou23] has proved a comparison theorem between the étale cohomology and the syntomic cohomology of certain large algebras over perfectoid rings, by showing that valuation rings over perfectoid rings have a regularity property called F -smoothness, introduced in [BM23, Definition 4.1]. Therefore, it is natural to consider possible non-Noetherian generalizations of Theorem 1.1. Unfortunately, we currently have no idea what a non-Noetherian version of Theorem 1.1 will be, if there is one. Nevertheless, we are able to prove the following theorem, which is an analog of the reduction step carried out in [Stacks, 07F1]:

Theorem 1.2 (cf. [Stacks, 07F5]). *Let R be a ring, Λ be a flat R -algebra, and r be an element of R . If $\Lambda[1/r]$ is ind-smooth over $R[1/r]$ and $\Lambda/r\Lambda$ is ind-smooth over R/r , then Λ is ind-smooth over R .*

Corollary 1.3. *Let R be a ring and Λ be a flat R -algebra. Let r_1, \dots, r_n be elements of R and set $r_{n+1} = 1$. If $(\Lambda/(r_1, \dots, r_i)\Lambda)[1/r_{i+1}]$ is ind-smooth over $(R/(r_1, \dots, r_i))[1/r_{i+1}]$ for $i = 0, 1, \dots, n$, then Λ is ind-smooth over R .*

The following consequence of Theorem 1.2 seems unknown in the literature, although Gabber, in his talk [Gab24] on [GZ24], has announced the special case of Theorem 1.2 with $R/r = \Lambda/r\Lambda$ that suffices to imply it:

Corollary 1.4. *Let \mathcal{O} be a valuation ring of rank 1 with fraction field of characteristic 0, and let $\pi \in \mathcal{O}$ be a pseudo-uniformizer. Let R be a finitely presented \mathcal{O} -algebra. Then the map from R to its π -completion is ind-smooth.*

It is nontrivial to deduce Corollary 1.4 from our main theorems; this uses several standard facts on affinoid algebras. We will address it at the beginning of §3.

Our proof of Theorem 1.2 uses in an essential way the theory of animated rings as is developed in [SAG, §25] and is recalled and used in [ČS24, §5]. Working with animated rings, one no longer worries about zerodivisors as one does in [Stacks, 07CN, 07CQ]; instead, one introduces higher homotopies when taking quotient by a zerodivisor so that the “kernel” stays free, and tries to control the resulting higher homotopies. Specifically, we generalize the definition of smoothness and the statement of Theorem 1.2 as follow:

Definition 1.5 (Definition 2.18). Let R be an animated ring and d be a natural number. An animated R -algebra A is *d-smooth* if it is a compact object in the category of animated R -algebras, and its cotangent complex has tor-amplitude concentrated in $[0, d]$. It is *ind-d-smooth* if it is a filtered colimit of d -smooth algebras.

Remark 1.6. Here compactness is the animated analog of the classical notion of finite presentation. Quillen showed [Qui70, Theorem 5.5] that for maps of classical rings, 0-smoothness is smoothness and 1-smoothness is lci. He also conjectured [Qui70, Conjectures 5.6, 5.7] that over a Noetherian classical ring, any d -smooth classical algebra is 2-smooth, and is 1-smooth if its tor-amplitude is finite. The latter conjecture has been solved by Avramov in [Avr99]; see also [BI23].

Theorem 1.7. *Let R be an animated ring and $S \subseteq R$ be a multiplicative submonoid. Let d be a natural number and Λ be an animated R -algebra. Then Λ is ind-d-smooth over R if:*

- $S^{-1}\Lambda$ is ind-d-smooth over $S^{-1}R$;
- For all $r \in S$, $\Lambda/r\Lambda$ is ind-d-smooth over R/r .

Here all quotients are animated, namely the animated tensor product $- \otimes_{\mathbb{Z}[x]} \mathbb{Z}$ where $x \mapsto r$ on the left and $x \mapsto 0$ on the right. Note that Theorem 1.7 does not immediately imply Theorem 1.2, as it is not clear that ind-smoothness of $R/r \rightarrow \Lambda/r\Lambda$ implies ind-smoothness of $R/r^n \rightarrow \Lambda/r^n\Lambda$, and the quotient in Theorem 1.2 is not animated. We will address this implication in §3.1.

1.2. Convention. By “rings” we mean commutative rings. Everything is animated unless otherwise stated, that is:

- By “categories” we mean ∞ -categories, unless we say “ordinary categories”;
- By “rings” we mean animated rings, unless we say “classical rings”;
- By “modules” we mean animated modules, unless we say “classical modules”;
- ...

But by “sets” we still mean 0-groupoids, because we have “animas” for ∞ -groupoids.

In any category, we say that a map $A \rightarrow B$ is an *injection* or that A is a *subobject* of B , if the fiber product $A \times_B A$ exists and the diagonal $A \rightarrow A \times_B A$ is an isomorphism; *surjections* refer to the class of maps left orthogonal to the class of injections, whenever this exists.

We do not notationally distinguish between a ring or a module and its underlying anima, just as in classical commutative algebra we do not notationally distinguish between a ring or a module and its underlying set.

We denote by \mathbf{Ani} the category of animas, namely the category of spaces in [HTT, Definition 1.2.16.1], and by \mathbf{Fun} the $(\infty, 1)$ -category of functors, as in [HTT, Notation 1.2.7.2]. For a ring R and R -algebras A and B , we will consider their maps both as R -algebras and as R -modules; to avoid confusion, we denote by $\mathrm{Hom}_R(A, B)$ the anima of R -algebra maps, by $\mathrm{map}_R(A, B)$ the R -complex of R -module maps, and by $\mathrm{Map}_R(A, B)$ the R -module or the anima of R -module maps.

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2. ANIMATED RING THEORY

2.1. Recollection on animated rings. We first recall the theory of animated rings very concisely, and derive some useful facts. For a more detailed treatment on basics of animated rings, see [ČS24, §5.1] or [SAG, §25].

Definition 2.1 (Animated rings). Let \mathbf{fPoly} denote the category of finitely generated polynomial \mathbb{Z} -algebras. *Animated rings* are functors $\mathbf{fPoly}^{\mathrm{op}} \rightarrow \mathbf{Ani}$ preserving finite products. Denote by \mathbf{Ring} the category of animated rings. The *underlying anima* of an animated ring is its value on $\mathbb{Z}[x]$.

Definition 2.2 (Modules). The inclusion functor from \mathbf{fPoly} to the category of connective \mathbb{E}_∞ - \mathbb{Z} -algebras left Kan extends to \mathbf{Ring} ; we call the extended functor taking *underlying \mathbb{E}_∞ -rings*. For a ring R , let $(\mathbf{D}(R), \otimes_R)$ denote the symmetric monoidal category of modules of its underlying \mathbb{E}_∞ -ring, and we call its objects *R -complexes*; this comes equipped with a t-structure, and we call its connective objects *R -modules*. Alternatively, one can animate the category of modules as in the end of [ČS24, §5.1.7], and define the category of complexes as the stabilization [HA, Definition 1.4.2.8] of the category of modules.

Definition 2.3 (Perfect complexes). For a ring R , let $\mathrm{Perf}(R) \subseteq \mathbf{D}(R)$ denote the full subcategory of compact objects in $\mathbf{D}(R)$, whose objects are called *perfect R -complexes*; for $c, d \in \mathbb{Z}$, let $\mathrm{Perf}_{\geq c}(R) = \mathrm{Perf}(R) \cap \mathbf{D}_{\geq c}(R)$,

$$\mathrm{Perf}_{\leq d}(R) = \{M \in \mathrm{Perf}(R) \mid \forall N \in \mathbf{D}_{>d}(R), \pi_0 \mathrm{Map}_R(M, N) = 0\},$$

and $\mathrm{Perf}_{[c,d]}(R) = \mathrm{Perf}_{\geq c}(R) \cap \mathrm{Perf}_{\leq d}(R)$; let $\mathbf{cProj}(R) = \mathrm{Perf}_{[0,0]}(R)$ and call its objects *finite projective R -modules*. These subcategories constitute a bounded weight structure defined in [ES21, Definition 2.2.1].

The following proposition collects basic properties of $\mathrm{Perf}_{[c,d]}(R)$.

Proposition 2.4. *Let R be a ring and c and d be integers.*

- (1) *For any $M \in \mathrm{Perf}_{\geq c}(R)$, there exists a finite free R -module P and a map $P[c] \rightarrow M$ with cofiber $C \in \mathrm{Perf}_{\geq c+1}(R)$. If $d > c$ and $M \in \mathrm{Perf}_{[c,d]}(R)$, then in fact $C \in \mathrm{Perf}_{[c+1,d]}(R)$.*
- (2) *$\mathrm{Perf}_{[c,d]}(R)$ is the smallest full subcategory of $\mathrm{Perf}(R)$ that is closed under extensions and contains $P[n]$ for any $P \in \mathbf{cProj}(R)$ and integer $n \in [c, d]$.*
- (3) *$\mathbf{cProj}(R)$ consists of exactly retracts of finite free R -modules.*
- (4) *$\mathrm{Perf}(R)$ consists of exactly the dualizable objects in $(\mathbf{D}(R), \otimes_R)$, and the dual of $\mathrm{Perf}_{[c,d]}(R)$ is $\mathrm{Perf}_{[-d,-c]}(R)$.*

Proof. Note first that by definition it is easy to check that $\text{Perf}_{\geq c}(R)$, $\text{Perf}_{\leq d}(R)$, and $\text{Perf}_{[c,d]}(R)$ are all closed under extensions.

- (1) Translating, we may assume $c = 0$. Since the t-structure on $D(R)$ is compatible with filtered colimits by [HA, Proposition 7.1.1.13], $\text{Perf}_{\geq 0}(R)$ consists of compact objects of $D_{\geq 0}(R)$. Consider the functor $\pi_0: D_{\geq 0}(R) \rightarrow D_{\heartsuit}(R) = D_{\heartsuit}(\pi_0(R))$. It is left adjoint to the natural inclusion functor which preserves filtered colimits, so it preserves compact objects. This means that for $M \in \text{Perf}_{\geq 0}(R)$, $\pi_0(M)$ is a finitely presented classical $\pi_0(R)$ -module. Choose generators m_1, \dots, m_r for $\pi_0(M)$ and lift them to M . They define a map $P = R^{\oplus r} \rightarrow M$, and by construction its cofiber C has trivial π_0 and hence lies in $\text{Perf}_{\geq 1}(R)$. If $d > 0$, then $P[1] \in \text{Perf}_{\leq d}(R)$, so by the fiber sequence $M \rightarrow C \rightarrow P[1]$ we can see that $C \in \text{Perf}_{\leq d}(R)$.
- (2) It is obvious that $P[n] \in \text{Perf}_{[c,d]}(R)$ for $P \in \text{cProj}(R)$ and $n \in [c, d]$. It remains to see the other direction, which for $d = c$ follows immediately from the definition, and otherwise follows from (1) by induction.
- (3) Take $M \in \text{cProj}(R)$ and take a map $P \rightarrow M$ from a finite free R -module P with cofiber $C \in \text{Perf}_{\geq 1}(R)$ as in (1). By definition, $\pi_0 \text{Map}_R(M, C) = 0$, so the map $M \rightarrow C$ is nullhomotopic and thus M is a retract of P .
- (4) Since the tensor unit $R \in D(R)$ is compact, dualizable objects are compact. Since $D(R)$ is generated under colimits by the translations of R which are dualizable, compact objects are dualizable. The rest follows from (2). \square

Remark 2.5. It is not hard to see that $\text{Perf}_{\leq d}(R)$ consists of exactly those perfect R -complexes with tor-amplitude $\leq d$ as defined in [HA, Definition 7.2.4.21]. We choose not to include the proof of this fact as we will not use it in the following.

Definition 2.6 (Stably free complexes). For a ring R , we say that a perfect R -complex M is *stably free* if its class $[M] \in K_0(R) := K_0(\text{Perf}(R))$ is an integer multiple of $[R]$. Here K_0 of a small stable category means the classical abelian group generated by objects of the category subject to the relations $[Y] = [X] + [Z]$ for all fiber sequences $X \rightarrow Y \rightarrow Z$.

The following proposition is actually a simple special case of [ES21, Theorem 6.0.1], but we give an elementary proof here for the convenience of the reader.

Proposition 2.7. *Let R be a ring. Then $K_0(R)$ is the group completion of the monoid of isomorphism classes of $\text{cProj}(R)$ under the direct sum.*

Proof. Denote the group completion by $K_0^{\oplus}(R)$. It has an obvious map to $K_0(R)$ which is surjective by Proposition 2.4(2). It suffices to define a left inverse $K_0(R) \rightarrow K_0^{\oplus}(R)$ to this map. In other words, we need to assign an element $[X]^{\oplus}$ for $X \in \text{Perf}(R)$ such that $[Y]^{\oplus} = [X]^{\oplus} + [Z]^{\oplus}$ for fiber sequences $X \rightarrow Y \rightarrow Z$ in $\text{Perf}(R)$.

First fix $d \in \mathbb{Z}$, and do downward induction on c to define $[X]^{\oplus}$ for $X \in \text{Perf}_{[c,d]}(R)$ and to show $[Y]^{\oplus} = [X]^{\oplus} + [Z]^{\oplus}$ for fiber sequences $X \rightarrow Y \rightarrow Z$ in $\text{Perf}_{[c,d]}(R)$. If $c = d$, then $X = P[d]$ for $P \in \text{cProj}(R)$, and we just define $[X]^{\oplus} = (-1)^d[P]$. Its compatibility with fiber sequences follows from the fact that fiber sequences in $\text{cProj}(R)$ split. Otherwise, take a map $f: P[c] \rightarrow X$ with $P \in \text{cProj}(R)$ and cofiber in $\text{Perf}_{[c+1,d]}(R)$, and define $[X]^{\oplus} = (-1)^c[P] + [\text{cofib}(f)]^{\oplus}$. We first show that this is independent of f : take another such map $g: Q[c] \rightarrow X$

and let $(f, g): P[c] \oplus Q[c] \rightarrow X$ be the direct sum. Then the commutative diagram

$$\begin{array}{ccccc}
 & P[c] & \xlongequal{\quad} & P[c] & \\
 & \downarrow & & \downarrow & \\
 Q[c] & \longrightarrow & X & \longrightarrow & \text{cofib}(g) \\
 \parallel & & \downarrow & & \downarrow \\
 Q[c] & \longrightarrow & \text{cofib}(f) & \longrightarrow & \text{cofib}(f, g)
 \end{array}$$

shows $(-1)^c[P] + [\text{cofib}(f)]^\oplus = (-1)^c[P \oplus Q] + [\text{cofib}(f, g)]^\oplus = (-1)^c[Q] + [\text{cofib}(g)]^\oplus$ by induction hypothesis. In particular, it agrees with the existing definition on $\text{Perf}_{[c+1, d]}(R)$. For a fiber sequence $X \rightarrow Y \rightarrow Z$ in $\text{Perf}_{[c, d]}(R)$, take $f: P[c] \rightarrow X$ and $g: Q[c] \rightarrow Z$ with $P, Q \in \text{cProj}(R)$ and cofibers in $\text{Perf}_{[c+1, d]}(R)$. Since $\text{cofib}(Y \rightarrow Z) = X[1] \in \text{D}_{>c}(R)$, one can lift g to a map $Q[c] \rightarrow Y$ and form the commutative diagram

$$\begin{array}{ccccc}
 P[c] & \longrightarrow & P[c] \oplus Q[c] & \longrightarrow & Q[c] \\
 \downarrow & & \downarrow & & \downarrow \\
 X & \longrightarrow & Y & \longrightarrow & Z
 \end{array}$$

Taking vertical cofibers, the desired equality for $X \rightarrow Y \rightarrow Z$ follows by induction.

It remains to show that the above definition is independent of d . This boils down to showing that for $X = P[d] \in \text{Perf}_{[d, d]}(R)$ and $f: F[d] \rightarrow X$ with $F \in \text{cProj}(R)$ and cofiber $\text{cofib}(f) = Q[d+1] \in \text{Perf}_{[d+1, d+1]}(R)$, we have $(-1)^d[P] = (-1)^d[F] + (-1)^{d+1}[Q]$. This is obvious, as in fact $F = P \oplus Q$. \square

The following corollary is obvious.

Corollary 2.8. *Let R be a ring and $c < d$ be integers.*

- (1) *If $P \in \text{cProj}(R)$ is stably free, then there exists a finite free module Q such that $P \oplus Q$ is also finite free.*
- (2) *Every stably free complex $M \in \text{Perf}_{[c, d]}(R)$ is a finite extension of modules of the form $R[n]$ for integers $n \in [c, d]$.*

Definition 2.9 (Connectivity). For $c \in \mathbb{N}$, a ring map is called c -connective if it is c -connective as an anima map. That is, a ring map $A \rightarrow B$ is c -connective if:

- For $i \leq c$, the map $\pi_i(A) \rightarrow \pi_i(B)$ is a surjection.
- For $i < c$, the map $\pi_i(A) \rightarrow \pi_i(B)$ is an isomorphism.

Note that a 0-connective map is the same as a surjection.

Proposition 2.10. *Let $c \in \mathbb{N}$ and $A \rightarrow B$ be a c -connective ring map. Then:*

- (1) *For all $n \in \mathbb{N}$, the functor $\text{Perf}_{[0, n]}(A) \rightarrow \text{Perf}_{[0, n]}(B)$ is $(c - n)$ -connective on mapping animas.*
- (2) *The functor $\text{Perf}_{[0, c+1]}(A) \rightarrow \text{Perf}_{[0, c+1]}(B)$ hits every object whose K_0 class is hit. In particular, it hits every stably free complex.*

Proof. (1) Let $X, Y \in \text{Perf}_{[0, n]}(A)$ and let $M = \text{map}_A(X, Y)$. By Proposition 2.4(4), $M = X^\vee \otimes_A Y \in \text{Perf}_{[-n, n]}(A)$, and we want to show that $\tau_{\geq 0}(M) \rightarrow \tau_{\geq 0}(M \otimes_A B)$ is $(c - n)$ -connective. For this we only need to show that $M \rightarrow M \otimes_A B$ is a $(c - n)$ -connective map of spectra, which is clear as M is $(-n)$ -connective and by assumption $\text{fib}(A \rightarrow B)$ is c -connective.

- (2) Do induction on c . For $c > 0$, the claim easily follows from the $n = c$ case of (1) and Proposition 2.4(1). For $c = 0$, we need to lift any module $\bar{X} \in \text{Perf}_{[0,1]}(B)$ to $\text{Perf}_{[0,1]}(A)$. By Proposition 2.4(1), we can write $\bar{X} = \text{cofib}(\bar{P} \rightarrow \bar{Q})$ where $\bar{P}, \bar{Q} \in \text{cProj}(B)$ and \bar{Q} is finite free. Now the K_0 class of \bar{P} is hit, so by Proposition 2.7 and Proposition 2.4(3), we can add a finite free module to both \bar{P} and \bar{Q} , and assume that they come from $P, Q \in \text{cProj}(A)$, and then the claim follows from (1). \square

The following construction is crucial.

Definition 2.11 (Quotient algebra). Let R be a ring and M be an R -module with a map $f: M \rightarrow R$. Form the pushout square

$$\begin{array}{ccc} \text{LSym}_R(M) & \xrightarrow{\text{LSym}(0)} & R \\ f \downarrow & & \downarrow \\ R & \xrightarrow{q} & R//M \end{array}$$

of R -algebras. We call $q: R \rightarrow R//M$ the *quotient algebra* of R by M . Sometimes we will use the notation $R//f$ or $R//(M, f)$ instead of $R//M$ to avoid confusion.

Remark 2.12. The formation of $R//M$ defines a functor from the slice category of R -modules over R to the category of R -algebras that preserves all colimits. Moreover, it satisfies base change, meaning that for a ring map $R \rightarrow R'$, if we set $M' = M \otimes_R R'$ and $f' = f \otimes_R R'$, then $R'//M' = (R//M) \otimes_R R'$.

Remark 2.13. Since LSym_R is the free functor left adjoint to the forgetful functor from R -algebras to R -modules, it is easy to see that a map from $R//M$ to an R -algebra A is just a nullhomotopy of the R -module map $M \rightarrow R \rightarrow A$. More formally, $R//M$ represents the functor $A \mapsto * \times_{\text{Map}(M, A)} *$ on Alg_R , where the two points are the zero map and the map $M \rightarrow R \rightarrow A$.

We recall some finiteness conditions on algebras.

Definition 2.14 (Compact algebras). We say that a ring map $R \rightarrow A$ is:

- *compact*, if A is a compact object in the category of R -algebras.
- *classically compact*, if $\pi_0(R) \rightarrow \pi_0(A)$ is finitely presented as a map of classical rings.

Remark 2.15. “Compact” is the analog of “locally of finite presentation” in [HA, Definition 7.2.4.26] in the animated setting. We choose this term simply because it has far fewer syllables.

2.2. Cotangent complexes. The cotangent complex is the higher analog of the Kähler differential module. Here we will not recall the definitions of square-zero extensions, cotangent complexes, and the basic deformation theory of animated rings; we refer the reader to [ČS24, §5.1.8, §5.1.9] and [SAG, §25.3]. Instead, we will control algebras with their cotangent complexes, just as Lurie has done in [HA, §7.4.3] for \mathbb{E}_∞ -rings. We will see that the cotangent complex has much more control on higher homotopy than on π_0 ; this can be seen as evidence of the intuition that higher homotopy should be thought of as thickening, and is something nearly linear.

Proposition 2.16. *Let notations be as in Definition 2.11, and let $I = \text{fib}(R \rightarrow R//M)$ with a natural R -module map $M \rightarrow I$ factorizing $f: M \rightarrow R$. Then*

$$L_{(R//M)/R} = M[1] \otimes_R (R//M),$$

and the composition $M[1] \rightarrow I[1] \rightarrow L_{(R//M)/R} = M[1] \otimes_R (R//M)$ of the natural map and the universal derivation can be naturally identified with the base change map of $M[1]$ along $R \rightarrow R//M$; in other words, the map $M \rightarrow I$ has a retract after base change to $R//M$ given by the scalar extension of the universal derivation.

Proof. Recall that $L_{\text{LSym}_R(M)/R} = M \otimes_R \text{LSym}_R(M)$. Consider the composition $R \rightarrow \text{LSym}_R(M) \rightarrow R$, where the second arrow is $\text{LSym}_R(M \rightarrow 0)$. Its cotangent fiber sequence reads

$$M \rightarrow 0 \rightarrow L_{R/\text{LSym}_R(M)},$$

which implies $L_{R/\text{LSym}_R(M)} = M[1]$, so the equality follows by base change. To give the identification, since both $M[1]$ and $L_{(R//M)/R} = M[1] \otimes_R (R//M)$ preserve colimits with respect to (M, f) , we are reduced to the case $M = R$, and then f can be seen as an element in R ; now by base change we are reduced to the universal such case, namely $R = \mathbb{Z}[x]$ and $f = x$, and we can compute

$$L_{\mathbb{Z}/\mathbb{Z}[x]} = \text{cofib}(L_{\mathbb{Z}[x]/\mathbb{Z}} \otimes_{\mathbb{Z}[x]} \mathbb{Z} \rightarrow L_{\mathbb{Z}/\mathbb{Z}}) = \text{cofib}(\mathbb{Z}dx \rightarrow 0) = \mathbb{Z}dx[1],$$

where the universal derivation $d: \text{cofib}(\mathbb{Z}[x] \rightarrow \mathbb{Z}) \rightarrow L_{\mathbb{Z}/\mathbb{Z}[x]}$ takes $x[1]$ to $dx[1]$ and $x^n[1]$ to 0 for $n > 1$, as expected. \square

Corollary 2.17. *Let $R \rightarrow A$ be a ring map with cofiber C . Let M be an R -module with a map $g: M \rightarrow C[-1]$. Then $R \rightarrow A$ naturally factors through $R//M$, and*

$$L_{A/(R//M)} = \text{cofib}(M[1] \otimes_R A \rightarrow L_{A/R}),$$

where the map $M[1] \otimes_R A \rightarrow L_{A/R}$ is the scalar extension along $R \rightarrow A$ of the composition $M[1] \rightarrow C \rightarrow L_{A/R}$ of $g[1]$ and the universal derivation.

Proof. That $R \rightarrow A$ naturally factors through $R//M$ is Remark 2.13. The rest follows from the proposition and the cotangent fiber sequence of $R \rightarrow R//M \rightarrow A$, since the universal derivation is functorial with respect to maps of R -algebras. \square

The following notion is central in the paper.

Definition 2.18 (d -smoothness). Let d be a natural number. A ring map $R \rightarrow A$ is d -smooth if it is compact and $L_{A/R} \in \text{Perf}_{[0,d]}(A)$. A ring map is $\text{ind-}d$ -smooth if it is a filtered colimit of d -smooth ring maps.

Remark 2.19. Since the cotangent complex of a compact map is perfect, every compact map is d -smooth for some $d \in \mathbb{N}$. For any $d \in \mathbb{N}$ and any ring R , it is easy to see that:

- d -smooth maps are closed under base change and finite composition.
- $\text{Ind-}d$ -smooth maps are closed under base change, composition, and filtered colimits.
- Maps of d -smooth R -algebras are $(d+1)$ -smooth.
- Maps of $\text{ind-}d$ -smooth R -algebras are $\text{ind-}(d+1)$ -smooth.
- For $M \in \text{Perf}_{[0,d]}(R)$ with a map to R , $R//M$ is $(d+1)$ -smooth over R .

To control d -smooth maps, we will use the following proposition in [SAG], whose statement is reproduced below for the convenience of the reader:

Proposition 2.20 ([SAG, Proposition 25.3.6.1]). *Let $\varphi: R \rightarrow A$ be a ring map. The scalar extension $\mathrm{cofib}(\varphi) \otimes_R A \rightarrow L_{A/R}$ of the universal derivation is:*

- *surjective;*
- *2-connective, if φ is surjective;*
- *$(c+3)$ -connective, if φ is c -connective for some $c \in \mathbb{Z}_+$.*

Corollary 2.21. *Let c be a natural number and $\varphi: R \rightarrow A$ be a c -connective ring map. Then $L_{A/R}$ is $(c+1)$ -connective. The converse holds providing that φ is a classically compact surjection and that there is no nontrivial idempotent $e \in R$ with A an $R[e^{-1}]$ -algebra; in particular, in this case if $L_{A/R} = 0$ then $A = R$.*

Proof. The proposition implies that the cofiber of $\mathrm{cofib}(\varphi) \rightarrow L_{A/R}$ is $(c+1)$ -connective, so the $(c+1)$ -connectivity of $L_{A/R}$ is equivalent to that of $\mathrm{cofib}(\varphi) \rightarrow L_{A/R}$. If φ is c -connective, then $\mathrm{cofib}(\varphi)$ is $(c+1)$ -connective, and so is $\mathrm{cofib}(\varphi) \otimes_R A$. Conversely if $\mathrm{cofib}(\varphi) \otimes_R A$ is $(c+1)$ -connective, we want to see that so is $\mathrm{cofib}(\varphi)$ under the assumptions above. If $c = 0$, there is nothing to prove, as we have assumed that φ is surjective. If $c > 0$, then since $\mathrm{fib}(\varphi) \otimes_R A$ is c -connective, we have $\pi_0(\mathrm{fib}(\pi_0(R) \rightarrow \pi_0(A)) \otimes_{\pi_0(R)} \pi_0(A)) = 0$, so $\pi_0(A)$ is the quotient of $\pi_0(R)$ by a classically idempotent ideal. Now the assumptions force $\pi_0(A) = \pi_0(R)$, so the map $\mathrm{cofib}(\varphi) \rightarrow \mathrm{cofib}(\varphi) \otimes_R A$ induces an isomorphism on the lowest homotopy group, which implies that $\mathrm{cofib}(\varphi)$ is $(c+1)$ -connective and hence φ is c -connective. \square

Lemma 2.22. *Let R be a ring and A be a compact R -algebra. Then there is a smooth ring map $A \rightarrow B$ admitting a retract, such that $L_{B/R}$ is stably free. We can also replace the “smooth” in the previous sentence by “ d -connective $(d+1)$ -smooth” for any natural number d .*

Proof. By Proposition 2.7, there is some $M \in \mathbf{cProj}(A)$ such that $M \oplus L_{A/R}$ is stably free. Then one can easily see that $B = \mathrm{LSym}_A(M)$ has the desired properties. Similarly, if one takes $M \in \mathbf{cProj}(A)$ with $M[d+1] \oplus L_{A/R}$ stably free, then $\mathrm{LSym}_A(M)$ has the desired properties for the final claim. \square

Lemma 2.23. *Let c be a natural number and $R \rightarrow A$ be a c -connective ring map with fiber I . Let $N \in \mathrm{Perf}_{[0, \min\{c, 2\}]}(R)$ with a map $\bar{g}: N[c] \otimes_R A \rightarrow L_{A/R}[-1]$. Then there is a map $g: N[c] \rightarrow I$ such that the natural R -algebra map $R//N[c] \rightarrow A$ induces $\bar{g}[1]$ on cotangent complexes.*

Proof. By Proposition 2.16, it suffices to produce a dashed arrow making

$$\begin{array}{ccc} N[c] & \xrightarrow{\quad\quad\quad} & N[c] \otimes_R A \\ g \downarrow & & \downarrow \bar{g} \\ I & \xrightarrow{\quad\quad\quad} & I \otimes_R A \longrightarrow L_{A/R}[-1] \end{array}$$

commute. Since both I and $R \rightarrow A$ are c -connective, the lower left map is $2c$ -connective; by Proposition 2.20, the lower right map is $(c+2)$ -connective when $c > 0$ and 1-connective when $c = 0$; so the cofiber of their composition is $(c+1+\min\{c, 2\})$ -connective, and thus the map from $N[c]$ to the cofiber has a nullhomotopy, giving the desired dashed arrow. \square

The following theorem is the analog of [HA, Theorem 7.4.3.18] in the setting of animated rings, saying that a classically compact ring map with a perfect cotangent complex is compact.

Theorem 2.24. *Let c and d be natural numbers. A ring map $R \rightarrow A$ is d -smooth if and only if it is classically compact and $L_{A/R} \in \text{Perf}_{[0,d]}(A)$. Assume that it is c -connective, that $d > 0$, and if $c = 0$ and $d = 1$ assume further that $L_{A/R}[-1] \in \text{cProj}(A)$ can be lifted to $\text{cProj}(R)$. Then there is a factorization*

$$R = A_c \rightarrow A_{c+1} \rightarrow \cdots \rightarrow A_d = A,$$

where $A_{i+1} = A_i // M_i[i]$ for some $M_i \in \text{cProj}(A_i)$ and some map $M_i[i] \rightarrow A_i$. One can arrange so that each M_i except M_{d-1} is free; if $c < d - 1$ and $L_{A/R}$ is stably free, then one can arrange so that each M_i is free.

Proof. To prove the first claim, by definition we only need to show that a classically compact ring map with a perfect cotangent complex is compact. Therefore, replacing R by its finitely generated polynomial algebra, we can assume that $R \rightarrow A$ is surjective, which reduces the first claim to the second claim. Moreover, we can assume that there is no nontrivial idempotent $e \in R$ with A an $R[e^{-1}]$ -algebra, by passing to an idempotent quotient of R .

Now do downward induction on c . If $c = d$, by Remark 2.19, $L_{A/R}$ is trivial, so by Corollary 2.21 we have $A = R$. If $c = d - 1$, we can take $N \in \text{cProj}(R)$, an isomorphism $N[c] \otimes_R A \rightarrow L_{A/R}[-1]$, and apply Lemma 2.23. Then the resulting map $R // N[c] \rightarrow A$ is an isomorphism by Corollary 2.16 and Corollary 2.21, which implies the claim. If $c < d - 1$, by Proposition 2.4(1) there is a finite free R -module N and a map $N[c+1] \otimes_R A \rightarrow L_{A/R}$ with cofiber in $\text{Perf}_{[c+2,d]}(A)$, and if $c = d - 2$, by Proposition 2.7 we can make this cofiber a translation of a free module. Then the claim follows from Lemma 2.23 and Corollary 2.16 by induction. \square

Corollary 2.25. *Let R be a ring, $S \subseteq R$ be a multiplicative submonoid, and A' be a compact $S^{-1}R$ -algebra with $L_{A'/S^{-1}R}$ stably free. Then there is a compact R -algebra A with $L_{A/R}$ stably free and $S^{-1}A = A'$.*

Proof. Take integer $d > 1$ such that A' is d -smooth over $S^{-1}R$. By the theorem, there is a factorization

$$S^{-1}R \rightarrow A'_0 \rightarrow A'_1 \rightarrow \cdots \rightarrow A'_d = A',$$

where A'_0 is a finitely generated polynomial $S^{-1}R$ -algebra, and $A'_{i+1} = A'_i // A'_i[i]^{\oplus r}$ for some $r \in \mathbb{N}$ and some map $A'_i[i]^{\oplus r} \rightarrow A'_i$. Now it suffices to lift each individual step of the factorization. For $S^{-1}R \rightarrow A'_0$ this is trivial. For other steps, it suffices to lift the module map $A'_i[i]^{\oplus r} \rightarrow A'_i$ to A_i , up to multiplication by an element in S . This is also trivial because

$$\text{map}_{A'}(A'_i[i]^{\oplus r}, A'_i) = A'_i[-i]^{\oplus r} = S^{-1}A_i[-i]^{\oplus r} = S^{-1}\text{map}_A(A_i[i]^{\oplus r}, A_i),$$

and $S^{-1}(-)$, as a filtered colimit, commutes with taking π_0 . \square

The following theorem shows that we can do better for a compact surjection with cotangent complex in a small range, writing it as just one quotient.

Theorem 2.26. *Let c be a natural number and $R \rightarrow A$ be a c -connective $(c + 2 + \min\{c, 1\})$ -smooth ring map. If $c = 0$, assume that the K_0 class of $L_{A/R}$ can be lifted to $K_0(R)$. Then there is an $M \in \text{Perf}_{[0, 1 + \min\{c, 1\}]}(R)$ with a map $f: M[c] \rightarrow R$ such that $A = R // M[c]$.*

Proof. Without loss of generality, we can assume that there is no nontrivial idempotent $e \in R$ with A an $R[e^{-1}]$ -algebra. Set $I = \text{fib}(R \rightarrow A)$. By Definition 2.18

and Corollary 2.21, $L_{A/R}[-1] \in \text{Perf}_{[c, c+1+\min\{c, 1\}]}(A)$. When $c > 1$, by Proposition 2.10 we can take $M \in \text{Perf}_{[0, 2]}(R)$ such that $M[c] \otimes_R A = L_{A/R}[-1]$. Take $g: M[c] \rightarrow I$ as in Lemma 2.23; then the final claim of Corollary 2.21 implies that $A = R//M[c]$ and we are done. When $c \leq 1$, take $N \in \text{Perf}_{[0, c]}(R)$ and $\bar{g}: N[c] \otimes_R A \rightarrow L_{A/R}[-1]$ such that $\bar{K} = \text{cofib}(\bar{g})[-2c-1] \in \text{cProj}(A)$ can be lifted to some $K \in \text{cProj}(R)$. Take $g: N[c] \rightarrow I$ as in Lemma 2.23; then

$$L_{A/(R//N[c])} = \text{cofib}(\bar{g}[1]) = \bar{K}[2c+2].$$

Use Lemma 2.23 again, now with $R//N[c] \rightarrow A$ and K , to get a map $h: K[2c+1] \rightarrow R//N[c]$. The final claim of Corollary 2.21 implies that $A = (R//N[c])//K[2c+1]$; to present A as one quotient instead of two, we have to splice K and N together. So set $J = \text{fib}(R \rightarrow R//N[c])$ and look at the cofiber C of the shift of the natural map $N[c] \rightarrow J$ as in the diagram

$$\begin{array}{ccc} K[2c+1] & \xrightarrow{h} & R//N[c] \\ \downarrow \text{---} & & \downarrow \\ N[c+1] & \longrightarrow & J[1] \longrightarrow C \end{array}$$

When $c = 1$, by the final claim of Proposition 2.16 and Proposition 2.20, $C \otimes_R (R//N[1])$ is 4-connective, so C must also be 4-connective. When $c = 0$, since $N \rightarrow J$ is surjective, $C = \text{cofib}(N[1] \rightarrow J[1])$ is 2-connective. In either case the map from $K[2c+1]$ to C has a nullhomotopy, and we can fill in the dash arrow. Now let $M = \text{cofib}(K[c] \rightarrow N)$ be the cofiber of a shift of the dashed arrow. Taking vertical fibers gives a map $f: M[c] \rightarrow R$, and it is easy to see that $R//M[c] = (R//N[c])//K[2c+1]$ by taking quotient algebras horizontally in the following diagram and computing cotangent complexes.

$$\begin{array}{ccc} M[c] & \xrightarrow{f} & R \\ \downarrow & & \downarrow \\ K[2c+1] & \xrightarrow{h} & R//N[c] \end{array}$$

□

2.3. Blowup algebras. We also need the basic theory of the animated analog of blowup algebras, also known as Rees algebras (cf. [Stacks, 052P]). Throughout the subsection, let $R \rightarrow A$ be a surjective ring map, and let $R[t]$ be the graded polynomial algebra with $\deg(t) = 1$. The following is [Complex, Proposition 13.3], up to the obvious equivalence between filtered R -algebras and graded $R[t]$ -algebras.

Proposition 2.27 (cf. [Mao21, Corollary 3.54]). *There is an initial \mathbb{Z} -graded $R[t]$ -algebra $\mathcal{R}_{A/R}$, whose reduction modulo t is an A -algebra. Moreover, this reduction is $\text{LSym}_A(L_{A/R}[-1])$ with $L_{A/R}[-1]$ on grade -1 . The formation of $\mathcal{R}_{A/R}$ is functorial with respect to $R \rightarrow A$ and commutes with base change in R . If $r \in R$ and $A = R/r$, then $\mathcal{R}_{A/R} = R[t, u]/(tu - r)$ with u on grade -1 , and the image of u under reduction modulo t is the generator $dr \in L_{(R/r)/R}[-1]$.*

Definition 2.28 (Blowup algebras).

- The *blowup algebra*, or the *Rees algebra*, associated to the ring map $R \rightarrow A$, is the graded $R[t]$ -algebra $\mathcal{R}_{A/R}$ appearing in Proposition 2.27.

- Let $r \in \text{fib}(R \rightarrow A)$. Then functoriality gives a map $R[t, u]/(tu - r) \rightarrow \mathcal{R}_{A/R}$. The *affine blowup algebra* of $R \rightarrow A$ with respect to r is the 0^{th} grade part of the base change of $\mathcal{R}_{A/R}$ along $R[t, u]/(tu - r) \rightarrow R[u^{\pm}]$.

Note that the Rees algebra here is usually called the extended Rees algebra and is different from the one in [Stacks, 052Q]. From now on till the end of the subsection, let $r \in \text{fib}(R \rightarrow A)$ and B be the affine blowup algebra of $R \rightarrow A$ with respect to r .

Remark 2.29. After inverting r , the maps $R/r \rightarrow A \rightarrow 0$ become isomorphisms. By functoriality, so do the maps $R[t, u]/(tu - r) \rightarrow \mathcal{R}_{A/R} \rightarrow R[t^{\pm}]$, and hence so does the map $R \rightarrow B$.

Proposition 2.30. *B is the initial R -algebra with a factorization $R/r \rightarrow A \rightarrow B/rB$ of the structure map modulo r .*

Proof. Recall that the category of R -algebras and that of graded $R[u^{\pm}]$ -algebras are equivalent. Therefore, for any R -algebra C ,

$$\text{Hom}_R(B, C) = \text{Hom}_{R[u^{\pm}]}(B[u^{\pm}], C[u^{\pm}]) = \text{Hom}_{R[t, u]/(tu - r)}(\mathcal{R}_{A/R}, C[u^{\pm}]).$$

By Proposition 2.27, the right hand side is the anima of A -algebra structures on the graded R/r -algebra $C[u^{\pm}] \otimes_{R[t]} R = (C/rC)[u^{\pm}]$, which is the same as the anima of A -algebra structures on the R/r -algebra C/rC . This proves the proposition. \square

Proposition 2.31. *Let d be a natural number. If $R/r \rightarrow A$ is $(d+1)$ -smooth, then so is $R \rightarrow B$, and moreover $A \rightarrow B/rB$ is d -smooth.*

Proof. By definition $R/r \rightarrow A$ is compact, so by Proposition 2.30 it is easy to see that $R \rightarrow B$ is also compact. Therefore, to show its $(d+1)$ -smoothness, it suffices to check $L_{B/R} \in \text{Perf}_{[0, d+1]}(B)$, which can be done after inverting r and reducing modulo r . Now $R[1/r] \rightarrow B[1/r]$ is an isomorphism by Remark 2.29, and $R/r \rightarrow B/rB$ factors through A , so it remains to show d -smoothness of $A \rightarrow B/rB$.

By definition, $B[u^{\pm}]$ is obtained by inverting $u \in \mathcal{R}_{A/R}$, where $ut = r$. Thus

$$(B/rB)[u^{\pm}] = \mathcal{R}_{A/R}[u^{-1}]/r = \mathcal{R}_{A/R}[u^{-1}]/t = \text{LSym}_A(L_{A/R}[-1])[u^{-1}],$$

where u goes to the image of $dr \in L_{(R/r)/R}[-1]$ in $L_{A/R}[-1]$. Therefore, to show that $A \rightarrow B/rB$ is d -smooth, it suffices to show the same for $A[u] \rightarrow \text{LSym}_A(L_{A/R}[-1])$. It is clearly compact, and by the cotangent fiber sequence of $A \rightarrow A[u] \rightarrow \text{LSym}_A(L_{A/R}[-1])$, its cotangent complex is easily seen to be

$$\begin{aligned} & \text{cofib}(L_{(R/r)/R}[-1] \otimes_{R/r} A \rightarrow L_{A/R}[-1]) \otimes_A \text{LSym}_A(L_{A/R}[-1]) \\ &= L_{A/(R/r)}[-1] \otimes_A \text{LSym}_A(L_{A/R}[-1]), \end{aligned}$$

which lies in $\text{Perf}_{[0, d]}(\text{LSym}_A(L_{A/R}[-1]))$ as $R/r \rightarrow A$ is $(d+1)$ -smooth. \square

3. PROOFS OF THE MAIN RESULTS

In this section we will prove our main theorems, Theorem 1.2 and Theorem 1.7, but before that let us address Corollary 1.4.

Proof of Corollary 1.4. Firstly, by [Stacks, 07BV], the map from \mathcal{O} to its completion is ind-smooth on fraction fields, so by Theorem 1.7 it is ind-smooth. Therefore, by base change, we can reduce to the case where \mathcal{O} is complete.

Then note that R has bounded π^{∞} -torsion: by [Stacks, 053G], the quotient of R by its π^{∞} -torsion part is of finite presentation, so the π^{∞} -torsion part is finitely generated, and hence is killed by some π^n for $n \in \mathbb{N}$ large enough.

Let Λ denote the classical π -completion of R . Then by the above, Λ is also the derived π -completion of R , so $R/\pi^n = \Lambda/\pi^n\Lambda$ holds derivedly for all $n \in \mathbb{N}$. By Theorem 1.7, it suffices to prove that $R[1/\pi] \rightarrow \Lambda[1/\pi]$ is ind-smooth. Now both rings are Noetherian and even excellent as explained in [Con99, §1.1], so by Theorem 1.1 it suffices to check regularity of this map, and by [Stacks, 07NT] it suffices to show that every maximal ideal of $\Lambda[1/\pi]$ gets pulled back to a maximal ideal in $R[1/\pi]$, and their completions coincide. This follows from [BGR84, §7.1.1]. \square

From now on, let d be a natural number and $R \rightarrow \bar{R}$ be a surjective ring map with fiber I . Whenever we have an R -algebra A and an A -module M , we use \bar{A} and \bar{M} to denote their base change to \bar{R} . In contrast, when we use \bar{B} to denote an \bar{R} -algebra, we do not mean that it can be lifted to an R -algebra B .

3.1. The lifting problem. This subsection is logically independent of the proof of 1.7 and aims at deducing Theorem 1.2 from Theorem 1.7. To this end, one needs to solve the lifting problem analogous to [Stacks, 07CJ]. Throughout the subsection, assume that $R \rightarrow \bar{R}$ is a square-zero extension and that the fiber $I = \text{fib}(R \rightarrow \bar{R})$ is d -connective. For any \bar{R} -algebra \bar{B} , denote by $I_{\bar{B}}$ the base change $I \otimes_{\bar{R}} \bar{B}$, no matter whether \bar{B} can be lifted to an R -algebra B .

Proposition 3.1 (cf. [Stacks, 07CM]). *Let Λ be an R -algebra with $\bar{\Lambda} = \Lambda \otimes_R \bar{R}$ ind- d -smooth over \bar{R} . Then Λ is ind- d -smooth over R .*

Proof. It suffices to prove that, for every factorization $R \rightarrow A \rightarrow \Lambda$ with A a compact R -algebra, one can find a further factorization $R \rightarrow A \rightarrow B \rightarrow \Lambda$ with a d -smooth R -algebra B . By assumption, over \bar{R} , there is a factorization $\bar{R} \rightarrow \bar{A} \rightarrow \bar{B} \rightarrow \bar{\Lambda}$ with \bar{B} a d -smooth \bar{R} -algebra. We want to find a lift of it to R after possibly enlarging \bar{B} . By deformation theory, this is a linear algebra problem. Let $L_{\bar{R}} \rightarrow I[1]$ classify the square-zero extension $R \rightarrow \bar{R}$, and similarly $L_{\bar{A}} \rightarrow I_A[1]$ and $L_{\bar{\Lambda}} \rightarrow I_{\Lambda}[1]$. They fit into the commutative diagram

$$\begin{array}{ccccccc} L_{\bar{R}} & \longrightarrow & L_{\bar{A}} & \longrightarrow & L_{\bar{B}} & \longrightarrow & L_{\bar{\Lambda}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ I[1] & \longrightarrow & I_A[1] & \longrightarrow & I_B[1] & \longrightarrow & I_{\Lambda}[1] \end{array}$$

and we want to find a dashed arrow making the squares commute, after possibly enlarging \bar{B} .

To this end, we first find a d -smooth R -algebra \bar{C} factoring $\bar{B} \rightarrow \bar{\Lambda}$ and a dashed arrow making

$$\begin{array}{ccccccc} L_{\bar{R}} & \longrightarrow & L_{\bar{A}} & \longrightarrow & L_{\bar{B}} & \longrightarrow & L_{\bar{\Lambda}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ I[1] & \longrightarrow & I_A[1] & \longrightarrow & I_C[1] & \longrightarrow & I_{\Lambda}[1] \end{array}$$

commute. Such a dashed arrow amounts to a nullhomotopy of the \bar{B} -linear map

$$L_{\bar{B}/\bar{A}}[-1] \rightarrow L_{\bar{A}} \otimes_{\bar{A}} \bar{B} \rightarrow I_A[1] \otimes_{\bar{A}} \bar{B} = I_B[1] \rightarrow I_C[1]$$

which, after composing with $I_C[1] \rightarrow I_{\Lambda}[1]$, coincides with the nullhomotopy given by the diagram. Remember that since $\bar{R} \rightarrow \bar{\Lambda}$ is ind- d -smooth, the category of such \bar{C} is filtered with colimit $\bar{\Lambda}$, so the desired nullhomotopy follows from the fact that $L_{\bar{B}/\bar{A}}[-1] = \text{fib}(L_{\bar{A}/\bar{R}} \otimes_{\bar{A}} \bar{B} \rightarrow L_{\bar{B}/\bar{R}})$ is a perfect \bar{B} -complex.

Now it remains to find a dashed arrow making

$$\begin{array}{ccccccc} L_{\bar{R}} & \longrightarrow & L_{\bar{B}} & \longrightarrow & L_{\bar{C}} & \longrightarrow & L_{\bar{\Lambda}} \\ \downarrow & & & \searrow & \downarrow & & \downarrow \\ I[1] & \longrightarrow & I_C[1] & \longrightarrow & I_{\Lambda}[1] & & \end{array}$$

commute, after possibly enlarging \bar{C} . Similarly, such a dashed arrow amounts to a nullhomotopy of the \bar{C} -linear map

$$L_{\bar{C}/\bar{B}}[-1] \rightarrow L_{\bar{B}} \otimes_{\bar{B}} \bar{C} \rightarrow I_C[1]$$

which, after composing with $I_C[1] \rightarrow I_{\Lambda}[1]$, coincides with the nullhomotopy given by the diagram. Since $L_{\bar{C}/\bar{B}}[-1] = \text{fib}(L_{\bar{B}/\bar{R}} \otimes_{\bar{B}} \bar{C} \rightarrow L_{\bar{C}/\bar{R}}) \in \text{Perf}_{[-1,d]}(\bar{C})$ while $I_C[1]$ is $(d+1)$ -connective, the above map is indeed zero. Choose a nullhomotopy of it, corresponding to a dashed arrow $f: L_{\bar{C}} \rightarrow I_C[1]$ making the middle triangle commute, defining a lift C of \bar{C} . This nullhomotopy, after composing with $I_C[1] \rightarrow I_{\Lambda}[1]$, may not coincide with the preferred nullhomotopy; however, it does after composing with $L_{\bar{C}/\bar{R}}[-1] \rightarrow L_{\bar{C}/\bar{B}}[-1]$, because $\text{Map}_{\bar{C}}(L_{\bar{C}/\bar{R}}[-1], I_{\Lambda}[1])$ is now 2-connective.

The above analysis implies that, ignoring $L_{\bar{B}}$, we can make the diagram commute; in other words, we can make the right square commute, but the composition of the commutations of the square and the triangle may not coincide with the original commutation of the right trapezoid, though it does after composing with the left trapezoid. Now the difference between the two commutations constitutes a loop in $\text{Map}_{\bar{B}}(L_{\bar{B}}, I_{\Lambda}[1])$ which becomes trivial in $\text{Map}_{\bar{R}}(L_{\bar{R}}, I_{\Lambda}[1])$, so it is actually a loop in $\text{Map}_{\bar{B}}(L_{\bar{B}/\bar{R}}, I_{\Lambda}[1])$, or equivalently a point in $\text{Map}_{\bar{B}}(L_{\bar{B}/\bar{R}}, I_{\Lambda})$. Note that $L_{\bar{B}/\bar{R}} \in \text{Perf}_{[0,d]}(\bar{B})$, so any point in

$$\text{Map}_{\bar{B}}(L_{\bar{B}/\bar{R}}, I_{\Lambda}) = L_{\bar{B}/\bar{R}}^{\vee}[d] \otimes_{\bar{B}} I_{\Lambda}[-d] = L_{\bar{B}/\bar{R}}^{\vee}[d] \otimes_{\bar{B}} \bar{\Lambda} \otimes_{\bar{R}} I[-d]$$

is represented by a finite sum of elementary tensors up to a homotopy, as all the tensor factors are connective. Therefore, we can find a finitely generated polynomial C -algebra D , a map $D \rightarrow \Lambda$, and a lift of the above point to $\text{Map}_{\bar{B}}(L_{\bar{B}/\bar{R}}, I_D)$. Now D corresponds to an arrow $g: L_{\bar{D}} \rightarrow I_D[1]$ inserting into the commutative square on the right, with the difference between the two commutations of the right trapezoid lifted to a loop $h \in \Omega \text{Map}_{\bar{B}}(L_{\bar{B}/\bar{R}}, I_D[1])$. Hence we can just use the map g as follows, with the commutation of the middle trapezoid modified by h , to conclude.

$$\begin{array}{ccccccccc} L_{\bar{R}} & \longrightarrow & L_{\bar{B}} & \longrightarrow & L_{\bar{C}} & \longrightarrow & L_{\bar{D}} & \longrightarrow & L_{\bar{\Lambda}} \\ \downarrow & & & \searrow & \downarrow f & & \downarrow g & & \downarrow \\ I[1] & \longrightarrow & I_C[1] & \longrightarrow & I_D[1] & \longrightarrow & I_{\Lambda}[1] & & \end{array}$$

□

The following corollary is obvious, noticing that when $d = 0$ the connectivity condition on I is vacuous.

Corollary 3.2. *The following hold:*

- (1) *If R has only finitely many nonzero homotopy groups, then an R -algebra is ind- d -smooth if its base change to $\tau_{\leq \max\{0, d-1\}} R$ is ind- d -smooth.*

- (2) For $n \in \mathbb{N}$ and $r \in R$, an R/r^n -algebra is ind-smooth if its base change to R/r is ind-smooth.
- (3) For $n \in \mathbb{N}$ and $r \in R$, if R is a classical ring, then an R/r -algebra is ind-smooth if its base change to $\pi_0(R/r)$ is ind-smooth.
- (4) Theorem 1.7 implies Theorem 1.2.

3.2. The lifting lemma. This subsection deals with the following lemma. Here, thanks to animated rings, one no longer needs to be fiendishly clever as in [Stacks, 07CN]. Throughout the subsection, fix $r \in R$ and set $\bar{R} = R/r$.

Lemma 3.3 (cf. [Stacks, 07CP]). *Let Λ be an R -algebra. Let \bar{C} be a $(d+1)$ -smooth \bar{R} -algebra with a factorization $\bar{R} \rightarrow \bar{C} \rightarrow \bar{\Lambda}$. Then there is an R -algebra D and a commutative diagram*

$$\begin{array}{ccccc} R & \longrightarrow & D & \longrightarrow & \Lambda \\ \downarrow & & \downarrow & & \downarrow \\ \bar{R} & \longrightarrow & \bar{C} & \longrightarrow & \bar{\Lambda} \end{array}$$

satisfying that $D[1/r]$ is smooth over $R[1/r]$ and that \bar{D} is d -smooth over \bar{C} . In fact, if $\bar{R} \rightarrow \bar{C}$ is surjective, there is an initial R -algebra D with a factorization $\bar{R} \rightarrow \bar{C} \rightarrow \bar{D}$; this D satisfies the above conditions with $D[1/r] = R[1/r]$.

Proof. Replacing R by a finitely generated polynomial R -algebra, we can assume that $\bar{R} \rightarrow \bar{C}$ is surjective. In this case everything reduces to the final claim, which follows from Remark 2.29 and Propositions 2.30 and 2.31. \square

3.3. The desingularization lemma. This subsection deals with the desingularization lemma as in [Stacks, 07CQ]. Here animated rings do not help that much; one still needs the fiendishly clever construction given there. Throughout the subsection, fix $r \in R$ and set $\bar{R} = R/r$.

Lemma 3.4. *Let M be an R -module with a map $f: M \rightarrow R$ and an R -algebra map $R//f \rightarrow \bar{R}$. Then there is a natural $g \in \text{Map}_R(M, R)$ with $f = rg$, and $R//g$ is the initial R -algebra Λ with a commutative diagram*

$$\begin{array}{ccccc} R & \longrightarrow & R//f & \longrightarrow & \Lambda \\ \downarrow & \swarrow & & & \downarrow \\ \bar{R} & \longrightarrow & & \longrightarrow & \bar{\Lambda} \end{array}$$

Proof. By Remark 2.13, an R -algebra map $R//f \rightarrow \bar{R}$ is a nullhomotopy of the composition $M \rightarrow R \rightarrow \bar{R}$, which is the same as a lift of f along $r: R \rightarrow \bar{R}$, giving the g in the statement. By the same reason, for a fixed Λ , such a commutative diagram is a nullhomotopy z of the composition $M \rightarrow R \rightarrow \Lambda$ whose further composition to $M \rightarrow R \rightarrow \bar{\Lambda}$ is the given nullhomotopy, which is the same thing as a lift of z along $r: \Lambda \rightarrow \bar{\Lambda}$. This is by definition represented by $R//g$. \square

Lemma 3.5. *For any $n \in \mathbb{N}$, the initial R -algebra Λ with a commutative diagram*

$$\begin{array}{ccccc} R & \longrightarrow & R[x_1, \dots, x_n] & \longrightarrow & \Lambda \\ \downarrow & \swarrow & & & \downarrow \\ \bar{R} & \longrightarrow & & \longrightarrow & \bar{\Lambda} \end{array} \quad \begin{array}{c} (0, \dots, 0) \end{array}$$

is given by $\Lambda = R[y_1, \dots, y_n]$, with the upper right arrow sending x_i to ry_i .

Proof. Omitted. It is similar to and simpler than Lemma 3.4. \square

Lemma 3.6 (cf. [Stacks, 07CR]). *Let A and Λ be R -algebras with a commutative diagram*

$$\begin{array}{ccccc} R & \longrightarrow & A & \longrightarrow & \Lambda \\ \downarrow & \swarrow & & & \downarrow \\ \bar{R} & \longrightarrow & & & \bar{\Lambda} \end{array}$$

Let $p \in R$ with $p^4 = r$. Suppose that $R \rightarrow A$ is 1-smooth, that $[L_{A/R}] = [A^{\oplus k}] \in K_0(A)$ for some $k \in \mathbb{N}$, and that there are maps $s: A^{\oplus k} \rightarrow L_{A/R}$ and $t: L_{A/R} \rightarrow A^{\oplus k}$ with $s \circ t = p \in \text{Map}_A(L_{A/R}, L_{A/R})$. Then there is a smooth R -algebra B factorizing $A \rightarrow \Lambda$, with $B[1/r]$ smooth over $A[1/r]$.

Proof. By assumption, we can take a surjection $P = R[x_1, \dots, x_n] \rightarrow A$ with $L_{A/P}[-1]$ finite free of rank $m = n - k$. Translating, we can assume that every x_i is mapped to 0 under $A \rightarrow \bar{R}$. By Theorem 2.24, there is a map $f: P^{\oplus m} \rightarrow P$ with $A = P//f$. Combining Lemmas 3.4 and 3.5, we see that the initial such commutative diagram is given by

$$\begin{array}{ccccc} R & \longrightarrow & A = P//f & \longrightarrow & \Lambda = Q//g \\ \downarrow & \swarrow & & & \downarrow \\ \bar{R} & \longrightarrow & & & \bar{\Lambda} \end{array}$$

where $Q = R[y_1, \dots, y_n]$, the map $P \rightarrow Q$ sends x_i to ry_i , and $g: Q^{\oplus m} \rightarrow Q$ satisfies $rg = f \otimes_P Q$, or informally $g(y) = f(ry)/r$. It suffices to prove the lemma for this diagram. Now write

$$(1) \quad g = g(0) + y \cdot a + (\text{higher order terms}),$$

where $g(0): R^{\oplus m} \rightarrow R$ and $a: R^{\oplus m} \rightarrow R^{\oplus n}$ are certain matrices; then

$$(2) \quad f = rg(0) + x \cdot a + (\text{higher order terms}).$$

Note that $L_{A/R}$ can be presented as

$$\text{cofib}(P^{\oplus m} \otimes_P A \rightarrow L_{P/R} \otimes_P A) = \text{cofib}(A^{\oplus m} \rightarrow A^{\oplus n}),$$

with the map given by $\partial f / \partial x$. Thus we can lift the map $s: A^{\oplus k} \rightarrow L_{A/R}$ to a matrix $s: A^{\oplus k} \rightarrow A^{\oplus n}$, view the map $t: L_{A/R} \rightarrow A^{\oplus k}$ as a matrix $t: A^{\oplus n} \rightarrow A^{\oplus k}$ with a nullhomotopy of $t \circ \partial f / \partial x$, and finally write the homotopy $s \circ t = p$ as a matrix $h: A^{\oplus n} \rightarrow A^{\oplus m}$ with homotopies $h \circ \partial f / \partial x = p$ and $\partial f / \partial x \circ h = s \circ t - p$. Lift h to a matrix $h: P^{\oplus n} \rightarrow P^{\oplus m}$ and consider its constant term $h(0): R^{\oplus n} \rightarrow R^{\oplus m}$. Recall that the constant term of $\partial f / \partial x$ is $a: R^{\oplus m} \rightarrow R^{\oplus n}$ as in (2). We apply the map $A \rightarrow \bar{R}$ on the homotopy $h \circ \partial f / \partial x = p$; in view of the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{(0, \dots, 0)} & R \\ \downarrow & & \downarrow \\ P//f = A & \longrightarrow & \bar{R} \end{array}$$

this gives a homotopy in $\text{Map}_R(R^{\oplus m}, R^{\oplus m})$ between $h(0) \circ a$ and p after base change to \bar{R} , or equivalently a point $q \in \text{Map}_R(R^{\oplus m}, R^{\oplus m})$ with

$$(3) \quad h(0) \circ a = p + rq = p(1 + p^3 q).$$

Now consider the polynomial ring $S = R[v_1, \dots, v_m, w_1, \dots, w_n]$ and the ring map $P \rightarrow S$ given by $x \mapsto p^2(v \cdot h(0) + pw)$. Then by (3), the base change of $f: P^{\oplus m} \rightarrow P$ along $P \rightarrow S$ is of the form

$$p^3(v + w \cdot a) + p^4(\text{other terms});$$

this amounts to some $c: S^{\oplus m} \rightarrow S$ that becomes $v + w \cdot a$ after base change to S/pS and satisfies $p^3c = f \otimes_P S$. Let $B = S//c$. It clearly receives a map from $A = P//f$. To define the map $B \rightarrow \Lambda$, consider the map $S \rightarrow Q$ taking all the v_j to 0 and w_i to py_i . By construction, the base change of c along $S \rightarrow Q$ is nothing but pg , and the identifications $pg = c \otimes_S Q$ and $p^3c = f \otimes_P S$ compose to the original identification $p^4g = f \otimes_P Q$. This gives the desired map $B = S//c \rightarrow \Lambda//g = \Lambda$ as well as the desired factorization $A \rightarrow B \rightarrow \Lambda$.

We now verify the claimed properties of B . Since $p^3c = f \otimes_P S$, the map $A[1/p] \rightarrow B[1/p]$ is a base change of the map $P[1/p] \rightarrow S[1/p]$ given by $x \mapsto p^2(v \cdot h(0) + pw)$, which is clearly smooth: in fact $S[1/p] = P[1/p][v_1, \dots, v_m]$ is polynomial, as we can write $w = p^{-1}(p^{-2}x - v \cdot h(0))$. By assumption, $L_{A[1/p]/R[1/p]} \cong A[1/p]^{\oplus k}$, so $A[1/p]$ is smooth over $R[1/p]$, and hence so is $B[1/p]$. Thus it remains to verify that B/pB is smooth over R/p . This is also clear: recall that $c: S^{\oplus m} \rightarrow S$ reduces to $v + w \cdot a$ over S/pS , so $B/pB = (S/pS)//(v + w \cdot a) = R/p[v_1, \dots, v_m, w_1, \dots, w_n]//(v + w \cdot a) = R/p[w_1, \dots, w_n]$ is in fact polynomial. \square

Lemma 3.7 (cf. [Stacks, 07F0]). *Let A and Λ be R -algebras and \bar{C} be an \bar{R} -algebra, with a commutative diagram*

$$\begin{array}{ccccccc} R & \longrightarrow & A & \longrightarrow & \Lambda \\ \downarrow & & \downarrow & & \downarrow \\ \bar{R} & \longrightarrow & \bar{A} & \longrightarrow & \bar{C} & \longrightarrow & \bar{\Lambda} \end{array}$$

Let $p \in R$ with $p^4 = r$. Suppose that $\bar{R} \rightarrow \bar{C}$ is 1-smooth, and that $R \rightarrow A$ satisfies the assumptions in Lemma 3.6. Then there is an R -algebra B factorizing $A \rightarrow \Lambda$ with $B[1/r]$ smooth over $A[1/r]$, such that \bar{B} factorizes $\bar{C} \rightarrow \bar{\Lambda}$ and is smooth over \bar{C} . In particular, if $\bar{R} \rightarrow \bar{C}$ is smooth, then so is $R \rightarrow B$.

Proof. Apply Lemma 3.3 to obtain a commutative diagram

$$\begin{array}{ccccccc} R & \longrightarrow & D & \longrightarrow & \Lambda \\ \downarrow & & \downarrow & & \downarrow \\ \bar{R} & \longrightarrow & \bar{A} & \longrightarrow & \bar{C} & \longrightarrow & \bar{D} & \longrightarrow & \bar{\Lambda} \end{array}$$

where $D[1/r]$ is smooth over $R[1/r]$ and \bar{D} is smooth over \bar{C} . Using the map $\bar{A} \rightarrow \bar{D}$, we obtain another diagram

$$\begin{array}{ccccc} D & \longrightarrow & A \otimes_R D & \longrightarrow & \Lambda \\ \downarrow & \swarrow & & & \downarrow \\ \bar{D} & \longrightarrow & & & \bar{\Lambda} \end{array}$$

that falls into the assumptions of Lemma 3.6 by base change. Hence there is a smooth D -algebra B factorizing $A \otimes_R D \rightarrow \Lambda$, such that $B[1/r]$ is smooth over $(A \otimes_R D)[1/r]$. It is easy to see that this B fulfills our requirements. \square

We also need higher analogs of the above lemmas.

Lemma 3.8. *Let A and Λ be R -algebras with a commutative diagram*

$$\begin{array}{ccccc} R & \longrightarrow & A & \longrightarrow & \Lambda \\ \downarrow & \swarrow & & & \downarrow \\ \bar{R} & \longrightarrow & & & \bar{\Lambda} \end{array}$$

Suppose that $R \rightarrow A$ is d -connective and $(d+2)$ -smooth, and that there is some $N \in \mathbf{cProj}(R)$ and maps $s: N[d+1] \otimes_R A \rightarrow L_{A/R}$ and $t: L_{A/R} \rightarrow N[d+1] \otimes_R A$ that are isomorphisms over $A[1/r]$, such that $s \circ t = r \in \text{Map}_A(L_{A/R}, L_{A/R})$. If $d = 0$, assume that the K_0 class of $L_{A/R}$ can be lifted to $K_0(R)$. Then there is a $(d+1)$ -smooth R -algebra B factorizing $A \rightarrow \Lambda$ with $B[1/r] = A[1/r]$.

Proof. By Theorem 2.26, there is an $M \in \text{Perf}_{[0, 1+\min\{d, 1\}]}(R)$ and an $f: M[d] \rightarrow R$ such that $A = R//M[d]$. By Proposition 2.16, $L_{A/R} = M[d+1] \otimes_R A$, so in fact $M \in \text{Perf}_{[0, 1]}(R)$. Lift $s[-d-1]$ to a map $N \rightarrow M$, which is possible because $N \in \mathbf{cProj}(R)$ and M is connective. This gives a factorization $R \rightarrow R//N[d] \rightarrow R//M[d]$; by Corollary 2.17, we can replace R by $R//N[d]$ and then the lemma is reduced to the case $N = 0$, namely $r = 0 \in \text{Map}_A(L_{A/R}, L_{A/R})$. Now $L_{A[1/r]/R[1/r]} = 0$, so after replacing R by a Zariski localization, by Corollary 2.21 we can assume that $A[1/r] = R[1/r]$.

By Lemma 3.4, we see that the initial such commutative diagram is given by

$$\begin{array}{ccccc} R & \longrightarrow & A = R//f & \longrightarrow & \Lambda = R//g \\ \downarrow & \swarrow & & & \downarrow \\ \bar{R} & \longrightarrow & & & \bar{\Lambda} \end{array}$$

where $g: M[d] \rightarrow R$ satisfies $rg = f$. By the above reduction, we know that $r: M \otimes_R A \rightarrow M \otimes_R A$ is zero, namely that $M \otimes_R A \in \text{Perf}(A)$ has the structure of an \bar{A} -module. Base change it along the map $\bar{A} \rightarrow \bar{R}$ and consider the R -module map $M \rightarrow (M \otimes_R A) \otimes_{\bar{A}} \bar{R}$. Note that it is an isomorphism after base change along either $R \rightarrow \bar{R}$ or $R \rightarrow R[1/r] = A[1/r]$; this implies that it is an isomorphism. Therefore M acquires an \bar{R} -module structure, so $r = 0 \in \text{Map}_R(M, M)$, and hence the map $A \rightarrow \Lambda$ factors through R since $rg = f$. We just take $B = R$: trivially R is $(d+1)$ -smooth over itself, and we have seen that $R[1/r] = A[1/r]$. \square

Lemma 3.9. *Let A and Λ be R -algebras and \bar{C} be an \bar{R} -algebra, with a commutative diagram*

$$\begin{array}{ccccccc} R & \longrightarrow & A & \longrightarrow & \Lambda \\ \downarrow & & \downarrow & & \downarrow \\ \bar{R} & \longrightarrow & \bar{A} & \longrightarrow & \bar{C} & \longrightarrow & \bar{\Lambda} \end{array}$$

Suppose that $\bar{R} \rightarrow \bar{C}$ is $(d+2)$ -smooth, and that $R \rightarrow A$ satisfies the assumptions in Lemma 3.8. Then there is an R -algebra B factorizing $A \rightarrow \Lambda$ with $B[1/r] = A[1/r]$, such that \bar{B} factorizes $\bar{C} \rightarrow \bar{\Lambda}$ and is $(d+1)$ -smooth over \bar{C} . In particular, if $\bar{R} \rightarrow \bar{C}$ is $(d+1)$ -smooth, then so is $R \rightarrow B$.

Proof. Apply Lemma 3.3 to obtain a commutative diagram

$$\begin{array}{ccccccc} R & \xrightarrow{\quad\quad\quad} & D & \longrightarrow & \Lambda \\ \downarrow & & \downarrow & & \downarrow \\ \bar{R} & \longrightarrow & \bar{A} & \longrightarrow & \bar{C} & \longrightarrow & \bar{D} & \longrightarrow & \bar{\Lambda} \end{array}$$

where $D[1/r] = R[1/r]$ and \bar{D} is $(d+1)$ -smooth over \bar{C} . Using the map $\bar{A} \rightarrow \bar{D}$, we obtain another diagram

$$\begin{array}{ccccc} D & \longrightarrow & A \otimes_R D & \longrightarrow & \Lambda \\ \downarrow & \swarrow & & & \downarrow \\ \bar{D} & \xrightarrow{\quad\quad\quad} & & & \bar{\Lambda} \end{array}$$

that falls into the assumptions of Lemma 3.8 by base change. Hence there is a $(d+1)$ -smooth D -algebra B factorizing $A \otimes_R D \rightarrow \Lambda$ with $B[1/r] = (A \otimes_R D)[1/r]$. It is easy to see that this B fulfills our requirements. \square

3.4. Proof of the main theorem. With all the preparations done, we now prove Theorem 1.7. Let notations be the same as there. To show that Λ is ind- d -smooth over R , it suffices to show that any R -homomorphism $A \rightarrow \Lambda$ from any compact R -algebra A factors through some d -smooth R -algebra B .

Since $S^{-1}\Lambda$ is ind- d -smooth over $S^{-1}R$, there is a d -smooth $S^{-1}R$ -algebra C' factorizing $S^{-1}A \rightarrow S^{-1}\Lambda$. As both $S^{-1}A$ and C' are compact $S^{-1}R$ -algebras, the map $S^{-1}A \rightarrow C'$ is compact. By Lemma 2.22 and Corollary 2.25, there is a compact A -algebra D , such that $S^{-1}A \rightarrow S^{-1}D$ factors through C' with $C' \rightarrow S^{-1}D$ smooth admitting a retract. Using this retract we see that $S^{-1}A \rightarrow S^{-1}\Lambda$ factors through $S^{-1}D$. Replacing A by D , we arrive at the case where $S^{-1}R \rightarrow S^{-1}A$ is d -smooth.

As $R \rightarrow A$ is compact, there is a $d' \in \mathbb{N}$ such that it is d' -smooth. If $d' \leq d$ then we are already done; if $d' > d$, by induction we only need to prove that $A \rightarrow \Lambda$ factors through some $(d' - 1)$ -smooth R -algebra B with $S^{-1}B$ smooth over $S^{-1}A$, as then $S^{-1}B$ will remain d -smooth over $S^{-1}R$. Replacing d by $d' - 1$, we can assume that A is $(d+1)$ -smooth over R . Use Lemma 2.22 again we can assume that $L_{A/R}$ is stably free, and that $L_{S^{-1}A/S^{-1}R}$ is free in case $d = 0$.

If $d > 0$ and we have a factorization $R \rightarrow R' \rightarrow A$ where $R \rightarrow R'$ is $(d-1)$ -smooth, then we can replace $R \rightarrow A \rightarrow \Lambda$ by $R' \rightarrow A \rightarrow \Lambda$ and S by its image in R' , and by Remark 2.19 both the assumptions in the theorem and those created along the reductions above remain true. Therefore, we can first take R' to be a polynomial R -algebra, and then apply Theorem 2.24, to arrive at a factorization such that $R' \rightarrow A$ is $(d-1)$ -connective. In other words, when $d > 0$, we can further assume that $R \rightarrow A$ is $(d-1)$ -connective. Note that along the way we can let $L_{A/R}$ remain stably free and make $L_{S^{-1}A/S^{-1}R}[-d]$ free.

At this point, we can take a finite free A -module N with $S^{-1}N[d] = S^{-1}L_{A/R}$, and eliminate denominators to get an element $p \in S$ and maps $s: N[d] \rightarrow L_{A/R}$ and $t: L_{A/R} \rightarrow N[d]$ with $s \circ t = p \in \text{Map}_A(L_{A/R}, L_{A/R})$. Let $r = p^4$ and set $\bar{R} = R/r$. By assumption $\bar{R} \rightarrow \bar{\Lambda}$ is ind- d -smooth, so there is a d -smooth \bar{R} -algebra \bar{C} with a factorization $\bar{A} \rightarrow \bar{C} \rightarrow \bar{\Lambda}$. Now we are in the situation of either Lemma 3.7 or Lemma 3.9 where the “in particular” assumption holds, so we win. \square

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