Abstracting Effect Systems for Algebraic Effect Handlers

TAKUMA YOSHIOKA, Kyoto University, Japan

TARO SEKIYAMA, National Institute of Informatics & SOKENDAI, Japan

ATSUSHI IGARASHI, Kyoto University, Japan

Many effect systems for algebraic effect handlers are designed to guarantee that all invoked effects are handled adequately. However, respective researchers have developed their own effect systems that differ in how to represent the collections of effects that may happen. This situation results in blurring what is required for the representation and manipulation of effect collections in a safe effect system.

In this work, we present a language λ_{EA} equipped with an effect system that abstracts the existing effect systems for algebraic effect handlers. The effect system of λ_{EA} is parameterized over *effect algebras*, which abstract the representation and manipulation of effect collections in safe effect systems. We prove the type-and-effect safety of λ_{EA} by assuming that a given effect algebra meets certain properties called *safety conditions*. As a result, we can obtain the safety properties of a concrete effect system by proving that an effect algebra corresponding to the concrete system meets the safety conditions. We also show that effect algebras meeting the safety conditions are expressive enough to accommodate some existing effect systems, each of which represents effect collections in a different style. Our framework can also differentiate the safety aspects of the effect collections of the existing effect systems. To this end, we extend λ_{EA} and the safety conditions to *lift coercions* and *type-erasure semantics*, propose other effect algebras including ones for which no effect system has been studied in the literature, and compare which effect algebra is safe and which is not for the extensions.

1 INTRODUCTION

1.1 Background: Effect Systems for Algebraic Effect Handlers

Algebraic effect handlers [Plotkin and Pretnar 2009, 2013] enable implementing user-defined computational effects, such as mutable states, exceptions, backtracking, and generators, and structuring programs with them in a modular way. A significant aspect of algebraic effect handlers is compositionality. Because of the algebraicity inherited from algebraic effects [Kammar et al. 2013; Plotkin and Power 2003], they allow composing multiple effects easily, unlike some other approaches to user-defined effects, such as monads [Moggi 1991; Wadler 1998]. Another benefit of algebraic effect handlers is to separate the interfaces and implementations of effects. For example, the manipulation of mutable states is expressed by two operations to set a new state and get the current state. While a program manipulates states via these operations, their implementation can be determined dynamically by installing *effect handlers*. This separation of interfaces from implementations allows writing effectful programs in a modular manner.

A key property expected in a statically typed language with algebraic effect handlers is *type-and-effect safety*. In the presence of effect handlers, type safety ensures that the type of an operation is matched with that of its implementation provided by an effect handler. *Effect safety* [Brachthäuser et al. 2020][1] states that every operation call is handled appropriately (i.e., it is performed under an effect handler that provides the called operation with an implementation). Ensuring effect safety

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[1]The notion of effect safety itself and its importance have been recognized before the name was coined [Kammar et al. 2013].

Authors' addresses: Takuma Yoshioka, Kyoto University, Japan, yoshioka@fos.kuis.kyoto-u.ac.jp; Taro Sekiyama, National Institute of Informatics & SOKENDAI, Japan, sekiyama@nii.ac.jp; Atsushi Igarashi, Kyoto University, Japan, igarashi@kuis.kyoto-u.ac.jp.

is crucial to guarantee the safety of programs as an "unhandled operation" makes programs get stuck.

Several researchers have proposed type-and-effect systems (effect systems for short) to guarantee type-and-effect safety. The effect systems in the literature are classified roughly into two groups according to how they represent collections of effects that programs may invoke. Certain effect systems adapt *sets* to represent such collections [Bauer and Pretnar 2013; Forster et al. 2017; Kammar et al. 2013; Kammar and Pretnar 2017; Saleh et al. 2018; Sekiyama et al. 2020]. Another approach is using *rows* [Biernacki et al. 2019; Hillerström and Lindley 2016; Leijen 2017; Xie et al. 2022], which allow manipulating the collections of effects in a more structured manner. For example, the effect system of Hillerström and Lindley [2016] can represent the presence and absence of effects in rows, and that of Leijen [2017] allows the duplication of effects with the same name in one row.

However, several issues are posed by the current situation that the effect systems in the different styles have been studied independently. First, it blurs what manipulation of effect collections are indispensable to give an effect system. Second, it is unclear what property an effect system requires for effect collections and their manipulation to guarantee effect safety. These unclarities cause the problem that designers of new effect systems grope in the dark for the representations of effect collections, and even if they come up with an appropriate representation, they need to prove the desired properties, such as effect safety, from scratch. The third issue is that, when extending languages with new features, one needs to build the metatheory for each of the representations.

1.2 Our Work

This work aims to reveal the essence of safe effect systems for effect handlers. Because we are interested in the shared nature of such effect systems, we avoid choosing one concrete representation of effect collections. Instead, we provide an effect system that abstracts over the representations of effect collections and can derive concrete effect systems by instantiating them.

More specifically, our effect system is parameterized over constructors and manipulations of effect collections. In general, effect systems for algebraic effect handlers require two kinds of manipulation. One is subeffecting, which overapproximates effects to adjust the effects of different expressions. The other is the removal of effects. An installed effect handler removes the effect it handles and forwards the remaining effects to outer effect handlers. We formulate such manipulation of effect collections required by effect systems as *effect algebras*[2] and ensure that our effect system relies only on the manipulations allowed on them.

However, some effect algebras make the effect system unsound. For instance, the effect system with an effect algebra that allows subeffecting to remove some effects may typecheck unsafe programs (e.g., ones that cause unhandled effects). To prevent the use of such effect algebras, we formalize *safety conditions*, which are sufficient conditions on effect algebras to guarantee effect safety; we call effect algebras meeting the conditions *safe*. We prove that the effect system instantiated with any safe effect algebra enjoys effect safety as well as type safety—therefore, ones can ensure the safety of their effect systems only by showing the safety of the corresponding effect algebras. Furthermore, we also show what kind of unsafe programs each condition excludes.

To show that our framework is expressive enough to capture the essence shared among sound effect systems in the literature, we provide three instances of our effect system. The instances represent effect collections by sets and two styles of rows—called *simple rows* [Hillerström and Lindley 2016] and *scoped rows* [Leijen 2017]. We define effect algebras for these three instances and prove

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[2]The name "effect algebra" has been used to specify algebraic structures found in quantum mechanics [Foulis and Bennett 1994], but we decided to use this name because the present work is far from quantum mechanics.

their safety, which means that all the instances satisfy type-and-effect safety. We also show that these instances indeed model the existing effect systems [Hillerström and Lindley 2016; Leijen 2017; Pretnar 2015].

Once it turns out that all the instances satisfy the desired property, what are differences among them? How can they be compared? To answer these questions, we make two changes on the language: introduction of *lift coercions* [Biernacki et al. 2018, 2019] and employment of a *type-erasure semantics* [Biernacki et al. 2019].

Lift coercions are a construct to prevent an operation call from being handled by the closest effect handler, introduced to avoid *accidental handling*, that is, unintended handling of operation calls. To reason about the effect of lift coercions soundly, effect collections should be able to express how many effect handlers need to be installed on effectful computation. Effect collections represented by sets or simple rows cannot express it because they collapse multiple occurrences of the same effect into one. Thus, the instances with them result in being unsound. By contrast, scoped rows can encode the number of necessary effect handlers due to the ability to duplicate effects. To enhance the importance of being able to represent the number of necessary effect handlers in the presence of lift coercions, we propose a new instance where effect collections are represented by *multisets*. Because multisets record the multiplicities of the elements they contain, it is expected that the instance with multisets, as well as that with scoped rows, satisfies type-and-effect safety even in the presence of lift coercions. We show that it is the case by providing an additional safety condition for lift coercions, proving that any instance of the effect system enjoys type-and-effect safety if it meets the new safety condition as well as the original ones, and showing that the effect algebra for scoped rows and the one for multisets meet both the additional and original safety conditions.

The second change is to adopt a type-erasure semantics, which differs from the original semantics in the effect comparison in the dynamic search for effect handlers: the original semantics takes into account what type parameters effects accompany to identify effects, while the type-erasure semantics does not. This nature of type-erasure semantics makes the instances with sets and multisets unsound because it is in conflict with the nature of sets and multisets that the order of elements is ignored. The row-based instances can be adapted to the type-erasure semantics by restricting the commutativity in rows. Even for sets and multisets, we can give type-and-effect safe instances based on them if we admit restriction on swapping elements.

The contributions of this work are summarized as follows.
- We introduce an abstract effect system for algebraic effect handlers. It abstracts over effect algebras, which characterize the representation and manipulation of effect collections in the effect system.
- We define safety conditions that enforce the effect manipulation allowed by effect algebras to be safe.
- We prove that effect systems instantiated by safe effect algebras are type-and-effect safe.
- We extend the effect system to lift coercions and type-erasure semantics, define an additional safety condition for each of them, and prove that an instance of each extension is type-and-effect safe provided that the effect algebra in the instance meets the specified conditions.
- We give four examples of safe effect algebras and their variants for the type-erasure semantics.

The effect system presented in this paper supposes *deep* effect handlers, but we also have adapted the system to *shallow* effect handlers [Kammar et al. 2013]; readers interested in the formulation for shallow effect handlers are referred to the supplementary material.

The rest of this paper is organized as follows. Section 2 reviews algebraic effect handlers and the existing effect systems, and overviews our approach. Section 3 introduces our type-and-effect

language and effect algebras. We also show the instances based on sets and rows as their examples. Section 4 presents our calculus with the abstract effect system. Section 5 states safety conditions, explains their necessities, and proves the type-and-effect safety of the calculus under the safe conditions. Section 6 shows that some existing effect systems can be modeled soundly by the corresponding instances of our calculus. Section 7 extends our language and the safety conditions to lift coercions and type-erasure semantics and Section 8 compares the effect algebras given in the paper. Section 9 describes additional related works and Section 10 concludes this paper with future works. This paper only states certain key properties. All the auxiliary lemmas, proofs, and full definition are given in the supplementary material.

2 OVERVIEW

This section reviews algebraic effect handlers and the existing effect systems for them, and provides an overview of our approach to abstracting the effect systems.

2.1 Review: Algebraic Effects and Handlers

Algebraic effect handlers are a means to implement user-defined effects in a modular way. The interface of effects consists of operations, and their behavior is specified by effect handlers.

For example, consider the following program that uses effect Choice (this paper uses ML-like syntax to describe programs):

effect Choice :: {decide : Unit $\Rightarrow$ Bool}

handle$_{\text{Choice}}$
    let $x =$ **if** decide () **then** 20 **else** 10 **in let** $y =$ **if** decide () **then** 5 **else** 0 **in** $x - y$
with { **return** $z \mapsto z$ } $\uplus$ {decide $z\,k \mapsto$ max (k true, k false)}

The first line declares *effect label* Choice with only one operation decide. As indicated by its type, decide takes the unit value and returns a Boolean. The program invokes the operation in the third line, determines numbers x and y depending on the results, and returns $x - y$ finally. To install an effect handler, we use the handling construct **handle–with**.

In general, an expression **handle**$_l$ e **with** h means that an expression e is executed under effect handler h, which interprets the operations of effect label l invoked by e; we call e a *handled expression*. An effect handler consists of one return clause and possibly several operation clauses. A return clause { **return** $x \mapsto e_r$ }, which corresponds to { **return** $z \mapsto z$ } in the example, is executed when a handled expression evaluates to a value, which the body e_r references by x. An operation clause takes the form {op $x\,k \mapsto e$}, which determines the implementation of operation op. When an operation op is called with an argument v under an effect handler with operation clause {op $x\,k \mapsto e$}, the reduction proceeds as follows. First, the remaining computation from the point of the operation call up to the **handle–with** construct installing the effect handler is captured; such a computation is called a *delimited continuation*. Then, the body e of the corresponding operation clause is executed by passing the argument v as x and the delimited continuation as k.

In the example, the delimited continuation for the first call to decide is

handle$_{\text{Choice}}$
    let $x =$ **if** $\square$ **then** 20 **else** 10 **in let** $y =$ **if** decide () **then** 5 **else** 0 **in** $x - y$
with { **return** $z \mapsto z$ } $\uplus$ {decide $z\,k \mapsto$ max (k true, k false)},

where $\square$ denotes a hole. The functional form v_1 of this delimited continuation is bound to variable k in the operation clause of decide, and the program evaluates to max (v_1 true, v_1 false). The function

application v_1 true fills the hole of the delimited continuation with argument true. Thus, it reduces **handle**$_{\text{Choice}}$

    let $x =$ **if** $\boxed{\text{true}}$ **then** 20 **else** 10 **in let** $y =$ **if** decide $()$ **then** 5 **else** 0 **in** $x - y$

with $\{\,$**return** $z \mapsto z\} \uplus \{$decide $z\,k \mapsto$ max $(k\,\text{true}, k\,\text{false})\}$,

where $\boxed{\text{true}}$ comes from the argument. Then, it substitutes 20 for x, and then calls decide again. The operation clause invokes the delimited continuation v_2 captured by the second call with arguments true and false. The applications v_2 true and v_2 false choose 5 and 0 as y and return the results of $20 - 5$ and $20 - 0$ (that is, 15 and 20), respectively. Then, the operation clause return max $(15, 20)$ as the result of v_1 true. Similarly, the function application v_1 false results in max $(5, 10)$. Thus, the entire program evaluates to max $($max $(15, 20),$ max $(5, 10))$ and then to 20 finally.

While the operation clause in the above example uses captured continuations, effect handlers can also discard them. Using this ability, we can implement exception handling, as the following program that divides x by y if y is nonzero:

effect Exc :: {raise : Unit $\Rightarrow$ Empty}

let $g = \lambda x :$ Int.$\lambda y :$ Int. **handle**$_{\text{Exc}}$ (**if** $y = 0$ **then** raise $()$ **else** x/y)

                            with $\{\,$**return** $z \mapsto$ `int_to_string`$z\} \uplus \{$raise $p\,k \mapsto$ `"divided by 0"`$\}$

In this example, Exc is an effect label consisting of one operation raise with type Unit $\Rightarrow$ Empty. Here, Empty is a type having no inhabitant, and we assume that an expression of this type can be regarded as that of any type. The return clause of the effect handler means that, when the handled expression evaluates to an integer, the handling construct returns its string version. Because the operation clause for raise discards the continuations, the handling construct returns the string `"divided by 0"` immediately once raise is called. Therefore, the operation call and effect handling in this example correspond to excepting raising and handling, respectively.

2.2 Effect Systems for Algebraic Effects and Handlers

This section briefly explains a role of effect systems for algebraic effect handlers and summarizes the existing systems.

2.2.1 A Role of Effect Systems. A property ensured by many effect systems in the literature is *effect safety*, which means that there is no unhandled operation. A simple example that breaks effect safety is op v, which just invokes operation op. Because no effect handler for op is given—thus, there is no way to interpret it—the program gets stuck. However, even if an operation call is enclosed by handling constructs, effect safety can be broken. For example, consider the following program:

effect Exc :: {raise : Unit $\Rightarrow$ Empty}

effect State :: {set : Int $\Rightarrow$ Unit, get : Unit $\Rightarrow$ Int}

let $g = \lambda x :$ Int. **handle**$_{\text{Exc}}$ (**if** $x = 0$ **then** raise $()$ **else** (**let** $y =$ get $()/x$ **in** set $y; y$))

                   with $\{\,$**return** $z \mapsto$ `int_to_string`$z\} \uplus \{$raise $p\,k \mapsto$ `"divided by 0"`$\}$

g 42 2

The effect label State is for mutable state, providing two operations set and get to update and get the current values in the state. The function g divides the current value of the state (returned by get) by x, sets the result to the state, and returns it if x is nonzero. All the operation calls in the application g 42 2 at the last line are performed under the effect handler, but the call to get is not handled. Hence, this example is not effect safe.

In general, the effect systems enjoying effect safety need to track which effect each expression may invoke and which effect an effect handler targets. However, there are choices to represent the effects caused by expressions. Thus far, mainly two styles of formalization of effect systems have been studied: one is based on *sets* [Bauer and Pretnar 2013; Forster et al. 2017; Kammar et al. 2013; Kammar and Pretnar 2017; Saleh et al. 2018; Sekiyama et al. 2020], and the other is based on *rows* [Biernacki et al. 2019; Hillerström and Lindley 2016; Leijen 2017; Xie et al. 2022].

2.2.2 Set-Based Effect Systems. Set-based effect systems assign to an expression a set of effect labels that the expression may invoke. For example, they assign to an operation call a set that includes the effect label of the called operation. This is formalized as follows, where typing judgment $\Gamma \vdash e : A \mid s$ means that expression e is of type A under typing context Γ and may invoke effects in set s:

$$\frac{\text{Operation op} : A \Rightarrow B \text{ belongs to effect } l \quad \Gamma \vdash v : A \mid \{\}}{\Gamma \vdash \text{op } v : B \mid \{l\}}$$

Subeffecting, which is supported to unify the effects of multiple expressions (such as branches in conditional expressions), is implemented by allowing the expansion of sets:

$$\frac{\Gamma \vdash e : A \mid s \quad s \subseteq s'}{\Gamma \vdash e : A \mid s'}$$

In the presence of algebraic effect handlers, sets not only expand but also may shrink. Such manipulation is performed in handling constructs:

$$\frac{\Gamma \vdash e : A \mid s \quad \{l\} \cup s' = s \quad \cdots}{\Gamma \vdash \mathbf{handle}_l \, e \, \mathbf{with} \, h : B \mid s'}$$

where the omitted premise states that h is a handler for effect l, translating a computation of type A to type B. This inference rule is matched with the behavior of the handling constructs because they can make handled effects l "unobservable." The set-based effect systems defined in such a way can soundly overapproximate the observable effects of programs and guarantee the effect safety of expressions to which the empty set can be assigned.

For instance, consider the example in Section 2.2.1. A set-based effect system would assign the set $\{\text{Exc}, \text{State}\}$ to the handled expression **if** $x = 0$ **then** raise () **else** (**let** $y = \text{get}()/x$ **in** set y; y) because it calls operation raise of Exc or get and set of State. Because this expression is only placed under the effect handler for Exc, the entire program $g \, 42 \, 2$ could have set $\{\text{State}\}$. As this set indicates that effect State may not be handled—and it *is not* actually—the effect system would conclude that the program may not be effect safe. If the program were wrapped by a handling construct with an effect handler for State, the empty set could be assigned to it; then, we could conclude that the program is effect safe.

2.2.3 Row-Based Effect Systems. Rows express collections of effect labels in a more structured way. In a monomorphic setting, they are just sequences of effect labels, as $\langle l_1, \dots, l_n \rangle$, which is the row consisting only of labels $l_1, \dots, l_n$. Rows are identified up to the reordering of labels. For example, $\langle l_1, l_2 \rangle$ equals $\langle l_2, l_1 \rangle$.[3]

Rows are often adapted in languages with effect polymorphism [Biernacki et al. 2019; Hillerström and Lindle 2016; Leijen 2017]. In such languages, rows are allowed to end with effect variables ρ, such as $\langle l_1, \dots, l_n, \rho \rangle$, which means that an expression may invoke effects $l_1, \dots, l_n$ as well as those in an

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[3]The label reordering might need to be restricted if effect labels are parameterized over, e.g., types, as discussed in Section 7.2.

instance of effect variable ρ. This extension enables abstraction over rows by universally quantifying effect variables. For example, consider function filtered_set, which, given an integer list and a function f from integers to Booleans, filters out the elements of the list using function f and then calls operation set of effect State on the remaining elements. Assume that the type of functions from type A to type B with effects in row r is described as $A \rightarrow_r B$. Then, filtered_set can be given type $\forall \rho.(\text{Int List} \times (\text{Int} \rightarrow_\rho \text{Bool})) \rightarrow_{\langle \text{State}, \rho \rangle} \text{Unit}$. By instantiating ρ with $\langle l_1, \ldots, l_n \rangle$, this type can express that, when passed a function f that may cause effects $l_1, \cdots, l_n$, filtered_set may also cause them via the application of f.

Inference rules of the row-based effect systems are similar to those of set-based ones, except that subeffecting allows enlarging rows only when they do not end with effect variables (such rows are called *closed*, while rows ending with effect variables are *open* [Hillerström and Lindley 2016]):

$$\frac{\Gamma \vdash e : A \mid \langle l_1, \ldots, l_n \rangle}{\Gamma \vdash e : A \mid \langle l_1, \ldots, l_n, r \rangle}$$

Rows shrink in handling constructs where handled effects are removed:

$$\frac{\Gamma \vdash e : A \mid r \qquad \langle l, r' \rangle = r \qquad \cdots}{\Gamma \vdash \mathbf{handle}_l \, e \, \mathbf{with} \, h : B \mid r'}$$

Similar to set-based ones, the row-based effect systems also ensure the effect safety of expressions to which the empty row $\langle \rangle$ can be assigned. The reasoning about the example in Section 2.2.1 can be done similarly to the case with simple rows.

These are the common core of the row-based effect systems, but they can be further classified into two groups depending on the formalism of rows. One is simple rows [Hillerström and Lindley 2016], where each label can appear at most once in one row. In this formalism, any l_i in row $\langle l_1, \ldots, l_n \rangle$ must be different from l_j for any $j \neq i$. The other is scoped rows [Leijen 2017], where the same label can appear in one row multiple times. Therefore, given a scoped row $\langle l_1, \ldots, l_n \rangle$, any l_i is allowed to be equivalent to some l_j, unlike simple rows.

2.3 Our Work: Abstracting Effect Systems

All effect systems based on sets, simple rows, or scoped rows exploit the structures of the respective representations to augment and shrink the information about effects. However, it is not clear which part of these structures essentially contributes to type-and-effect safety. To reveal it, we provide an abstract model of effect collections and their manipulation and give an effect system relying only on the abstract model. We also state sufficient conditions on the abstract model to guarantee the safety of our effect system. With the effect system depending only on the abstract nature of effect collections, we reveal the essence of safe effect systems for algebraic effect handlers.

We abstract the effect collections and manipulation in the effect systems for algebraic effect handlers by an *effect algebra*, which consists of an equivalence relation $\sim$ and a partial binary operation $\odot$, which mean the equivalence over effects and effect concatenation, respectively (these notations come from Morris and McKinna [2019]). For example, $\varepsilon_1 \odot \varepsilon_2 \sim \varepsilon_3$ intends to state that the concatenation of effects ε_1 and ε_2 is equal to ε_3. Our effect system is parameterized by effect algebras and manipulate effect collections only through the operation $\odot$ of a given effect algebra; hence, it does not suppose any concrete effect manipulation.

To abstract over the representations of effect collections, our effect system assumes two effect constructors. One is $\mathbb{0}$, which represents the empty collection and corresponds to the empty set and row in the set- and row-based effect systems, respectively. The other constructor is $(l)^\uparrow$, which constructs the effect collection composed only of effect label l.

With these abstractions, the inference rules that manipulate effect collections—i.e., those for operation calls, subeffecting, and handling constructs—are given as follows:

$$\frac{\text{Operation op} : A \Rightarrow B \text{ belongs to effect } l \quad \Gamma \vdash v : A \mid \mathbb{0}}{\Gamma \vdash \text{op } v : B \mid (l)^{\uparrow}}$$

$$\frac{\Gamma \vdash e : A \mid \varepsilon \quad \varepsilon \odot \varepsilon_0 \sim \varepsilon'}{\Gamma \vdash e : A \mid \varepsilon'} \qquad \frac{\Gamma \vdash e : A \mid \varepsilon \quad (l)^{\uparrow} \odot \varepsilon' \sim \varepsilon \quad \cdots}{\Gamma \vdash \textbf{handle}_l \, e \, \textbf{with} \, h : B \mid \varepsilon'}$$

The rule for operation call op v simply injects the corresponding effect label into the effect collection. The subsumption rule with subeffecting means that the effect ε of an expression can be expanded to ε' by appending some effects ε_0. The rule for handling constructs means that, if a handled expression may invoke effects in ε, only the remaining ε' of excluding the handled effect l from ε is observable from the outer context.

It is noteworthy that the above usage of effect algebras pays attention to the order of effects appearing in effect collections. Specifically, the subsumption rule only allows appending extra effects ε_0 and does not allow prepending them, and the rule for handling constructs removes only the handled effect label that occurs first in ε. This mirrors the nature of the effect handling that an operation call is handled by the effect handler closest to the call. The importance of considering the order of effects is confirmed in, e.g., adopting a type-erasure semantics: as discussed in Section 7.2, our effect system becomes unsound under the type-erasure semantics if a given effect algebra is equipped with commutative $\odot$, which makes the effect system *in*sensitive to the order of effects.

While effect algebras are expressive enough to represent the manipulation of effect collections, some effect algebras make the effect system unsafe. For example, consider an effect algebra where $(l)^{\uparrow} \odot \varepsilon \sim \mathbb{0}$ holds. Given an operation op of the effect label l, the subsumption rule allows coercing the effect $(l)^{\uparrow}$ of an operation call op v to $\mathbb{0}$. It means that the effect system can state that op v invokes no unhandled operation, so the effect system with such an effect algebra is unsafe.

To prevent the use of such effect algebras, we establish conditions on effect algebras; we call them *safety conditions* and also call effect algebras meeting them *safe*. We prove that, given a safe effect algebra, our effect system satisfies type and effect safety. We also demonstrate the expressibility of our framework by providing effect algebras for sets, simple rows, and scoped rows from the literature, as well as one for multisets, which are a new representation of effect collections.

3   ABSTRACTING EFFECTS

This section introduces the core notions of our effect system: effect algebras, an abstract model of effect collections and their manipulations. Because we aim at a formal effect system, we need to decide the syntactic representation of effect collections manipulated by the effect system. However, relying on specific representations prevents accommodating a variety of effect systems in the literature. To address this problem, we parameterize our effect system over the representations of effect collections and assume that the interface of their constructs is given by an *effect signature*.

Throughout this paper, we use the notation $\boldsymbol{\alpha}^I$ for a finite sequence $\alpha_0, \ldots, \alpha_n$ with an index set $I = \{0, \ldots, n\}$, where α is any metavariable. We also write $\{\boldsymbol{\alpha}^I\}$ for the set consisting of the elements of $\boldsymbol{\alpha}^I$. Index sets are designated by I, J, and N. We omit index sets and write $\boldsymbol{\alpha}$ simply when they are not important (e.g., all the sequences of interest have the same length).

3.1   Syntax

We start by defining *label* and *effect signatures*, which specify available *label names* (the names of effects) and effect collection constructors as well as their kinds, respectively. We then introduce

f, g, x, y, z, p, k (variables) $\alpha, \beta, \gamma, \tau, \iota, \rho$ (typelike variables)   op (operation names)

$l \in \mathrm{dom}(\Sigma_{\mathrm{lab}})$ (label names) $\mathcal{F} \in \mathrm{dom}(\Sigma_{\mathrm{eff}})$ (effect constructors) $C \in \mathrm{dom}(\Sigma_{\mathrm{lab}}) \cup \mathrm{dom}(\Sigma_{\mathrm{eff}})$

$$
\begin{array}{rcll}
K & ::= & \mathbf{Typ} \mid \mathbf{Lab} \mid \mathbf{Eff} & \text{(kinds)} \\
A, B, C & ::= & \tau \mid A \to_\varepsilon B \mid \forall \alpha : K.A^\varepsilon & \text{(types)} \\
\varepsilon & ::= & \rho \mid \mathcal{F}\, S^I & \text{(effects)} \\
\sigma & ::= & \{\} \mid \sigma \uplus \{\mathrm{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K}^J.A \Rightarrow B\} & \text{(operation signatures)} \\
\Gamma & ::= & \emptyset \mid \Gamma, x : A \mid \Gamma, \alpha : K & \text{(typing contexts)}
\end{array}
$$

$$
\begin{array}{rcll}
S, T & ::= & A \mid L \mid \varepsilon & \text{(typelikes)} \\
L & ::= & \iota \mid l\, S^I & \text{(labels)} \\
\Xi & ::= & \emptyset \mid \Xi, l :: \forall \boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma & \text{(effect contexts)}
\end{array}
$$

Fig. 1. Typelike syntax over an label signature Σ_{lab} and an effect signature Σ_{eff}.

the syntax of types, effect labels, and effect collections using a given label and effect signature. Kinds, ranged by K, are **Typ** for types, **Lab** for effect labels, or **Eff** for effect collections.

Definition 3.1 (Signatures). *Given a set S of label names, a label signature Σ_{lab} is a functional relation whose domain $\mathrm{dom}(\Sigma_{\mathrm{lab}})$ is S. The codomain of Σ_{lab} is the set of functional kinds of the form $\Pi_{i \in I} K_i \to \mathbf{Lab}$ for some I and $K_i^{i \in I}$ (if $I = \emptyset$, it means $\mathbf{Lab}$ simply). Similarly, given a set S of effect constructors, an effect signature Σ_{eff} is a functional relation whose domain $\mathrm{dom}(\Sigma_{\mathrm{eff}})$ is S and its codomain is the set of functional kinds of the form $\Pi_{i \in I} K_i \to \mathbf{Eff}$ for some I and $K_i^{i \in I}$. A signature Σ is the disjoint union of a label signature and an effect signature. We write $\Pi \boldsymbol{K}^I \to K$, and more simply, $\Pi \boldsymbol{K} \to K$ as an abbreviation for $\Pi_{i \in I} K_i \to K$.*

We write $C : \Pi \boldsymbol{K} \to K$ to denote the pair $\langle C, \Pi \boldsymbol{K} \to K \rangle$ for label name or effect constructor C.

Example 3.2 (Label Signatures of Exc and State). The label signature for label names Exc and State used in Section 2.2.1 are given as $\{\mathrm{Exc} : \mathbf{Lab}, \; \mathrm{State} : \mathbf{Lab}\}$. The label State in Section 2.2.1 assumes the values of state to be integers, but, if one wants to parameterize label State over the types of the state values, the signature of State changes to State : $\mathbf{Typ} \to \mathbf{Lab}$. This signature indicates that State can take a type argument A that represents the type of the state values. We call parameterized label names, as State of kind $\mathbf{Typ} \to \mathbf{Lab}$, *parametric effects*, which facilitate the reuse of program components as explained later.

The following is an effect signature for *effect sets*, effect collections implemented by sets.

Example 3.3 (Effect Signature of Effect Sets). The effect signature $\Sigma_{\mathrm{eff}}^{\mathrm{Set}}$ of effect sets consists of the pairs $\{\} : \mathbf{Eff}$ (for the empty set), $\{-\} : \mathbf{Lab} \to \mathbf{Eff}$ (for singleton sets), and $- \underline{\cup} - : \mathbf{Eff} \times \mathbf{Eff} \to \mathbf{Eff}$ (for set unions).[4]

Given a signature $\Sigma = \Sigma_{\mathrm{lab}} \uplus \Sigma_{\mathrm{eff}}$, the syntax of types, ranged over by A, B, and C, effect labels (or labels for short), ranged over by L, and effect collections (or effects for short), ranged over by ε, is defined as in Figure 1. This work allows three kinds of polymorphism, that is, type, label, and effect polymorphism. To simplify their presentation, we introduce a syntactic category that unifies types, labels, and effects; we call its entities *typelikes* [Biernacki et al. 2019], which are ranged over by S and T. Typelikes are classified into types, labels, and effects using the kind system presented in Section 3.2. We use τ, ι, and ρ to designate type, label, and effect variables (i.e., typelike variables with kind $\mathbf{Typ}$, $\mathbf{Eff}$, and $\mathbf{Lab}$), respectively, and α, β, and γ in a general context.

Types consist of: type variables; function types $A \to_\varepsilon B$, which represent functions from type A to B with effect ε; and polymorphic types $\forall \alpha : K.A^\varepsilon$, which represent (suspended) computation

---

[4]We use "$-$" for unnamed arguments. Multiple occurrences of "$-$" are distinguished from each other; the i-th occurrence from the left represents the i-th argument.

Kinding $\boxed{\Gamma \vdash S : K}$ $\boxed{\Gamma \vdash S^I : K^I}$ $\iff \forall i \in I.(\Gamma \vdash S_i : K_i)$

$$\frac{\vdash \Gamma \quad \alpha : K \in \Gamma}{\Gamma \vdash \alpha : K} \text{ K\_Var} \qquad \frac{\vdash \Gamma \quad C : \Pi K \to K_0 \in \Sigma \quad \Gamma \vdash S : K}{\Gamma \vdash C\,S : K_0} \text{ K\_Cons}$$

$$\frac{\Gamma \vdash A : \textbf{Typ} \quad \Gamma \vdash \varepsilon : \textbf{Eff} \quad \Gamma \vdash B : \textbf{Typ}}{\Gamma \vdash A \to_\varepsilon B : \textbf{Typ}} \text{ K\_Fun} \qquad \frac{\Gamma, \alpha : K \vdash A : \textbf{Typ} \quad \Gamma, \alpha : K \vdash \varepsilon : \textbf{Eff}}{\Gamma \vdash \forall \alpha : K.A^\varepsilon : \textbf{Typ}} \text{ K\_Poly}$$

Fig. 2. Kinding rules.

with effect ε abstracting over typelikes of kind K. We omit base types such as Int for simplification, but assume them and some operations on them (such as + for integers) in giving examples.

A label is a label variable or a label name, ranged over by l, possibly with type arguments. For example, consider State : **Typ** $\to$ **Lab** given in Example 3.2. A label State A represents mutable state possessing the values of the type A. We can implement State A using a state-passing effect handler, which abstracts over type arguments A [Leijen 2017]. Thus, the effect handler can be reused for different type arguments.

Effects are composed of effect variables and effect constructors, ranged over by $\mathcal{F}$, given by Σ_{eff}. As label names, effect constructors can take typelikes as arguments. For example, effect set {Exc} is represented by $\mathcal{F}$ Exc where $\mathcal{F}$ is the constructor $\{-\}$ for singleton sets.

Effect contexts, ranged over by Ξ, are finite sequences of declarations of effect label names. Each label name l is associated with a type scheme of the form $\forall \boldsymbol{\alpha} : \boldsymbol{K}.\sigma$, where σ is an operation signature parameterized over typelike variables $\boldsymbol{\alpha}$ of kinds $\boldsymbol{K}$. In general, the functional kind $\Pi K' \to \textbf{Lab}$ of l in Σ_{lab} needs to be consistent with the kind of the type scheme, that is, $K' = K$; we will formalize this requirement in Section 5.2. An *operation signature* is a set of pairs of an operation name op and its type $\forall \boldsymbol{\beta} : \boldsymbol{K}.A \Rightarrow B$. Here, A and B are the argument and return types of the operation, respectively, and they are parameterized over $\boldsymbol{\beta}$ of kinds $\boldsymbol{K}$. Namely, not only effect labels but also operations can be parametric. For example, the effect context for nonparametric effect labels Exc and State in Section 2.1 is given as

$$\text{Exc} :: \{\text{raise} : \text{Unit} \Rightarrow \text{Empty}\}, \text{State} :: \{\text{set} : \text{Int} \Rightarrow \text{Unit}, \text{get} : \text{Unit} \Rightarrow \text{Int}\} .$$

If one wants to parameterize label State over the types of the state values, and operation raise of label Exc over return types (because it returns no value actually), the effect context can change to

$$\text{Exc} :: \{\text{raise} : \forall \alpha : \textbf{Typ}.\text{Unit} \Rightarrow \alpha\}, \text{State} :: \forall \alpha : \textbf{Typ}.\{\text{set} : \alpha \Rightarrow \text{Unit}, \text{get} : \text{Unit} \Rightarrow \alpha\} .$$

A difference between parametric effects and operations is that, while effect handlers for parametric effects can be typechecked depending on given type arguments, ones for parametric operations must abstract over type arguments. See Sekiyama and Igarashi [2019] for detail.

Typing contexts, ranged over by Γ, are finite sequences of bindings of the form $x : A$ or $\alpha : K$.

3.2   Kind System

We show our kind system in Figure 2. We omit the rules for well-formedness of typing contexts because they are defined as usual [Kawamata et al. 2024; Sekiyama et al. 2020]. The rules other than K_Cons are standard or straightforward. When signature Σ assigns $\Pi K \to K_0$ to label name or effect constructor C, and typelike arguments S are of the kinds K, respectively, the rule K_Cons assigns kind K_0 to the typelike $C\,S$.

3.3 Effect Algebras

Now, we define effect algebras. In short, an effect algebra provides an effect signature Σ_{eff}, a partial monoid on effects defined over Σ_{eff}, and a function $(-)^{\uparrow}$ that injects labels to effects, but more formally, it also requires that each involved operation preserve well-formedness and kind-aware typelike substitution make a homomorphism. In what follows, we denote the sets of types, effect labels, and effect collections over a signature Σ by $\textbf{Typ}(\Sigma)$, $\textbf{Lab}(\Sigma)$, and $\textbf{Eff}(\Sigma)$, respectively (we refer to the set of entities at kind K by $K(\Sigma)$).

Definition 3.4 (Well-Formedness-Preserving Functions). *Given a signature Σ, a (possibly partial) function $f \in K_i(\Sigma)^{i \in \{1,\dots,n\}} \rightharpoonup K(\Sigma)$ preserves well-formedness if*

$$\forall \Gamma, S_1, \dots, S_n.\, \Gamma \vdash S_1 : K_1 \wedge \cdots \wedge \Gamma \vdash S_n : K_n \wedge f(S_1, \dots, S_n) \in K(\Sigma) \implies \Gamma \vdash f(S_1, \dots, S_n) : K\,.$$

Similarly, $f \in K(\Sigma)$ preserves well-formedness if $\Gamma \vdash f : K$ for any Γ.

In what follows, we write $\alpha \mapsto T \vdash S : K_0$ for a quadruple $\langle \alpha, T, S, K_0 \rangle$ such that $\exists \Gamma_1, K, \Gamma_2.\,(\forall S_0 \in S.\,\Gamma_1, \alpha : K, \Gamma_2 \vdash S_0 : K_0) \wedge \Gamma_1 \vdash T : K$; it means that typelikes S are well formed at kind K_0 and substituting typelike T for typelike variable α in S preserves their well-formedness.

Definition 3.5 (Effect algebras). *Given a label signature Σ_{lab}, an effect algebra is a quintuple $\langle \Sigma_{\text{eff}}, \odot, \mathbb{0}, (-)^{\uparrow}, \sim \rangle$ satisfying the following, where we let $\Sigma = \Sigma_{\text{lab}} \uplus \Sigma_{\text{eff}}$.*

- *$\odot \in \textbf{Eff}(\Sigma) \times \textbf{Eff}(\Sigma) \rightharpoonup \textbf{Eff}(\Sigma)$, $\mathbb{0} \in \textbf{Eff}(\Sigma)$, and $(-)^{\uparrow} \in \textbf{Lab}(\Sigma) \to \textbf{Eff}(\Sigma)$ preserve well-formedness. Furthermore, $\sim$ is an equivalence relation on $\textbf{Eff}(\Sigma)$ and preserves well-formedness, that is, $\forall \varepsilon_1, \varepsilon_2.\, \varepsilon_1 \sim \varepsilon_2 \implies (\forall \Gamma.\, \Gamma \vdash \varepsilon_1 : \textbf{Eff} \iff \Gamma \vdash \varepsilon_2 : \textbf{Eff})$.*
- *$\langle \textbf{Eff}(\Sigma), \odot, \mathbb{0} \rangle$ is a partial monoid under $\sim$, that is, the following holds:*
  - *$\forall \varepsilon \in \textbf{Eff}(\Sigma).\, \varepsilon \odot \mathbb{0} \sim \varepsilon \wedge \mathbb{0} \odot \varepsilon \sim \varepsilon$; and*
  - *$\forall \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \textbf{Eff}(\Sigma).$*
    $(\varepsilon_1 \odot \varepsilon_2) \odot \varepsilon_3 \in \textbf{Eff}(\Sigma) \vee \varepsilon_1 \odot (\varepsilon_2 \odot \varepsilon_3) \in \textbf{Eff}(\Sigma) \implies (\varepsilon_1 \odot \varepsilon_2) \odot \varepsilon_3 \sim \varepsilon_1 \odot (\varepsilon_2 \odot \varepsilon_3)$.
- *Typelike substitution respecting well-formedness is a homomorphism for $\odot$, $(-)^{\uparrow}$, and $\sim$, that is, the following holds:*
  - *$\forall \alpha, S, \varepsilon_1, \varepsilon_2.\, \alpha \mapsto S \vdash \varepsilon_1, \varepsilon_2 : \textbf{Eff} \wedge \varepsilon_1 \odot \varepsilon_2 \in \textbf{Eff}(\Sigma) \implies (\varepsilon_1 \odot \varepsilon_2)[S/\alpha] = \varepsilon_1[S/\alpha] \odot \varepsilon_2[S/\alpha]$;*
  - *$\forall \alpha, S, L.\, \alpha \mapsto S \vdash L : \textbf{Lab} \implies (L)^{\uparrow}[S/\alpha] = (L[S/\alpha])^{\uparrow}$; and*
  - *$\forall \alpha, S, \varepsilon_1, \varepsilon_2.\, \alpha \mapsto S \vdash \varepsilon_1, \varepsilon_2 : \textbf{Eff} \wedge \varepsilon_1 \sim \varepsilon_2 \implies \varepsilon_1[S/\alpha] \sim \varepsilon[S/\alpha]$.*

For example, an effect algebra for effect sets can be given as follows.

Example 3.6 (Effect Sets). An effect algebra EA_{Set} for effect sets is a tuple $\langle \Sigma_{\text{eff}}^{\text{Set}}, - \underline{\cup} -, \{\}, \{-\}, \sim_{\text{Set}} \rangle$ where $\sim_{\text{Set}}$ is the least equivalence relation satisfying the following rules:

$$\frac{}{\varepsilon \underline{\cup} \{\} \sim_{\text{Set}} \varepsilon} \qquad \frac{}{\varepsilon_1 \underline{\cup} \varepsilon_2 \sim_{\text{Set}} \varepsilon_2 \underline{\cup} \varepsilon_1} \qquad \frac{}{\varepsilon \underline{\cup} \varepsilon \sim_{\text{Set}} \varepsilon}$$

$$\frac{}{(\varepsilon_1 \underline{\cup} \varepsilon_2) \underline{\cup} \varepsilon_3 \sim_{\text{Set}} \varepsilon_1 \underline{\cup} (\varepsilon_2 \underline{\cup} \varepsilon_3)} \qquad \frac{\varepsilon_1 \sim_{\text{Set}} \varepsilon_2 \quad \varepsilon_3 \sim_{\text{Set}} \varepsilon_4}{\varepsilon_1 \underline{\cup} \varepsilon_3 \sim_{\text{Set}} \varepsilon_2 \underline{\cup} \varepsilon_4}$$

These rules reflect that the union operator in sets has the identity element $\{\}$ and satisfies commutativity, idempotence, associativity, and compatibility.

We also show an instance for simple rows and scoped rows.

Example 3.7 (Simple Rows). The effect signature $\Sigma_{\text{eff}}^{\text{Row}}$ for simple rows is the set of $\langle \rangle : \textbf{Eff}$ and $\langle - \mid - \rangle : \textbf{Lab} \times \textbf{Eff} \to \textbf{Eff}$. An effect algebra EA_{SimpR} for them is $\langle \Sigma_{\text{eff}}^{\text{Row}}, \odot_{\text{SimpR}}, \langle \rangle, \langle - \mid \langle \rangle \rangle, \sim_{\text{SimpR}} \rangle$ where

$$\varepsilon_1 \odot_{\text{SimpR}} \varepsilon_2 \overset{\text{def}}{=} \begin{cases} \langle L_1 \mid \langle \cdots \langle L_n \mid \varepsilon_2 \rangle \rangle \rangle & (\text{if } \varepsilon_1 = \langle L_1 \mid \langle \cdots \langle L_n \mid \langle \rangle \rangle \rangle \rangle) \\ \varepsilon_1 & (\text{if } \varepsilon_1 = \langle L_1 \mid \langle \cdots \langle L_n \mid \rho \rangle \rangle \rangle \text{ and } \varepsilon_2 = \langle \rangle) \end{cases}$$

$$
\begin{array}{llll}
e & ::= & v \mid v_1\,v_2 \mid v\,S \mid \mathbf{let}\,x = e_1\,\mathbf{in}\,e_2 \mid \mathbf{handle}_{l\,S^I}\,e\,\mathbf{with}\,h & \text{(expressions)} \\
v & ::= & x \mid \mathbf{fun}\,(f, x, e) \mid \Lambda\alpha : K.e \mid \mathrm{op}_{l\,S^I}\,T^{\mathcal{J}} & \text{(values)} \\
h & ::= & \{\,\mathbf{return}\,x \mapsto e\,\} \mid h \uplus \{\mathrm{op}\,\boldsymbol{\beta}^{\mathcal{J}} : K^{\mathcal{J}}\,p\,k \mapsto e\} & \text{(handlers)} \\
E & ::= & \square \mid \mathbf{let}\,x = E\,\mathbf{in}\,e \mid \mathbf{handle}_{l\,S^I}\,E\,\mathbf{with}\,h & \text{(evaluation contexts)}
\end{array}
$$

Fig. 3. Program syntax of λ_{EA}.

and $\sim_{\mathrm{SimpR}}$ is the least equivalence relation satisfying the following.

$$
\frac{\varepsilon_1 \sim_{\mathrm{SimpR}} \varepsilon_2}{\langle L \mid \varepsilon_1 \rangle \sim_{\mathrm{SimpR}} \langle L \mid \varepsilon_2 \rangle} \qquad \frac{L_1 \neq L_2}{\langle L_1 \mid \langle L_2 \mid \varepsilon \rangle \rangle \sim_{\mathrm{SimpR}} \langle L_2 \mid \langle L_1 \mid \varepsilon \rangle \rangle} \qquad \frac{}{\langle L \mid \varepsilon \rangle \sim_{\mathrm{SimpR}} \langle L \mid \langle L \mid \varepsilon \rangle \rangle}
$$

Note that the definition of $\varepsilon_1 \odot_{\mathrm{SimpR}} \varepsilon_2$ depends on whether effect ε_1 ends with an effect variable. If it does, ε_2 must be empty because simple rows ending with effect variables cannot be extended. Otherwise, $\varepsilon_1 \odot_{\mathrm{SimpR}} \varepsilon_2$ simply concatenates ε_1 and ε_2.

The first rule of $\sim_{\mathrm{SimpR}}$ means that the results of adding the same label to equivalent effects are also equivalent. The remaining two rules allow reordering different labels and collapsing multiple occurrences of the same label into one, respectively. The collapsing of multiple occurrences reflects the characteristic of simple rows that the same label appears at most once in a row because it means that two or more occurrences of a label cannot be distinguished from one occurrence of it.

Example 3.8 (Scoped Rows). An effect algebra $\mathrm{EA}_{\mathrm{ScpR}}$ for scoped rows is defined in a way similar to that for simple rows. The only difference is in the definition of equivalence $\sim_{\mathrm{ScpR}}$. The equivalence $\sim_{\mathrm{ScpR}}$ for scoped rows is defined as the least equivalence relation satisfying the following rules:

$$
\frac{\varepsilon_1 \sim_{\mathrm{ScpR}} \varepsilon_2}{\langle L \mid \varepsilon_1 \rangle \sim_{\mathrm{ScpR}} \langle L \mid \varepsilon_2 \rangle} \qquad \frac{L_1 \neq L_2}{\langle L_1 \mid \langle L_2 \mid \varepsilon \rangle \rangle \sim_{\mathrm{ScpR}} \langle L_2 \mid \langle L_1 \mid \varepsilon \rangle \rangle}
$$

Unlike simple rows, scoped rows are distinguished if they have different numbers of occurrences of some label.

4 λ_{EA}: A CALCULUS WITH ABSTRACT EFFECT SYSTEM

This section shows the syntax, semantics, and type-and-effect system of our language λ_{EA}. It is similar to the call-by-value polymorphic λ-calculi with algebraic effect handlers in the literature [Biernacki et al. 2018; Leijen 2017; Sekiyama et al. 2020] except that it is parameterized over effect algebras. Throughout this and the next sections, we fix a label signature Σ_{lab}, effect algebra $\langle \Sigma_{\mathrm{eff}}, \odot, \mathbb{0}, (-)^{\uparrow}, \sim \rangle$ over Σ_{lab}, and effect context Ξ, which are given as parameters.

4.1 Syntax

We show the program syntax of λ_{EA} in Figure 3.

Expressions, ranged over e, are composed of: values; function applications $v_1\,v_2$; typelike applications $v\,S$; let-bindings $\mathbf{let}\,x = e_1\,\mathbf{in}\,e_2$; and handling expressions $\mathbf{handle}_{l\,S^I}\,e\,\mathbf{with}\,h$. Values are: variables x; recursive functions $\mathbf{fun}\,(f, x, e)$; typelike abstractions $\Lambda\alpha : K.e$; or operations $\mathrm{op}_{l\,S^I}\,T^{\mathcal{J}}$. An operation $\mathrm{op}_{l\,S^I}\,T^{\mathcal{J}}$ accompanies two typelike sequences S^I and $T^{\mathcal{J}}$, which are parameters of effect label l and operation op, respectively. We write $\lambda x.e$ for $\mathbf{fun}\,(f, x, e)$ when variable f does not occur free in expression e.

An effect handler for label name l possesses one return clause and clauses for the operations of l. For a return clause $\{\,\mathbf{return}\,x \mapsto e\,\}$, the body e is executed once a handled expression evaluates

Freeness of labels $\boxed{n{-}\mathrm{free}(L, E)}$

$$\frac{}{0{-}\mathrm{free}(L, \square)} \qquad \frac{n{-}\mathrm{free}(L, E)}{n{-}\mathrm{free}(L, \mathbf{let}\ x = E\ \mathbf{in}\ e)} \qquad \frac{n{-}\mathrm{free}(L, E) \quad L \neq L'}{n{-}\mathrm{free}(L, \mathbf{handle}_{L'}\ E\ \mathbf{with}\ h)}$$

Reduction $\boxed{e \longmapsto e'}$

$$\frac{}{\mathbf{fun}\ (f, x, e)\ v \longmapsto e[\mathbf{fun}\ (f, x, e)/f][v/x]}\ \text{R\_App} \qquad \frac{}{(\Lambda\alpha : K.e)\ S \longmapsto e[S/\alpha]}\ \text{R\_TApp}$$

$$\frac{}{\mathbf{let}\ x = v\ \mathbf{in}\ e \longmapsto e[v/x]}\ \text{R\_Let} \qquad \frac{\mathbf{return}\ x \mapsto e_r \in h}{\mathbf{handle}_{l\,S^I}\ v\ \mathbf{with}\ h \longmapsto e_r[v/x]}\ \text{R\_Handle1}$$

$$\frac{\mathrm{op}\ \boldsymbol{\beta}^J : K^J\ p\ k \mapsto e \in h \quad v_{cont} = \lambda z.\mathbf{handle}_{l\,S^I}\ E[z]\ \mathbf{with}\ h \quad 0{-}\mathrm{free}(l\,S^I, E)}{\mathbf{handle}_{l\,S^I}\ E[\mathrm{op}_{l\,S^I}\ T^J\ v]\ \mathbf{with}\ h \longmapsto e[T^J/\boldsymbol{\beta}^J][v/p][v_{cont}/k]}\ \text{R\_Handle2}$$

Evaluation $\boxed{e \longrightarrow e'}$

$$\frac{e_1 \longmapsto e_2}{E[e_1] \longrightarrow E[e_2]}\ \text{E\_Eval}$$

Fig. 4. Operational semantics of λ_{EA}.

to a value v; x is used to refer to the value v. For an operation clause $\{\mathrm{op}\ \boldsymbol{\beta} : K\ p\ k \mapsto e\}$, the body e is executed once a handled expression calls operation op. Typelike variables $\boldsymbol{\beta}$, variable p, and variable k are replaced by typelike parameters attached to the operation call, the argument of the call, and the delimited continuation from the call up to the handling expression installing the effect handler, respectively.

Evaluation contexts, ranged over by E, are defined in a standard manner. They may wrap a hole $\square$ by let-constructs and handling constructs.

4.2 Operational Semantics

The operational semantics of λ_{EA} is defined in Figure 4. Following Biernacki et al. [2018], it uses the notion of *freeness*, which helps define the operational semantics of lift coercions in Section 7.1. Figure 4 defines 0-*freeness* of labels [Biernacki et al. 2018]. The judgment $0{-}\mathrm{free}(L, E)$, which is read as "an label L is 0-free in an evaluation context E," means that any operation of L called under E is not handled. The operational semantics of λ_{EA} uses this notion to ensure that every call to an operation of effect label L is handled by the innermost L's effect handler enclosing the operation call. We generalize 0-freeness to n-freeness for an arbitrary natural number n in introducing lift coercions (see Section 7.1 for detail).

We show the operational semantics of λ_{EA} in Figure 4. The semantics comprises two binary relations: the reduction relation $\longmapsto$ and the evaluation relation $\longrightarrow$. The reduction relation defines the basic computation; in contrast, the evaluation relation gives a way of reducing subexpressions.

The reduction relation is defined by five rules. Function applications, typelike applications, let-bindings are reduced as usual. The remaining are the standard rules to reduce handling expressions. Consider an expression $\mathbf{handle}_{l\,S^I}\ e\ \mathbf{with}\ h$. If the handled expression e is a value v, the rule R_Handle1 reduces the handling expression to the body e_r of the return clause $\{\mathbf{return}\ x \mapsto e_r\}$ of h by substituting v for x in e_r. The other rule R_Handle2 is used when e calls an operation op of label name l, that is, e takes the form $E[\mathrm{op}_{l\,S^I}\ T^J\ v]$ for some E, T^J, and v (it is guaranteed by

the type-and-effect system that the typelike arguments to l in the operation call are S^I). The reduction rule R_HANDLE2 assumes $0-\text{free}(l\,S^I, E)$, which ensures that h is the effect handler closest to the operation call among the ones for $l\,S^I$. After substituting the argument typelikes T^J, the argument value v, and the captured delimited continuation v_{cont} (which installs the effect handler h on the captured evaluation context E because effect handlers in λ_{EA} are *deep*) for the corresponding variables of op's operation clause in h, the evaluation proceeds to reducing the clause's body.

The evaluation relation only has one rule E_EVAL. It means that the evaluation of an entire program proceeds by decomposing it into a redex e and an evaluation context E, reducing e to an expression e', and then filling the hole of E with the reduction result e'.

4.3　Type-and-Effect System

We show the type-and-effect system of λ_{EA} in Figure 5. Typing judgments are of the form $\Gamma \vdash e : A \mid \varepsilon$, meaning that an expression e is typed at A under a typing context Γ and the evaluation of e may cause effect ε. The rules for variables, function abstractions, function applications, typelike abstractions, typelike applications, and let-bindings are standard.

The rule T_SUB allows subsumption by subtyping. We show the subtyping relation $\Gamma \vdash A <: B$ for values and the one $\Gamma \vdash A \mid \varepsilon_1 <: B \mid \varepsilon_2$ for computations at the bottom of Figure 5. The subtyping rules are standard except for the subeffecting $\Gamma \vdash \varepsilon_1 \oslash \varepsilon_2$, which is used in the rule ST_COMP for the second subtyping relation. The subeffecting is defined via the given effect algebra:

$$\Gamma \vdash \varepsilon_1 \oslash \varepsilon_2 \stackrel{\text{def}}{=} \exists\varepsilon.\, \varepsilon_1 \odot \varepsilon \sim \varepsilon_2 \wedge (\forall\varepsilon' \in \{\varepsilon_1, \varepsilon_2, \varepsilon\}.\, \Gamma \vdash \varepsilon' : \textbf{Eff})\ .$$

The rule T_OP typecheckes operation $\text{op}_{l\,S^I}\, T^J$ if op belongs to effect label l, and if the kinds of typelike arguments S^I and T^J are matched with those of parameters of l in the effect context Ξ. The operation is given a function type determined by the argument and return type of op in Ξ and typelike arguments S^I and T^J. Because every call to the operation only invokes effect label $l\,S^I$, the latent effect of the function type is given by injecting $l\,S^I$ via $(-)^\uparrow$.

The rule T_HANDLING is for handling expressions. Assume that a handled expression e is of type A and has effect ε'. If it is handled by an effect handler for effect label $l\,S^I$, the operations of $l\,S^I$ become unobservable from the outer context. Thus, the effect ε of the handling expression is the result of removing label $l\,S^I$ from effect ε'. This "label-removing manipulation" is represented as $\Gamma \vdash (l\,S^I)^\uparrow \odot \varepsilon \sim \varepsilon'$. Therefore, the result ε of the label-removing manipulation depends on the given effect algebra. For example, if the effect algebra EA_{SimpR} for simple rows is given, the result of removing the label Exc from the effect $\langle \text{Exc} \mid \langle \text{Exc} \mid \langle \text{Choice} \mid \langle\rangle\rangle\rangle\rangle$ can be $\langle \text{Choice} \mid \langle\rangle\rangle$ because $\langle \text{Exc} \mid \langle\rangle\rangle \odot_{\text{SimpR}} \langle \text{Choice} \mid \langle\rangle\rangle \sim_{\text{SimpR}} \langle \text{Exc} \mid \langle \text{Exc} \mid \langle \text{Choice} \mid \langle\rangle\rangle\rangle\rangle$ holds (recall that simple rows can collapse multiple occurrences of the same label into one). On the contrary, the removing result in the algebra EA_{ScpR} for scoped rows can be $\langle \text{Exc} \mid \langle \text{Choice} \mid \langle\rangle\rangle\rangle$ but cannot be $\langle \text{Choice} \mid \langle\rangle\rangle$.

The type B of the handling expression is determined by handler h: typing judgments for handlers take the form $\Gamma \vdash_\sigma h : A \Rightarrow^\varepsilon B$, which means that handler h transforms computation of type A involving an effect label with operation signature σ to that of type B with effect ε. The rules H_RETURN and H_OP are for return and operation clauses and reflect the reduction rules R_HANDLE1 and R_HANDLE2, respectively. Note that the return type of a continuation variable k equals the type B of the handling expression as the effect handlers in λ_{EA} are deep [Kammar et al. 2013].

Typing $\boxed{\Gamma \vdash e : A \mid \varepsilon}$

$$\frac{\vdash \Gamma \quad x : A \in \Gamma}{\Gamma \vdash x : A \mid \mathbb{0}} \ \text{T\_Var} \qquad \frac{\Gamma, f : A \to_\varepsilon B, x : A \vdash e : B \mid \varepsilon}{\Gamma \vdash \mathbf{fun}\,(f, x, e) : A \to_\varepsilon B \mid \mathbb{0}} \ \text{T\_Abs}$$

$$\frac{\Gamma \vdash v_1 : A \to_\varepsilon B \mid \mathbb{0} \quad \Gamma \vdash v_2 : A \mid \mathbb{0}}{\Gamma \vdash v_1\,v_2 : B \mid \varepsilon} \ \text{T\_App} \qquad \frac{\Gamma, \alpha : K \vdash e : A \mid \varepsilon}{\Gamma \vdash \Lambda \alpha : K.e : \forall \alpha : K.A^\varepsilon \mid \mathbb{0}} \ \text{T\_TAbs}$$

$$\frac{\Gamma \vdash v : \forall \alpha : K.A^\varepsilon \mid \mathbb{0} \quad \Gamma \vdash S : K}{\Gamma \vdash v\,S : A[S/\alpha] \mid \varepsilon[S/\alpha]} \ \text{T\_TApp} \qquad \frac{\Gamma \vdash e_1 : A \mid \varepsilon \quad \Gamma, x : A \vdash e_2 : B \mid \varepsilon}{\Gamma \vdash \mathbf{let}\ x = e_1\ \mathbf{in}\ e_2 : B \mid \varepsilon} \ \text{T\_Let}$$

$$\frac{\Gamma \vdash e : A \mid \varepsilon \quad \Gamma \vdash A \mid \varepsilon <: A' \mid \varepsilon'}{\Gamma \vdash e : A' \mid \varepsilon'} \ \text{T\_Sub}$$

$$\frac{l :: \forall \boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi \quad \mathrm{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K'}^J.A \Rightarrow B \in \sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]}{\vdash \Gamma \quad \Gamma \vdash \boldsymbol{S}^I : \boldsymbol{K}^I \quad \Gamma \vdash \boldsymbol{T}^J : \boldsymbol{K'}^J}{\Gamma \vdash \mathrm{op}_{l\boldsymbol{S}^I}\,\boldsymbol{T}^J : (A[\boldsymbol{T}^J/\boldsymbol{\beta}^J]) \to_{(l\boldsymbol{S}^I)^\uparrow} (B[\boldsymbol{T}^I/\boldsymbol{\beta}^I]) \mid \mathbb{0}} \ \text{T\_Op}$$

$$\frac{\Gamma \vdash e : A \mid \varepsilon' \quad l :: \forall \boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi \quad \Gamma \vdash \boldsymbol{S}^I : \boldsymbol{K}^I}{\Gamma \vdash_{\sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} h : A \Rightarrow^\varepsilon B \quad (l\boldsymbol{S}^I)^\uparrow \odot \varepsilon \sim \varepsilon'}{\Gamma \vdash \mathbf{handle}_{l\boldsymbol{S}^I}\,e\,\mathbf{with}\ h : B \mid \varepsilon} \ \text{T\_Handling}$$

Handler Typing $\boxed{\Gamma \vdash_\sigma h : A \Rightarrow^\varepsilon B}$

$$\frac{\Gamma, x : A \vdash e_r : B \mid \varepsilon}{\Gamma \vdash_{\{\}} \{\mathbf{return}\ x \mapsto e_r\} : A \Rightarrow^\varepsilon B} \ \text{H\_Return}$$

$$\frac{\sigma = \sigma' \uplus \{\mathrm{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K}^J.A' \Rightarrow B'\}}{\Gamma \vdash_{\sigma'} h : A \Rightarrow^\varepsilon B \quad \Gamma, \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A', k : B' \to_\varepsilon B \vdash e : B \mid \varepsilon}{\Gamma \vdash_\sigma h \uplus \{\mathrm{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e\} : A \Rightarrow^\varepsilon B} \ \text{H\_Op}$$

Subtyping $\boxed{\Gamma \vdash A <: B}$ $\boxed{\Gamma \vdash A \mid \varepsilon_1 <: B \mid \varepsilon_2}$

$$\frac{\Gamma \vdash A : \mathbf{Typ}}{\Gamma \vdash A <: A} \ \text{ST\_Refl} \qquad \frac{\Gamma \vdash A_2 <: A_1 \quad \Gamma \vdash B_1 \mid \varepsilon_1 <: B_2 \mid \varepsilon_2}{\Gamma \vdash A_1 \to_{\varepsilon_1} B_1 <: A_2 \to_{\varepsilon_2} B_2} \ \text{ST\_Fun}$$

$$\frac{\Gamma, \alpha : K \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2}{\Gamma \vdash \forall \alpha : K.A_1{}^{\varepsilon_1} <: \forall \alpha : K.A_2{}^{\varepsilon_2}} \ \text{ST\_Poly} \qquad \frac{\Gamma \vdash A_1 <: B \quad \Gamma \vdash \varepsilon_1 \oslash \varepsilon_2}{\Gamma \vdash A \mid \varepsilon_1 <: B \mid \varepsilon_2} \ \text{ST\_Comp}$$

Fig. 5. Type-and-effect system of λ_{EA}.

5 SAFETY PROPERTIES

This section shows the safety properties of λ_{EA}. The proofs rely on safety conditions, which are requirements on effect algebras. Under the assumption that a given effect algebra meets the safety conditions, we prove type-and-effect safety of λ_{EA}.

5.1 Safety Conditions

To prove type-and-effect safety, a given effect algebra must meet safety conditions shown in the following. We write $\varepsilon_1 \oslash \varepsilon_2$ to state that $\varepsilon_1 \odot \varepsilon \sim \varepsilon_2$ for some ε.

Definition 5.1 (Safety Conditions).
　(1) For any L, $(L)^\uparrow \oslash \mathbb{0}$ does not hold.
　(2) If $(L)^\uparrow \oslash \varepsilon$ and $(L')^\uparrow \odot \varepsilon' \sim \varepsilon$ and $L \neq L'$, then $(L)^\uparrow \oslash \varepsilon'$.

Condition (1) disallows the subeffecting to hide an invoked effect label L as if it were not performed. Condition (2) means that, if an expression invoking a label L is given an effect ε, and an effect handler for a different label L' handles the expression, then the information of L still remains in the effect ε' assigned to the handling expression (that is, it is observable from the outer context).

To understand problems excluded by safety conditions (1) and (2), we consider effect algebras that violate one of the conditions, and then show unsafe programs being typeable under the algebras.

Example 5.2 (Unsafe Effect Algebras).
　Effect algebra violating safety condition (1) Consider an effect algebra such that $\mathbb{0} \vdash (l)^\uparrow \oslash \mathbb{0}$ holds for some l. Clearly, this effect algebra violates safety condition (1). In this case, $\mathbb{0} \vdash \mathrm{op}_l\, v : A \mid \mathbb{0}$ can be derived for some A (if $\mathrm{op}_l\, v$ is well typed) because $\mathrm{op}_l\, v$ is given the effect $(l)^\uparrow$ and the subeffecting $\mathbb{0} \vdash (l)^\uparrow \oslash \mathbb{0}$ holds. However, the operation call is not handled.
　Effect algebra violating safety condition (2) Consider an effect algebra such that safety condition (1), $(l)^\uparrow \oslash (l')^\uparrow$, and $(l')^\uparrow \odot \mathbb{0} \sim (l')^\uparrow$ hold for some l and l' such that $l \neq l'$. This effect algebra must violate safety condition (2): if safety condition (2) were met, we would have $(l)^\uparrow \oslash \mathbb{0}$, but it is contradictory with safety condition (1).
　This effect algebra allows assigning the empty effect $\mathbb{0}$ to the expression $\mathbf{handle}_{l'}\, \mathrm{op}_l\, v\, \mathbf{with}\, h$ as illustrated by the following typing derivation:

$$
\cfrac{\cdots \quad (l')^\uparrow \odot \mathbb{0} \sim (l')^\uparrow \quad \cfrac{\mathbb{0} \vdash \mathrm{op}_l\, v : A \mid (l)^\uparrow \quad \mathbb{0} \vdash A \mid (l)^\uparrow <: A \mid (l')^\uparrow}{\mathbb{0} \vdash \mathrm{op}_l\, v : A \mid (l')^\uparrow}\ \text{T\_Sub}}{\mathbb{0} \vdash \mathbf{handle}_{l'}\, \mathrm{op}_l\, v\, \mathbf{with}\, h : B \mid \mathbb{0}}\ \text{T\_Handling}
$$

However, the operation call in it is not handled.

5.2 Type-and-Effect Safety

This section shows type-and-effect safety. To prove it, we assume that an effect algebra meets the safety conditions and an effect context is proper, which means that it is consistent with a given label signature Σ_{lab} and the types of operations in it are well formed.

Definition 5.3 (Proper Effect Contexts). *An effect context Ξ is* proper *if, for any $l :: \forall \boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi$, the following holds:*
　• $l : \Pi \boldsymbol{K}^I \rightarrow \mathbf{Lab} \in \Sigma_{\mathrm{lab}}$;
　• *the type schemes $\forall \boldsymbol{\alpha_0}^{I_0} : \boldsymbol{K_0}^{I_0}.\sigma_0$ associated with l by Ξ are uniquely determined; and*
　• *for any* $\mathrm{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K_0}^J.A \Rightarrow B \in \sigma$ *and* $C \in \{A, B\}$, $\boldsymbol{\alpha}^I : \boldsymbol{K}^I, \boldsymbol{\beta}^J : \boldsymbol{K_0}^J \vdash C : \mathbf{Typ}$.

5.2.1 Type Safety. The statement of type safety is as follows. We write $\longrightarrow^*$ for the reflexive, transitive closure of $\longrightarrow$ and $e \nrightarrow$ to denote that there is no e' such that $e \longrightarrow e'$.

Lemma 5.4 (Type Safety). *If $\mathbb{0} \vdash e : A \mid \varepsilon$ and $e \longrightarrow^* e' \nrightarrow$, then one of the following holds:*
　• *$e' = v$ for some value v such that $\mathbb{0} \vdash v : A \mid \varepsilon$; or*
　• *$e' = E[\mathrm{op}_{l\,S^I}\, \boldsymbol{T}^J\, v]$ for some E, l, S^I, op, $\boldsymbol{T}^J$, and v such that $0\text{-}\mathrm{free}(l\,S^I, E)$.*

Table 1. Comparison of the effectful aspects in λ_{EA} and the existing works. The mark ✗ means "not supported," and "explicit*" in the column "polymorphism" for Links indicates that Links supports not only explicit type and effect polymorphism, but also row polymorphism in the style of Rémy [1994] at the effect-level.

	effect collections	collected effects	effect contexts' assignment	polymorphism
λ_{EA}	effect algebras	label	global	explicit
Eff	sets	operation	global	✗
Links	simple rows	operation	local	explicit*
Koka	scoped rows	label	global	implicit

While the type safety guarantees that the result of a program, if any, has the same type as the program, it does not ensure that all operations are handled even if the effect $\mathbb{0}$, which denotes that no unhandled operation remains, is assigned to the program: as shown shortly, the latter property is guaranteed by effect safety.

Type safety is proven via progress and preservation as usual [Wright and Felleisen 1994].

Lemma 5.5 (Progress). *If $\emptyset \vdash e : A \mid \varepsilon$, then one of the following holds: e is a value; $e \longrightarrow e'$ for some e'; or $e = E[\text{op}_{l\,S^l}\,T^J\,v]$ for some E, l, S^I, op, T^J, and v such that $\mathbb{0}{-}\text{free}(l\,S^I, E)$.*

Lemma 5.6 (Preservation). *If $\emptyset \vdash e : A \mid \varepsilon$ and $e \longrightarrow e'$, then $\emptyset \vdash e' : A \mid \varepsilon$.*

5.2.2 Effect Safety. Effect safety is stated as follows.

Lemma 5.7 (Effect Safety). *If $\Gamma \vdash e : A \mid \mathbb{0}$, then there exist no E, l, S^I, op, T^J, and v such that both $e = E[\text{op}_{l\,S^l}\,T^J\,v]$ and $\mathbb{0}{-}\text{free}(l\,S^I, E)$ hold.*

This lemma means, if an expression is assigned to $\mathbb{0}$, no unhandled operation call remains there.

5.2.3 Type-and-Effect Safety. We obtain type-and-effect safety—terminating programs with effect $\mathbb{0}$ always evaluates to values—as a corollary from type safety and effect safety.

Theorem 5.8 (Type-and-Effect Safety). *If $\emptyset \vdash e : A \mid \mathbb{0}$ and $e \longrightarrow^* e' \not\longrightarrow$, then $e' = v$ for some v.*

5.2.4 The Safety of Instances. The three effect algebras EA_{Set}, EA_{SimpR}, and EA_{ScpR} presented thus far meet the safety conditions as stated below, which derives that the effect systems with these algebras enjoy the type-and-effect safety just as corollaries of Theorem 5.8.

Theorem 5.9. *The effect algebras EA_{Set}, EA_{SimpR}, and EA_{ScpR} meet safety conditions (1) and (2).*

6 FORMAL RELATIONSHIPS BETWEEN λ_{EA} AND THE EXISTING SYSTEMS

This section shows that λ_{EA} soundly models the key aspects of the existing effect systems. As targets, we select the effect systems of Pretnar [2015], Hillerström et al. [2017], and Leijen [2017], which employ sets, simple rows, and scoped rows, respectively, to represent effect collections. We call them Eff, Links, and Koka because they model the core part of the programming languages Eff [Bauer and Pretnar 2021], Links [Lindley et al. 2023], and Koka [Leijen 2024], respectively.[5]

6.1 Differences between λ_{EA} and The Selected Systems

We aim to establish the formal connection between each of the existing systems and λ_{EA}, but there exist some gaps between them. First, the existing systems adopt their own syntax not only for

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[5]The core effect system of Links was first presented by Hillerström and Lindley [2016], but it seems to have a minor flaw in the typing of sequential composition. We thus refer to Hillerström et al. [2017] where the flaw is fixed.

effects but also for types and programs, which hinders the formal comparison. To address this problem, we define a syntactic translation $\mathbb{T}_{\mathcal{E}}$ from each $\mathcal{E}$ of the selected systems to the instance of λ_{EA} with the corresponding effect algebra. For example, operation calls in EFF take the form $\text{op}(v, y.c)$, carrying continuations $y.c$. The translator $\mathbb{T}_{\text{EFF}}$ converts it to the expression $\textbf{let } y = \text{op}_l \, \mathbb{T}_{\text{EFF}}(v) \textbf{ in } \mathbb{T}_{\text{EFF}}(c)$ in λ_{EA} using some appropriate label l. Readers interested in the complete definitions of the translations are referred to the supplementary material.

The remaining gaps between λ_{EA} and the existing systems are summarized in Table 1. Because addressing the gaps other than the representation of effect collections is beyond the scope of the present work, we impose certain assumptions on the existing systems for the comparison. In what follows, we detail the gaps and how we address them.

Collected effects. In λ_{EA}, effect collections gather effect labels, which are sets of operations of some specific effects. For example, the effect for state can be expressed by a label State equipped with operations get and set for getting and updating, respectively, the current state. In this style, which we call *label-based*, an operation call is given an effect collection including the effect label to which the called operation belongs, and a handler is required to handle all the operations of a specified label. λ_{EA} and KOKA employ the label-based style. By contrast, EFF and LINKS adopt the *operation-based* style, where effect collections gather operations. In this style, an operation call is given an effect collection including the called operation (not labels), and effect handlers can implement any operation freely. To address this difference, when translating EFF and LINKS in the operation-based style to λ_{EA} in the label-based style, we assume that some labels are given and any effect collection appearing in EFF and LINKS can be decomposed into a subset of the given labels.

Effect contexts' assignment. Our language λ_{EA} supposes that an effect context Ξ is fixed during typechecking one program. We call this assignment of Ξ *global*. EFF and KOKA employ the same assignment style for effect contexts. In contrast, in LINKS, effect contexts can change during the typechecking. For example, consider the following program.

$$\textbf{handle } (\textbf{if } \text{ask} \, () \textbf{ then } 0 \textbf{ else } (\textbf{handle } \text{ask} \, () + 1 \textbf{ with } \{ \textbf{ return } x \mapsto x\} \uplus \{\text{ask } z \, k \mapsto k \, 2\}))$$

$$\textbf{with } \{ \textbf{ return } x \mapsto x\} \uplus \{\text{ask } z \, k \mapsto k \, \text{true}\}$$

In this program, both ask operation calls take the unit value, but the first and second ones return Booleans and integers, respectively. This program cannot be typechecked if an effect context is globally fixed. LINKS can typecheck it because LINKS allows enclosing handlers to modify effect contexts; namely, effect contexts are assigned *locally*. To address the local assignment of effect contexts, we assume that every operation has a unique, closed type in LINKS, which enables determining the types of operations globally.

Polymorphism. The languages λ_{EA}, LINKS, and KOKA support type and effect polymorphism. Among them, only the polymorphism in KOKA is *implicit*, that is, no term constructor for type abstraction and application is given. Unfortunately, it is not straightforward to translate a program (or its typing derivation) with implicit polymorphism in KOKA to one with *explicit* polymorphism in λ_{EA} while preserving the meaning of the program because KOKA does not adopt *value restriction* [Tofte 1990; Wright 1995]. Our approach to this difference in polymorphism is simply to forbid the use of implicit polymorphism in KOKA and instead introduce explicit polymorphism by equipping KOKA with term constructors for type abstraction and application as in λ_{EA} and LINKS. It is also noteworthy that LINKS supports more advanced polymorphism, inspired by row polymorphism proposed by Rémy [1994]. It introduces *presence types*, which can state that a specific label is present or absent in a row, *presence polymorphism*, and effect variables constrained by which labels are present or absent. This form of polymorphism facilitates solving unification problems in the

composition of effect handlers [Hillerström and Lindley 2016]. Our translation from Links to λ_{EA} addresses these unique features in Links as follows: first, present labels remain in the translated row but labels with the absent flag do not; second, the constraints on effect variables are ignored; third, we assume that programs to be translated do not use presence polymorphism. We left the support for presence polymorphism as future work: it seems to be motivated by unification and type inference, which are beyond the scope of the present work.

6.2 Type-and-Effect Preservation of Translations

We show that the translations preserve well-typedness under the aforementioned assumptions.

Theorem 6.1. *Let* $(\mathcal{E}, \mathcal{A}) \in \{(\text{Eff}, \text{EA}_{\text{Set}}), (\text{Links}, \text{EA}_{\text{SimpR}}), (\text{Koka}, \text{EA}_{\text{ScpR}})\}$. *If a program c in the system* $\mathcal{E}$ *is well typed at an effect* ϵ, *then* $\mathbb{T}_{\mathcal{E}}(c)$ *is well typed at effect* $\mathbb{T}_{\mathcal{E}}(\epsilon)$ *in* λ_{EA} *with* $\mathcal{A}$.

This result guarantees that, for each $\mathcal{E}$ of the selected systems, the programs in $\mathcal{E}$ can be safely executed in the semantics of λ_{EA}. In other words, λ_{EA} can work as an intermediate language that ensures type-and-effect safety. Note that the equivalence relation on scoped rows in Koka is more restrictive than $\sim_{\text{ScpR}}$ in EA_{ScpR} because the row equivalence in Koka allows swapping effect labels $l_1 \, S_1$ and $l_2 \, S_2$ only if $l \neq l'$, whereas $\sim_{\text{ScpR}}$ allows their swapping if the label names l_1 and l_2, *or the* type arguments S_1 and S_2 are different. This gap does not prevent proving Theorem 6.1 because it only means that λ_{EA} with EA_{ScpR} may accept more programs than Koka. We will show an effect algebra with the row equivalence in Koka in Section 7.2.

7 EXTENSIONS OF λ_{EA}

This section extends λ_{EA} and safety conditions to lift coercions and type-erasure semantics. We also introduce effect algebras safe for these extensions (including a new one based on multisets) and discuss how adaptable each effect representation addressed in this paper—sets, multisets, simple rows, and scoped rows—is for the extensions.

7.1 Lift Coercions

This section shows an extension to *lift coercions* [Biernacki et al. 2018, 2019] (also known as in-jection [Leijen 2018] or masking [Leijen 2024]). Given an effect label, a lift coercion forbids the innermost handler for the label to handle any operation of the label. They can prevent *accidental handling*, a situation that an effect handler handles an operation call against the programmer's intention. This paper focuses on how λ_{EA} is extended with lift coercions; see the prior work [Biernacki et al. 2018, 2019; Leijen 2018] for the detail of the accidental handling and how lift coercions work to address it. We also show that the effect algebras EA_{Set} and EA_{SimpR} are unsafe in the extension and that EA_{ScpR} and a new effect algebra for *multisets* are safe. Note that Biernacki et al. [2019] introduce coercions in other forms. We do not support them because they can be encoded with lift coercions (if label polymorphism is not used) [Biernacki et al. 2018, 2019].

7.1.1 Extending λ_{EA} to Lift Coercions. We show the extended part of λ_{EA} in Figure 6. Expressions and evaluation contexts are extended with lift coercions $[-]_L$. To define the semantics of lift coercions, we generalize 0-freeness to *n-freeness* for an arbitrary natural number n by following Biernacki et al. [2018]. The predicate $n-\text{free}(L, E)$ is defined by the rules in Figure 6 in addition to the ones given previously (Figure 4). Intuitively, $n-\text{free}(L, E)$ means that, for an operation op of L, the operation call in $E[\text{op}_L \, T^{\overline{j}} \, v]$ will be handled by the $(n+1)$-th innermost enclosing handler for L. For example, $1-\text{free}(L, [\square]_L)$ and $0-\text{free}(L, \textbf{handle}_L \, [\square]_L \, \textbf{with} \, h_1)$ hold. Because the semantics of the effect handling (specifically, the reduction rule R_Handle2 in Figure 4) requires the label of the handled operation call to be 0-free in the evaluation context enclosing the operation call, the

$$e ::= \cdots \mid [e]_L \quad \text{(expressions)} \qquad E ::= \cdots \mid [E]_L \quad \text{(evaluation contexts)}$$

Freeness of labels $\boxed{n-\text{free}(L, E)}$

$$\frac{(n+1)-\text{free}(L, E)}{n-\text{free}(L, \mathbf{handle}_L\, E \text{ with } h)} \qquad \frac{n-\text{free}(L, E)}{n+1-\text{free}(L, [E]_L)} \qquad \frac{n-\text{free}(L, E) \quad L \neq L'}{n-\text{free}(L, [E]_{L'})}$$

Reduction $\boxed{e \longmapsto e'}$ **Typing** $\boxed{\Gamma \vdash e : A \mid \varepsilon}$

$$\frac{}{[v]_L \longmapsto v}\ \text{R\_Lift} \qquad\qquad \frac{\Gamma \vdash e : A \mid \varepsilon' \quad \Gamma \vdash L : \mathbf{Lab} \quad (L)^{\uparrow} \odot \varepsilon' \sim \varepsilon}{\Gamma \vdash [e]_L : A \mid \varepsilon}\ \text{T\_Lift}$$

Fig. 6. The extension for lift coercions.

operation call in $\mathbf{handle}_L\, \mathbf{handle}_L\, [\mathsf{op}_L\, v]_L$ with h_1 with h_2 will be handled by h_2. If a lift coercion is given a value, it returns the value as it is (R_Lift). The type-and-effect system is extended with the rule T_Lift, which allows the information ε' of effects of an expression e to pass through the innermost effect handler for a label L by prepending L to ε'.

7.1.2 Safety Conditions and Type-and-Effect Safety. To ensure the safety of programs in the presence of lift coercions, we introduce a new safety condition in addition to the ones given in Section 5.

Definition 7.1 (Safety Condition for Lift Coercions). *The safety condition added for lift coercions is:*
(3) If $(L)^{\uparrow} \odot \varepsilon_1 \sim (L_1)^{\uparrow} \odot \cdots \odot (L_n)^{\uparrow} \odot (L)^{\uparrow} \odot \varepsilon_2$ *and* $L \notin \{L_1, \ldots, L_n\}$, *then* $\varepsilon_1 \sim (L_1)^{\uparrow} \odot \cdots \odot (L_n)^{\uparrow} \odot \varepsilon_2$.

This new condition can be understood as follows. First, let ε_2 be an effect of an expression e. Then, the effect of the expression $[\cdots [[e]_L]_{L_n} \cdots]_{L_1}$ is given as $(L_1)^{\uparrow} \odot \cdots \odot (L_n)^{\uparrow} \odot (L)^{\uparrow} \odot \varepsilon_2$. Assume that the expression is handled by an effect handler for L and the remaining effect is ε_1. Then, ε_1 should retain the information that e is surrounded by lift coercions for $L_1, \cdots, L_n$ because the handling expression may be enclosed by effect handlers for $L_1, \cdots, L_n$. Such information is described by $(L_1)^{\uparrow} \odot \cdots \odot (L_n)^{\uparrow} \odot \varepsilon_2$. Thus, safety condition (3) requires $\varepsilon_1 \sim (L_1)^{\uparrow} \odot \cdots \odot (L_n)^{\uparrow} \odot \varepsilon_2$.

To see the importance of the new safety condition more concretely, we show that the effect algebras EA_{Set} and EA_{SimpR} violate this new condition and then present how they make some unsafe programs typeable.

Theorem 7.2 (Unsafe Effect Algebras with Lift Coercions). *The effect algebras* EA_{Set} *and* EA_{SimpR} *do not meet safety condition (3). Furthermore, there exists an expression that is well typed under* EA_{Set} *and* EA_{SimpR} *and gets stuck.*

Proof. We consider only EA_{Set} here; a similar discussion can be applied to EA_{SimpR}. Recall that the operation $\odot$ in EA_{Set} is implemented by the set union, so it meets idempotence: $\{L\} \cup \{L\} \sim \{L\}$. Furthermore, we can use the empty set as the identity element, so $\{L\} \cup \{L\} \sim \{L\} \cup \{\}$. If safety condition (3) was met, $\{L\} \sim \{\}$ (where $\{L\}$, $\{\}$, and 0 are taken as ε_1, ε_2, and n, respectively, in Definition 7.1). However, the equivalence does not hold.

As a program that is typeable under EA_{Set}, consider $\mathbf{handle}_{\text{Exc}}\, [\mathsf{raise}_{\text{Exc}}\, \mathsf{Unit}\, ()]_{\text{Exc}}$ with h where Exc :: $\{\mathsf{raise} : \forall \alpha : \mathbf{Typ}.\mathsf{Unit} \Rightarrow \alpha\}$. This program can be typechecked under an appropriate assumption as illustrated by the following typing derivation:

$$\cfrac{\cdots \quad \{\mathsf{Exc}\} \cup \{\} \sim \{\mathsf{Exc}\} \qquad \cfrac{\emptyset \vdash \mathsf{raise}_{\mathsf{Exc}}\, \mathsf{Unit}\, () : A \mid \{\mathsf{Exc}\} \quad \{\mathsf{Exc}\} \cup \{\mathsf{Exc}\} \sim \{\mathsf{Exc}\}}{\emptyset \vdash [\mathsf{raise}_{\mathsf{Exc}}\, \mathsf{Unit}\, ()]_{\mathsf{Exc}} : A \mid \{\mathsf{Exc}\}}\ \text{T\_Lift}}{\emptyset \vdash \mathbf{handle}_{\mathsf{Exc}}\, [\mathsf{raise}_{\mathsf{Exc}}\, \mathsf{Unit}\, ()]_{\mathsf{Exc}} \text{ with } h : B \mid \{\}}\ \text{T\_Handling}$$

Freeness of label names $\boxed{n-\text{free}(l, E)}$

$$\frac{}{0-\text{free}(l, \square)} \qquad \frac{n-\text{free}(l, E)}{n-\text{free}(l, \mathbf{let}\ x = E\ \mathbf{in}\ e)} \qquad \frac{n-\text{free}(l, E) \quad l \neq l'}{n-\text{free}(l, \mathbf{handle}_{l'\ S^I}\ E\ \mathbf{with}\ h)}$$

Reduction $\boxed{e \longmapsto e'}$

$$\frac{\mathsf{op}\ \boldsymbol{\beta}^{\mathcal{J}} : K^{\mathcal{J}}\ p\ k \mapsto e \in h \quad v_{cont} = \lambda z.\mathbf{handle}_{l\ S^I}\ E[z]\ \mathbf{with}\ h \quad 0-\text{free}(l, E)}{\mathbf{handle}_{l\ S^I}\ E[\mathsf{op}_{l\ S'^I}\ T^{\mathcal{J}}\ v]\ \mathbf{with}\ h \longmapsto e[T^{\mathcal{J}}/\boldsymbol{\beta}^{\mathcal{J}}][v/p][v_{cont}/k]} \quad \text{R\_Handle2'}$$

Fig. 7. Type-erasure semantics.

However, the call to raise is not handled as it needs to be handled by the *second* closest handler. ∎

In contrast, the effect algebra EA_{ScpR} for scoped rows satisfies safety condition (3). The point is that $\odot_{\text{ScpR}}$ in EA_{ScpR} is *not* idempotent. Therefore, they can represent how many lift coercions are used and how many effect handlers are necessary to handle expressions as the information of effects. This observation gives us a new effect algebra with *multisets*. Multisets can have multiple instances of the same element and their sum operation is also nonidempotent. Thus, we can expect—and it is the case—that the algebra for multisets meets safety condition (3) as well as the other conditions.

Example 7.3 (Effect Multisets). The effect signature $\Sigma_{\text{eff}}^{\text{MSet}}$ of effect multisets is given by $\{\} : \mathbf{Eff}$, $\{-\} : \mathbf{Lab} \to \mathbf{Eff}$, and $- \sqcup - : \mathbf{Eff} \times \mathbf{Eff} \to \mathbf{Eff}$ (which is the sum operation for multisets). An effect algebra EA_{MSet} for multisets is defined by $\langle \Sigma_{\text{eff}}^{\text{MSet}}, - \sqcup -, \{\}, \{-\}, \sim_{\text{MSet}} \rangle$ where $\sim_{\text{MSet}}$ is the least equivalence relation satisfying the same rules as $\sim_{\text{Set}}$ except for the idempotence rule.

Theorem 7.4. *The effect algebras* EA_{ScpR} *and* EA_{MSet} *meet safety conditions (1)–(3).*

The type-and-effect safety of λ_{EA} with lift coercions is proven similarly to Theorem 5.8 provided that an effect algebra meets safety conditions (1)–(3).

Theorem 7.5 (Type-and-Effect Safety). *Assume that a given effect algebra meets safety conditions (1)–(3). If* $\emptyset \vdash e : A \mid \mathbb{0}$ *and* $e \longrightarrow^* e' \not\longrightarrow$, *then* $e' = v$ *for some* v.

7.2 Type-Erasure Semantics

This section shows an adaption of λ_{EA} to type-erasure semantics, which is different from those given in Sections 4 and 7.1 in that it does not rely on type arguments of label names in seeking effect handlers matching with called operations. Type erasure semantics is helpful to develop efficient implementations of effect handlers with parametric effects [Biernacki et al. 2019].

7.2.1 Formal Definition of Type-Erasure Semantics. The part modified to support the type-erasure semantics is shown in Figure 7. The label freeness in the type-erasure semantics refers only to label names, while the original definition in Figure 4 refers to entire labels. The only change in the semantics is that the reduction rule R_Handle2 is replaced by R_Handle2' presented in Figure 7. For instance, consider an expression $\mathbf{handle}_{\text{State Int}}\ (\mathbf{handle}_{\text{State Bool}}\ (\mathsf{set}_{\text{State}\ A}\ v)\ \mathbf{with}\ h_1)\ \mathbf{with}\ h_2$. In the original semantics, it depends on the type argument A which of h_1 and h_2 handles the operation call. By contrast, in the type-erasure semantics, the handler h_1 will be chosen regardless of A. The type-and-effect system is not changed.

7.2.2 Safety Conditions and Type-and-Effect Safety. To ensure the safety in the type-erasure semantics, we need an additional safety condition.

Definition 7.6 (Safety Condition for Type-Erasure). *The safety condition added for the type-erasure semantics is: (4) If $(l\,S_1{}^I)^\uparrow \oslash \varepsilon$ and $(l\,S_2{}^I)^\uparrow \odot \varepsilon' \sim \varepsilon$, then $S_1{}^I = S_2{}^I$.*

To understand this condition, assume that an operation of label name l is called with typelike parameters $S_1{}^I$ and some effect ε_1 such that $(l\,S_1{}^I)^\uparrow \oslash \varepsilon$ is assigned to the operation call via subtyping. When the operation call is handled by an effect handler for effect label $l\,S_2{}^I$, the typelike parameters $S_1{}^I$ for the operation call and $S_2{}^I$ for the handler must be matched. None of the effect algebras EA_{Set}, EA_{SimpR}, EA_{ScpR}, and EA_{MSet} presented thus far meets this new condition, and, even worse, they can accept some programs unsafe in the type-erasure semantics.

Theorem 7.7 (Unsafe Effect Algebras in Type-Erasure Semantics). *The effect algebras EA_{Set}, EA_{MSet}, EA_{SimpR}, and EA_{ScpR} do not meet safety condition (4). Furthermore, there exists an expression that is well typed under these algebras and gets stuck.*

PROOF. Here we focus on the effect algebra EA_{Set}, but a similar discussion can be applied to the other algebras. Recall that $\odot$ in EA_{Set} is implemented by the union operation for sets, and therefore it is commutative (i.e., it allows exchanging labels in a set no matter what label names and what type arguments are in the labels). Hence, for example, $\{l\,\text{Int}\} \cup \{l\,\text{Bool}\} \sim_{\text{Set}} \{l\,\text{Bool}\} \cup \{l\,\text{Int}\}$ for a label name l taking one type parameter. It means that EA_{Set} violates safety condition (4).

To give a program that is typeable under EA_{Set} but unsafe in the type-erasure semantics, consider the following which uses an effect label $\text{Writer} :: \forall\alpha : \textbf{Typ}.\{\text{tell} : \alpha \Rightarrow \text{Unit}\}$:

handle$_{\text{Writer Int}}$ **handle**$_{\text{Writer Bool}}$

    tell$_{\text{Writer Int}}$ 42

  with { **return** $x \mapsto 0$} $\uplus$ {tell $p\,k \mapsto$ **if** p **then** 0 **else** 42}

with { **return** $x \mapsto x$} $\uplus$ {tell $p\,k \mapsto p$}

This program is well typed because
- the operation call tell$_{\text{Writer Int}}$ 42 can have effect $\{\text{Writer Bool}\} \cup \{\text{Writer Int}\}$ via subeffecting $\{\text{Writer Int}\} \oslash \{\text{Writer Bool}\} \cup \{\text{Writer Int}\}$ (which holds because Writer Int and Writer Bool are exchangeable),
- the inner handling expression is well typed and its effect is $\{\text{Writer Int}\}$, and
- the outer one is well typed and its effect is $\{\}$.

Note that this typing rests on the fact that the inner handler assumes that the argument variable p of its tell clause will be replaced by Boolean values as indicated by the type argument Bool to Writer. However, the variable p will be replaced by integer 42 and the program will get stuck. ∎

The proof of Theorem 7.7 relies on the commutativity of $\odot$ in each effect algebra. This observation indicates that an effect algebra with *non*commutative $\odot$ can be safe even in the type-erasure semantics. In fact, the previous work [Biernacki et al. 2019; Leijen 2017, 2018] has given an instance of such an effect algebra. By following it, we can adapt the effect algebras defined thus far to be safe in the type-erasure semantics; we call the effect collections in such effect algebras *erasable*.

Example 7.8 (Erasable Effect Algebras). An effect algebra EA_{ESet} for erasable sets is defined similarly to EA_{Set}. The only difference is that the equivalence relation $\sim_{\text{ESet}}$ of EA_{ESet} is defined as $\sim_{\text{Set}}$, but the commutativity rule used in the definition of $\sim_{\text{Set}}$ is replaced with

$$\frac{l_1 \neq l_2}{\{l_1\,S_1{}^{I_1}\} \cup \{l_2\,S_2{}^{I_2}\} \sim_{\text{ESet}} \{l_2\,S_2{}^{I_2}\} \cup \{l_1\,S_1{}^{I_1}\}}$$

Table 2. Comparison of the effect algebras.

	Lift coercions	Adaptable to type-erasure	Multiple effect variables
EA_{Set}	✗	✓	✓
EA_{MSet}	✓	✓	✓
EA_{SimpR}	✗	✓	✗
EA_{ScpR}	✓	✓	✗

which only allows exchanging labels with different names. Effect algebras EA_{ESet}, EA_{EMSet}, and EA_{EScpR} for erasable sets, multisets, and scoped rows, respectively, are defined similarly.

Theorem 7.9. *The effect algebras* EA_{ESet}, EA_{EMSet}, $\text{EA}_{\text{ESimpR}}$, *and* EA_{EScpR} *meet safety conditions (1), (2), and (4).*

Note that some equivalence properties holding on nonerasable effect collections do not hold on erasable ones. For instance, $\{\text{Writer Int}\} \cup \{\text{Writer Bool}\} \sim \{\text{Writer Bool}\} \cup \{\text{Writer Int}\}$ and $\rho_1 \cup \rho_2 \sim \rho_2 \cup \rho_1$ do not hold in erasable sets. The latter equivalence is not allowed because ρ_1 and ρ_2 may be replaced with, e.g., $\{\text{Writer Int}\}$ and $\{\text{Writer Bool}\}$, respectively. This limitation could be relaxed by supporting qualified types [Jones 1992].

Finally, we can prove the type-and-effect safety of λ_{EA} with the type-erasure semantics as Theorem 5.8 provided that an effect algebra meets safety conditions (1), (2), and (4).

Theorem 7.10 (Type-and-Effect Safety). *Assume that a given effect algebra meets safety conditions (1), (2), and (4). If $\emptyset \vdash e : A \mid \emptyset$ and $e \longrightarrow^* e' \nrightarrow$, then $e' = v$ for some v.*

7.3 Mixing Lift Coercions and Type-Erasure Semantics

It is easy to extend λ_{EA} with both lift coercions and type-erasure semantics and prove its type-and-effect safety if a given effect algebra is assumed to meet safety conditions (1)–(4). Among the effect algebras presented in the paper, only EA_{EScpR} satisfies these conditions, and so λ_{EA} instantiated with it is type-and-effect safe. See the supplementary material for the detail of the combination.

8 COMPARISON OF EFFECT ALGEBRAS

In this section, we discuss how different the effect algebras EA_{Set}, EA_{MSet}, EA_{SimpR}, and EA_{ScpR} are; it is summarized in Table 2. The first column in Table 2 presents whether the effect algebras are safe in the presence of lift coercions. As shown in Section 7.1, EA_{Set} and EA_{SimpR} are unsafe and EA_{MSet} and EA_{ScpR} are safe. The second column indicates whether the effect algebras can be adapted to the type-erasure semantics. As discussed in Section 7.2, none of the compared effect algebras is safe as it is, but all of them become safe if we can admit restricting the commutativity of the concatenation on effect collections. The third column shows whether each effect algebra allows multiple effect variables to appear in one effect collection. While EA_{SimpR} and EA_{ScpR} disallow it because effect variables can appear only at the end of rows, neither EA_{Set} nor EA_{MSet} has such a restriction.

Allowing multiple effect variables in one effect collection in EA_{Set} and EA_{MSet} leads to more powerful abstraction of effect collections. For example, consider a module interface IntSet for integer sets, which is given using EA_{Set}:

$$\exists \alpha : \textbf{Typ}. \exists \rho : \textbf{Eff}.\{ \quad empty : \alpha, \quad add : \text{Int} \rightarrow_{\{\}} \alpha \rightarrow_{\{\}} \alpha, \quad \cdots, \quad choose : \alpha \rightarrow_\rho \text{Int},$$
$$accumulate : \forall \beta : \textbf{Typ}. \forall \rho' : \textbf{Eff}.(\text{Unit} \rightarrow_{\rho \cup \rho'} \beta) \rightarrow_{\rho'} \beta \, \text{List} \quad \}$$

In this type, type variable α is an abstract type representing integer sets, and the fields represent the operations on integer sets. The interface IntSet requires modules to implement, in addition to

the basic operations on sets (e.g., the empty set *empty* and the addition of integers to sets *add*), two additional functions for nondeterministic computation. Function *choose* takes an integer set, performs some abstract effect ρ, and returns one element of it. Intuitively, the abstract effect ρ has a role of notifying that *choose* is called. The other function *accumulate* is a higher-order function, taking a function that may perform effects ρ and ρ'. Thus, the argument function may call *choose*. Intuitively, function *accumulate* collects all the results of the computation that the argument function performs with some value in a set passed to *choose* during its execution. The argument function may invoke any effect ρ', which leaks to the call side of *accumulate*. The type of the argument function represents that it may invoke two *abstract* effects ρ and ρ' using the union operation $\underline{\cup}$ in EA$_{\text{Set}}$. The use of effect variable ρ enables abstracting module implementations over not only what effect labels are used in the implementations but also *how many labels are used there*. We provide some implementation examples of IntSet with different numbers of effect labels in the supplementary material. Note that the type interface IntSet cannot be expressed in EA$_{\text{SimpR}}$ nor EA$_{\text{ScpR}}$ because only one effect variable may appear in a row.

However, this is not the end of the story: some existing works have discussed benefits of using rows as effect collections. Hillerström and Lindley [2016] demonstrated that simple rows with row polymorphism in the style of Rémy [1994] are useful to solve the unification occurring in the composition of effect handlers. Leijen [2017] implemented a sound and complete type inference for the effect system with polymorphism by utilizing scoped rows. The current form of our theoretical framework, effect algebras, does not provide a means to discuss unification and type inference for algebraic effects and handlers, and it is left open how we can address it in an abstract manner.

9 RELATED WORK

We have explained the existing effect systems for effect handlers in Section 2, compared some of them with the instances of our effect system in Section 6. We will also discuss what aspect of effect handlers our framework does not support in Section 10.

Although, as far as we know, there is no prior work on abstracting effect systems for effect handlers with nor without algebraic structures, the research on generic effect systems that can reason about the use of a wide range of effects (such as file resource usage, memory usage and management, and exception checking) has been conducted. Marino and Millstein [2009] proposed a monomorphic type-and-effect system that tracks a set of capabilities (or privileges) to perform effectful operations such as memory manipulation and exception raising. Their effect system is generic in that it is parameterized over the forms of capabilities as well as the adjustments and checkings of capabilities per context. It assumes that capabilities are gathered into a set and its typing discipline relies on the set operations (e.g., the subeffecting is implemented by set inclusion). Rytz et al. [2012] generalized Marino and Millstein's effect system by allowing the use of a join semilattice to represent collections of capabilities and introducing effect polymorphism. Join-semilattices are underlying structures of effects in effect systems for *may analysis*. In such a system, the join operation $\sqcup$ and the ordering relation $\sqsubseteq$ in a join semilattice are used to merge multiple effects into one and to introduce effect overapproximation as subeffecting, respectively. As $\sqsubseteq$ can be induced by $\sqcup$ ($x \sqsubseteq y \iff x \sqcup y = y$), we define the subeffecting $\oslash$ using $\odot$ in an effect algebra ($\varepsilon_1 \oslash \varepsilon_2 \iff \exists \varepsilon. \varepsilon_1 \odot \varepsilon \sim \varepsilon_2$). Thus, the role of $\odot$ is similar to that of $\sqcup$, but $\odot$ is not required to be commutative nor idempotent, unlike $\sqcup$ (note that join operations are characterized by associativity, commutativity, and idempotence). In fact, $\odot$ in the effect algebra EA$_{\text{ScpR}}$ or EA$_{\text{MSet}}$ is nonidempotent, which is key to support lift coercions (Section 7.1.2), and $\odot$ in each of the effect algebras being safe in the type-erasure semantics is noncommutative (Section 7.2.2).

Recent developments of generic effect systems have focused on *sequential effect systems* [Atkey 2009; Gordon 2017, 2021; Katsumata 2014; Mycroft et al. 2016; Tate 2013], which aim to reason

about the properties where the order of effects matters (e.g., whether no closed file will be read nor written). An approach common in the prior work on sequential effect systems is to introduce sequential composition $\rhd$, an operation to compose effects happening sequentially. For example, given expressions e_1 with effect ε_1 and e_2 with ε_2, the effect of a let-expression $\textbf{let } x = e_1 \textbf{ in } e_2$ is given by $\varepsilon_1 \rhd \varepsilon_2$. The sequential composition can be characterized as a (partial) monoid. Thus, it might look similar to $\odot$ in an effect algebra, but their roles are significantly different: $\odot$ is used to expand (i.e., overapproximate) effects and remove specific labels from effects, whereas the sequential composition $\rhd$ is used to compose the effects of expressions executed sequentially. In fact, if we were use $\odot$ to sequence effects, the safety of λ_{EA} in the type-erasure semantics would not hold even in the effect algebra EA_{EScpR} for erasable scoped rows. For example, assume that an expression $\textbf{let } x = e_1 \textbf{ in } e_2$ is given effect $\varepsilon_1 \odot \varepsilon_2$ if the effects ε_1 and ε_2 are assigned to e_1 and e_2. Then, the expression $e \overset{\text{def}}{=} \textbf{let } x = \text{tell}_{\text{Writer Bool}} \text{ true } \textbf{in } \text{tell}_{\text{Writer Int}} 1$ could have the effect $\{\text{Writer Bool}\} \sqcup \{\text{Writer Int}\}$ under EA_{EScpR}. Thus, the expression $\textbf{handle}_{\text{Writer Int}} (\textbf{handle}_{\text{Writer Bool}} e \textbf{ with } h_1) \textbf{ w}$ would be well typed (for some appropriate effect handlers h_1 and h_2), although it may get stuck in the type-erasure semantics because the operation call $\text{tell}_{\text{Writer Int}} 1$ will be handled by the effect handler h_1 for Writer Bool. Readers might wonder why $\odot$ cannot work as a sequential composition despite the fact that join operations, which are also used to overapproximate effects, can. We think that this is because the assumptions on $\odot$ are weaker than those on join operations as discussed above. Making $\odot$ in effect algebras and $\rhd$ in sequential effect systems coexist is a promising future direction, motivated by the recent study on sequential effect systems for control operators [Gordon 2020; Sekiyama and Unno 2023; Song et al. 2022].

10 CONCLUSION AND FUTURE WORK

In this paper, we give λ_{EA} equipped with the abstract effect system that can be instantiated to concrete effect systems, define safety conditions on effect algebras, and prove the type-and-effect safety of λ_{EA} by assuming that a given effect algebra satisfies the conditions. As far as we know, no research formalizes the differences among effect systems for effect handlers nor the requirements for the effect systems to prove safety properties. We reveal these essences via the abstraction of effect systems by effect algebras, and the formalization of the safety conditions. The safety conditions added for lift coercions or type-erasure semantics clarify the differences among effect algebras. In the rest of the paper, we discuss possible directions for future work.

Abstraction of handling mechanisms. Although the framework in the paper targets deep effect handlers, adapting it to shallow effect handlers is easy. In fact, we have provided this adaption and proved its safety under the same safety conditions as the ones given in the main paper; interested readers are referred to the supplementary material. In the literature, there are other proposals of the effect handling, especially for resolving the problem with *accidental handling* without relying on lift coercions. For instance, local effects [Biernacki et al. 2019], tunneling [Zhang and Myers 2019], and lexically scoped effect handlers [Biernacki et al. 2020; Brachthäuser et al. 2020] have been proposed. These approaches can be applied to address the accidental handling, but they employ significantly different styles. For example, lexically scoped effect handlers can enable a new notion of effect polymorphism, called *contextual polymorphism* [Brachthäuser et al. 2020]. Exploring abstraction to accommodate all of these mechanisms is a challenging but interesting direction.

Abstraction for unification and type inference. As mentioned in Section 6, our framework has not yet exposed the essential roles of rows in their main application—unification and type inference. One of our ambitious goals for future research is to give a theoretical framework that can discover differences among effect representations in unification and type-inference, which

have been well explored with concrete effect representations, such as sets [Pretnar 2014], simple rows [Hillerström and Lindley 2016], and scoped rows [Leijen 2017], but not in an abstract manner.

Abstraction of constrained effect collections. Another interesting direction is to abstract *constrained* effect collections. For example, Hillerström and Lindley [2016] introduce Rémy's row polymorphism, which can state that some labels are present or absent in row variables, for effective unification and Tang et al. [2024] propose an effect system that allows type abstraction over subtyping constraints on row variables. Row constraints have been extensively studied for programming with records and variants [Cardelli and Mitchell 1989; Harper and Pierce 1991; Jones 1992; Rémy 1994]. Morris and McKinna [2019] proposed a type system which treats rows and constraints on them abstractly. Integrating the idea of their work with our framework is a promising approach.

Abstraction of implementation techniques. One approach to implementing effect handlers is to apply type-directed translation into an intermediate language [Hillerström et al. 2017; Leijen 2017; Schuster et al. 2022; Xie et al. 2020]. Exploring the type-directed translations and optimization techniques proposed thus far, such as a selective translation into continuation passing style (CPS) [Leijen 2017], in an abstract manner may lead to a common implementation infrastructure for languages with different effect systems or give an insight into the influence of effect representations on efficiency.

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1 Definitions

Remark 1.1 (Notation)**.** *We write $\boldsymbol{\alpha}^I$ for a finite sequence $\alpha_0, \ldots, \alpha_n$ with an index set $I = \{0, \ldots, n\}$, where α is any metavariable. We also write $\{\boldsymbol{\alpha}^I\}$ for the set consisting of the elements of $\boldsymbol{\alpha}^I$.*

Definition 1.2 (Kinds)**.** *Kinds are defined as $K ::= \mathbf{Typ} \mid \mathbf{Lab} \mid \mathbf{Eff}$.*

Definition 1.3 (Signatures)**.** *Given a set S of label names, a label signature Σ_{lab} is a functional relation whose domain $\mathrm{dom}(\Sigma_{\mathrm{lab}})$ is S. The codomain of Σ_{lab} is the set of functional kinds of the form $\Pi_{i \in I} K_i \to \mathbf{Lab}$ for some I and $K_i^{i \in I}$ (if $I = \emptyset$, it means $\mathbf{Lab}$ simply). Similarly, given a set S of effect constructors, an effect signature Σ_{eff} is a function relation whose domain $\mathrm{dom}(\Sigma_{\mathrm{eff}})$ is S and its codomain is the set of functional kinds of the form $\Pi_{i \in I} K_i \to \mathbf{Eff}$ for some I and $K_i^{i \in I}$. A signature Σ is the disjoint union of a label signature and an effect signature. We write $\Pi \boldsymbol{K}^I \to K$, and more simply, $\Pi \boldsymbol{K} \to K$ as an abbreviation for $\Pi_{i \in I} K_i \to K$.*

Remark 1.4. *We write $\mathcal{C} : \Pi \boldsymbol{K} \to K$ to denote the pair $\langle \mathcal{C}, \Pi \boldsymbol{K} \to K \rangle$ for label name or effect constructor $\mathcal{C}$.*

Definition 1.5 (The Syntax of λ_{EA})**.** *Given a signature $\Sigma = \Sigma_{\mathrm{lab}} \uplus \Sigma_{\mathrm{eff}}$, the syntax of λ_{EA} is defined as follows.*

I, J, N	(index sets)	i, j, n, r	(indices)
f, g, x, y, z, p, k	(variables)	$\alpha, \beta, \gamma, \tau, \iota, \rho$	(typelike variables)
op	(operation names)	$l \in \mathrm{dom}(\Sigma_{\mathrm{lab}})$	(label names)
$\mathcal{F} \in \mathrm{dom}(\Sigma_{\mathrm{eff}})$	(effect constructors)	$\mathcal{C} \in \mathrm{dom}(\Sigma_{\mathrm{lab}}) \cup \mathrm{dom}(\Sigma_{\mathrm{eff}})$	

$$
\begin{array}{rcll}
K & ::= & \mathbf{Typ} \mid \mathbf{Lab} \mid \mathbf{Eff} & \text{(kinds)} \\
S, T & ::= & A \mid L \mid \varepsilon & \text{(typelikes)} \\
A, B, C & ::= & \tau \mid A \to_\varepsilon B \mid \forall \alpha : K.A^\varepsilon & \text{(types)} \\
L & ::= & \iota \mid l\, \boldsymbol{S}^I & \text{(labels)} \\
\varepsilon & ::= & \rho \mid \mathcal{F}\, \boldsymbol{S}^I & \text{(effects)} \\
\sigma & ::= & \{\} \mid \sigma \uplus \{\mathsf{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K}^J.A \Rightarrow B\} & \text{(operation signatures)} \\
\Xi & ::= & \emptyset \mid \Xi, l :: \forall \boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma & \text{(effect contexts)} \\
\Gamma & ::= & \emptyset \mid \Gamma, x : A \mid \Gamma, \alpha : K & \text{(typing contexts)} \\
e & ::= & v \mid v_1\, v_2 \mid v\, S \mid \mathbf{let}\, x = e_1\, \mathbf{in}\, e_2 \mid \mathbf{handle}_{l\, \boldsymbol{S}^I}\, e\, \mathbf{with}\, h & \text{(expressions)} \\
v & ::= & x \mid \mathbf{fun}\,(f, x, e) \mid \Lambda \alpha : K.e \mid \mathsf{op}_{l\, \boldsymbol{S}^I}\, \boldsymbol{T}^J & \text{(values)} \\
h & ::= & \{\, \mathbf{return}\, x \mapsto e\} \mid h \uplus \{\mathsf{op}\, \boldsymbol{\beta}^J : \boldsymbol{K}^J\, p\, k \mapsto e\} & \text{(handlers)} \\
E & ::= & \square \mid \mathbf{let}\, x = E\, \mathbf{in}\, e \mid \mathbf{handle}_{l\, \boldsymbol{S}^I}\, E\, \mathbf{with}\, h & \text{(evaluation contexts)}
\end{array}
$$

Remark 1.6. *We write $\lambda x.e$ for $\mathbf{fun}\,(f, x, e)$ if variable f does not occur in expression e.*

Definition 1.7 (Free Variables)**.** *The notion of free variables is defined as usual. We write $\mathrm{FV}(e)$ for the set of free variables in expression e.*

Definition 1.8 (Free Typelike Variables)**.** *The notion of free typelike variables is defined as usual. We write $\mathrm{FTV}(e)$ and $\mathrm{FTV}(S)$ for the sets of free typelike variables in expression e and typelike S, respectively.*

Definition 1.9 (Value Substitution)**.** *Substitution $e[v/x]$ and $h[v/x]$ of value v for variable x in expression e and handler h, respectively, are defined as follows:*

$$
\begin{aligned}
x[v/x] &= v \\
y[v/x] &= y \quad (\text{if } x \neq y) \\
\mathbf{fun}\,(f, y, e)[v/x] &= \mathbf{fun}\,(f, y, e[v/x]) \quad (\text{if } f, y \notin \mathrm{FV}(v) \cup \{x\}) \\
(\Lambda \alpha : K.e)[v/x] &= \Lambda \alpha : K.e[v/x] \quad (\text{if } \alpha \notin \mathrm{FTV}(v)) \\
\mathsf{op}_{l\, \boldsymbol{S}^I}\, \boldsymbol{T}^J[v/x] &= \mathsf{op}_{l\, \boldsymbol{S}^I}\, \boldsymbol{T}^J \\
(v_1\, v_2)[v/x] &= (v_1[v/x])\,(v_2[v/x]) \\
(v'\, S)[v/x] &= (v'[v/x])\, S \\
(\mathbf{handle}_{l\, \boldsymbol{S}^N}\, e\, \mathbf{with}\, h)[v/x] &= \mathbf{handle}_{l\, \boldsymbol{S}^N}\, e[v/x]\, \mathbf{with}\, (h[v/x]) \\
(\mathbf{let}\, y = e_1\, \mathbf{in}\, e_2)[v/x] &= \mathbf{let}\, y = e_1[v/x]\, \mathbf{in}\, e_2[v/x] \\
& \qquad (\text{if } y \neq x \text{ and } y \notin \mathrm{FV}(v)) \\
([e]_L)[v/x] &= [e[v/x]]_L \\
\{\, \mathbf{return}\, y \mapsto e_r\}[v/x] &= \{\, \mathbf{return}\, y \mapsto e_r[v/x]\}
\end{aligned}
$$

$$(\text{if } y \neq x \text{ and } y \notin \text{FV}(v))$$

$$(h \uplus \{\text{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e\})[v/x] = h[v/x] \uplus \{\text{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e[v/x]\}$$

$$(\text{if } x \neq p, k \text{ and } p, k \notin \text{FV}(v) \text{ and } \{\boldsymbol{\beta}^J\} \cap \text{FTV}(v) = \emptyset)$$

Definition 1.10 (Typelike Substitution). *Substitution $e[S/\alpha]$, $h[S/\alpha]$, $T[S/\alpha]$, and $\Gamma[S/\alpha]$ of typelike S for typelike variable α in expression e, handler h, typelike T, and typing context Γ, respectively, are defined as follows:*

$$x[S/\alpha] = x$$
$$(\mathbf{fun}\,(f, x, e))[S/\alpha] = \mathbf{fun}\,(f, x, e[S/\alpha])$$
$$(\Lambda\beta : K.e)[S/\alpha] = \Lambda\beta : K.(e[S/\alpha]) \quad (\text{if } \alpha \neq \beta \text{ and } \beta \notin \text{FTV}(S))$$
$$(\text{op}_{l\,\boldsymbol{S'}^I}\,\boldsymbol{T}^J)[S/\alpha] = \text{op}_{l\,\boldsymbol{S'}[\boldsymbol{S}/\boldsymbol{\alpha}]^I}\,\boldsymbol{T}[\boldsymbol{S}/\boldsymbol{\alpha}]^J$$
$$(v_1\,v_2)[S/\alpha] = (v_1[S/\alpha])\,(v_2[S/\alpha])$$
$$(v\,T)[S/\alpha] = (v[S/\alpha])\,(T[S/\alpha])$$
$$(\mathbf{handle}_{l\,\boldsymbol{T}^N}\,e\,\mathbf{with}\,h)[S/\alpha] = \mathbf{handle}_{l\,\boldsymbol{T}[\boldsymbol{S}/\boldsymbol{\alpha}]^N}\,e[S/\alpha]\,\mathbf{with}\,(h[S/\alpha])$$
$$(\mathbf{let}\,x = e_1\,\mathbf{in}\,e_2)[S/\alpha] = \mathbf{let}\,x = e_1[S/\alpha]\,\mathbf{in}\,e_2[S/\alpha]$$
$$([e]_L)[S/\alpha] = [e[S/\alpha]]_{L[S/\alpha]}$$
$$\{\,\mathbf{return}\,x \mapsto e_r\}[S/\alpha] = \{\,\mathbf{return}\,x \mapsto e_r[S/\alpha]\}$$
$$(h \uplus \{\text{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e\})[S/\alpha] = h[S/\alpha] \uplus \{\text{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e[S/\alpha]\}$$
$$(\text{if } \{\boldsymbol{\beta}^J\} \cap (\{\alpha\} \cup \text{FTV}(S)) = \emptyset)$$

$$\alpha[S/\alpha] = S$$
$$\beta[S/\alpha] = \beta \quad (\text{if } \alpha \neq \beta)$$
$$(A \to_\varepsilon B)[S/\alpha] = (A[S/\alpha]) \to_{\varepsilon[S/\alpha]} (B[S/\alpha])$$
$$(\forall\beta : K.A^\varepsilon)[S/\alpha] = \forall\beta : K.A[S/\alpha]^{(\varepsilon[S/\alpha])}$$
$$(\text{if } \alpha \neq \beta \text{ and } \beta \notin \text{FTV}(S))$$
$$(\mathcal{C}\,\boldsymbol{T}^I)[S/\alpha] = \mathcal{C}\,\boldsymbol{T}[\boldsymbol{S}/\boldsymbol{\alpha}]^I$$

$$\{\}[S/\alpha] = \{\}$$
$$(\sigma \uplus \{\text{op} : \forall\boldsymbol{\beta}^J : \boldsymbol{K}^J.A \Rightarrow B\})[S/\alpha] = \sigma[S/\alpha] \uplus \{\text{op} : \forall\boldsymbol{\beta}^J : \boldsymbol{K}^J.A[S/\alpha] \Rightarrow B[S/\alpha]\}$$
$$(\text{if } \{\boldsymbol{\beta}^J\} \cap \text{FTV}(S) = \emptyset)$$

$$\emptyset[S/\alpha] = \emptyset$$
$$(\Gamma, x : A)[S/\alpha] = \Gamma[S/\alpha], x : A[S/\alpha]$$
$$(\Gamma, \beta : K)[S/\alpha] = \Gamma[S/\alpha], \beta : K \quad (\text{if } \alpha \neq \beta)$$

Definition 1.11 (Typelike Extraction Function). *A typelike context $\Delta(\Gamma)$ extracted from a typing context Γ is defined as follows:*

$$\Delta(\emptyset) = \emptyset \qquad \Delta(\Gamma, x : A) = \Delta(\Gamma) \qquad \Delta(\Gamma, \alpha : K) = \Delta(\Gamma), \alpha : K\ .$$

Definition 1.12 (Domains of Typing Contexts). *The set $\text{dom}(\Gamma)$ of variables and typelike variables bound by a typing context Γ is defined as follows:*

$$\text{dom}(\emptyset) = \emptyset \qquad \text{dom}(\Gamma, x : A) = \text{dom}(\Gamma) \cup \{x\} \qquad \text{dom}(\Gamma, \alpha : K) = \text{dom}(\Gamma) \cup \{\alpha\}\ .$$

Definition 1.13 (Context Well-formedness and Kinding Rules).
Contexts Well-formedness $\boxed{\vdash \Gamma}$

$$\frac{}{\vdash \emptyset}\ \text{C\_Empty} \qquad \frac{x \notin \text{dom}(\Gamma) \quad \Gamma \vdash A : \mathbf{Typ}}{\vdash \Gamma, x : A}\ \text{C\_Var} \qquad \frac{\vdash \Gamma \quad \alpha \notin \text{dom}(\Gamma)}{\vdash \Gamma, \alpha : K}\ \text{C\_TVar}$$

Kinding $\boxed{\Gamma \vdash S : K}$ $\qquad$ $\boxed{\Gamma \vdash \boldsymbol{S}^I : \boldsymbol{K}^I}$ $\iff \forall i \in I.(\Gamma \vdash S_i : K_i)$

$$\frac{\vdash \Gamma \quad \alpha : K \in \Gamma}{\Gamma \vdash \alpha : K} \text{ K\_VAR} \qquad \frac{\vdash \Gamma \quad \mathcal{C} : \Pi\boldsymbol{K}^I \to K \in \Sigma \quad \Gamma \vdash \boldsymbol{S}^I : \boldsymbol{K}^I}{\Gamma \vdash \mathcal{C}\,\boldsymbol{S}^I : K} \text{ K\_CONS}$$

$$\frac{\Gamma \vdash A : \mathbf{Typ} \quad \Gamma \vdash \varepsilon : \mathbf{Eff} \quad \Gamma \vdash B : \mathbf{Typ}}{\Gamma \vdash A \to_\varepsilon B : \mathbf{Typ}} \text{ K\_FUN} \qquad \frac{\Gamma, \alpha : K \vdash A : \mathbf{Typ} \quad \Gamma, \alpha : K \vdash \varepsilon : \mathbf{Eff}}{\Gamma \vdash \forall \alpha : K.A^\varepsilon : \mathbf{Typ}} \text{ K\_POLY}$$

Definition 1.14 (Proper Effect Contexts). *An effect context Ξ is proper if, for any $l :: \forall\boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi$, the following holds:*

- *$l : \Pi\boldsymbol{K}^I \to \mathbf{Lab} \in \Sigma_{\text{lab}}$;*

- *for any $\boldsymbol{\alpha_0}^{I_0}$, $\boldsymbol{K_0}^{I_0}$, and σ_0, if $l :: \forall\boldsymbol{\alpha_0}^{I_0} : \boldsymbol{K_0}^{I_0}.\sigma_0 \in \Xi$, then $\boldsymbol{\alpha}^I : \boldsymbol{K}^I = \boldsymbol{\alpha_0}^{I_0} : \boldsymbol{K_0}^{I_0}$ and $\sigma = \sigma_0$; and*

- *for any $\mathsf{op} : \forall\boldsymbol{\beta}^J : \boldsymbol{K_0}^J.A \Rightarrow B \in \sigma$,*

$$\boldsymbol{\alpha}^I : \boldsymbol{K}^I, \boldsymbol{\beta}^J : \boldsymbol{K_0}^J \vdash A : \mathbf{Typ} \quad\text{ and }\quad \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \boldsymbol{\beta}^J : \boldsymbol{K_0}^J \vdash B : \mathbf{Typ}\ .$$

Definition 1.15 (Well-Formedness-Preserving Functions). *Given a signature Σ, a (possibly partial) function $f \in K_i(\Sigma)^{i \in \{1,\dots,n\}} \rightharpoonup K(\Sigma)$ preserves well-formedness if*

$$\forall\Gamma, S_1, \dots, S_n.\, \Gamma \vdash S_1 : K_1 \wedge \cdots \wedge \Gamma \vdash S_n : K_n \wedge f(S_1, \dots, S_n) \in K(\Sigma) \implies \Gamma \vdash f(S_1, \dots, S_n) : K\ .$$

Similarly, $f \in K(\Sigma)$ preserves well-formedness if $\Gamma \vdash f : K$ for any Γ.

Definition 1.16. *We write $\alpha \mapsto T \vdash \boldsymbol{S} : K_0$ for a quadruple $\langle \alpha, T, \boldsymbol{S}, K_0 \rangle$ such that $\exists\Gamma_1, K, \Gamma_2.\,(\forall S_0 \in \boldsymbol{S}.\,\Gamma_1, \alpha : K, \Gamma_2 \vdash S_0 : K_0) \wedge \Gamma_1 \vdash T : K$.*

Definition 1.17 (Effect algebras). *Given a label signature Σ_{lab}, an effect algebra is a quintuple $\langle \Sigma_{\text{eff}}, \odot, \mathbb{0}, (-)^\uparrow, \sim \rangle$ satisfying the following, where we let $\Sigma = \Sigma_{\text{lab}} \uplus \Sigma_{\text{eff}}$.*

- *$\odot \in \mathbf{Eff}(\Sigma) \times \mathbf{Eff}(\Sigma) \rightharpoonup \mathbf{Eff}(\Sigma)$, $\mathbb{0} \in \mathbf{Eff}(\Sigma)$, and $(-)^\uparrow \in \mathbf{Lab}(\Sigma) \to \mathbf{Eff}(\Sigma)$ preserve well-formedness. Furthermore, $\sim$ is an equivalence relation on $\mathbf{Eff}(\Sigma)$ and preserves well-formedness, that is, $\forall\varepsilon_1, \varepsilon_2.\,\varepsilon_1 \sim \varepsilon_2 \implies (\forall\Gamma.\,\Gamma \vdash \varepsilon_1 : \mathbf{Eff} \iff \Gamma \vdash \varepsilon_2 : \mathbf{Eff})$.*

- *$\langle \mathbf{Eff}(\Sigma), \odot, \mathbb{0} \rangle$ is a partial monoid under $\sim$, that is, the following holds:*
  - *$\forall\varepsilon \in \mathbf{Eff}(\Sigma).\,\varepsilon \odot \mathbb{0} \sim \varepsilon \wedge \mathbb{0} \odot \varepsilon \sim \varepsilon$; and*
  - *$\forall\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbf{Eff}(\Sigma).$*
    $(\varepsilon_1 \odot \varepsilon_2) \odot \varepsilon_3 \in \mathbf{Eff}(\Sigma) \vee \varepsilon_1 \odot (\varepsilon_2 \odot \varepsilon_3) \in \mathbf{Eff}(\Sigma) \implies (\varepsilon_1 \odot \varepsilon_2) \odot \varepsilon_3 \sim \varepsilon_1 \odot (\varepsilon_2 \odot \varepsilon_3)$.

- *Typelike substitution respecting well-formedness is a homomorphism for $\odot$, $(-)^\uparrow$, and $\sim$, that is, the following holds:*
  - *$\forall\alpha, S, \varepsilon_1, \varepsilon_2.\,\alpha \mapsto S \vdash \varepsilon_1, \varepsilon_2 : \mathbf{Eff} \wedge \varepsilon_1 \odot \varepsilon_2 \in \mathbf{Eff}(\Sigma) \implies (\varepsilon_1 \odot \varepsilon_2)[S/\alpha] = \varepsilon_1[S/\alpha] \odot \varepsilon_2[S/\alpha]$;*
  - *$\forall\alpha, S, L.\,\alpha \mapsto S \vdash L : \mathbf{Lab} \implies (L)^\uparrow[S/\alpha] = (L[S/\alpha])^\uparrow$; and*
  - *$\forall\alpha, S, \varepsilon_1, \varepsilon_2.\,\alpha \mapsto S \vdash \varepsilon_1, \varepsilon_2 : \mathbf{Eff} \wedge \varepsilon_1 \sim \varepsilon_2 \implies \varepsilon_1[S/\alpha] \sim \varepsilon[S/\alpha]$.*

Remark 1.18. *For readability, we introduce the following abbreviations.*

- *$\varepsilon_1 \oslash \varepsilon_2 \overset{\text{def}}{=} \exists\varepsilon.\,\varepsilon_1 \odot \varepsilon \sim \varepsilon_2$ and*

- *$\Gamma \vdash \varepsilon_1 \oslash \varepsilon_2 \overset{\text{def}}{=} \exists\varepsilon.\,\varepsilon_1 \odot \varepsilon \sim \varepsilon_2 \wedge (\forall\varepsilon' \in \{\varepsilon_1, \varepsilon_2, \varepsilon\}.\,\Gamma \vdash \varepsilon' : \mathbf{Eff})$.*

Remark 1.19 (Parameters of λ_{EA}). *λ_{EA} takes a label signature in Definition 1.3, an effect algebra over that label signature in Definition 1.17, and an effect context as parameters.*

Example 1.20 (Effect Signature for Effect Sets). The effect signature $\Sigma_{\text{eff}}^{\text{Set}}$ for effect sets consists of the pairs $\{\} : \mathbf{Eff}$, $\{-\} : \mathbf{Lab} \to \mathbf{Eff}$, and $- \underline{\cup} - : \mathbf{Eff} \times \mathbf{Eff} \to \mathbf{Eff}$.

Example 1.21 (Effect Signature for Effect Multisets). The effect signature $\Sigma_{\text{eff}}^{\text{MSet}}$ for effect multisets consists of the pairs $\{\} : \mathbf{Eff}$, $\{-\} : \mathbf{Lab} \to \mathbf{Eff}$, and $- \underline{\sqcup} - : \mathbf{Eff} \times \mathbf{Eff} \to \mathbf{Eff}$.

Example 1.22 (Effect Signature for Rows). The effect signature $\Sigma_{\text{eff}}^{\text{Row}}$ for both simple rows and scoped rows consists of the pairs $\langle\rangle : \mathbf{Eff}$ and $\langle - \mid - \rangle : \mathbf{Lab} \times \mathbf{Eff} \to \mathbf{Eff}$.

Example 1.23 (Effect Sets). An effect algebra EA_{Set} for effect sets is defined by $\langle \Sigma_{\text{eff}}^{\text{Set}}, - \underline{\cup} -, \{\}, \{-\}, \sim_{\text{Set}} \rangle$ where $\sim_{\text{Set}}$ is the least equivalence relation satisfying the following rules:

$$\overline{\varepsilon \underline{\cup} \{\} \sim_{\text{Set}} \varepsilon} \qquad \overline{\varepsilon_1 \underline{\cup} \varepsilon_2 \sim_{\text{Set}} \varepsilon_2 \underline{\cup} \varepsilon_1} \qquad \overline{\varepsilon \underline{\cup} \varepsilon \sim_{\text{Set}} \varepsilon} \qquad \overline{(\varepsilon_1 \underline{\cup} \varepsilon_2) \underline{\cup} \varepsilon_3 \sim_{\text{Set}} \varepsilon_1 \underline{\cup} (\varepsilon_2 \underline{\cup} \varepsilon_3)}$$

$$\frac{\varepsilon_1 \sim_{\text{Set}} \varepsilon_2 \quad \varepsilon_3 \sim_{\text{Set}} \varepsilon_4}{\varepsilon_1 \underline{\cup} \varepsilon_3 \sim_{\text{Set}} \varepsilon_2 \underline{\cup} \varepsilon_4}$$

Example 1.24 (Effect Multisets). An effect algebra EA_{MSet} for effect multisets is defined by $\langle \Sigma_{\text{eff}}^{\text{MSet}}, - \underline{\sqcup} -, \{\}, \{-\}, \sim_{\text{MSet}} \rangle$ where $\sim_{\text{MSet}}$ is the least equivalence relation satisfying the following rules:

$$\overline{\varepsilon \underline{\sqcup} \{\} \sim_{\text{MSet}} \varepsilon} \qquad \overline{\varepsilon_1 \underline{\sqcup} \varepsilon_2 \sim_{\text{MSet}} \varepsilon_2 \underline{\sqcup} \varepsilon_1} \qquad \overline{(\varepsilon_1 \underline{\sqcup} \varepsilon_2) \underline{\sqcup} \varepsilon_3 \sim_{\text{MSet}} \varepsilon_1 \underline{\sqcup} (\varepsilon_2 \underline{\sqcup} \varepsilon_3)} \qquad \frac{\varepsilon_1 \sim_{\text{MSet}} \varepsilon_2 \quad \varepsilon_3 \sim_{\text{MSet}} \varepsilon_4}{\varepsilon_1 \underline{\sqcup} \varepsilon_3 \sim_{\text{MSet}} \varepsilon_2 \underline{\sqcup} \varepsilon_4}$$

Example 1.25 (Simple Rows). An effect algebra EA_{SimpR} for simple rows is defined by $\langle \Sigma_{\text{eff}}^{\text{Row}}, \odot_{\text{SimpR}}, \langle \rangle, \langle - \mid \langle \rangle \rangle, \sim_{\text{SimpR}} \rangle$ where

$$\varepsilon_1 \odot_{\text{SimpR}} \varepsilon_2 \overset{\text{def}}{=} \begin{cases} \langle L_1 \mid \langle \cdots \langle L_n \mid \varepsilon_2 \rangle \rangle \rangle & (\text{if } \varepsilon_1 = \langle L_1 \mid \langle \cdots \langle L_n \mid \langle \rangle \rangle \rangle \rangle) \\ \varepsilon_1 & (\text{if } \varepsilon_1 = \langle L_1 \mid \langle \cdots \langle L_n \mid \rho \rangle \rangle \rangle \text{ and } \varepsilon_2 = \langle \rangle) \\ \textit{undefined} & (\text{otherwise}) \end{cases}$$

and $\sim_{\text{SimpR}}$ is the least equivalence relation satisfying the following.

$$\frac{\varepsilon_1 \sim_{\text{SimpR}} \varepsilon_2}{\langle L \mid \varepsilon_1 \rangle \sim_{\text{SimpR}} \langle L \mid \varepsilon_2 \rangle} \qquad \frac{L_1 \neq L_2}{\langle L_1 \mid \langle L_2 \mid \varepsilon \rangle \rangle \sim_{\text{SimpR}} \langle L_2 \mid \langle L_1 \mid \varepsilon \rangle \rangle} \qquad \overline{\langle L \mid \varepsilon \rangle \sim_{\text{SimpR}} \langle L \mid \langle L \mid \varepsilon \rangle \rangle}$$

Example 1.26 (Scoped Rows). An effect algebra EA_{ScpR} for scoped rows is defined by $\langle \Sigma_{\text{eff}}^{\text{Row}}, \odot_{\text{ScpR}}, \langle \rangle, \langle - \mid \langle \rangle \rangle, \sim_{\text{ScpR}} \rangle$ where

$$\varepsilon_1 \odot_{\text{ScpR}} \varepsilon_2 \overset{\text{def}}{=} \begin{cases} \langle L_1 \mid \langle \cdots \langle L_n \mid \varepsilon_2 \rangle \rangle \rangle & (\text{if } \varepsilon_1 = \langle L_1 \mid \langle \cdots \langle L_n \mid \langle \rangle \rangle \rangle \rangle) \\ \varepsilon_1 & (\text{if } \varepsilon_1 = \langle L_1 \mid \langle \cdots \langle L_n \mid \rho \rangle \rangle \rangle \text{ and } \varepsilon_2 = \langle \rangle) \\ \textit{undefined} & (\text{otherwise}) \end{cases}$$

and $\sim_{\text{ScpR}}$ is the least equivalence relation satisfying the following.

$$\frac{\varepsilon_1 \sim_{\text{ScpR}} \varepsilon_2}{\langle L \mid \varepsilon_1 \rangle \sim_{\text{ScpR}} \langle L \mid \varepsilon_2 \rangle} \qquad \frac{L_1 \neq L_2}{\langle L_1 \mid \langle L_2 \mid \varepsilon \rangle \rangle \sim_{\text{ScpR}} \langle L_2 \mid \langle L_1 \mid \varepsilon \rangle \rangle}$$

Example 1.27 (Erasable Sets). An effect algebra EA_{ESet} for effect sets is defined by $\langle \Sigma_{\text{eff}}^{\text{Set}}, - \underline{\cup} -, \{\}, \{-\}, \sim_{\text{ESet}} \rangle$ where $\sim_{\text{ESet}}$ is the least equivalence relation satisfying the following rules:

$$\frac{l_1 \neq l_2}{\{l_1 \, \boldsymbol{S_1}^{I_1}\} \underline{\cup} \{l_2 \, \boldsymbol{S_2}^{I_2}\} \sim_{\text{ESet}} \{l_2 \, \boldsymbol{S_2}^{I_2}\} \underline{\cup} \{l_1 \, \boldsymbol{S_1}^{I_1}\}} \qquad \overline{\{l \, \boldsymbol{S_1}^{I_1}\} \underline{\cup} \{l \, \boldsymbol{S_2}^{I_2}\} \sim_{\text{ESet}} \{l \, \boldsymbol{S_1}^{I_1}\}} \qquad \overline{\varepsilon \underline{\cup} \{\} \sim_{\text{ESet}} \varepsilon}$$

$$\overline{\{\} \underline{\cup} \varepsilon \sim_{\text{ESet}} \varepsilon} \qquad \overline{(\varepsilon_1 \underline{\cup} \varepsilon_2) \underline{\cup} \varepsilon_3 \sim_{\text{ESet}} \varepsilon_1 \underline{\cup} (\varepsilon_2 \underline{\cup} \varepsilon_3)} \qquad \frac{\varepsilon_1 \sim_{\text{ESet}} \varepsilon_2 \quad \varepsilon_3 \sim_{\text{ESet}} \varepsilon_4}{\varepsilon_1 \underline{\cup} \varepsilon_3 \sim_{\text{ESet}} \varepsilon_2 \underline{\cup} \varepsilon_4}$$

Example 1.28 (Erasable Multisets). An effect algebra EA_{EMSet} for effect multisets is defined by $\langle \Sigma_{\text{eff}}^{\text{MSet}}, - \underline{\sqcup} -, \{\}, \{-\}, \sim_{\text{EMSet}} \rangle$ where $\sim_{\text{EMSet}}$ is the least equivalence relation satisfying the following rules:

$$\frac{l_1 \neq l_2}{\{l_1 \, \boldsymbol{S_1}^{I_1}\} \underline{\sqcup} \{l_2 \, \boldsymbol{S_2}^{I_2}\} \sim_{\text{EMSet}} \{l_2 \, \boldsymbol{S_2}^{I_2}\} \underline{\sqcup} \{l_1 \, \boldsymbol{S_1}^{I_1}\}} \qquad \overline{\varepsilon \underline{\sqcup} \{\} \sim_{\text{EMSet}} \varepsilon} \qquad \overline{\{\} \underline{\sqcup} \varepsilon \sim_{\text{EMSet}} \varepsilon}$$

$$\overline{(\varepsilon_1 \underline{\sqcup} \varepsilon_2) \underline{\sqcup} \varepsilon_3 \sim_{\text{EMSet}} \varepsilon_1 \underline{\sqcup} (\varepsilon_2 \underline{\sqcup} \varepsilon_3)} \qquad \frac{\varepsilon_1 \sim_{\text{EMSet}} \varepsilon_2 \quad \varepsilon_3 \sim_{\text{EMSet}} \varepsilon_4}{\varepsilon_1 \underline{\sqcup} \varepsilon_3 \sim_{\text{EMSet}} \varepsilon_2 \underline{\sqcup} \varepsilon_4}$$

Example 1.29 (Erasable Simple Rows). An effect algebra $\text{EA}_{\text{ESimpR}}$ for erasable simple rows is defined by $\langle \Sigma_{\text{eff}}^{\text{Row}}, \odot_{\text{ESimpR}}, \langle \rangle, \langle - \mid \langle \rangle \rangle, \sim_{\text{ESimpR}} \rangle$ where

$$\varepsilon_1 \odot_{\text{ESimpR}} \varepsilon_2 \overset{\text{def}}{=} \begin{cases} \langle L_1 \mid \langle \cdots \langle L_n \mid \varepsilon_2 \rangle \rangle \rangle & (\text{if } \varepsilon_1 = \langle L_1 \mid \langle \cdots \langle L_n \mid \langle \rangle \rangle \rangle \rangle) \\ \varepsilon_1 & (\text{if } \varepsilon_1 = \langle L_1 \mid \langle \cdots \langle L_n \mid \rho \rangle \rangle \rangle \text{ and } \varepsilon_2 = \langle \rangle) \\ \textit{undefined} & (\text{otherwise}) \end{cases}$$

and $\sim_{\text{ESimpR}}$ is the least equivalence relation satisfying the following.

$$\frac{\varepsilon_1 \sim_{\text{SimpR}} \varepsilon_2}{\langle L \mid \varepsilon_1 \rangle \sim_{\text{SimpR}} \langle L \mid \varepsilon_2 \rangle} \qquad \frac{l_1 \neq l_2}{\langle l_1 \, \boldsymbol{S_1}^{I_1} \mid \langle l_2 \, \boldsymbol{S_2}^{I_2} \mid \varepsilon \rangle \rangle \sim_{\text{SimpR}} \langle l_2 \, \boldsymbol{S_2}^{I_2} \mid \langle l_1 \, \boldsymbol{S_1}^{I_1} \mid \varepsilon \rangle \rangle}$$

$$\overline{\langle l \, \boldsymbol{S_1}^{I_1} \mid \varepsilon \rangle \sim_{\text{SimpR}} \langle l \, \boldsymbol{S_1}^{I_1} \mid \langle l \, \boldsymbol{S_2}^{I_2} \mid \varepsilon \rangle \rangle}$$

Example 1.30 (Erasable Scoped Rows). An effect algebra EA_{EScpR} for scoped rows is defined by $\langle \Sigma_{\text{eff}}^{\text{Row}}, \odot_{\text{EScpR}}, \langle \rangle, \langle - \mid \langle \rangle \rangle, \sim_{\text{EScpR}} \rangle$ where

$$\varepsilon_1 \odot_{\text{EScpR}} \varepsilon_2 \overset{\text{def}}{=} \begin{cases} \langle L_1 \mid \langle \cdots \langle L_n \mid \varepsilon_2 \rangle \rangle \rangle & (\text{if } \varepsilon_1 = \langle L_1 \mid \langle \cdots \langle L_n \mid \langle \rangle \rangle \rangle \rangle) \\ \varepsilon_1 & (\text{if } \varepsilon_1 = \langle L_1 \mid \langle \cdots \langle L_n \mid \rho \rangle \rangle \rangle \text{ and } \varepsilon_2 = \langle \rangle) \\ \textit{undefined} & (\text{otherwise}) \end{cases}$$

and $\sim_{\text{EScpR}}$ is the least equivalence relation satisfying the following.

$$\frac{\varepsilon_1 \sim_{\text{EScpR}} \varepsilon_2}{\langle L \mid \varepsilon_1 \rangle \sim_{\text{EScpR}} \langle L \mid \varepsilon_2 \rangle} \qquad \frac{l_1 \neq l_2}{\langle l_1 \, \boldsymbol{S_1}^{I_1} \mid \langle l_2 \, \boldsymbol{S_2}^{I_2} \mid \varepsilon \rangle \rangle \sim_{\text{EScpR}} \langle l_2 \, \boldsymbol{S_2}^{I_2} \mid \langle l_1 \, \boldsymbol{S_1}^{I_1} \mid \varepsilon \rangle \rangle}$$

Definition 1.31 (Freeness of Labels).

Freeness of labels $\boxed{n-\text{free}(L, E)}$

$$\overline{0-\text{free}(L, \square)} \qquad \frac{n-\text{free}(L, E)}{n-\text{free}(L, \textbf{let } x = E \textbf{ in } e)} \qquad \frac{n+1-\text{free}(L, E)}{n-\text{free}(L, \textbf{handle}_L \, E \textbf{ with } h)} \qquad \frac{n-\text{free}(L, E) \quad L \neq L'}{n-\text{free}(L, \textbf{handle}_{L'} \, E \textbf{ with } h)}$$

Definition 1.32 (Operational Semantics).

Reduction $\boxed{e \longmapsto e'}$

$$\frac{}{\textbf{fun} \, (f, x, e) \, v \longmapsto e[\textbf{fun} \, (f, x, e)/f][v/x]} \text{ R\_App} \qquad \frac{}{(\Lambda \alpha : K.e) \, S \longmapsto e[S/\alpha]} \text{ R\_TApp}$$

$$\frac{}{\textbf{let } x = v \textbf{ in } e \longmapsto e[v/x]} \text{ R\_Let} \qquad \frac{\textbf{return } x \mapsto e_r \in h}{\textbf{handle}_{l \, \boldsymbol{S}^I} \, v \textbf{ with } h \longmapsto e_r[v/x]} \text{ R\_Handle1}$$

$$\frac{\textsf{op} \, \boldsymbol{\beta}^J : \boldsymbol{K}^J \, p \, k \mapsto e \in h \quad v_{cont} = \lambda z.\textbf{handle}_{l \, \boldsymbol{S}^I} \, E[z] \textbf{ with } h \quad 0-\text{free}(l \, \boldsymbol{S}^I, E)}{\textbf{handle}_{l \, \boldsymbol{S}^I} \, E[\textsf{op}_{l \, \boldsymbol{S}^I} \, \boldsymbol{T}^J \, v] \textbf{ with } h \longmapsto e[\boldsymbol{T}^J/\boldsymbol{\beta}^J][v/p][v_{cont}/k]} \text{ R\_Handle2}$$

Evaluation $\boxed{e \longrightarrow e'}$

$$\frac{e_1 \longmapsto e_2}{E[e_1] \longrightarrow E[e_2]} \text{ E\_Eval}$$

Definition 1.33. *We write* $\longrightarrow^*$ *for the reflexive, transitive closure of* $\longrightarrow$. *We also write* $e \longarrownot\longrightarrow$ *to denote that there is no* e' *such that* $e \longrightarrow e'$.

Definition 1.34 (Typing and Subtyping Rules).

Typing $\boxed{\Gamma \vdash e : A \mid \varepsilon}$

$$\frac{\vdash \Gamma \quad x : A \in \Gamma}{\Gamma \vdash x : A \mid \mathbb{0}} \text{ T\_Var} \qquad \frac{\Gamma, f : A \rightarrow_\varepsilon B, x : A \vdash e : B \mid \varepsilon}{\Gamma \vdash \textbf{fun} \, (f, x, e) : A \rightarrow_\varepsilon B \mid \mathbb{0}} \text{ T\_Abs}$$

$$\frac{\Gamma \vdash v_1 : A \rightarrow_\varepsilon B \mid \mathbb{0} \quad \Gamma \vdash v_2 : A \mid \mathbb{0}}{\Gamma \vdash v_1 \, v_2 : B \mid \varepsilon} \text{ T\_App} \qquad \frac{\Gamma, \alpha : K \vdash e : A \mid \varepsilon}{\Gamma \vdash \Lambda \alpha : K.e : \forall \alpha : K.A^\varepsilon \mid \mathbb{0}} \text{ T\_TAbs}$$

$$\frac{\Gamma \vdash v : \forall \alpha : K.A^\varepsilon \mid \mathbb{0} \quad \Gamma \vdash S : K}{\Gamma \vdash v \, S : A[S/\alpha] \mid \varepsilon[S/\alpha]} \text{ T\_TApp} \qquad \frac{\Gamma \vdash e_1 : A \mid \varepsilon \quad \Gamma, x : A \vdash e_2 : B \mid \varepsilon}{\Gamma \vdash \textbf{let } x = e_1 \textbf{ in } e_2 : B \mid \varepsilon} \text{ T\_Let}$$

$$\frac{\Gamma \vdash e : A \mid \varepsilon \quad \Gamma \vdash A \mid \varepsilon <: A' \mid \varepsilon'}{\Gamma \vdash e : A' \mid \varepsilon'} \text{ T\_Sub} \qquad \frac{\begin{array}{c} l :: \forall \boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi \quad \textsf{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K'}^J.A \Rightarrow B \in \sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \\ \vdash \Gamma \quad \Gamma \vdash \boldsymbol{S}^I : \boldsymbol{K}^I \quad \Gamma \vdash \boldsymbol{T}^J : \boldsymbol{K'}^J \end{array}}{\Gamma \vdash \textsf{op}_{l \, \boldsymbol{S}^I} \, \boldsymbol{T}^J : (A[\boldsymbol{T}^J/\boldsymbol{\beta}^J]) \rightarrow_{(l \, \boldsymbol{S}^I)^\uparrow} (B[\boldsymbol{T}^I/\boldsymbol{\beta}^I]) \mid \mathbb{0}} \text{ T\_Op}$$

$$\frac{\begin{array}{c} \Gamma \vdash e : A \mid \varepsilon' \quad l :: \forall \boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi \quad \Gamma \vdash \boldsymbol{S}^I : \boldsymbol{K}^I \\ \Gamma \vdash_{\sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} h : A \Rightarrow^\varepsilon B \quad (l \, \boldsymbol{S}^I)^\uparrow \odot \varepsilon \sim \varepsilon' \end{array}}{\Gamma \vdash \textbf{handle}_{l \, \boldsymbol{S}^I} \, e \textbf{ with } h : B \mid \varepsilon} \text{ T\_Handling}$$

Handler Typing $\boxed{\Gamma \vdash_\sigma h : A \Rightarrow^\varepsilon B}$

$$\frac{\Gamma, x : A \vdash e_r : B \mid \varepsilon}{\Gamma \vdash_{\{\}} \{\, \mathbf{return}\, x \mapsto e_r \,\} : A \Rightarrow^\varepsilon B} \ \text{H\_Return}$$

$$\frac{\begin{array}{c} \sigma = \sigma' \uplus \{\mathsf{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K}^J.A' \Rightarrow B'\} \\ \Gamma \vdash_{\sigma'} h : A \Rightarrow^\varepsilon B \quad \Gamma, \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A', k : B' \to_\varepsilon B \vdash e : B \mid \varepsilon \end{array}}{\Gamma \vdash_\sigma h \uplus \{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\, p\, k \mapsto e\} : A \Rightarrow^\varepsilon B} \ \text{H\_Op}$$

Subtyping $\boxed{\Gamma \vdash A <: B}$

$$\frac{\Gamma \vdash A : \mathbf{Typ}}{\Gamma \vdash A <: A} \ \text{ST\_Refl} \qquad \frac{\Gamma \vdash A_2 <: A_1 \quad \Gamma \vdash B_1 \mid \varepsilon_1 <: B_2 \mid \varepsilon_2}{\Gamma \vdash A_1 \to_{\varepsilon_1} B_1 <: A_2 \to_{\varepsilon_2} B_2} \ \text{ST\_Fun}$$

$$\frac{\Gamma, \alpha : K \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2}{\Gamma \vdash \forall \alpha : K.A_1{}^{\varepsilon_1} <: \forall \alpha : K.A_2{}^{\varepsilon_2}} \ \text{ST\_Poly} \qquad \frac{\Gamma \vdash A_1 <: B \quad \Gamma \vdash \varepsilon_1 \oslash \varepsilon_2}{\Gamma \vdash A \mid \varepsilon_1 <: B \mid \varepsilon_2} \ \text{ST\_Comp}$$

Definition 1.35 (Semantics of Shallow Handlers). *The semantics for shallow handlers consists of the reduction and evaluation relations defined by the following rule* R_SHandle *and those in Definition 1.32 except for* R_Handle2*.*

$$\frac{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\, p\, k \mapsto e \in h \quad v_{cont} = \lambda z.E[z] \quad 0-\mathrm{free}(l\,\boldsymbol{S}^I, E)}{\mathbf{handle}_{l\,\boldsymbol{S}^I}\, E[\mathsf{op}_{l\,\boldsymbol{S}^I}\, \boldsymbol{T}^J\, v]\, \mathbf{with}\, h \longmapsto e[\boldsymbol{T}^J/\boldsymbol{\beta}^J][v/p][v_{cont}/k]} \ \text{R\_SHandle}$$

Definition 1.36 (Typing of Shallow Handlers). *The typing rules of shallow handlers consist of the rules defined by the following rules* T_SHandling, SH_Return, *and* SH_Op, *and those in Definition 1.34 except for* T_Handling, H_Return, *and* H_Op*.*

Typing $\boxed{\Gamma \vdash e : A \mid \varepsilon}$

$$\frac{\begin{array}{c} \Gamma \vdash e : A \mid \varepsilon' \quad l :: \forall \boldsymbol{\alpha}^N : \boldsymbol{K}^N.\sigma \in \Xi \quad \Gamma \vdash \boldsymbol{S}^N : \boldsymbol{K}^N \\ \Gamma \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : A^{\varepsilon'} \Rightarrow^\varepsilon B \quad (l\,\boldsymbol{S}^N)^\uparrow \odot \varepsilon \sim \varepsilon' \end{array}}{\Gamma \vdash \mathbf{handle}_{l\,\boldsymbol{S}^N}\, e\, \mathbf{with}\, h : B \mid \varepsilon} \ \text{T\_SHandling}$$

Shallow Handler Typing $\boxed{\Gamma \vdash_\sigma h : A^{\varepsilon'} \Rightarrow^\varepsilon B}$

$$\frac{\Gamma, x : A \vdash e_r : B \mid \varepsilon \quad \Gamma \vdash \varepsilon' : \mathbf{Eff}}{\Gamma \vdash_{\{\}} \{\, \mathbf{return}\, x \mapsto e_r \,\} : A^{\varepsilon'} \Rightarrow^\varepsilon B} \ \text{SH\_Return}$$

$$\frac{\begin{array}{c} \sigma = \sigma' \uplus \{\mathsf{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K}^J.A' \Rightarrow B'\} \\ \Gamma \vdash_{\sigma'} h : A^{\varepsilon'} \Rightarrow^\varepsilon B \quad \Gamma, \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A', k : B' \to_{\varepsilon'} A \vdash e : B \mid \varepsilon \end{array}}{\Gamma \vdash_\sigma h \uplus \{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\, p\, k \mapsto e\} : A^{\varepsilon'} \Rightarrow^\varepsilon B} \ \text{SH\_Op}$$

Definition 1.37 (The Syntax of λ_{EA} with Lift Coercions). *The syntax of λ_{EA} extended by lift coercions is the same as Definition 1.5 except for the following.*

$$e \quad ::= \quad \cdots \mid [e]_L \quad \text{(expressions)} \qquad E \quad ::= \quad \cdots \mid [E]_L \quad \text{(evaluation contexts)}$$

Definition 1.38 (Freeness of Labels with Lift Coercions). *The rules of freeness of labels for λ_{EA} extended by lift coercions consist of the rules in Definition 1.31 and the following rules.*

Freeness of labels $\boxed{n-\mathrm{free}(L, E)}$

$$\frac{n-\mathrm{free}(L, E)}{n + 1-\mathrm{free}(L, [E]_L)} \qquad \frac{n-\mathrm{free}(L, E) \quad L \neq L'}{n-\mathrm{free}(L, [E]_{L'})}$$

Definition 1.39 (Semantics with Lift Coercions). *The semantics for λ_{EA} extended by lift coercions consists of the reduction and evaluation relations defined by the following rule* R_Lift *and those in Definition 1.32 except for* R_Handle2*.*

Reduction $\boxed{e \longmapsto e'}$

$$\frac{}{[v]_L \longmapsto v} \ \text{R\_Lift}$$

Definition 1.40 (Typing of Lift Coercions)**.** *The typing rules of* λ_{EA} *extended by lift coercions consist of the rules in Definition* 1.34 *and the following rule.*

$$\frac{\Gamma \vdash e : A \mid \varepsilon' \quad \Gamma \vdash L : \mathbf{Lab} \quad (L)^\uparrow \odot \varepsilon' \sim \varepsilon}{\Gamma \vdash [e]_L : A \mid \varepsilon} \ \text{T\_LIFT}$$

Definition 1.41 (Freeness of Label Names)**.**

Freeness of label names $\boxed{n-\mathrm{free}(L, E)}$

$$\frac{}{0-\mathrm{free}(l, \square)} \qquad \frac{n-\mathrm{free}(l, E)}{n-\mathrm{free}(l, \mathbf{let}\ x = E\ \mathbf{in}\ e)} \qquad \frac{n+1-\mathrm{free}(l, E)}{n-\mathrm{free}(l, \mathbf{handle}_{l\ \boldsymbol{S}^I}\ E\ \mathbf{with}\ h)}$$

$$\frac{n-\mathrm{free}(l, E) \quad l \neq l'}{n-\mathrm{free}(l, \mathbf{handle}_{l'\ \boldsymbol{S}^I}\ E\ \mathbf{with}\ h)}$$

Definition 1.42 (Operational Semantics with Type-Erasure)**.** *The type-erasure semantics consists of the reduction and evaluation relations defined by the following rule* R_HANDLE2' *and those in Definition* 1.32 *except for* R_HANDLE2.

$$\frac{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\ p\ k \mapsto e \in h \quad v_{cont} = \lambda z.\mathbf{handle}_{l\ \boldsymbol{S}^I}\ E[z]\ \mathbf{with}\ h \quad 0-\mathrm{free}(l, E)}{\mathbf{handle}_{l\ \boldsymbol{S}^I}\ E[\mathsf{op}_{l\ \boldsymbol{S}'^I}\ \boldsymbol{T}^J\ v]\ \mathbf{with}\ h \longmapsto e[\boldsymbol{T}^J/\boldsymbol{\beta}^J][v/p][v_{cont}/k]} \ \text{R\_HANDLE2'}$$

Definition 1.43 (Freeness of Label Names with Lift Coercions)**.**
The rules of freeness of label names for λ_{EA} *extended by lift coercions consist of the rules in Definition* 1.41 *and the following rules.*

Freeness of label names $\boxed{n-\mathrm{free}(l, E)}$

$$\frac{n-\mathrm{free}(l, E)}{n+1-\mathrm{free}(l, [E]_{l\ \boldsymbol{S}^I})} \qquad \frac{n-\mathrm{free}(l, E) \quad L \neq l\ \boldsymbol{S}^I}{n-\mathrm{free}(l, [E]_L)}$$

Definition 1.44 (Semantics with Lift Coercions and Type-Erasure)**.** *The semantics for lift coercions consists of the reduction and evaluation relations defined by the rule* R_HANDLE2' *defined in Definition* 1.42 *and those in Definition* 1.39 *except for* R_HANDLE2.

Definition 1.45 (Safety Conditions)**.**
(1) *For any* L, $(L)^\uparrow \oslash \mathbb{0}$ *does not hold.*

(2) *If* $(L)^\uparrow \oslash \varepsilon$ *and* $(L')^\uparrow \odot \varepsilon' \sim \varepsilon$ *and* $L \neq L'$, *then* $(L)^\uparrow \oslash \varepsilon'$.

Definition 1.46 (Safety Condition for Lift Coercions)**.** *The safety condition added for lift coercions is the following:*
(3) *If* $(L)^\uparrow \odot \varepsilon_1 \sim (L_1)^\uparrow \odot \cdots \odot (L_n)^\uparrow \odot (L)^\uparrow \odot \varepsilon_2$ *and* $L \notin \{L_1, \ldots, L_n\}$, *then* $\varepsilon_1 \sim (L_1)^\uparrow \odot \cdots \odot (L_n)^\uparrow \odot \varepsilon_2$.

Definition 1.47 (Safety Condition for Type-Erasure)**.** *The safety condition added for the type-erasure semantics is the following:*
(4) *If* $(l\ \boldsymbol{S_1}^{I_1})^\uparrow \oslash \varepsilon$ *and* $(l\ \boldsymbol{S_2}^{I_2})^\uparrow \oslash \varepsilon$, *then* $\boldsymbol{S_1}^{I_1} = \boldsymbol{S_2}^{I_2}$.

Example 1.48 (Unsafe Effect Algebras)**.**

Effect algebra violating safety condition (1) Consider an effect algebra such that $\emptyset \vdash (l)^\uparrow \oslash \mathbb{0}$ holds for some l. Clearly, this effect algebra violates safety condition (1). In this case, $\emptyset \vdash \mathsf{op}_l\ v : A \mid \mathbb{0}$ can be derived for some A (if $\mathsf{op}_l\ v$ is well typed) because $\mathsf{op}_l\ v$ is given the effect $(l)^\uparrow$ and the subeffecting $\emptyset \vdash (l)^\uparrow \oslash \mathbb{0}$ holds. However, the operation call is not handled.

Effect algebra violating safety condition (2) Consider an effect algebra such that safety condition (1), $(l)^\uparrow \oslash (l')^\uparrow$, and $(l')^\uparrow \odot \mathbb{0} \sim (l')^\uparrow$ hold for some l and l' such that $l \neq l'$. This effect algebra violates safety condition (2): if safety condition (2) is met, we would have $(l)^\uparrow \oslash \mathbb{0}$, but it is contradictory with safety condition (1).

This effect algebra allows assigning the empty effect $\mathbb{0}$ to the expression $\mathbf{handle}_{l'}\ \mathsf{op}_l\ v\ \mathbf{with}\ h$ as illustrated by the following typing derivation:

$$\frac{\cdots \quad (l')^\uparrow \odot \mathbb{0} \sim (l')^\uparrow \quad \dfrac{\dfrac{\emptyset \vdash \mathsf{op}_l\ v : A \mid (l)^\uparrow \quad \emptyset \vdash A \mid (l)^\uparrow <: A \mid (l')^\uparrow}{\emptyset \vdash \mathsf{op}_l\ v : A \mid (l')^\uparrow} \ \text{T\_SUB}}{}}{\emptyset \vdash \mathbf{handle}_{l'}\ \mathsf{op}_l\ v\ \mathbf{with}\ h : B \mid \mathbb{0}} \ \text{T\_HANDLING}$$

However, the operation call in it is not handled.

Supplementary Material for "Abstracting Effect Systems for Algebraic Effect Handlers"

Takuma Yoshioka[1], Taro Sekiyama[2], and Atsushi Igarashi[1]

[1]*Kyoto University, Japan*
[2]*National Institute of Informatics & SOKENDAI, Japan*

1 Definitions

Remark 1.1 (Notation). *We write $\boldsymbol{\alpha}^I$ for a finite sequence $\alpha_0, \ldots, \alpha_n$ with an index set $I = \{0, \ldots, n\}$, where α is any metavariable. We also write $\{\boldsymbol{\alpha}^I\}$ for the set consisting of the elements of $\boldsymbol{\alpha}^I$.*

Definition 1.2 (Kinds). *Kinds are defined as $K ::= \mathbf{Typ} \mid \mathbf{Lab} \mid \mathbf{Eff}$.*

Definition 1.3 (Signatures). *Given a set S of label names, a label signature Σ_{lab} is a functional relation whose domain $\mathrm{dom}(\Sigma_{\mathrm{lab}})$ is S. The codomain of Σ_{lab} is the set of functional kinds of the form $\Pi_{i \in I} K_i \to \mathbf{Lab}$ for some I and $K_i^{i \in I}$ (if $I = \emptyset$, it means $\mathbf{Lab}$ simply). Similarly, given a set S of effect constructors, an effect signature Σ_{eff} is a function relation whose domain $\mathrm{dom}(\Sigma_{\mathrm{eff}})$ is S and its codomain is the set of functional kinds of the form $\Pi_{i \in I} K_i \to \mathbf{Eff}$ for some I and $K_i^{i \in I}$. A signature Σ is the disjoint union of a label signature and an effect signature. We write $\Pi \boldsymbol{K}^I \to K$, and more simply, $\Pi \boldsymbol{K} \to K$ as an abbreviation for $\Pi_{i \in I} K_i \to K$.*

Remark 1.4. *We write $\mathcal{C} : \Pi \boldsymbol{K} \to K$ to denote the pair $\langle \mathcal{C}, \Pi \boldsymbol{K} \to K \rangle$ for label name or effect constructor C.*

Definition 1.5 (The Syntax of λ_{EA}). *Given a signature $\Sigma = \Sigma_{\mathrm{lab}} \uplus \Sigma_{\mathrm{eff}}$, the syntax of λ_{EA} is defined as follows.*

I, J, N	(index sets)	i, j, n, r	(indices)
f, g, x, y, z, p, k	(variables)	$\alpha, \beta, \gamma, \tau, \iota, \rho$	(typelike variables)
op	(operation names)	$l \in \mathrm{dom}(\Sigma_{\mathrm{lab}})$	(label names)
$\mathcal{F} \in \mathrm{dom}(\Sigma_{\mathrm{eff}})$	(effect constructors)	$\mathcal{C} \in \mathrm{dom}(\Sigma_{\mathrm{lab}}) \cup \mathrm{dom}(\Sigma_{\mathrm{eff}})$	

$$
\begin{array}{rcll}
K & ::= & \mathbf{Typ} \mid \mathbf{Lab} \mid \mathbf{Eff} & \text{(kinds)} \\
S, T & ::= & A \mid L \mid \varepsilon & \text{(typelikes)} \\
A, B, C & ::= & \tau \mid A \to_\varepsilon B \mid \forall \alpha : K.A^\varepsilon & \text{(types)} \\
L & ::= & \iota \mid l\, \boldsymbol{S}^I & \text{(labels)} \\
\varepsilon & ::= & \rho \mid \mathcal{F}\, \boldsymbol{S}^I & \text{(effects)} \\
\sigma & ::= & \{\} \mid \sigma \uplus \{\mathsf{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K}^J.A \Rightarrow B\} & \text{(operation signatures)} \\
\Xi & ::= & \emptyset \mid \Xi, l :: \forall \boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma & \text{(effect contexts)} \\
\Gamma & ::= & \emptyset \mid \Gamma, x : A \mid \Gamma, \alpha : K & \text{(typing contexts)} \\
e & ::= & v \mid v_1\, v_2 \mid v\, S \mid \mathbf{let}\, x = e_1\, \mathbf{in}\, e_2 \mid \mathbf{handle}_{l\, \boldsymbol{S}^I}\, e\, \mathbf{with}\, h & \text{(expressions)} \\
v & ::= & x \mid \mathbf{fun}\,(f, x, e) \mid \Lambda \alpha : K.e \mid \mathsf{op}_{l\, \boldsymbol{S}^I}\, \boldsymbol{T}^J & \text{(values)} \\
h & ::= & \{\mathbf{return}\, x \mapsto e\} \mid h \uplus \{\mathsf{op}\, \boldsymbol{\beta}^J : \boldsymbol{K}^J\, p\, k \mapsto e\} & \text{(handlers)} \\
E & ::= & \Box \mid \mathbf{let}\, x = E\, \mathbf{in}\, e \mid \mathbf{handle}_{l\, \boldsymbol{S}^I}\, E\, \mathbf{with}\, h & \text{(evaluation contexts)}
\end{array}
$$

Remark 1.6. *We write $\lambda x.e$ for $\mathbf{fun}\,(f, x, e)$ if variable f does not occur in expression e.*

Definition 1.7 (Free Variables). *The notion of free variables is defined as usual. We write $\mathrm{FV}(e)$ for the set of free variables in expression e.*

Definition 1.8 (Free Typelike Variables). *The notion of free typelike variables is defined as usual. We write $\mathrm{FTV}(e)$ and $\mathrm{FTV}(S)$ for the sets of free typelike variables in expression e and typelike S, respectively.*

Definition 1.9 (Value Substitution). *Substitution $e[v/x]$ and $h[v/x]$ of value v for variable x in expression e and handler h, respectively, are defined as follows:*

$$x[v/x] = v$$

$$y[v/x] = y \quad (\text{if } x \neq y)$$
$$\mathbf{fun}\,(f, y, e)[v/x] = \mathbf{fun}\,(f, y, e[v/x]) \quad (\text{if } f, y \notin \mathrm{FV}(v) \cup \{x\})$$
$$(\Lambda\alpha : K.e)[v/x] = \Lambda\alpha : K.e[v/x] \quad (\text{if } \alpha \notin \mathrm{FTV}(v))$$
$$\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J[v/x] = \mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J$$
$$(v_1\,v_2)[v/x] = (v_1[v/x])\,(v_2[v/x])$$
$$(v'\,S)[v/x] = (v'[v/x])\,S$$
$$(\mathbf{handle}_{l\,\boldsymbol{S}^N}\,e\,\mathbf{with}\,h)[v/x] = \mathbf{handle}_{l\,\boldsymbol{S}^N}\,e[v/x]\,\mathbf{with}\,(h[v/x])$$
$$(\mathbf{let}\,y = e_1\,\mathbf{in}\,e_2)[v/x] = \mathbf{let}\,y = e_1[v/x]\,\mathbf{in}\,e_2[v/x]$$
$$(\text{if } y \neq x \text{ and } y \notin \mathrm{FV}(v))$$
$$([e]_L)[v/x] = [e[v/x]]_L$$
$$\{\,\mathbf{return}\,y \mapsto e_r\}[v/x] = \{\,\mathbf{return}\,y \mapsto e_r[v/x]\}$$
$$(\text{if } y \neq x \text{ and } y \notin \mathrm{FV}(v))$$
$$(h \uplus \{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e\})[v/x] = h[v/x] \uplus \{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e[v/x]\}$$
$$(\text{if } x \neq p, k \text{ and } p, k \notin \mathrm{FV}(v) \text{ and } \{\boldsymbol{\beta}^J\} \cap \mathrm{FTV}(v) = \emptyset)$$

Definition 1.10 (Typelike Substitution). *Substitution $e[S/\alpha]$, $h[S/\alpha]$, $T[S/\alpha]$, and $\Gamma[S/\alpha]$ of typelike S for typelike variable α in expression e, handler h, typelike T, and typing context Γ, respectively, are defined as follows:*

$$x[S/\alpha] = x$$
$$(\mathbf{fun}\,(f, x, e))[S/\alpha] = \mathbf{fun}\,(f, x, e[S/\alpha])$$
$$(\Lambda\beta : K.e)[S/\alpha] = \Lambda\beta : K.(e[S/\alpha]) \quad (\text{if } \alpha \neq \beta \text{ and } \beta \notin \mathrm{FTV}(S))$$
$$(\mathsf{op}_{l\,\boldsymbol{S'}^I}\,\boldsymbol{T}^J)[S/\alpha] = \mathsf{op}_{l\,\boldsymbol{S'}[\boldsymbol{S}/\boldsymbol{\alpha}]^I}\,\boldsymbol{T}[\boldsymbol{S}/\boldsymbol{\alpha}]^J$$
$$(v_1\,v_2)[S/\alpha] = (v_1[S/\alpha])\,(v_2[S/\alpha])$$
$$(v\,T)[S/\alpha] = (v[S/\alpha])\,(T[S/\alpha])$$
$$(\mathbf{handle}_{l\,\boldsymbol{T}^N}\,e\,\mathbf{with}\,h)[S/\alpha] = \mathbf{handle}_{l\,\boldsymbol{T}[\boldsymbol{S}/\boldsymbol{\alpha}]^N}\,e[S/\alpha]\,\mathbf{with}\,(h[S/\alpha])$$
$$(\mathbf{let}\,x = e_1\,\mathbf{in}\,e_2)[S/\alpha] = \mathbf{let}\,x = e_1[S/\alpha]\,\mathbf{in}\,e_2[S/\alpha]$$
$$([e]_L)[S/\alpha] = [e[S/\alpha]]_{L[S/\alpha]}$$
$$\{\,\mathbf{return}\,x \mapsto e_r\}[S/\alpha] = \{\,\mathbf{return}\,x \mapsto e_r[S/\alpha]\}$$
$$(h \uplus \{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e\})[S/\alpha] = h[S/\alpha] \uplus \{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e[S/\alpha]\}$$
$$(\text{if } \{\boldsymbol{\beta}^J\} \cap (\{\alpha\} \cup \mathrm{FTV}(S)) = \emptyset)$$

$$\alpha[S/\alpha] = S$$
$$\beta[S/\alpha] = \beta \quad (\text{if } \alpha \neq \beta)$$
$$(A \to_\varepsilon B)[S/\alpha] = (A[S/\alpha]) \to_{\varepsilon[S/\alpha]} (B[S/\alpha])$$
$$(\forall\beta : K.A^\varepsilon)[S/\alpha] = \forall\beta : K.A[S/\alpha]^{(\varepsilon[S/\alpha])}$$
$$(\text{if } \alpha \neq \beta \text{ and } \beta \notin \mathrm{FTV}(S))$$
$$(\mathcal{C}\,\boldsymbol{T}^I)[S/\alpha] = \mathcal{C}\,\boldsymbol{T}[\boldsymbol{S}/\boldsymbol{\alpha}]^I$$

$$\{\}[S/\alpha] = \{\}$$
$$(\sigma \uplus \{\mathsf{op} : \forall\boldsymbol{\beta}^J : \boldsymbol{K}^J.A \Rightarrow B\})[S/\alpha] = \sigma[S/\alpha] \uplus \{\mathsf{op} : \forall\boldsymbol{\beta}^J : \boldsymbol{K}^J.A[S/\alpha] \Rightarrow B[S/\alpha]\}$$
$$(\text{if } \{\boldsymbol{\beta}^J\} \cap \mathrm{FTV}(S) = \emptyset)$$

$$\emptyset[S/\alpha] = \emptyset$$
$$(\Gamma, x : A)[S/\alpha] = \Gamma[S/\alpha], x : A[S/\alpha]$$
$$(\Gamma, \beta : K)[S/\alpha] = \Gamma[S/\alpha], \beta : K \quad (\text{if } \alpha \neq \beta)$$

Definition 1.11 (Typelike Extraction Function). *A typelike context $\Delta(\Gamma)$ extracted from a typing context Γ is defined as follows:*

$$\Delta(\emptyset) = \emptyset \qquad \Delta(\Gamma, x : A) = \Delta(\Gamma) \qquad \Delta(\Gamma, \alpha : K) = \Delta(\Gamma), \alpha : K \ .$$

Definition 1.12 (Domains of Typing Contexts). *The set $\mathrm{dom}(\Gamma)$ of variables and typelike variables bound by a typing context Γ is defined as follows:*

$$\mathrm{dom}(\emptyset) = \emptyset \qquad \mathrm{dom}(\Gamma, x : A) = \mathrm{dom}(\Gamma) \cup \{x\} \qquad \mathrm{dom}(\Gamma, \alpha : K) = \mathrm{dom}(\Gamma) \cup \{\alpha\} \ .$$

Definition 1.13 (Context Well-formedness and Kinding Rules).

Contexts Well-formedness $\boxed{\vdash \Gamma}$

$$\frac{}{\vdash \emptyset} \ \text{C\_Empty} \qquad \frac{x \notin \mathrm{dom}(\Gamma) \quad \Gamma \vdash A : \mathbf{Typ}}{\vdash \Gamma, x : A} \ \text{C\_Var} \qquad \frac{\vdash \Gamma \quad \alpha \notin \mathrm{dom}(\Gamma)}{\vdash \Gamma, \alpha : K} \ \text{C\_TVar}$$

Kinding $\boxed{\Gamma \vdash S : K}$ $\quad \boxed{\Gamma \vdash \boldsymbol{S}^I : \boldsymbol{K}^I} \iff \forall i \in I.(\Gamma \vdash S_i : K_i)$

$$\frac{\vdash \Gamma \quad \alpha : K \in \Gamma}{\Gamma \vdash \alpha : K} \ \text{K\_Var} \qquad \frac{\vdash \Gamma \quad \mathcal{C} : \Pi \boldsymbol{K}^I \to K \in \Sigma \quad \Gamma \vdash \boldsymbol{S}^I : \boldsymbol{K}^I}{\Gamma \vdash \mathcal{C} \, \boldsymbol{S}^I : K} \ \text{K\_Cons}$$

$$\frac{\Gamma \vdash A : \mathbf{Typ} \quad \Gamma \vdash \varepsilon : \mathbf{Eff} \quad \Gamma \vdash B : \mathbf{Typ}}{\Gamma \vdash A \to_\varepsilon B : \mathbf{Typ}} \ \text{K\_Fun} \qquad \frac{\Gamma, \alpha : K \vdash A : \mathbf{Typ} \quad \Gamma, \alpha : K \vdash \varepsilon : \mathbf{Eff}}{\Gamma \vdash \forall \alpha : K.A^\varepsilon : \mathbf{Typ}} \ \text{K\_Poly}$$

Definition 1.14 (Proper Effect Contexts). *An effect context Ξ is proper if, for any $l :: \forall \boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi$, the following holds:*

- $l : \Pi \boldsymbol{K}^I \to \mathbf{Lab} \in \Sigma_{\mathrm{lab}}$;

- *for any $\boldsymbol{\alpha_0}^{I_0}$, $\boldsymbol{K_0}^{I_0}$, and σ_0, if $l :: \forall \boldsymbol{\alpha_0}^{I_0} : \boldsymbol{K_0}^{I_0}.\sigma_0 \in \Xi$, then $\boldsymbol{\alpha}^I : \boldsymbol{K}^I = \boldsymbol{\alpha_0}^{I_0} : \boldsymbol{K_0}^{I_0}$ and $\sigma = \sigma_0$; and*

- *for any $\mathrm{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K_0}^J.A \Rightarrow B \in \sigma$,*

$$\boldsymbol{\alpha}^I : \boldsymbol{K}^I, \boldsymbol{\beta}^J : \boldsymbol{K_0}^J \vdash A : \mathbf{Typ} \quad and \quad \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \boldsymbol{\beta}^J : \boldsymbol{K_0}^J \vdash B : \mathbf{Typ} \ .$$

Definition 1.15 (Well-Formedness-Preserving Functions). *Given a signature Σ, a (possibly partial) function $f \in K_i(\Sigma)^{i \in \{1,\ldots,n\}} \rightharpoonup K(\Sigma)$ preserves well-formedness if*

$$\forall \Gamma, S_1, \ldots, S_n. \Gamma \vdash S_1 : K_1 \wedge \cdots \wedge \Gamma \vdash S_n : K_n \wedge f(S_1, \ldots, S_n) \in K(\Sigma) \implies \Gamma \vdash f(S_1, \ldots, S_n) : K \ .$$

Similarly, $f \in K(\Sigma)$ preserves well-formedness if $\Gamma \vdash f : K$ for any Γ.

Definition 1.16. *We write $\alpha \mapsto T \vdash \boldsymbol{S} : K_0$ for a quadruple $\langle \alpha, T, \boldsymbol{S}, K_0 \rangle$ such that $\exists \Gamma_1, K, \Gamma_2. (\forall S_0 \in \boldsymbol{S}. \Gamma_1, \alpha : K, \Gamma_2 \vdash S_0 : K_0) \wedge \Gamma_1 \vdash T : K$.*

Definition 1.17 (Effect algebras). *Given a label signature Σ_{lab}, an effect algebra is a quintuple $\langle \Sigma_{\mathrm{eff}}, \odot, \mathbb{0}, (-)^\uparrow, \sim \rangle$ satisfying the following, where we let $\Sigma = \Sigma_{\mathrm{lab}} \uplus \Sigma_{\mathrm{eff}}$.*

- $\odot \in \mathbf{Eff}(\Sigma) \times \mathbf{Eff}(\Sigma) \rightharpoonup \mathbf{Eff}(\Sigma)$, $\mathbb{0} \in \mathbf{Eff}(\Sigma)$, and $(-)^\uparrow \in \mathbf{Lab}(\Sigma) \to \mathbf{Eff}(\Sigma)$ *preserve well-formedness. Furthermore, $\sim$ is an equivalence relation on $\mathbf{Eff}(\Sigma)$ and preserves well-formedness, that is, $\forall \varepsilon_1, \varepsilon_2. \varepsilon_1 \sim \varepsilon_2 \implies (\forall \Gamma. \Gamma \vdash \varepsilon_1 : \mathbf{Eff} \iff \Gamma \vdash \varepsilon_2 : \mathbf{Eff})$.*

- $\langle \mathbf{Eff}(\Sigma), \odot, \mathbb{0} \rangle$ *is a partial monoid under $\sim$, that is, the following holds:*

  - $\forall \varepsilon \in \mathbf{Eff}(\Sigma). \varepsilon \odot \mathbb{0} \sim \varepsilon \wedge \mathbb{0} \odot \varepsilon \sim \varepsilon$; *and*

  - $\forall \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbf{Eff}(\Sigma).$
    $(\varepsilon_1 \odot \varepsilon_2) \odot \varepsilon_3 \in \mathbf{Eff}(\Sigma) \vee \varepsilon_1 \odot (\varepsilon_2 \odot \varepsilon_3) \in \mathbf{Eff}(\Sigma) \implies (\varepsilon_1 \odot \varepsilon_2) \odot \varepsilon_3 \sim \varepsilon_1 \odot (\varepsilon_2 \odot \varepsilon_3)$.

- *Typelike substitution respecting well-formedness is a homomorphism for $\odot$, $(-)^\uparrow$, and $\sim$, that is, the following holds:*

  - $\forall \alpha, S, \varepsilon_1, \varepsilon_2. \alpha \mapsto S \vdash \varepsilon_1, \varepsilon_2 : \mathbf{Eff} \wedge \varepsilon_1 \odot \varepsilon_2 \in \mathbf{Eff}(\Sigma) \implies (\varepsilon_1 \odot \varepsilon_2)[S/\alpha] = \varepsilon_1[S/\alpha] \odot \varepsilon_2[S/\alpha]$;

  - $\forall \alpha, S, L. \alpha \mapsto S \vdash L : \mathbf{Lab} \implies (L)^\uparrow[S/\alpha] = (L[S/\alpha])^\uparrow$; *and*

  - $\forall \alpha, S, \varepsilon_1, \varepsilon_2. \alpha \mapsto S \vdash \varepsilon_1, \varepsilon_2 : \mathbf{Eff} \wedge \varepsilon_1 \sim \varepsilon_2 \implies \varepsilon_1[S/\alpha] \sim \varepsilon[S/\alpha]$.

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Remark 1.18. *For readability, we introduce the following abbreviations.*

- $\varepsilon_1 \oslash \varepsilon_2 \overset{\text{def}}{=} \exists \varepsilon.\, \varepsilon_1 \odot \varepsilon \sim \varepsilon_2$ *and*

- $\Gamma \vdash \varepsilon_1 \oslash \varepsilon_2 \overset{\text{def}}{=} \exists \varepsilon.\, \varepsilon_1 \odot \varepsilon \sim \varepsilon_2 \wedge (\forall \varepsilon' \in \{\varepsilon_1, \varepsilon_2, \varepsilon\}.\, \Gamma \vdash \varepsilon' : \mathbf{Eff})$.

Remark 1.19 (Parameters of λ_{EA}). λ_{EA} *takes a label signature in Definition 1.3, an effect algebra over that label signature in Definition 1.17, and an effect context as parameters.*

Example 1.20 (Effect Signature for Effect Sets). The effect signature $\Sigma_{\text{eff}}^{\text{Set}}$ for effect sets consists of the pairs $\{\} : \mathbf{Eff}$, $\{-\} : \mathbf{Lab} \to \mathbf{Eff}$, and $-\underline{\cup}- : \mathbf{Eff} \times \mathbf{Eff} \to \mathbf{Eff}$.

Example 1.21 (Effect Signature for Effect Multisets). The effect signature $\Sigma_{\text{eff}}^{\text{MSet}}$ for effect multisets consists of the pairs $\{\} : \mathbf{Eff}$, $\{-\} : \mathbf{Lab} \to \mathbf{Eff}$, and $-\underline{\sqcup}- : \mathbf{Eff} \times \mathbf{Eff} \to \mathbf{Eff}$.

Example 1.22 (Effect Signature for Rows). The effect signature $\Sigma_{\text{eff}}^{\text{Row}}$ for both simple rows and scoped rows consists of the pairs $\langle\rangle : \mathbf{Eff}$ and $\langle - \mid - \rangle : \mathbf{Lab} \times \mathbf{Eff} \to \mathbf{Eff}$.

Example 1.23 (Effect Sets). An effect algebra EA_{Set} for effect sets is defined by $\langle \Sigma_{\text{eff}}^{\text{Set}}, -\underline{\cup}-, \{\}, \{-\}, \sim_{\text{Set}}\rangle$ where $\sim_{\text{Set}}$ is the least equivalence relation satisfying the following rules:

$$\overline{\varepsilon \underline{\cup} \{\} \sim_{\text{Set}} \varepsilon} \qquad \overline{\varepsilon_1 \underline{\cup} \varepsilon_2 \sim_{\text{Set}} \varepsilon_2 \underline{\cup} \varepsilon_1} \qquad \overline{\varepsilon \underline{\cup} \varepsilon \sim_{\text{Set}} \varepsilon} \qquad \overline{(\varepsilon_1 \underline{\cup} \varepsilon_2) \underline{\cup} \varepsilon_3 \sim_{\text{Set}} \varepsilon_1 \underline{\cup} (\varepsilon_2 \underline{\cup} \varepsilon_3)}$$

$$\frac{\varepsilon_1 \sim_{\text{Set}} \varepsilon_2 \quad \varepsilon_3 \sim_{\text{Set}} \varepsilon_4}{\varepsilon_1 \underline{\cup} \varepsilon_3 \sim_{\text{Set}} \varepsilon_2 \underline{\cup} \varepsilon_4}$$

Example 1.24 (Effect Multisets). An effect algebra EA_{MSet} for effect multisets is defined by $\langle \Sigma_{\text{eff}}^{\text{MSet}}, -\underline{\sqcup}-, \{\},$ $\{-\}, \sim_{\text{MSet}}\rangle$ where $\sim_{\text{MSet}}$ is the least equivalence relation satisfying the following rules:

$$\overline{\varepsilon \underline{\sqcup} \{\} \sim_{\text{MSet}} \varepsilon} \qquad \overline{\varepsilon_1 \underline{\sqcup} \varepsilon_2 \sim_{\text{MSet}} \varepsilon_2 \underline{\sqcup} \varepsilon_1} \qquad \overline{(\varepsilon_1 \underline{\sqcup} \varepsilon_2) \underline{\sqcup} \varepsilon_3 \sim_{\text{MSet}} \varepsilon_1 \underline{\sqcup} (\varepsilon_2 \underline{\sqcup} \varepsilon_3)} \qquad \frac{\varepsilon_1 \sim_{\text{MSet}} \varepsilon_2 \quad \varepsilon_3 \sim_{\text{MSet}} \varepsilon_4}{\varepsilon_1 \underline{\sqcup} \varepsilon_3 \sim_{\text{MSet}} \varepsilon_2 \underline{\sqcup} \varepsilon_4}$$

Example 1.25 (Simple Rows). An effect algebra EA_{SimpR} for simple rows is defined by $\langle \Sigma_{\text{eff}}^{\text{Row}}, \odot_{\text{SimpR}}, \langle\rangle, \langle - \mid \langle\rangle\rangle, \sim_{\text{SimpR}}\rangle$ where

$$\varepsilon_1 \odot_{\text{SimpR}} \varepsilon_2 \overset{\text{def}}{=} \begin{cases} \langle L_1 \mid \langle \cdots \langle L_n \mid \varepsilon_2 \rangle\rangle\rangle & (\text{if } \varepsilon_1 = \langle L_1 \mid \langle \cdots \langle L_n \mid \langle\rangle\rangle\rangle\rangle) \\ \varepsilon_1 & (\text{if } \varepsilon_1 = \langle L_1 \mid \langle \cdots \langle L_n \mid \rho \rangle\rangle\rangle \text{ and } \varepsilon_2 = \langle\rangle) \\ \textit{undefined} & (\text{otherwise}) \end{cases}$$

and $\sim_{\text{SimpR}}$ is the least equivalence relation satisfying the following.

$$\frac{\varepsilon_1 \sim_{\text{SimpR}} \varepsilon_2}{\langle L \mid \varepsilon_1 \rangle \sim_{\text{SimpR}} \langle L \mid \varepsilon_2 \rangle} \qquad \frac{L_1 \neq L_2}{\langle L_1 \mid \langle L_2 \mid \varepsilon\rangle\rangle \sim_{\text{SimpR}} \langle L_2 \mid \langle L_1 \mid \varepsilon\rangle\rangle} \qquad \overline{\langle L \mid \varepsilon\rangle \sim_{\text{SimpR}} \langle L \mid \langle L \mid \varepsilon\rangle\rangle}$$

Example 1.26 (Scoped Rows). An effect algebra EA_{ScpR} for scoped rows is defined by $\langle \Sigma_{\text{eff}}^{\text{Row}}, \odot_{\text{ScpR}}, \langle\rangle, \langle - \mid \langle\rangle\rangle, \sim_{\text{ScpR}}\rangle$ where

$$\varepsilon_1 \odot_{\text{ScpR}} \varepsilon_2 \overset{\text{def}}{=} \begin{cases} \langle L_1 \mid \langle \cdots \langle L_n \mid \varepsilon_2 \rangle\rangle\rangle & (\text{if } \varepsilon_1 = \langle L_1 \mid \langle \cdots \langle L_n \mid \langle\rangle\rangle\rangle\rangle) \\ \varepsilon_1 & (\text{if } \varepsilon_1 = \langle L_1 \mid \langle \cdots \langle L_n \mid \rho \rangle\rangle\rangle \text{ and } \varepsilon_2 = \langle\rangle) \\ \textit{undefined} & (\text{otherwise}) \end{cases}$$

and $\sim_{\text{ScpR}}$ is the least equivalence relation satisfying the following.

$$\frac{\varepsilon_1 \sim_{\text{ScpR}} \varepsilon_2}{\langle L \mid \varepsilon_1 \rangle \sim_{\text{ScpR}} \langle L \mid \varepsilon_2 \rangle} \qquad \frac{L_1 \neq L_2}{\langle L_1 \mid \langle L_2 \mid \varepsilon\rangle\rangle \sim_{\text{ScpR}} \langle L_2 \mid \langle L_1 \mid \varepsilon\rangle\rangle}$$

Example 1.27 (Erasable Sets). An effect algebra EA_{ESet} for effect sets is defined by $\langle \Sigma_{\text{eff}}^{\text{Set}}, -\underline{\cup}-, \{\}, \{-\}, \sim_{\text{ESet}}\rangle$ where $\sim_{\text{ESet}}$ is the least equivalence relation satisfying the following rules:

$$\frac{l_1 \neq l_2}{\{l_1 \, \boldsymbol{S_1}^{I_1}\} \underline{\cup} \{l_2 \, \boldsymbol{S_2}^{I_2}\} \sim_{\text{ESet}} \{l_2 \, \boldsymbol{S_2}^{I_2}\} \underline{\cup} \{l_1 \, \boldsymbol{S_1}^{I_1}\}} \qquad \overline{\{l \, \boldsymbol{S_1}^{I_1}\} \underline{\cup} \{l \, \boldsymbol{S_2}^{I_2}\} \sim_{\text{ESet}} \{l \, \boldsymbol{S_1}^{I_1}\}} \qquad \overline{\varepsilon \underline{\cup} \{\} \sim_{\text{ESet}} \varepsilon}$$

$$\overline{\{\} \underline{\cup} \varepsilon \sim_{\text{ESet}} \varepsilon} \qquad \overline{(\varepsilon_1 \underline{\cup} \varepsilon_2) \underline{\cup} \varepsilon_3 \sim_{\text{ESet}} \varepsilon_1 \underline{\cup} (\varepsilon_2 \underline{\cup} \varepsilon_3)} \qquad \frac{\varepsilon_1 \sim_{\text{ESet}} \varepsilon_2 \quad \varepsilon_3 \sim_{\text{ESet}} \varepsilon_4}{\varepsilon_1 \underline{\cup} \varepsilon_3 \sim_{\text{ESet}} \varepsilon_2 \underline{\cup} \varepsilon_4}$$

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Example 1.28 (Erasable Multisets). An effect algebra $\mathrm{EA}_{\mathrm{EMSet}}$ for effect multisets is defined by $\langle \Sigma_{\mathrm{eff}}^{\mathrm{MSet}}, - \sqcup -, \{\}, \{-\}, \sim_{\mathrm{EMSet}} \rangle$ where $\sim_{\mathrm{EMSet}}$ is the least equivalence relation satisfying the following rules:

$$\frac{l_1 \neq l_2}{\{l_1 \, \boldsymbol{S_1}^{I_1}\} \sqcup \{l_2 \, \boldsymbol{S_2}^{I_2}\} \sim_{\mathrm{EMSet}} \{l_2 \, \boldsymbol{S_2}^{I_2}\} \sqcup \{l_1 \, \boldsymbol{S_1}^{I_1}\}} \qquad \frac{}{\varepsilon \sqcup \{\} \sim_{\mathrm{EMSet}} \varepsilon} \qquad \frac{}{\{\} \sqcup \varepsilon \sim_{\mathrm{EMSet}} \varepsilon}$$

$$\frac{}{(\varepsilon_1 \sqcup \varepsilon_2) \sqcup \varepsilon_3 \sim_{\mathrm{EMSet}} \varepsilon_1 \sqcup (\varepsilon_2 \sqcup \varepsilon_3)} \qquad \frac{\varepsilon_1 \sim_{\mathrm{EMSet}} \varepsilon_2 \quad \varepsilon_3 \sim_{\mathrm{EMSet}} \varepsilon_4}{\varepsilon_1 \sqcup \varepsilon_3 \sim_{\mathrm{EMSet}} \varepsilon_2 \sqcup \varepsilon_4}$$

Example 1.29 (Erasable Simple Rows). An effect algebra $\mathrm{EA}_{\mathrm{ESimpR}}$ for erasable simple rows is defined by $\langle \Sigma_{\mathrm{eff}}^{\mathrm{Row}}, \odot_{\mathrm{ESimpR}}, \langle \rangle, \langle - \mid \langle \rangle \rangle, \sim_{\mathrm{ESimpR}} \rangle$ where

$$\varepsilon_1 \odot_{\mathrm{ESimpR}} \varepsilon_2 \stackrel{\mathrm{def}}{=} \begin{cases} \langle L_1 \mid \langle \cdots \langle L_n \mid \varepsilon_2 \rangle \rangle \rangle & (\text{if } \varepsilon_1 = \langle L_1 \mid \langle \cdots \langle L_n \mid \langle \rangle \rangle \rangle \rangle) \\ \varepsilon_1 & (\text{if } \varepsilon_1 = \langle L_1 \mid \langle \cdots \langle L_n \mid \rho \rangle \rangle \rangle \text{ and } \varepsilon_2 = \langle \rangle) \\ undefined & (\text{otherwise}) \end{cases}$$

and $\sim_{\mathrm{ESimpR}}$ is the least equivalence relation satisfying the following.

$$\frac{\varepsilon_1 \sim_{\mathrm{SimpR}} \varepsilon_2}{\langle L \mid \varepsilon_1 \rangle \sim_{\mathrm{SimpR}} \langle L \mid \varepsilon_2 \rangle} \qquad \frac{l_1 \neq l_2}{\langle l_1 \, \boldsymbol{S_1}^{I_1} \mid \langle l_2 \, \boldsymbol{S_2}^{I_2} \mid \varepsilon \rangle \rangle \sim_{\mathrm{SimpR}} \langle l_2 \, \boldsymbol{S_2}^{I_2} \mid \langle l_1 \, \boldsymbol{S_1}^{I_1} \mid \varepsilon \rangle \rangle}$$

$$\frac{}{\langle l \, \boldsymbol{S_1}^{I_1} \mid \varepsilon \rangle \sim_{\mathrm{SimpR}} \langle l \, \boldsymbol{S_1}^{I_1} \mid \langle l \, \boldsymbol{S_2}^{I_2} \mid \varepsilon \rangle \rangle}$$

Example 1.30 (Erasable Scoped Rows). An effect algebra $\mathrm{EA}_{\mathrm{EScpR}}$ for scoped rows is defined by $\langle \Sigma_{\mathrm{eff}}^{\mathrm{Row}}, \odot_{\mathrm{EScpR}}, \langle \rangle, \langle - \mid \langle \rangle \rangle, \sim_{\mathrm{EScpR}} \rangle$ where

$$\varepsilon_1 \odot_{\mathrm{EScpR}} \varepsilon_2 \stackrel{\mathrm{def}}{=} \begin{cases} \langle L_1 \mid \langle \cdots \langle L_n \mid \varepsilon_2 \rangle \rangle \rangle & (\text{if } \varepsilon_1 = \langle L_1 \mid \langle \cdots \langle L_n \mid \langle \rangle \rangle \rangle \rangle) \\ \varepsilon_1 & (\text{if } \varepsilon_1 = \langle L_1 \mid \langle \cdots \langle L_n \mid \rho \rangle \rangle \rangle \text{ and } \varepsilon_2 = \langle \rangle) \\ undefined & (\text{otherwise}) \end{cases}$$

and $\sim_{\mathrm{EScpR}}$ is the least equivalence relation satisfying the following.

$$\frac{\varepsilon_1 \sim_{\mathrm{EScpR}} \varepsilon_2}{\langle L \mid \varepsilon_1 \rangle \sim_{\mathrm{EScpR}} \langle L \mid \varepsilon_2 \rangle} \qquad \frac{l_1 \neq l_2}{\langle l_1 \, \boldsymbol{S_1}^{I_1} \mid \langle l_2 \, \boldsymbol{S_2}^{I_2} \mid \varepsilon \rangle \rangle \sim_{\mathrm{EScpR}} \langle l_2 \, \boldsymbol{S_2}^{I_2} \mid \langle l_1 \, \boldsymbol{S_1}^{I_1} \mid \varepsilon \rangle \rangle}$$

Definition 1.31 (Freeness of Labels).

Freeness of labels $\boxed{n-\mathrm{free}(L, E)}$

$$\frac{}{0-\mathrm{free}(L, \square)} \qquad \frac{n-\mathrm{free}(L, E)}{n-\mathrm{free}(L, \mathbf{let} \, x = E \, \mathbf{in} \, e)} \qquad \frac{n+1-\mathrm{free}(L, E)}{n-\mathrm{free}(L, \mathbf{handle}_L \, E \, \mathbf{with} \, h)} \qquad \frac{n-\mathrm{free}(L, E) \quad L \neq L'}{n-\mathrm{free}(L, \mathbf{handle}_{L'} \, E \, \mathbf{with} \, h)}$$

Definition 1.32 (Operational Semantics).

Reduction $\boxed{e \longmapsto e'}$

$$\frac{}{\mathbf{fun} \, (f, x, e) \, v \longmapsto e[\mathbf{fun} \, (f, x, e)/f][v/x]} \; \mathrm{R\_App} \qquad \frac{}{(\Lambda \alpha : K.e) \, S \longmapsto e[S/\alpha]} \; \mathrm{R\_TApp}$$

$$\frac{}{\mathbf{let} \, x = v \, \mathbf{in} \, e \longmapsto e[v/x]} \; \mathrm{R\_Let} \qquad \frac{\mathbf{return} \, x \mapsto e_r \in h}{\mathbf{handle}_{l \, \boldsymbol{S}^I} \, v \, \mathbf{with} \, h \longmapsto e_r[v/x]} \; \mathrm{R\_Handle1}$$

$$\frac{\mathsf{op} \, \boldsymbol{\beta}^J : \boldsymbol{K}^J \, p \, k \mapsto e \in h \quad v_{cont} = \lambda z.\mathbf{handle}_{l \, \boldsymbol{S}^I} \, E[z] \, \mathbf{with} \, h \quad 0-\mathrm{free}(l \, \boldsymbol{S}^I, E)}{\mathbf{handle}_{l \, \boldsymbol{S}^I} \, E[\mathsf{op}_{l \, \boldsymbol{S}^I} \, \boldsymbol{T}^J \, v] \, \mathbf{with} \, h \longmapsto e[\boldsymbol{T}^J/\boldsymbol{\beta}^J][v/p][v_{cont}/k]} \; \mathrm{R\_Handle2}$$

Evaluation $\boxed{e \longrightarrow e'}$

$$\frac{e_1 \longmapsto e_2}{E[e_1] \longrightarrow E[e_2]} \; \mathrm{E\_Eval}$$

Definition 1.33. *We write* $\longrightarrow^*$ *for the reflexive, transitive closure of* $\longrightarrow$. *We also write* $e \not\longrightarrow$ *to denote that there is no* e' *such that* $e \longrightarrow e'$.

Definition 1.34 (Typing and Subtyping Rules)**.**

Typing $\boxed{\Gamma \vdash e : A \mid \varepsilon}$

$$\frac{\vdash \Gamma \quad x : A \in \Gamma}{\Gamma \vdash x : A \mid \mathbb{0}} \ \text{T\_VAR} \qquad \frac{\Gamma, f : A \to_\varepsilon B, x : A \vdash e : B \mid \varepsilon}{\Gamma \vdash \mathbf{fun}\,(f, x, e) : A \to_\varepsilon B \mid \mathbb{0}} \ \text{T\_ABS}$$

$$\frac{\Gamma \vdash v_1 : A \to_\varepsilon B \mid \mathbb{0} \quad \Gamma \vdash v_2 : A \mid \mathbb{0}}{\Gamma \vdash v_1\,v_2 : B \mid \varepsilon} \ \text{T\_APP} \qquad \frac{\Gamma, \alpha : K \vdash e : A \mid \varepsilon}{\Gamma \vdash \Lambda\alpha : K.e : \forall\alpha : K.A^\varepsilon \mid \mathbb{0}} \ \text{T\_TABS}$$

$$\frac{\Gamma \vdash v : \forall\alpha : K.A^\varepsilon \mid \mathbb{0} \quad \Gamma \vdash S : K}{\Gamma \vdash v\,S : A[S/\alpha] \mid \varepsilon[S/\alpha]} \ \text{T\_TAPP} \qquad \frac{\Gamma \vdash e_1 : A \mid \varepsilon \quad \Gamma, x : A \vdash e_2 : B \mid \varepsilon}{\Gamma \vdash \mathbf{let}\,x = e_1\,\mathbf{in}\,e_2 : B \mid \varepsilon} \ \text{T\_LET}$$

$$\frac{\Gamma \vdash e : A \mid \varepsilon \quad \Gamma \vdash A \mid \varepsilon <: A' \mid \varepsilon'}{\Gamma \vdash e : A' \mid \varepsilon'} \ \text{T\_SUB} \qquad \frac{\begin{array}{c} l :: \forall\boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi \quad \mathsf{op} : \forall\boldsymbol{\beta}^J : \boldsymbol{K'}^J.A \Rightarrow B \in \sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \\ \vdash \Gamma \quad \Gamma \vdash \boldsymbol{S}^I : \boldsymbol{K}^I \quad \Gamma \vdash \boldsymbol{T}^J : \boldsymbol{K'}^J \end{array}}{\Gamma \vdash \mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J : (A[\boldsymbol{T}^J/\boldsymbol{\beta}^J]) \to_{(l\,\boldsymbol{S}^I)^\uparrow} (B[\boldsymbol{T}^I/\boldsymbol{\beta}^I]) \mid \mathbb{0}} \ \text{T\_OP}$$

$$\frac{\begin{array}{c} \Gamma \vdash e : A \mid \varepsilon' \quad l :: \forall\boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi \quad \Gamma \vdash \boldsymbol{S}^I : \boldsymbol{K}^I \\ \Gamma \vdash_{\sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} h : A \Rightarrow^\varepsilon B \quad (l\,\boldsymbol{S}^I)^\uparrow \odot \varepsilon \sim \varepsilon' \end{array}}{\Gamma \vdash \mathbf{handle}_{l\,\boldsymbol{S}^I}\,e\,\mathbf{with}\,h : B \mid \varepsilon} \ \text{T\_HANDLING}$$

Handler Typing $\boxed{\Gamma \vdash_\sigma h : A \Rightarrow^\varepsilon B}$

$$\frac{\Gamma, x : A \vdash e_r : B \mid \varepsilon}{\Gamma \vdash_{\{\}} \{\,\mathbf{return}\,x \mapsto e_r\,\} : A \Rightarrow^\varepsilon B} \ \text{H\_RETURN}$$

$$\frac{\sigma = \sigma' \uplus \{\mathsf{op} : \forall\boldsymbol{\beta}^J : \boldsymbol{K}^J.A' \Rightarrow B'\}}{\begin{array}{c} \Gamma \vdash_{\sigma'} h : A \Rightarrow^\varepsilon B \quad \Gamma, \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A', k : B' \to_\varepsilon B \vdash e : B \mid \varepsilon \\ \hline \Gamma \vdash_\sigma h \uplus \{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e\} : A \Rightarrow^\varepsilon B \end{array}} \ \text{H\_OP}$$

Subtyping $\boxed{\Gamma \vdash A <: B}$

$$\frac{\Gamma \vdash A : \mathbf{Typ}}{\Gamma \vdash A <: A} \ \text{ST\_REFL} \qquad \frac{\Gamma \vdash A_2 <: A_1 \quad \Gamma \vdash B_1 \mid \varepsilon_1 <: B_2 \mid \varepsilon_2}{\Gamma \vdash A_1 \to_{\varepsilon_1} B_1 <: A_2 \to_{\varepsilon_2} B_2} \ \text{ST\_FUN}$$

$$\frac{\Gamma, \alpha : K \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2}{\Gamma \vdash \forall\alpha : K.A_1{}^{\varepsilon_1} <: \forall\alpha : K.A_2{}^{\varepsilon_2}} \ \text{ST\_POLY} \qquad \frac{\Gamma \vdash A_1 <: B \quad \Gamma \vdash \varepsilon_1 \oslash \varepsilon_2}{\Gamma \vdash A \mid \varepsilon_1 <: B \mid \varepsilon_2} \ \text{ST\_COMP}$$

Definition 1.35 (Semantics of Shallow Handlers)**.** *The semantics for shallow handlers consists of the reduction and evaluation relations defined by the following rule* R_SHANDLE *and those in Definition 1.32 except for* R_HANDLE2.

$$\frac{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e \in h \quad v_{cont} = \lambda z.E[z] \quad 0{-}\mathrm{free}(l\,\boldsymbol{S}^I, E)}{\mathbf{handle}_{l\,\boldsymbol{S}^I}\,E[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v]\,\mathbf{with}\,h \longmapsto e[\boldsymbol{T}^J/\boldsymbol{\beta}^J][v/p][v_{cont}/k]} \ \text{R\_SHANDLE}$$

Definition 1.36 (Typing of Shallow Handlers)**.** *The typing rules of shallow handlers consist of the rules defined by the following rules* T_SHANDLING, SH_RETURN, *and* SH_OP, *and those in Definition 1.34 except for* T_HANDLING, H_RETURN, *and* H_OP.

Typing $\boxed{\Gamma \vdash e : A \mid \varepsilon}$

$$\frac{\begin{array}{c} \Gamma \vdash e : A \mid \varepsilon' \quad l :: \forall\boldsymbol{\alpha}^N : \boldsymbol{K}^N.\sigma \in \Xi \quad \Gamma \vdash \boldsymbol{S}^N : \boldsymbol{K}^N \\ \Gamma \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : A^{\varepsilon'} \Rightarrow^\varepsilon B \quad (l\,\boldsymbol{S}^N)^\uparrow \odot \varepsilon \sim \varepsilon' \end{array}}{\Gamma \vdash \mathbf{handle}_{l\,\boldsymbol{S}^N}\,e\,\mathbf{with}\,h : B \mid \varepsilon} \ \text{T\_SHANDLING}$$

Shallow Handler Typing $\boxed{\Gamma \vdash_\sigma h : A^{\varepsilon'} \Rightarrow^\varepsilon B}$

$$\frac{\Gamma, x : A \vdash e_r : B \mid \varepsilon \quad \Gamma \vdash \varepsilon' : \mathbf{Eff}}{\Gamma \vdash_{\{\}} \{\,\mathbf{return}\,x \mapsto e_r\,\} : A^{\varepsilon'} \Rightarrow^\varepsilon B} \ \text{SH\_RETURN}$$

$$\frac{\sigma = \sigma' \uplus \{\mathsf{op} : \forall\boldsymbol{\beta}^J : \boldsymbol{K}^J.A' \Rightarrow B'\}}{\begin{array}{c} \Gamma \vdash_{\sigma'} h : A^{\varepsilon'} \Rightarrow^\varepsilon B \quad \Gamma, \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A', k : B' \to_{\varepsilon'} A \vdash e : B \mid \varepsilon \\ \hline \Gamma \vdash_\sigma h \uplus \{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e\} : A^{\varepsilon'} \Rightarrow^\varepsilon B \end{array}} \ \text{SH\_OP}$$

Definition 1.37 (The Syntax of λ_{EA} with Lift Coercions). *The syntax of λ_{EA} extended by lift coercions is the same as Definition 1.5 except for the following.*

$$e \quad ::= \quad \cdots \mid [e]_L \quad \text{(expressions)} \qquad E \quad ::= \quad \cdots \mid [E]_L \quad \text{(evaluation contexts)}$$

Definition 1.38 (Freeness of Labels with Lift Coercions). *The rules of freeness of labels for λ_{EA} extended by lift coercions consist of the rules in Definition 1.31 and the following rules.*

Freeness of labels $\boxed{n-\mathrm{free}(L, E)}$

$$\frac{n-\mathrm{free}(L, E)}{n+1-\mathrm{free}(L, [E]_L)} \qquad \frac{n-\mathrm{free}(L, E) \quad L \neq L'}{n-\mathrm{free}(L, [E]_{L'})}$$

Definition 1.39 (Semantics with Lift Coercions). *The semantics for λ_{EA} extended by lift coercions consists of the reduction and evaluation relations defined by the following rule R_LIFT and those in Definition 1.32 except for R_HANDLE2.*

Reduction $\boxed{e \longmapsto e'}$

$$\frac{}{[v]_L \longmapsto v} \ \mathrm{R\_LIFT}$$

Definition 1.40 (Typing of Lift Coercions). *The typing rules of λ_{EA} extended by lift coercions consist of the rules in Definition 1.34 and the following rule.*

$$\frac{\Gamma \vdash e : A \mid \varepsilon' \quad \Gamma \vdash L : \mathbf{Lab} \quad (L)^{\uparrow} \odot \varepsilon' \sim \varepsilon}{\Gamma \vdash [e]_L : A \mid \varepsilon} \ \mathrm{T\_LIFT}$$

Definition 1.41 (Freeness of Label Names).

Freeness of label names $\boxed{n-\mathrm{free}(L, E)}$

$$\frac{}{0-\mathrm{free}(l, \square)} \qquad \frac{n-\mathrm{free}(l, E)}{n-\mathrm{free}(l, \mathbf{let}\ x = E\ \mathbf{in}\ e)} \qquad \frac{n+1-\mathrm{free}(l, E)}{n-\mathrm{free}(l, \mathbf{handle}_{l\ \boldsymbol{S}^I}\ E\ \mathbf{with}\ h)}$$

$$\frac{n-\mathrm{free}(l, E) \quad l \neq l'}{n-\mathrm{free}(l, \mathbf{handle}_{l'\ \boldsymbol{S}^I}\ E\ \mathbf{with}\ h)}$$

Definition 1.42 (Operational Semantics with Type-Erasure). *The type-erasure semantics consists of the reduction and evaluation relations defined by the following rule R_HANDLE2' and those in Definition 1.32 except for R_HANDLE2.*

$$\frac{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\ p\ k \mapsto e \in h \quad v_{cont} = \lambda z.\mathbf{handle}_{l\ \boldsymbol{S}^I}\ E[z]\ \mathbf{with}\ h \quad 0-\mathrm{free}(l, E)}{\mathbf{handle}_{l\ \boldsymbol{S}^I}\ E[\mathsf{op}_{l\ \boldsymbol{S}'^I}\ \boldsymbol{T}^J\ v]\ \mathbf{with}\ h \longmapsto e[\boldsymbol{T}^J/\boldsymbol{\beta}^J][v/p][v_{cont}/k]} \ \mathrm{R\_HANDLE2'}$$

Definition 1.43 (Freeness of Label Names with Lift Coercions).
The rules of freeness of label names for λ_{EA} extended by lift coercions consist of the rules in Definition 1.41 and the following rules.

Freeness of label names $\boxed{n-\mathrm{free}(l, E)}$

$$\frac{n-\mathrm{free}(l, E)}{n+1-\mathrm{free}(l, [E]_{l\ \boldsymbol{S}^I})} \qquad \frac{n-\mathrm{free}(l, E) \quad L \neq l\ \boldsymbol{S}^I}{n-\mathrm{free}(l, [E]_L)}$$

Definition 1.44 (Semantics with Lift Coercions and Type-Erasure). *The semantics for lift coercions consists of the reduction and evaluation relations defined by the rule R_HANDLE2' defined in Definition 1.42 and those in Definition 1.39 except for R_HANDLE2.*

Definition 1.45 (Safety Conditions).
(1) For any L, $(L)^{\uparrow} \oslash \mathbb{0}$ does not hold.

(2) If $(L)^{\uparrow} \oslash \varepsilon$ and $(L')^{\uparrow} \odot \varepsilon' \sim \varepsilon$ and $L \neq L'$, then $(L)^{\uparrow} \oslash \varepsilon'$.

Definition 1.46 (Safety Condition for Lift Coercions). *The safety condition added for lift coercions is the following:*

(3) If $(L)^{\uparrow} \odot \varepsilon_1 \sim (L_1)^{\uparrow} \odot \cdots \odot (L_n)^{\uparrow} \odot (L)^{\uparrow} \odot \varepsilon_2$ and $L \notin \{L_1, \ldots, L_n\}$, then $\varepsilon_1 \sim (L_1)^{\uparrow} \odot \cdots \odot (L_n)^{\uparrow} \odot \varepsilon_2$.

Definition 1.47 (Safety Condition for Type-Erasure). *The safety condition added for the type-erasure semantics is the following:*
(4) If $(l\,\boldsymbol{S_1}^{I_1})^{\uparrow} \oslash \varepsilon$ and $(l\,\boldsymbol{S_2}^{I_2})^{\uparrow} \oslash \varepsilon$, then $\boldsymbol{S_1}^{I_1} = \boldsymbol{S_2}^{I_2}$.

Example 1.48 (Unsafe Effect Algebras).

Effect algebra violating safety condition (1) Consider an effect algebra such that $\emptyset \vdash (l)^{\uparrow} \oslash \mathbb{0}$ holds for some l. Clearly, this effect algebra violates safety condition (1). In this case, $\emptyset \vdash \mathsf{op}_l\, v : A \mid \mathbb{0}$ can be derived for some A (if $\mathsf{op}_l\, v$ is well typed) because $\mathsf{op}_l\, v$ is given the effect $(l)^{\uparrow}$ and the subeffecting $\emptyset \vdash (l)^{\uparrow} \oslash \mathbb{0}$ holds. However, the operation call is not handled.

Effect algebra violating safety condition (2) Consider an effect algebra such that safety condition (1), $(l)^{\uparrow} \oslash (l')^{\uparrow}$, and $(l')^{\uparrow} \odot \mathbb{0} \sim (l')^{\uparrow}$ hold for some l and l' such that $l \neq l'$. This effect algebra violates safety condition (2): if safety condition (2) is met, we would have $(l)^{\uparrow} \oslash \mathbb{0}$, but it is contradictory with safety condition (1).

This effect algebra allows assigning the empty effect $\mathbb{0}$ to the expression $\mathbf{handle}_{l'}\, \mathsf{op}_l\, v\, \mathbf{with}\, h$ as illustrated by the following typing derivation:

$$
\cfrac{\cdots \quad (l')^{\uparrow} \odot \mathbb{0} \sim (l')^{\uparrow} \qquad \cfrac{\emptyset \vdash \mathsf{op}_l\, v : A \mid (l)^{\uparrow} \quad \emptyset \vdash A \mid (l)^{\uparrow} <: A \mid (l')^{\uparrow}}{\emptyset \vdash \mathsf{op}_l\, v : A \mid (l')^{\uparrow}}\ \text{T\_Sub}}{\emptyset \vdash \mathbf{handle}_{l'}\, \mathsf{op}_l\, v\, \mathbf{with}\, h : B \mid \mathbb{0}}\ \text{T\_Handling}
$$

However, the operation call in it is not handled.

$$\exists \alpha : \mathbf{Typ}.\exists \rho : \mathbf{Eff}.\{$$

$$empty : \alpha, \quad add : \mathsf{Int} \to_{\{\}} \alpha \to_{\{\}} \alpha, \quad size : \alpha \to_{\{\}} \mathsf{Int}, \quad find : \mathsf{Int} \to_{\{\}} \alpha \to_{\{\}} \mathsf{Bool},$$

$$filter : (\mathsf{Int} \to_{\{\}} \mathsf{Bool}) \to_{\{\}} \alpha \to_{\{\}} \alpha, \quad choose : \alpha \to_{\rho} \mathsf{Int},$$

$$accumulate : \forall \beta : \mathbf{Typ}.\forall \rho' : \mathbf{Eff}.(\mathsf{Unit} \to_{\rho \sqcup \rho'} \beta) \to_{\rho'} \beta \; \mathsf{List}$$

$$\}$$

Figure 1: Module Interface IntSet

$\mathbf{pack}(\mathsf{Int}\,\mathsf{List}, \{\mathsf{Selection}\,\mathsf{Int}\}, \{\cdots$

$\quad choose = \mathsf{select}_{\mathsf{Selection}\,\mathsf{Int}}$

$\quad accumulate = \Lambda\beta : \mathbf{Typ}.\Lambda\rho' : \mathbf{Eff}.\lambda f : \mathsf{Unit} \to_{\{\mathsf{Selection}\,\mathsf{Int}\} \sqcup \rho'} \beta.$

$\qquad\qquad \mathbf{handle}_{\mathsf{Selection}\,\mathsf{Int}} f\,()\,\mathbf{with}\,\{\,\mathbf{return}\,x \mapsto [x]\} \uplus \{\mathsf{select}\,xs\,k \mapsto \mathtt{concat}\,(\mathtt{map}\,k\,xs)\}$

$\})$

$\mathbf{pack}(\mathsf{Int}\,\mathsf{List}, \{\mathsf{Fail}\} \sqcup \{\mathsf{Choice}\}, \{\cdots$

$\quad choose = \mathbf{fun}(aux, xs, \mathbf{match}\,xs\,\mathbf{with}$

$\qquad\qquad\qquad |\,[] \to \mathsf{fail}_{\mathsf{Fail}}\,\mathsf{Int}\,()$

$\qquad\qquad\qquad |\,y :: ys \to \mathbf{if}\,\mathsf{decide}_{\mathsf{Choice}}\,()\,\mathbf{then}\,y\,\mathbf{else}\,aux\,ys),$

$\quad accumulate = \Lambda\beta : \mathbf{Typ}.\Lambda\rho' : \mathbf{Eff}.\lambda f : \mathsf{Unit} \to_{\{\mathsf{Fail}\} \sqcup \{\mathsf{Choice}\} \sqcup \rho'} \beta.$

$\qquad\qquad \mathbf{handle}_{\mathsf{Choice}}$

$\qquad\qquad\quad \mathbf{handle}_{\mathsf{Fail}}$

$\qquad\qquad\qquad f\,()$

$\qquad\qquad \mathbf{with}\,\{\,\mathbf{return}\,x \mapsto [x]\} \uplus \{\mathsf{fail}\,\alpha : \mathbf{Typ}\,\_\_ \mapsto []\}$

$\qquad\qquad \mathbf{with}\,\{\,\mathbf{return}\,x \mapsto x\} \uplus \{\mathsf{decide}\,\_\,k \mapsto k\,\mathsf{true}\,@\,k\,\mathsf{false}\}$

$\})$

Figure 2: Two implementation of IntSet

2 Example

We present a motivating example of allowing multiple effect variables in one effect collection. In this example, we use $\mathrm{EA}_{\mathrm{Set}}$ and offer two modules of type IntSet, which is an interface of implementations for integer sets defined in Figure 1.

We show two implementations of IntSet in Figure 2. The former implementation assumes the effect context Selection :: $\forall \alpha : \mathbf{Typ}.\{\mathsf{select} : \alpha\,\mathsf{List} \Rightarrow \alpha\}$, and concretizes ρ by Selection Int. The latter implementation assumes the effect context Choice :: $\{\mathsf{decide} : \mathsf{Unit} \Rightarrow \mathsf{Bool}\}$, Fail :: $\{\mathsf{fail} : \forall \alpha : \mathbf{Typ}.\mathsf{Unit} \Rightarrow \alpha\}$, and concretizes ρ by $\{\mathsf{Fail}\} \sqcup \{\mathsf{Choice}\}$. Because the concrete effect of the latter implementation consists of two labels, it needs to be abstracted by a row variable, not by a label variable.

We define the function *search_path* using this package as follows.

$search\_path = \lambda sets : \mathsf{IntSet}\,\mathsf{List}.\lambda s : \mathsf{Int}.\lambda t : \mathsf{Int}.$

$\qquad\qquad \mathbf{fun}(aux, p, \lambda path : \mathsf{Int}\,\mathsf{List}.$

$\qquad\qquad\qquad \mathbf{if}\,p = t\,\mathbf{then}\,path$

$\qquad\qquad\qquad \mathbf{else}\,\mathbf{let}\,x = choose\,(filter\,(\lambda y : \mathsf{Int}.not\,(\mathtt{exists}\,(\lambda z.z = y)\,path))\,(nth\,p\,sets))$

$\qquad\qquad\qquad\qquad \mathbf{in}\,aux\,x\,(x :: path))$

$\qquad\qquad\quad s\,[s]$

We show the example program using *search_path* as follows.

$graph = [add\,1\,(add\,2\,empty);\ add\,0\,(add\,2\,(add\,3\,empty));\ add\,0\,(add\,1\,(add\,4\,empty));$
$\qquad\quad add\,1\,(add\,4\,(add\,5\,empty));\ add\,2\,(add\,3\,(add\,6\,empty));\ add\,3\,empty;\ add\,4\,empty]$
$clean\,(\lambda\_ : \mathsf{Unit}.search\_path\,graph\,0\,5)$
$clean\,(\lambda\_ : \mathsf{Unit}.search\_path\,graph\,0\,5)$

The evaluation results are as follows.

[[5; 3; 4; 2; 1; 0]; [5; 3; 1; 0]; [5; 3; 4; 2; 0]; [5; 3; 1; 2; 0]]
[[6; 4; 2; 0; 1]; [6; 4; 2; 1]; [6; 4; 3; 1]]

3 Properties

3.1 Properties with Deep Handlers

This section assumes that the safety conditions in Definition 1.45 hold.

Lemma 3.1 (Well-formedness of context in judgement)**.** *If $\Gamma \vdash S : K$, then $\vdash \Gamma$.*

Proof. By induction on a derivation of $\Gamma \vdash S : K$. We proceed by case analysis on the kinding rule applied lastly to this derivation.
Case K_VAR**:** Clearly.

Case K_FUN**:** $S = A \rightarrow_\varepsilon B$, $\Gamma \vdash A : \textbf{Typ}$, $\Gamma \vdash \varepsilon : \textbf{Eff}$, and $\Gamma \vdash B : \textbf{Typ}$ are given. By the induction hypothesis, we have $\vdash \Gamma$.

Case K_POLY**:** $S = \forall \alpha : K.A^\varepsilon$, $\Gamma, \alpha : K \vdash A : \textbf{Typ}$, and $\Gamma, \alpha : K \vdash \varepsilon : \textbf{Eff}$ are given. By the induction hypothesis, we have $\vdash \Gamma, \alpha : K$. Since only C_TVAR can derive $\vdash \Gamma, \alpha : K$, the required result $\vdash \Gamma$ is achieved.

Case K_CONS**:** Clearly.

∎

Lemma 3.2.
  (1) If $\vdash \Gamma$, then $\vdash \Delta(\Gamma)$.

  (2) If $\Gamma \vdash S : K$, then $\Delta(\Gamma) \vdash S : K$.

Proof. By mutual induction on the derivations. We proceed by case analysis on the rule applied lastly to the derivation.
Case C_EMPTY**:** Clearly because of $\Delta(\emptyset) = \emptyset$.

Case C_VAR**:** For some Γ', x, and A, the following are given:

- $\Gamma = \Gamma', x : A$ and
- $\Gamma' \vdash A : \textbf{Typ}$.

By the induction hypothesis, we have $\Delta(\Gamma') \vdash A : \textbf{Typ}$. By Lemma 3.1, we have $\vdash \Delta(\Gamma')$. Thus, we get $\vdash \Delta(\Gamma)$ as required because of $\Delta(\Gamma', x : A) = \Delta(\Gamma')$.

Case C_TVAR**:** For some Γ', α, and K, the following are given:

- $\Gamma = \Gamma', \alpha : K$,
- $\vdash \Gamma'$, and
- $\alpha \notin \text{dom}(\Gamma')$.

By the induction hypothesis, we have $\vdash \Delta(\Gamma')$. By $\alpha \notin \text{dom}(\Gamma')$, we have $\alpha \notin \text{dom}(\Delta(\Gamma'))$ because $\text{dom}(\Delta(\Gamma')) \subseteq \text{dom}(\Gamma')$. Thus, C_TVAR derives $\vdash \Delta(\Gamma'), \alpha : K$ as required.

Case K_VAR**:** $\vdash \Gamma$, $\alpha : K \in \Gamma$, and $S = \alpha$ are given for some α. By the induction hypothesis, we have $\vdash \Delta(\Gamma)$. By Definition 1.11, we have $\alpha : K \in \Delta(\Gamma)$. Thus, K_VAR derives $\Delta(\Gamma) \vdash \alpha : K$.

Case K_CONS**:** For some $\mathcal{C}$, $\boldsymbol{S}^I$, and $\boldsymbol{K}^I$, the following are given:

- $S = \mathcal{C}\,\boldsymbol{S}^I$,
- $\vdash \Gamma$,
- $\mathcal{C} : \Pi \boldsymbol{K}^I \rightarrow K \in \Sigma$, and
- $\Gamma \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$.

By the induction hypothesis, we have $\vdash \Delta(\Gamma)$ and $\Delta(\Gamma) \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$. Thus, K_CONS derives $\Delta(\Gamma) \vdash \mathcal{C}\,\boldsymbol{S}^I : K$ as required.

Case K_FUN**:** For some A, ε, and B, the following are given:

- $S = A \rightarrow_\varepsilon B$,
- $K = \textbf{Typ}$,
- $\Gamma \vdash A : \textbf{Typ}$,

- $\Gamma \vdash \varepsilon : \mathbf{Eff}$, and

- $\Gamma \vdash B : \mathbf{Typ}$.

By the induction hypothesis, we have

- $\Delta(\Gamma) \vdash A : \mathbf{Typ}$,

- $\Delta(\Gamma) \vdash \varepsilon : \mathbf{Eff}$, and

- $\Delta(\Gamma) \vdash B : \mathbf{Typ}$.

Thus, K_FUN derives $\Delta(\Gamma) \vdash A \rightarrow_\varepsilon B : \mathbf{Typ}$.

Case K_POLY: For some α, K', A, and ε, the following are given:

- $S = \forall \alpha : K'.A^\varepsilon$,

- $K = \mathbf{Typ}$,

- $\Gamma, \alpha : K' \vdash A : \mathbf{Typ}$, and

- $\Gamma, \alpha : K' \vdash \varepsilon : \mathbf{Eff}$.

By the induction hypothesis, we have

- $\Delta(\Gamma, \alpha : K') \vdash A : \mathbf{Typ}$ and

- $\Delta(\Gamma, \alpha : K') \vdash \varepsilon : \mathbf{Eff}$.

By Definition 1.11, we have $\Delta(\Gamma, \alpha : K') = \Delta(\Gamma), \alpha : K'$. Thus, K_POLY derives $\Delta(\Gamma) \vdash \forall \alpha : K'.A^\varepsilon : \mathbf{Typ}$ as required.

∎

Lemma 3.3.

(1) For any Γ and ε, if $\Gamma \vdash \varepsilon : \mathbf{Eff}$, then $\Gamma \vdash \varepsilon \oslash \varepsilon$ holds.

(2) For any Γ, ε_1, ε_2, and ε_3, if $\Gamma \vdash \varepsilon_1 \oslash \varepsilon_2$ and $\Gamma \vdash \varepsilon_2 \oslash \varepsilon_3$, then $\Gamma \vdash \varepsilon_1 \oslash \varepsilon_3$.

Proof.

(1) Clearly because of Lemma 3.2(2) and because $\mathbb{0}$ is a unit element.

(2) Clearly because $\odot$ is associative and preserves well-formendness.

∎

Lemma 3.4 (Transitivity of Subtyping)**.**

(1) If $\Gamma \vdash A_1 <: A_2$ and $\Gamma \vdash A_2 <: A_3$, then $\Gamma \vdash A_1 <: A_3$.

(2) If $\Gamma \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2$ and $\Gamma \vdash A_2 \mid \varepsilon_2 <: A_3 \mid \varepsilon_3$, then $\Gamma \vdash A_1 \mid \varepsilon_1 <: A_3 \mid \varepsilon_3$.

Proof. By the structural induction on the summation of the sizes of A_1, A_2, and A_3. If either $\Gamma \vdash A_1 <: A_2$ or $\Gamma \vdash A_2 <: A_3$ is derived by ST_REFL, then we have $\Gamma \vdash A_1 <: A_3$ immediately. Thus, we suppose that neither $\Gamma \vdash A_1 <: A_2$ nor $\Gamma \vdash A_2 <: A_3$ is derived by ST_REFL in the following. We proceed by case analysis on what form A_1 has.

Case $A_1 = \tau$: No rules other than ST_REFL can derive $\Gamma \vdash A_1 <: A_2$.

Case $A_1 = B_1 \rightarrow_{\varepsilon_1} C_1$: Since only ST_FUN can derive $\Gamma \vdash B_1 \rightarrow_{\varepsilon_1} C_1 <: A_2$, we have $A_2 = B_2 \rightarrow_{\varepsilon_2} C_2$ for some B_2, ε_2, and C_2 such that

- $\Gamma \vdash B_2 <: B_1$ and

- $\Gamma \vdash C_1 \mid \varepsilon_1 <: C_2 \mid \varepsilon_2$.

Since only ST_FUN can derive $\Gamma \vdash B_2 \rightarrow_{\varepsilon_2} C_2 <: A_3$, we have $A_3 = B_3 \rightarrow_{\varepsilon_3} C_3$ for some B_3, ε_3, and C_3 such that

- $\Gamma \vdash B_3 <: B_2$ and

- $\Gamma \vdash C_2 \mid \varepsilon_2 <: C_3 \mid \varepsilon_3$.

Since only ST_COMP can derive $\Gamma \vdash C_2 \mid \varepsilon_2 <: C_3 \mid \varepsilon_3$ and $\Gamma \vdash C_1 \mid \varepsilon_1 <: C_2 \mid \varepsilon_2$, we have

- $\Gamma \vdash C_1 <: C_2$,

- $\Gamma \vdash C_2 <: C_3$,

- $\Gamma \vdash \varepsilon_1 \oslash \varepsilon_2$, and
- $\Gamma \vdash \varepsilon_2 \oslash \varepsilon_3$.

By the induction hypothesis and Lemma 3.3(2), we have

- $\Gamma \vdash B_3 <: B_1$,
- $\Gamma \vdash \varepsilon_1 \oslash \varepsilon_3$, and
- $\Gamma \vdash C_1 <: C_3$.

Thus, we have $\Gamma \vdash A_1 <: A_3$ by ST_Fun as required.

Case $A_1 = \forall \alpha : K.B_1{}^{\varepsilon_1}$**:** Since only ST_Poly can derive $\Gamma \vdash \forall \alpha : K.B_1{}^{\varepsilon_1} <: A_2$, we have $A_2 = \forall \alpha : K.B_2{}^{\varepsilon_2}$ for some B_2 and ε_2 such that $\Gamma, \alpha : K \vdash B_1 \mid \varepsilon_1 <: B_2 \mid \varepsilon_2$. Since only ST_Poly can derive $\Gamma \vdash \forall \alpha : K.B_2{}^{\varepsilon_2} <: A_3$, we have $A_3 = \forall \alpha : K.B_3{}^{\varepsilon_3}$ for some B_3 and ε_3 such that $\Gamma, \alpha : K \vdash B_2 \mid \varepsilon_2 <: B_3 \mid \varepsilon_3$. Since only ST_Comp can derive $\Gamma, \alpha : K \vdash B_1 \mid \varepsilon_1 <: B_2 \mid \varepsilon_2$ and $\Gamma, \alpha : K \vdash B_2 \mid \varepsilon_2 <: B_3 \mid \varepsilon_3$, we have

- $\Gamma, \alpha : K \vdash B_1 <: B_2$,
- $\Gamma, \alpha : K \vdash \varepsilon_1 \oslash \varepsilon_2$,
- $\Gamma, \alpha : K \vdash B_2 <: B_3$, and
- $\Gamma, \alpha : K \vdash \varepsilon_2 \oslash \varepsilon_3$.

By the induction hypothesis and Lemma 3.3(2), we have

- $\Gamma, \alpha : K \vdash B_1 <: B_3$ and
- $\Gamma, \alpha : K \vdash \varepsilon_1 \oslash \varepsilon_3$.

Thus, we have $\Gamma \vdash A_1 <: A_3$ by ST_Poly as required.

$\blacksquare$

Lemma 3.5 (Weakening). *Suppose that* $\vdash \Gamma_1, \Gamma_2$ *and* $\mathrm{dom}(\Gamma_2) \cap \mathrm{dom}(\Gamma_3) = \emptyset$.
(1) If $\vdash \Gamma_1, \Gamma_3$, *then* $\vdash \Gamma_1, \Gamma_2, \Gamma_3$.

(2) If $\Gamma_1, \Gamma_3 \vdash S : K$, *then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash S : K$.

(3) If $\Gamma_1, \Gamma_3 \vdash A <: B$, *then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash A <: B$.

(4) If $\Gamma_1, \Gamma_3 \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2$, *then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2$.

(5) If $\Gamma_1, \Gamma_3 \vdash e : A \mid \varepsilon$, *then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash e : A \mid \varepsilon$.

(6) If $\Gamma_1, \Gamma_3 \vdash_\sigma h : A \Rightarrow^\varepsilon B$, *then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash_\sigma h : A \Rightarrow^\varepsilon B$.

Proof.
(1)(2) By mutual induction on derivations of the judgments. We proceed by case analysis on the rule applied lastly to the derivation.

Case C_Empty**:** Clearly because of $\vdash \Gamma_1, \Gamma_2$ and $\Gamma_1 = \Gamma_3 = \emptyset$.

Case C_Var**:** If $\Gamma_3 = \emptyset$, then $\vdash \Gamma_1, \Gamma_2, \Gamma_3$ holds immediately. If $\Gamma_3 \neq \emptyset$, then for some Γ_3', x, and A, the following are given:
- $\Gamma_3 = \Gamma_3', x : A$,
- $x \notin \mathrm{dom}(\Gamma_1, \Gamma_3')$, and
- $\Gamma_1, \Gamma_3' \vdash A : \mathbf{Typ}$.

Since $\mathrm{dom}(\Gamma_2) \cap \mathrm{dom}(\Gamma_3') = \emptyset$ holds, we have $\Gamma_1, \Gamma_2, \Gamma_3' \vdash A : \mathbf{Typ}$ by the induction hypothesis. By $x \notin \mathrm{dom}(\Gamma_1, \Gamma_3')$ and $\mathrm{dom}(\Gamma_2) \cap \mathrm{dom}(\Gamma_3', x : A) = \emptyset$, we have $x \notin \mathrm{dom}(\Gamma_1, \Gamma_2, \Gamma_3')$. Thus, C_Var derives $\vdash \Gamma_1, \Gamma_2, \Gamma_3', x : A$.

Case C_TVar**:** If $\Gamma_3 = \emptyset$, then $\vdash \Gamma_1, \Gamma_2$ holds immediately. If $\Gamma_3 \neq \emptyset$, then for some Γ_3', α, and K, the following are given:
- $\Gamma_3 = \Gamma_3', \alpha : K$,
- $\alpha \notin \mathrm{dom}(\Gamma_1, \Gamma_3')$, and
- $\vdash \Gamma_1, \Gamma_3'$.

Since $\mathrm{dom}(\Gamma_2) \cap \mathrm{dom}(\Gamma_3') = \emptyset$, we have $\vdash \Gamma_1, \Gamma_2, \Gamma_3'$ by the induction hypothesis. By $\alpha \notin \mathrm{dom}(\Gamma_1, \Gamma_3')$ and $\mathrm{dom}(\Gamma_2) \cap \mathrm{dom}(\Gamma_3', \alpha : K) = \emptyset$, we have $\alpha \notin \mathrm{dom}(\Gamma_1, \Gamma_2, \Gamma_3')$ Thus, C_TVar derives $\vdash \Gamma_1, \Gamma_2, \Gamma_3', \alpha : K$.

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Case K_VAR: For some α, the following are given:

- $S = \alpha$,
- $\vdash \Gamma_1, \Gamma_3$, and
- $\alpha : K \in \Gamma_1, \Gamma_3$.

By the induction hypothesis, we have $\vdash \Gamma_1, \Gamma_2, \Gamma_3$. Thus, $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \alpha : K$ holds because of $\alpha : K \in \Gamma_1, \Gamma_2, \Gamma_3$.

Case K_FUN: For some A, B, and ε, the following are given:

- $S = A \rightarrow_\varepsilon B$,
- $K = \mathbf{Typ}$,
- $\Gamma_1, \Gamma_3 \vdash A : \mathbf{Typ}$,
- $\Gamma_1, \Gamma_3 \vdash \varepsilon : \mathbf{Eff}$, and
- $\Gamma_1, \Gamma_3 \vdash B : \mathbf{Typ}$.

By the induction hypothesis, we have

- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash A : \mathbf{Typ}$,
- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \varepsilon : \mathbf{Eff}$, and
- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash B : \mathbf{Typ}$.

Thus, K_FUN derives $\Gamma_1, \Gamma_2, \Gamma_3 \vdash A \rightarrow_\varepsilon B : \mathbf{Typ}$.

Case K_POLY: Without loss of generality, we can choose α such that $\alpha \notin \mathrm{dom}(\Gamma_2)$. For some K', A, and ε, the following are given:

- $S = \forall \alpha : K'.A^\varepsilon$,
- $K = \mathbf{Typ}$,
- $\Gamma_1, \Gamma_3, \alpha : K' \vdash A : \mathbf{Typ}$, and
- $\Gamma_1, \Gamma_3, \alpha : K' \vdash \varepsilon : \mathbf{Eff}$.

Since $\mathrm{dom}(\Gamma_2) \cap \mathrm{dom}(\Gamma_3, \alpha : K') = \emptyset$, we have

- $\Gamma_1, \Gamma_2, \Gamma_3, \alpha : K' \vdash A : \mathbf{Typ}$ and
- $\Gamma_1, \Gamma_2, \Gamma_3, \alpha : K' \vdash \varepsilon : \mathbf{Eff}$

by the induction hypothesis. Thus, K_POLY derives $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \forall \alpha : K'.A^\varepsilon : \mathbf{Typ}$.

Case K_CONS: For some $\mathcal{C}$, $\boldsymbol{S}^I$, and $\boldsymbol{K}^I$, the following are given:

- $S = \mathcal{C}\,\boldsymbol{S}^I$,
- $\mathcal{C} : \Pi \boldsymbol{K}^I \rightarrow K \in \Sigma$,
- $\vdash \Gamma_1, \Gamma_3$, and
- $\Gamma_1, \Gamma_3 \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$.

By the induction hypothesis, we have $\vdash \Gamma_1, \Gamma_2, \Gamma_3$ and $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$. Thus, K_CONS derives $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \mathcal{C}\,\boldsymbol{S}^I : K$.

(3)(4) By mutual induction on derivations of the judgments. We proceed by case analysis on the rule applied lastly to the derivation.

Case ST_REFL: $A = B$ and $\Gamma_1, \Gamma_3 \vdash A : \mathbf{Typ}$ are given. By case (2), we have $\Gamma_1, \Gamma_2, \Gamma_3 \vdash A : \mathbf{Typ}$. Thus, ST_REFL derives $\Gamma_1, \Gamma_2, \Gamma_3 \vdash A <: A$.

Case ST_FUN: For some A_1, ε_1, B_1, A_2, ε_2, and B_2, the following are given:

- $A = A_1 \rightarrow_{\varepsilon_1} B_1$,
- $B = A_2 \rightarrow_{\varepsilon_2} B_2$,
- $\Gamma_1, \Gamma_3 \vdash A_2 <: A_1$, and
- $\Gamma_1, \Gamma_3 \vdash B_1 \mid \varepsilon_1 <: B_2 \mid \varepsilon_2$.

By the induction hypothesis, we have $\Gamma_1, \Gamma_2, \Gamma_3 \vdash B_1 \mid \varepsilon_1 <: B_2 \mid \varepsilon_2$. Thus, ST_FUN derives

$$\Gamma_1, \Gamma_2, \Gamma_3 \vdash A_1 \rightarrow_{\varepsilon_1} B_1 <: A_2 \rightarrow_{\varepsilon_2} B_2$$

as required.

Case ST_POLY: Without loss of generality, we can choose α such that $\alpha \notin \mathrm{dom}(\Gamma_2)$. For some K, A_1, ε_1, A_2, and ε_2, the following are given:

- $A = \forall \alpha : K.A_1^{\varepsilon_1}$,

  – $B = \forall \alpha : K.A_2{}^{\varepsilon_2}$, and

  – $\Gamma_1, \Gamma_3, \alpha : K \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2$.

 By the induction hypothesis, we have $\Gamma_1, \Gamma_2, \Gamma_3, \alpha : K \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2$. Thus, ST_Poly derives

$$\Gamma_1, \Gamma_2, \Gamma_3 \vdash \forall \alpha : K.A_1{}^{\varepsilon_1} <: \forall \alpha : K.A_2{}^{\varepsilon_2}$$

 as required.

Case ST_Comp**:** We have $\Gamma_1, \Gamma_3 \vdash A_1 <: A_2$ and $\Gamma_1, \Gamma_3 \vdash \varepsilon_1 \oslash \varepsilon_2$. By the induction hypothesis, we have $\Gamma_1, \Gamma_2, \Gamma_3 \vdash A_1 <: A_2$. By case (2), we have $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \varepsilon_1 \oslash \varepsilon_2$. Thus, ST_Comp derives $\Gamma_1, \Gamma_2, \Gamma_3 \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2$ as required.

(5)(6) By mutual induction on derivations of the judgments. We proceed by case analysis on the rule applied lastly to the derivation.

Case T_Var**:** For some x, the following are given:

  – $e = x$,

  – $\varepsilon = \mathbb{0}$,

  – $\vdash \Gamma_1, \Gamma_3$, and

  – $x : A \in \Gamma_1, \Gamma_3$.

 By case (1), we have $\vdash \Gamma_1, \Gamma_2, \Gamma_3$. Thus, T_Var derives $\Gamma_1, \Gamma_2, \Gamma_3 \vdash x : A \mid \mathbb{0}$ because of $x : A \in \Gamma_1, \Gamma_2, \Gamma_3$.

Case T_Abs**:** Without loss of generality, we can choose f and x such that $f \notin \mathrm{dom}(\Gamma_2)$ and $x \notin \mathrm{dom}(\Gamma_2)$. For some e', A', B', and ε', the following are given:

  – $e = \mathbf{fun}\,(f, x, e')$,

  – $A = A' \rightarrow_{\varepsilon'} B'$,

  – $\varepsilon = \mathbb{0}$, and

  – $\Gamma_1, \Gamma_3, f : A' \rightarrow_{\varepsilon'} B', x : A' \vdash e' : B' \mid \varepsilon'$.

 By the induction hypothesis, we have $\Gamma_1, \Gamma_2, \Gamma_3, f : A' \rightarrow_{\varepsilon'} B', x : A' \vdash e' : B' \mid \varepsilon'$ because of $\mathrm{dom}(\Gamma_2) \cap \mathrm{dom}(\Gamma_3, f : A' \rightarrow_{\varepsilon'} B', x : A') = \emptyset$. Thus, T_Abs derives $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \mathbf{fun}\,(f, x, e') : A' \rightarrow_{\varepsilon'} B' \mid \mathbb{0}$.

Case T_App**:** For some v_1, v_2, and C, the following are given:

  – $e = v_1\,v_2$,

  – $\Gamma_1, \Gamma_3 \vdash v_1 : B \rightarrow_\varepsilon A \mid \mathbb{0}$, and

  – $\Gamma_1, \Gamma_3 \vdash v_2 : B \mid \mathbb{0}$.

 By the induction hypothesis, we have

  – $\Gamma_1, \Gamma_2, \Gamma_3 \vdash v_1 : B \rightarrow_\varepsilon A \mid \mathbb{0}$ and

  – $\Gamma_1, \Gamma_2, \Gamma_3 \vdash v_2 : B \mid \mathbb{0}$.

 Thus, T_App derives $\Gamma_1, \Gamma_2, \Gamma_3 \vdash v_1\,v_2 : A \mid \varepsilon$.

Case T_TAbs**:** Without loss of generality, we can choose α such that $\alpha \notin \mathrm{dom}(\Gamma_2)$. For some K, e', B', and ε', the following are given:

  – $e = \Lambda \alpha : K.e'$,

  – $A = \forall \alpha : K.A'^{\varepsilon'}$,

  – $\varepsilon = \mathbb{0}$, and

  – $\Gamma_1, \Gamma_3, \alpha : K \vdash e' : A' \mid \varepsilon'$.

 By the induction hypothesis, we have $\Gamma_1, \Gamma_2, \Gamma_3, \alpha : K \vdash e' : A' \mid \varepsilon'$ because of $\mathrm{dom}(\Gamma_2) \cap \mathrm{dom}(\Gamma_3, \alpha : K) = \emptyset$. Thus, T_TAbs derives $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Lambda \alpha : K.e' : \forall \alpha : K.A'^{\varepsilon'} \mid \mathbb{0}$.

Case T_TApp**:** For some v, S, α, A', ε', and K, the following are given:

  – $e = v\,S$,

  – $A = A'[S/\alpha]$,

  – $\varepsilon = \varepsilon'[S/\alpha]$,

  – $\Gamma_1, \Gamma_3 \vdash v : \forall \alpha : K.A'^{\varepsilon'} \mid \mathbb{0}$, and

  – $\Gamma_1, \Gamma_3 \vdash S : K$.

 By the induction hypothesis and case (2), we have

- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash v : \forall \alpha : K.A'^{\varepsilon'} \mid \mathbb{0}$ and
- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash S : K$.

Thus, T_TAPP derives $\Gamma_1, \Gamma_2, \Gamma_3 \vdash v\, S : A'[S/\alpha] \mid \varepsilon'[S/\alpha]$.

Case T_LET: Without loss of generality, we can choose x such that $x \notin \mathrm{dom}(\Gamma_2)$. For some e_1, e_2, and B, the following are given:

- $e = (\mathbf{let}\ x = e_1\ \mathbf{in}\ e_2)$,
- $\Gamma_1, \Gamma_3 \vdash e_1 : B \mid \varepsilon$, and
- $\Gamma_1, \Gamma_3, x : B \vdash e_2 : A \mid \varepsilon$.

By the induction hypothesis, we have

- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash e_1 : B \mid \varepsilon$ and
- $\Gamma_1, \Gamma_2, \Gamma_3, x : B \vdash e_2 : A \mid \varepsilon$

because of $\mathrm{dom}(\Gamma_2) \cap \mathrm{dom}(\Gamma_3, x : B) = \emptyset$. Thus, T_LET derives $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \mathbf{let}\ x = e_1\ \mathbf{in}\ e_2 : A \mid \varepsilon$.

Case T_SUB: For some A' and ε', the following are given:

- $\Gamma_1, \Gamma_3 \vdash e : A' \mid \varepsilon'$ and
- $\Gamma_1, \Gamma_3 \vdash A' \mid \varepsilon' <: A \mid \varepsilon$.

By the induction hypothesis and case (4), we have

- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash e : A' \mid \varepsilon'$ and
- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash A' \mid \varepsilon' <: A \mid \varepsilon$.

Thus, T_SUB derives $\Gamma_1, \Gamma_2, \Gamma_3 \vdash e : A \mid \varepsilon$.

Case T_OP: For some op, l, A', B', I, and J, the following are given:

- $e = \mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J$,
- $A = (A'[\boldsymbol{T}^J/\boldsymbol{\beta}^J]) \to_{(l\,\boldsymbol{S}^I)^\uparrow} (B'[\boldsymbol{T}^J/\boldsymbol{\beta}^J])$,
- $\varepsilon = \mathbb{0}$,
- $l :: \forall \boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi$,
- $\mathsf{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K'}^J.A' \Rightarrow B' \in \sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$,
- $\vdash \Gamma_1, \Gamma_3$,
- $\Gamma_1, \Gamma_3 \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$, and
- $\Gamma_1, \Gamma_3 \vdash \boldsymbol{T}^J : \boldsymbol{K'}^J$.

By cases (1) and (2), we have

- $\vdash \Gamma_1, \Gamma_2, \Gamma_3$,
- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$, and
- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \boldsymbol{T}^J : \boldsymbol{K'}^J$.

Thus, T_OP derives

$$\Gamma_1, \Gamma_2, \Gamma_3 \vdash \mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^I : (A'[\boldsymbol{T}^J/\boldsymbol{\beta}^J]) \to_{(l\,\boldsymbol{S}^I)^\uparrow} (B'[\boldsymbol{T}^J/\boldsymbol{\beta}^J]) \mid \mathbb{0}.$$

Case T_HANDLING: For some N, e', A', ε', l, $\boldsymbol{S}^N$, $\boldsymbol{K}^N$, h, and σ, the following are given:

- $e = \mathbf{handle}_{l\,\boldsymbol{S}^N}\,e'\,\mathbf{with}\,h$,
- $\Gamma_1, \Gamma_3 \vdash e' : A' \mid \varepsilon'$,
- $l :: \forall \boldsymbol{\alpha}^N : \boldsymbol{K}^N.\sigma \in \Xi$,
- $\Gamma_1, \Gamma_3 \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : A' \Rightarrow^\varepsilon A$,
- $\Gamma_1, \Gamma_3 \vdash \boldsymbol{S}^N : \boldsymbol{K}^N$, and
- $(l\,\boldsymbol{S}^N)^\uparrow \odot \varepsilon \sim \varepsilon'$.

By the induction hypothesis and case (2), we have

- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash e' : A' \mid \varepsilon'$,
- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : A' \Rightarrow^\varepsilon A$, and
- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \boldsymbol{S}^N : \boldsymbol{K}^N$.

Thus, T_HANDLING derives

$$\Gamma_1, \Gamma_2, \Gamma_3 \vdash \mathbf{handle}_{l\,\boldsymbol{S}^N}\,e\,\mathbf{with}\,h : A \mid \varepsilon.$$

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Case H_Return**:** Without loss of generality, we can choose x such that $x \notin \operatorname{dom}(\Gamma_2)$. For some e_r, the following are given:

- $h = \{\, \textbf{return}\, x \mapsto e_r \}$,
- $\sigma = \{\}$, and
- $\Gamma_1, \Gamma_3, x : A \vdash e_r : B \mid \varepsilon$.

By the induction hypothesis, we have $\Gamma_1, \Gamma_2, \Gamma_3, x : A \vdash e_r : B \mid \varepsilon$. Thus, H_Return derives $\Gamma_1, \Gamma_2, \Gamma_3 \vdash_{\{\}} \{\, \textbf{return}\, x \mapsto e_r \} : A \Rightarrow^\varepsilon B$.

Case H_Op**:** Without loss of generality, we can choose $\boldsymbol{\beta}^J$ and p and k such that:

- $\{\boldsymbol{\beta}^J\} \cap \operatorname{dom}(\Gamma_2) = \emptyset$,
- $p \notin \operatorname{dom}(\Gamma_2)$, and
- $k \notin \operatorname{dom}(\Gamma_2)$.

For some h', σ', op, A', B', and e, the following are given:

- $h = h' \uplus \{\mathsf{op}\, \boldsymbol{\beta}^J : \boldsymbol{K}^J\, p\, k \mapsto e\}$,
- $\sigma = \sigma' \uplus \{\mathsf{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K}^J.A' \Rightarrow B'\}$,
- $\Gamma_1, \Gamma_3 \vdash_{\sigma'} h' : A \Rightarrow^\varepsilon B$, and
- $\Gamma_1, \Gamma_3, \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A', k : B' \to_\varepsilon B \vdash e : B \mid \varepsilon$.

By the induction hypothesis, we have

- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash_{\sigma'} h' : A \Rightarrow^\varepsilon B$ and
- $\Gamma_1, \Gamma_2, \Gamma_3, \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A', k : B' \to_\varepsilon B \vdash e : B \mid \varepsilon$.

Thus, H_Op derives $\Gamma_1, \Gamma_2, \Gamma_3 \vdash_\sigma h' \uplus \{\mathsf{op}\, \boldsymbol{\beta}^J : \boldsymbol{K}^J\, p\, k \mapsto e\} : A \Rightarrow^\varepsilon B$. ∎

Lemma 3.6. *For any* Γ_1, Γ_2, S, *and* K, *if* $\Delta(\Gamma_1), \Gamma_2 \vdash S : K$ *and* $\vdash \Gamma_1$ *and* $\operatorname{dom}(\Gamma_1) \cap \operatorname{dom}(\Gamma_2) = \emptyset$, *then* $\Gamma_1, \Gamma_2 \vdash S : K$.

Proof. By induction on the size of Γ_1. We proceed by case analysis on the rule lastly applied to this derivation.
Case C_Empty**:** Clearly.

Case C_Var**:** For some Γ_1', x, and A, we have

- $\Gamma_1 = \Gamma_1', x : A$,
- $x \notin \operatorname{dom}(\Gamma_1')$, and
- $\Gamma_1' \vdash A : \textbf{Typ}$.

By Lemma 3.1, we have $\vdash \Gamma_1'$. By Definition 1.11, we have $\Delta(\Gamma_1'), \Gamma_2 \vdash S : K$. By $\operatorname{dom}(\Gamma_1') \subseteq \operatorname{dom}(\Gamma_1)$, we have $\operatorname{dom}(\Gamma_1') \cap \operatorname{dom}(\Gamma_2) = \emptyset$. By the induction hypothesis, we have $\Gamma_1', \Gamma_2 \vdash S : K$. By Lemma 3.5(2), we have $\Gamma_1', x : A, \Gamma_2 \vdash S : K$ as required.

Case C_TVar**:** For some Γ_1', α, and K', we have

- $\Gamma_1 = \Gamma_1', \alpha : K'$,
- $\vdash \Gamma_1'$, and
- $\alpha \notin \operatorname{dom}(\Gamma_1')$.

By Definition 1.11, we have $\Delta(\Gamma_1'), \alpha : K', \Gamma_2 \vdash S : K$. By $\alpha \notin \operatorname{dom}(\Gamma_1')$ and $\operatorname{dom}(\Gamma_1') \subseteq \operatorname{dom}(\Gamma_1)$, we have $\operatorname{dom}(\Gamma_1') \cap \operatorname{dom}(\alpha : K', \Gamma_2') = \emptyset$. By the induction hypothesis, we have $\Gamma_1', \alpha : K', \Gamma_2 \vdash S : K$ as required. ∎

Lemma 3.7 (Substitution of values). *Suppose that* $\Gamma_1 \vdash v : A \mid \mathbb{0}$.
(1) If $\vdash \Gamma_1, x : A, \Gamma_2$, *then* $\vdash \Gamma_1, \Gamma_2$.

(2) If $\Gamma_1, x : A, \Gamma_2 \vdash S : K$, *then* $\Gamma_1, \Gamma_2 \vdash S : K$.

(3) If $\Gamma_1, x : A, \Gamma_2 \vdash B <: C$, *then* $\Gamma_1, \Gamma_2 \vdash B <: C$.

(4) If $\Gamma_1, x : A, \Gamma_2 \vdash B_1 \mid \varepsilon_1 <: B_2 \mid \varepsilon_2$, *then* $\Gamma_1, \Gamma_2 \vdash B_1 \mid \varepsilon_1 <: B_2 \mid \varepsilon_2$.

(5) If $\Gamma_1, x : A, \Gamma_2 \vdash e : B \mid \varepsilon$, *then* $\Gamma_1, \Gamma_2 \vdash e[v/x] : B \mid \varepsilon$.

(6) If $\Gamma_1, x : A, \Gamma_2 \vdash_\sigma h : B \Rightarrow^\varepsilon C$, *then* $\Gamma_1, \Gamma_2 \vdash_\sigma h[v/x] : B \Rightarrow^\varepsilon C$.

Proof.

(1)(2) By mutual induction on derivations of the judgments. We proceed by case analysis on the rule applied lastly to the derivation.

Case C_EMPTY: Cannot happen.

Case C_VAR: If $\Gamma_2 = \emptyset$, then we have $\Gamma_1 \vdash A : \textbf{Typ}$. By Lemma 3.1, $\vdash \Gamma_1$ holds. If $\Gamma_2 \neq \emptyset$, then we have

- $\Gamma_2 = \Gamma_2', y : B$,
- $\Gamma_1, x : A, \Gamma_2' \vdash B : \textbf{Typ}$, and
- $y \notin \mathrm{dom}(\Gamma_1, x : A, \Gamma_2')$,

for some Γ_2', y, and B. By the induction hypothesis, we have $\Gamma_1, \Gamma_2' \vdash B : \textbf{Typ}$. Thus, C_VAR derives $\vdash \Gamma_1, \Gamma_2$ because $y \notin \mathrm{dom}(\Gamma_1, \Gamma_2')$.

Case C_TVAR: Since Γ_2 cannot be $\emptyset$, we have

- $\Gamma_2 = \Gamma_2', \alpha : K$,
- $\vdash \Gamma_1, x : A, \Gamma_2'$, and
- $\alpha \notin \mathrm{dom}(\Gamma_1, x : A, \Gamma_2')$,

for some Γ_2, α, and K. By the induction hypothesis, we have $\vdash \Gamma_1, \Gamma_2'$. Thus, C_TVAR derives $\vdash \Gamma_1, \Gamma_2$ because $\alpha \notin \mathrm{dom}(\Gamma_1, \Gamma_2')$.

Case K_VAR: For some α, the following are given:

- $S = \alpha$,
- $\vdash \Gamma_1, x : A, \Gamma_2$, and
- $\alpha : K \in \Gamma_1, x : A, \Gamma_2$.

By the induction hypothesis, we have $\vdash \Gamma_1, \Gamma_2$. Thus, K_VAR derives $\Gamma_1, \Gamma_2 \vdash \alpha : K$ because of $\alpha : K \in \Gamma_1, \Gamma_2$.

Case K_FUN: For some B, C, and ε, the following are given:

- $S = B \rightarrow_\varepsilon C$,
- $K = \textbf{Typ}$,
- $\Gamma_1, x : A, \Gamma_2 \vdash B : \textbf{Typ}$,
- $\Gamma_1, x : A, \Gamma_2 \vdash \varepsilon : \textbf{Eff}$, and
- $\Gamma_1, x : A, \Gamma_2 \vdash C : \textbf{Typ}$.

By the induction hypothesis, we have

- $\Gamma_1, \Gamma_2 \vdash B : \textbf{Typ}$,
- $\Gamma_1, \Gamma_2 \vdash \varepsilon : \textbf{Eff}$, and
- $\Gamma_1, \Gamma_2 \vdash C : \textbf{Typ}$.

Thus, K_FUN derives $\Gamma_1, \Gamma_2 \vdash B \rightarrow_\varepsilon C : \textbf{Typ}$.

Case K_POLY: For some α, K', A', and ε, the following are given:

- $S = \forall \alpha : K'.A'^\varepsilon$,
- $K = \textbf{Typ}$,
- $\Gamma_1, x : A, \Gamma_2, \alpha : K' \vdash A' : \textbf{Typ}$, and
- $\Gamma_1, x : A, \Gamma_2, \alpha : K' \vdash \varepsilon : \textbf{Eff}$.

By the induction hypothesis, we have

- $\Gamma_1, \Gamma_2, \alpha : K' \vdash A' : \textbf{Typ}$, and
- $\Gamma_1, \Gamma_2, \alpha : K' \vdash \varepsilon : \textbf{Eff}$.

Thus, K_POLY derives $\Gamma_1, \Gamma_2 \vdash \forall \alpha : K'.A'^\varepsilon : \textbf{Typ}$.

Case K_CONS: For some $\mathcal{C}$, $\boldsymbol{S}^I$ and $\boldsymbol{K}^I$, the following are given:

- $S = \mathcal{C}\,\boldsymbol{S}^I$,
- $\mathcal{C} : \Pi \boldsymbol{K}^I \rightarrow K \in \Sigma$,
- $\vdash \Gamma_1, x : A, \Gamma_2$, and
- $\Gamma_1, x : A, \Gamma_2 \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$.

By the induction hypothesis, we have $\vdash \Gamma_1, \Gamma_2$ and $\Gamma_1, \Gamma_2 \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$. Thus, K_CONS derives $\Gamma_1, \Gamma_2 \vdash \mathcal{C}\,\boldsymbol{S}^I : K$.

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(3)(4) By mutual induction on derivations of the judgments. We proceed by case analysis on the rule applied lastly to the derivation.

Case ST_Refl: $B = C$ and $\Gamma_1, x : A, \Gamma_2 \vdash B : \mathbf{Typ}$ are given. By case (2), we have $\Gamma_1, \Gamma_2 \vdash B : \mathbf{Typ}$. Thus, ST_Refl derives $\Gamma_1, \Gamma_2 \vdash B <: B$.

Case ST_Fun: For some $A_{11}, \varepsilon_1, A_{12}, A_{21}, \varepsilon_2$, and A_{22}, the following are given:
- $B = A_{11} \to_{\varepsilon_1} A_{12}$,
- $C = A_{21} \to_{\varepsilon_2} A_{22}$,
- $\Gamma_1, x : A, \Gamma_2 \vdash A_{21} <: A_{11}$, and
- $\Gamma_1, x : A, \Gamma_2 \vdash A_{12} \mid \varepsilon_1 <: A_{22} \mid \varepsilon_2$.

By the induction hypothesis, we have $\Gamma_1, \Gamma_2 \vdash A_{21} <: A_{11}$ and $\Gamma_1, \Gamma_2 \vdash A_{12} \mid \varepsilon_1 <: A_{22} \mid \varepsilon_2$. Thus, ST_Fun derives $\Gamma_1, \Gamma_2 \vdash A_{11} \to_{\varepsilon_1} A_{12} <: A_{21} \to_{\varepsilon_2} A_{22}$.

Case ST_Poly: For some $\alpha, K, A_1, \varepsilon_1, A_2$, and ε_2, the following are given:
- $B = \forall \alpha : K.A_1^{\varepsilon_1}$,
- $C = \forall \alpha : K.A_2^{\varepsilon_2}$, and
- $\Gamma_1, x : A, \Gamma_2, \alpha : K \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2$.

By the induction hypothesis, we have $\Gamma_1, \Gamma_2, \alpha : K \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2$. Thus, ST_Poly derives $\Gamma_1, \Gamma_2 \vdash \forall \alpha : K.A_1^{\varepsilon_1} <: \forall \alpha : K.A_2^{\varepsilon_2}$.

Case ST_Comp: We have $\Gamma_1, x : A, \Gamma_2 \vdash B_1 <: B_2$ and $\Gamma_1, x : A, \Gamma_2 \vdash \varepsilon_1 \oslash \varepsilon_2$. By the induction hypothesis, we have $\Gamma_1, \Gamma_2 \vdash B_1 <: B_2$. By case (2), we have $\Gamma_1, \Gamma_2 \vdash \varepsilon_1 \oslash \varepsilon_2$. Thus, ST_Comp derives $\Gamma_1, \Gamma_2 \vdash B_1 \mid \varepsilon_1 <: B_2 \mid \varepsilon_2$ as required.

(5)(6) By mutual induction on derivations of the judgments. We proceed by case analysis on the rule applied lastly to the derivation.

Case T_Var: For some y, the following are given:
- $e = y$,
- $\varepsilon = \mathbb{0}$,
- $\vdash \Gamma_1, x : A, \Gamma_2$, and
- $y : B \in \Gamma_1, x : A, \Gamma_2$.

By case (1), we have $\vdash \Gamma_1, \Gamma_2$.
If $y = x$, then $\Gamma_1, \Gamma_2 \vdash v : A \mid \mathbb{0}$ holds because of $\Gamma_1 \vdash v : A \mid \mathbb{0}$ and Lemma 3.5(5).
If $y \neq x$, then we have $y : B \in \Gamma_1, \Gamma_2$. Thus, T_Var derives $\Gamma_1, \Gamma_2 \vdash y : B \mid \mathbb{0}$.

Case T_Abs: Without loss of generality, we can choose f and y such that $f, y \neq x$ and $f, y \notin \mathrm{FV}(v)$. For some e', A', B', and ε', the following are given:
- $e = \mathbf{fun}\,(f, y, e')$,
- $B = A' \to_{\varepsilon'} B'$,
- $\varepsilon = \mathbb{0}$, and
- $\Gamma_1, x : A, \Gamma_2, f : A' \to_{\varepsilon'} B', y : A' \vdash e' : B' \mid \varepsilon'$.

By the induction hypothesis, we have $\Gamma_1, \Gamma_2, g : A' \to_{\varepsilon'} B', y : A' \vdash e'[v/x] : B' \mid \varepsilon'$. Thus, T_Abs derives $\Gamma_1, \Gamma_2 \vdash \mathbf{fun}\,(f, y, e'[v/x]) : A' \to_{\varepsilon'} B' \mid \mathbb{0}$, and since $(\mathbf{fun}\,(f, y, e'))[v/x] = \mathbf{fun}\,(f, y, e'[v/x])$, the required result is achieved.

Case T_App: For some v_1, v_2, and C, the following are given:
- $e = v_1\,v_2$,
- $\Gamma_1, x : A, \Gamma_2 \vdash v_1 : C \to_\varepsilon B \mid \mathbb{0}$, and
- $\Gamma_1, x : A, \Gamma_2 \vdash v_2 : C \mid \mathbb{0}$.

By the induction hypothesis, we have
- $\Gamma_1, \Gamma_2 \vdash v_1[v/x] : C \to_\varepsilon B \mid \mathbb{0}$
- and $\Gamma_1, \Gamma_2 \vdash v_2[v/x] : C \mid \mathbb{0}$.

Thus, T_App derives $\Gamma_1, \Gamma_2 \vdash (v_1[v/x])\,(v_2[v/x]) : B \mid \varepsilon$, and since $(v_1\,v_2)[v/x] = (v_1[v/x])\,(v_2[v/x])$, the required result is achieved.

Case T_TABS: Without loss of generality, we can choose α such that $\alpha \notin \mathrm{FTV}(v)$. For some K, e', B', and ε', the following are given:
- $e = \Lambda \alpha : K.e'$,

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- $B = \forall \alpha : K.B'^{\varepsilon'}$,
- $\varepsilon = \mathbb{0}$, and
- $\Gamma_1, x : A, \Gamma_2, \alpha : K \vdash e' : B' \mid \varepsilon'$.

By the induction hypothesis, we have $\Gamma_1, \Gamma_2, \alpha : K \vdash e'[v/x] : B' \mid \varepsilon'$. Thus, T_TABS derives $\Gamma_1, \Gamma_2 \vdash \Lambda \alpha : K.e'[v/x] : \forall \alpha : K.B'^{\varepsilon'} \mid \mathbb{0}$, and since $(\Lambda \alpha : K.e')[v/x] = \Lambda \alpha : K.e'[v/x]$, the required result is achieved.

Case T_TAPP: For some v', S, α, B', ε', and K, the following are given:
- $e = v' \, S$,
- $B = B'[S/\alpha]$,
- $\varepsilon = \varepsilon'[S/\alpha]$,
- $\Gamma_1, x : A, \Gamma_2 \vdash v' : \forall \alpha : K.B'^{\varepsilon'} \mid \mathbb{0}$, and
- $\Gamma_1, x : A, \Gamma_2 \vdash S : K$.

By the induction hypothesis and case (2), we have
- $\Gamma_1, \Gamma_2 \vdash v'[v/x] : \forall \alpha : K.B'^{\varepsilon} \mid \mathbb{0}$ and
- $\Gamma_1, \Gamma_2 \vdash S : K$.

Thus, T_TAPP derives $\Gamma_1, \Gamma_2 \vdash v'[v/x] \, S : B'[S/\alpha] \mid \varepsilon'[S/\alpha]$, and since $(v' \, S)[v/x] = v'[v/x] \, S$, the required result is achieved.

Case T_LET: Without loss of generality, we can choose y such that $y \neq x$ and $y \notin \mathrm{FV}(v)$. For some e_1, e_2, and C, the following are given:
- $e = (\textbf{let } y = e_1 \textbf{ in } e_2)$,
- $\Gamma_1, x : A, \Gamma_2 \vdash e_1 : C \mid \varepsilon$, and
- $\Gamma_1, x : A, \Gamma_2, y : C \vdash e_2 : B \mid \varepsilon$.

By the induction hypothesis, we have
- $\Gamma_1, \Gamma_2 \vdash e_1[v/x] : C \mid \varepsilon$ and
- $\Gamma_1, \Gamma_2, y : C \vdash e_2[v/x] : B \mid \varepsilon$.

Thus, T_LET derives $\Gamma_1, \Gamma_2 \vdash \textbf{let } y = e_1[v/x] \textbf{ in } e_2[v/x] : B \mid \varepsilon$, and since $(\textbf{let } y = e_1 \textbf{ in } e_2)[v/x] = \textbf{let } y = e_1[v/x] \textbf{ in } e_2[v/x]$, the required result is achieved.

Case T_SUB: For some B' and ε', the following are given:
- $\Gamma_1, x : A, \Gamma_2 \vdash e : B' \mid \varepsilon'$ and
- $\Gamma_1, x : A, \Gamma_2 \vdash B' \mid \varepsilon' <: B \mid \varepsilon$.

By the induction hypothesis and case (4), we have $\Gamma_1, \Gamma_2 \vdash e[v/x] : B' \mid \varepsilon'$ and $\Gamma_1, \Gamma_2 \vdash B' \mid \varepsilon' <: B \mid \varepsilon$. Thus, T_SUB derives $\Gamma_1, \Gamma_2 \vdash e[v/x] : B \mid \varepsilon$.

Case T_OP: For some $\mathsf{op}_{l\,\boldsymbol{S}^I} \, \boldsymbol{T}^J$, A', and B', the following are given:
- $e = \mathsf{op}_{l\,\boldsymbol{S}^I} \, \boldsymbol{T}^J$,
- $B = (A'[\boldsymbol{T}^J/\boldsymbol{\beta}^J]) \to_{(l\,\boldsymbol{S}^I)^\uparrow} (B'[\boldsymbol{T}^J/\boldsymbol{\beta}^J])$,
- $\varepsilon = \mathbb{0}$,
- $l :: \forall \boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi$
- $\mathsf{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K'}^J.A' \Rightarrow B' \in \sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$,
- $\vdash \Gamma_1, x : A, \Gamma_2$,
- $\Gamma_1, x : A, \Gamma_2 \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$, and
- $\Gamma_1, x : A, \Gamma_2 \vdash \boldsymbol{T}^J : \boldsymbol{K'}^J$.

By cases (1) and (2), we have
- $\vdash \Gamma_1, \Gamma_2$,
- $\Gamma_1, \Gamma_2 \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$, and
- $\Gamma_1, \Gamma_2 \vdash \boldsymbol{T}^J : \boldsymbol{K'}^J$.

Thus, T_OP derives

$$\Gamma_1, \Gamma_2 \vdash \mathsf{op}_{l\,\boldsymbol{S}^I} \, \boldsymbol{T}^J : (A'[\boldsymbol{T}^J/\boldsymbol{\beta}^J]) \to_{(l\,\boldsymbol{S}^I)^\uparrow} (B'[\boldsymbol{T}^J/\boldsymbol{\beta}^J]) \mid \mathbb{0}.$$

Case T_HANDLING: For some N, e', A', ε', l, $\boldsymbol{S}^N$, $\boldsymbol{\alpha}^N$, $\boldsymbol{K}^N$, h, and σ, the following are given:
- $e = \textbf{handle}_{l\,\boldsymbol{S}^N} \, e' \textbf{ with } h$,

- $\Gamma_1, x : A, \Gamma_2 \vdash e' : A' \mid \varepsilon'$,
- $l :: \forall \boldsymbol{\alpha}^N : \boldsymbol{K}^N.\sigma \in \Xi$,
- $\Gamma_1, x : A, \Gamma_2 \vdash \boldsymbol{S}^N : \boldsymbol{K}^N$,
- $\Gamma_1, x : A, \Gamma_2 \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : A' \Rightarrow^\varepsilon B$, and
- $(l\,\boldsymbol{S}^N)^\uparrow \odot \varepsilon \sim \varepsilon'$.

By the induction hypothesis and case (2), we have
- $\Gamma_1, \Gamma_2 \vdash \boldsymbol{S}^N : \boldsymbol{K}^N$,
- $\Gamma_1, \Gamma_2 \vdash e'[v/x] : A' \mid \varepsilon'$, and
- $\Gamma_1, \Gamma_2 \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h[v/x] : A' \Rightarrow^\varepsilon A$.

Thus, T_HANDLING derives

$$\Gamma_1, \Gamma_2 \vdash \mathbf{handle}_{l\,\boldsymbol{S}^N}\ e'[v/x]\,\mathbf{with}\,h[v/x] : B \mid \varepsilon.$$

Case H_RETURN**:** Without loss of generality, we can choose y such that $y \neq x$ and $y \notin \mathrm{FV}(v)$. For some e_r, the following are given:
- $h = \{\,\mathbf{return}\,y \mapsto e_r\}$,
- $\sigma = \{\}$, and
- $\Gamma_1, x : A, \Gamma_2, y : B \vdash e_r : C \mid \varepsilon$.

By the induction hypothesis, we have
- $\Gamma_1, \Gamma_2, y : B \vdash e_r[v/x] : C \mid \varepsilon$.

Thus, H_RETURN derives

$$\Gamma_1, \Gamma_2 \vdash_{\{\}} \{\,\mathbf{return}\,y \mapsto e_r[v/x]\} : B \Rightarrow^\varepsilon C.$$

Case H_OP**:** Without loss of generality, we can choose $\boldsymbol{\beta}^J$ and p and k such that:
- $p \neq x$,
- $k \neq x$,
- $p \notin \mathrm{FV}(v)$,
- $k \notin \mathrm{FV}(v)$, and
- $\{\boldsymbol{\beta}^J\} \cap \mathrm{FTV}(v) = \emptyset$.

For some h', σ', op, A', B', and e, the following are given:
- $h = h' \uplus \{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e\}$,
- $\sigma = \sigma' \uplus \{\mathsf{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K}^J.A' \Rightarrow B'\}$,
- $\Gamma_1, x : A, \Gamma_2 \vdash_{\sigma'} h' : B \Rightarrow^\varepsilon C$, and
- $\Gamma_1, x : A, \Gamma_2, \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A', k : B' \to_\varepsilon C \vdash e : C \mid \varepsilon$.

By the induction hypothesis, we have
- $\Gamma_1, \Gamma_2 \vdash_{\sigma'} h'[v/x] : A \Rightarrow^\varepsilon B$ and
- $\Gamma_1, \Gamma_2, \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A', k : B' \to_\varepsilon B \vdash e[v/x] : B \mid \varepsilon$.

Thus, H_OP derives

$$\Gamma_1, \Gamma_2 \vdash_\sigma h'[v/x] \uplus \{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e[v/x]\} : B \Rightarrow^\varepsilon C$$

.

$\blacksquare$

Lemma 3.8 (Well-formedness of contexts in subtyping judgments)**.**
- *If $\Gamma \vdash A_1 <: A_2$, then $\vdash \Gamma$.*

- *If $\Gamma \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2$, then $\vdash \Gamma$.*

Proof. Straightforward by mutual induction on the subtyping derivations with Lemma 3.1. $\blacksquare$

Lemma 3.9 (Well-formedness of contexts in typing judgments)**.**
- *If $\Gamma \vdash e : A \mid \varepsilon$, then $\vdash \Gamma$.*

- *If $\Gamma \vdash_\sigma h : A \Rightarrow^\varepsilon B$, then $\vdash \Gamma$.*

Proof. Straightforward by mutual induction on the derivations with Lemma 3.1. ∎

Lemma 3.10 (Substitution of Typelikes). *Suppose that* $\Gamma_1 \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$.

(1) *If* $\vdash \Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2$, *then* $\vdash \Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

(2) *If* $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash T : K$, *then* $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash T[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : K$.

(3) *If* $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash A <: B$, *then* $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] <: B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

(4) *If* $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2$, *then* $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash A_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] <: A_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

(5) *If* $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash e : A \mid \varepsilon$, *then* $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash e[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

(6) *If* $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash_\sigma h : A \Rightarrow^\varepsilon B$, *then* $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash_{\sigma[\boldsymbol{S}/\boldsymbol{\alpha}]} h[\boldsymbol{S}/\boldsymbol{\alpha}] : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \Rightarrow^{\varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

Proof.

(1)(2) By mutual induction on derivations of the judgments. We proceed by case analysis on the rule applied lastly to the derivations.

Case C_Empty: Cannot happen.

Case C_Var: Since Γ_2 cannot be $\emptyset$, for some Γ_2', x, and A, the following are given:
- $\Gamma_2 = \Gamma_2', x : A$,
- $x \notin \mathrm{dom}(\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2')$, and
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2' \vdash A : \mathbf{Typ}$.

By the induction hypothesis, we have $\Gamma_1, (\Gamma_2'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]) \vdash A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : \mathbf{Typ}$. By $x \notin \mathrm{dom}(\Gamma_1, (\Gamma_2'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]))$, C_Var derives $\vdash \Gamma_1, (\Gamma_2'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]), x : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$, and since $\Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = \Gamma_2'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I], x : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$ holds, the required result is achieved.

Case C_TVar: If $\Gamma_2 = \emptyset$, we have
- $\boldsymbol{\alpha}^I : \boldsymbol{K}^I = \boldsymbol{\alpha}^J : \boldsymbol{K}^J, \alpha_i : K_i$,
- $\vdash \Gamma_1, \boldsymbol{\alpha}^J : \boldsymbol{K}^J$, and
- $\alpha_i \notin \mathrm{dom}(\Gamma_1, \boldsymbol{\alpha}^J : \boldsymbol{K}^J)$,

for some J, $\boldsymbol{\alpha}^J$, $\boldsymbol{K}^J$, i, α_i, and K_i. By the induction hypothesis, we have $\vdash \Gamma_1$.

If $\Gamma_2 \neq \emptyset$, for some Γ_2, β, and K', the following are given:
- $\Gamma_2 = \Gamma_2', \beta : K'$,
- $\vdash \Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2'$, and
- $\beta \notin \mathrm{dom}(\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2')$.

By the induction hypothesis, we have $\vdash \Gamma_1, (\Gamma_2'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])$. Thus, C_TVar derives $\vdash \Gamma_1, (\Gamma_2'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]), \beta : K'$, and since

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = \Gamma_1, (\Gamma_2'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]), \beta : K'$$

holds, the required result is achieved.

Case K_Var: For some β, the following are given:
- $T = \beta$,
- $\vdash \Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2$, and
- $\beta : K \in \Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2$.

By the induction hypothesis, we have $\vdash \Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

If $\beta = \alpha_i$ for some $i \in I$, then $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash \beta[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : K$ holds because of the following:
- $\Gamma_1 \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$,
- Lemma 3.5(2),
- $S_i = \beta[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$, and
- $K_i = K$.

If $\beta \neq \alpha_i$ for any $i \in I$, then K_Var derives $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash \beta : K$ because of $\beta : K \in \Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$. Since $\beta = \beta[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$, the required result is achieved.

Case K_Fun: For some A, B, and ε, the following are given:
- $S = A \rightarrow_\varepsilon B$,

- $K = \mathbf{Typ}$,
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash A : \mathbf{Typ}$,
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash \varepsilon : \mathbf{Eff}$, and
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash B : \mathbf{Eff}$.

By the induction hypothesis, we have
- $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : \mathbf{Typ}$,
- $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash \varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : \mathbf{Eff}$, and
- $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : \mathbf{Eff}$.

Thus, K_FUN derives

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash (A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]) \to_{\varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} (B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]) : \mathbf{Typ},$$

and since

$$(A \to_\varepsilon B)[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = (A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]) \to_{\varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} (B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])$$

holds, the required result is achieved.

Case K_POLY: For some β, K', A, and ε, the following are given:
- $S = \forall\beta : K'.A^\varepsilon$,
- $K = \mathbf{Typ}$,
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2, \beta : K' \vdash A : \mathbf{Typ}$, and
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2, \beta : K' \vdash \varepsilon : \mathbf{Eff}$.

By the induction hypothesis, we have
- $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I], \beta : K' \vdash A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : \mathbf{Typ}$ and
- $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I], \beta : K' \vdash \varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : \mathbf{Eff}$.

Thus, K_POLY derives

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash \forall\beta : K'.A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^{(\varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])} : \mathbf{Typ} \ .$$

Since we can assume that β does not occur in $\boldsymbol{S}^I$ and $\boldsymbol{\alpha}^I$ without loss of generality, we have

$$(\forall\beta : K'.A^\varepsilon)[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = \forall\beta : K'.A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^{(\varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])} \ .$$

Therefore, the required result is achieved.

Case K_CONS: For some $\mathcal{C}$, $\boldsymbol{S'}^J$, and $\boldsymbol{K'}^J$, the following are given:
- $S = \mathcal{C}\, \boldsymbol{S'}^J$,
- $\mathcal{C} : \Pi\boldsymbol{K'}^J \to K \in \Sigma$,
- $\vdash \Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2$, and
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash \boldsymbol{S'}^J : \boldsymbol{K'}^J$

By the induction hypothesis, we have $\vdash \Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$ and $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash \boldsymbol{S'}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^J : \boldsymbol{K'}^J$. Thus, K_CONS derives $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash \mathcal{C}\, \boldsymbol{S'}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^J : K$.

(3)(4) By mutual induction on derivations of the judgments. We proceed by case analysis on the rule applied lastly to the derivation.

Case ST_REFL: $A = B$ and $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash A : \mathbf{Typ}$ are given. By case (2), we have $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : \mathbf{Typ}$. Thus, ST_REFL derives

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] <: A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I].$$

Case ST_FUN: For some A_{11}, ε_1, A_{12}, A_{21}, ε_2, B_{22}, the following are given:
- $A = A_{11} \to_{\varepsilon_1} A_{12}$,
- $B = A_{21} \to_{\varepsilon_2} A_{22}$,
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash A_{21} <: A_{11}$, and
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash A_{12} \mid \varepsilon_1 <: A_{22} \mid \varepsilon_2$.

By the induction hypothesis, we have

- $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash A_{21}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] <: A_{11}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$ and
- $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash A_{12}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] <: A_{22}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I].$

Thus, ST_FUN drives

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash (A_{11}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]) \to_{\varepsilon_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} (A_{12}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]) <: (A_{21}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]) \to_{\varepsilon_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} (A_{22}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])$$

and since, for any $i \in \{1, 2\}$,

$$(A_{i\,1} \to_{\varepsilon_i} A_{i\,2})[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = (A_{i\,1}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]) \to_{\varepsilon_i[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} (A_{i\,2}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])$$

holds, the required result is achieved.

Case ST_POLY**:** For some β, K, A_1, ε_1, A_2, and ε_2, the following are given:
- $A = \forall \beta : K.A_1{}^{\varepsilon_1}$,
- $B = \forall \beta : K.A_2{}^{\varepsilon_2}$, and
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2, \beta : K \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon.$

By the induction hypothesis, we have $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I], \beta : K \vdash A_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] <: A_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$. Thus, ST_POLY derives

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash \forall \beta : K.A_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^{(\varepsilon_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])} <: \forall \beta : K.A_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^{(\varepsilon_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])}$$

and since
- $(\forall \beta : K.A_1{}^{\varepsilon_1})[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = \forall \beta : K.A_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^{(\varepsilon_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])}$ and
- $(\forall \beta : K.A_2{}^{\varepsilon_2})[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = \forall \beta : K.A_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^{(\varepsilon_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])}$

hold, the required result is achieved.

Case ST_COMP**:** We have $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash A_1 <: A_2$ and $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash \varepsilon_1 \oslash \varepsilon_2$. By the induction hypothesis, we have $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash A_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] <: A_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$. By Lemma 3.8, $\vdash \Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2$. Then, by case (2) and the fact that a typelike substitution is homomorphism for $\oslash$ and $\sim$, we have $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash (\varepsilon_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]) \oslash (\varepsilon_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])$. Thus, ST_COMP derives $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash A_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] <: A_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I].$

(5)(6) By mutual induction on derivations of the judgments. We proceed by case analysis on the rule applied lastly to the derivation.

Case T_VAR**:** For some x, the following are given:
- $e = x$,
- $\varepsilon = \mathbb{0}$,
- $\vdash \Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2$, and
- $x : A \in \Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2.$

By case (1), we have $\vdash \Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

Case $x : A \in \Gamma_1$**:** Since $x : A \in \Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$ and $A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = A$ hold, T_VAR derives

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash x : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \mathbb{0}.$$

Case $x : A \in \boldsymbol{\alpha}^I : \boldsymbol{K}^I$**:** Cannot happen.

Case $x : A \in \Gamma_2$**:** Since $x : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \in \Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$ holds, T_VAR derives

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash x : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \mathbb{0}.$$

Thus, the required result is achieved because of $x[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = x$.

Case T_ABS**:** For some f, x, e', A', B', and ε', the following are given:
- $e = \mathbf{fun}\,(f, x, e')$,
- $A = A' \to_{\varepsilon'} B'$,
- $\varepsilon = \mathbb{0}$, and
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2, f : A' \to_{\varepsilon'} B', x : A' \vdash e' : B' \mid \varepsilon'.$

By the induction hypothesis, we have

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I], f : (A' \to_{\varepsilon'} B')[\boldsymbol{S}^I/\boldsymbol{\alpha}^I], x : A'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash e'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : B'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I].$$

Since

$$(A' \to_{\varepsilon'} B')[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = (A'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]) \to_{\varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} (B'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])$$

holds, T_Abs derives

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash \mathbf{fun}\,(f, x, e'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]) : (A'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]) \to_{\varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} (B'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]) \mid \mathbb{0}.$$

Thus, the required result is achieved because

$$(\mathbf{fun}\,(f, x, e'))[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = \mathbf{fun}\,(f, x, e'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])$$

holds.

Case T_App: For some v_1, v_2, and B, the following are given:
- $e = v_1\,v_2$,
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash v_1 : B \to_\varepsilon A \mid \mathbb{0}$, and
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash v_2 : B \mid \mathbb{0}$.

By the induction hypothesis, we have
- $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash v_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : (B \to_\varepsilon A)[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \mathbb{0}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$ and
- $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash v_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \mathbb{0}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

Since
- $(B \to_\varepsilon A)[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = (B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]) \to_{\varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} (A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])$ and
- $\mathbb{0}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = \mathbb{0}$

hold, T_App derives

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash (v_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])\,(v_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]) : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$$

as required.

Case T_TAbs: Without loss of generality, we can choose β such that $\beta \neq \alpha_i$ and $\beta \notin \mathrm{FTV}(S_i)$ for any $i \in I$. For some K, e', A', and ε', the following are given:
- $e = \Lambda\beta : K.e'$,
- $A = \forall\beta : K.A'^{\varepsilon'}$,
- $\varepsilon = \mathbb{0}$, and
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2, \beta : K \vdash e' : A' \mid \varepsilon'$.

By the induction hypothesis, we have

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I], \beta : K \vdash e'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : A'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$$

Thus, T_TAbs derives

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash \Lambda\beta : K.(e'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]) : \forall\beta : K.A'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^{(\varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])} \mid \mathbb{0}$$

and since

$$(\Lambda\beta : K.e')[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = \Lambda\beta : K.(e'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])$$

holds, the required result is achieved.

Case T_TApp: Without loss of generality, we can choose β such that $\beta \neq \alpha_i$ and $\beta \notin \mathrm{FTV}(S_i)$ for any $i \in I$. For some v, T, β, A', ε', and K, the following are given:
- $e = v\,T$,
- $A = A'[T/\beta]$,
- $\varepsilon = \varepsilon'[T/\beta]$,
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash v : \forall\beta : K.A'^{\varepsilon'} \mid \mathbb{0}$, and

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– $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash T : K$.

By the induction hypothesis, we have

– $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash v[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : (\forall\beta : K.A'^{\varepsilon'})[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \mathbb{0}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$ and
– $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash T[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : K$.

Since

– $(\forall\beta : K.A'^{\varepsilon'})[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = \forall\beta : K.A'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^{(\varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])}$ and
– $\mathbb{0}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = \mathbb{0}$

hold, T_TAPP derives

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash (v[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])\,(T[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]) : (A'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])[T[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]/\beta] \mid (\varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])[T[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]/\beta].$$

Finally, we have

– $(v\,T)[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = (v[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])\,(T[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])$,
– $(A'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])[T[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]/\beta] = (A'[T/\beta])[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$, and
– $(\varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])[T[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]/\beta] = (\varepsilon'[T/\beta])[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$

because $\forall i \in I.(\beta \notin \mathrm{FTV}(S_i))$. Thus, the required result is achieved.

Case T_LET: For some x, e_1, e_2, and B, the following are given:

– $e = (\mathbf{let}\,x = e_1\,\mathbf{in}\,e_2)$
– $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash e_1 : B \mid \varepsilon$, and
– $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2, x : B \vdash e_2 : A \mid \varepsilon$.

By the induction hypothesis, we have

– $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash e_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$ and
– $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I], x : B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash e_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

Thus, T_LET derives

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash \mathbf{let}\,x = e_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]\,\mathbf{in}\,(e_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]) : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$$

and since

$$(\mathbf{let}\,x = e_1\,\mathbf{in}\,e_2)[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = \mathbf{let}\,x = e_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]\,\mathbf{in}\,(e_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I])$$

holds, the required result is achieved.

Case T_SUB: For some A' and ε', the following are given:

– $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash e : A' \mid \varepsilon'$ and
– $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash A' \mid \varepsilon' <: A \mid \varepsilon$.

By the induction hypothesis and case (4), we have

– $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash e[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : A'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$ and
– $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash A'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] <: A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

Thus, T_SUB derives

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash e[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$$

as required.

Case T_OP: For some op, l, $\boldsymbol{S_0}^{I_0}$, $\boldsymbol{T}^J$, σ, $\boldsymbol{\alpha_0}^{I_0}$, $\boldsymbol{K_0}^{I_0}$, $\boldsymbol{\beta}^J$, $\boldsymbol{K''}^J$, A', and B', the following are given:

– $e = \mathsf{op}_{l\,\boldsymbol{S_0}^{I_0}}\,\boldsymbol{T}^J$,
– $A = (A'[\boldsymbol{T}^J/\boldsymbol{\beta}^J]) \rightarrow_{(l\,\boldsymbol{S_0}^{I_0})\uparrow} (B'[\boldsymbol{T}^J/\boldsymbol{\beta}^J])$,
– $\varepsilon = \mathbb{0}$,
– $l :: \forall\boldsymbol{\alpha_0}^{I_0} : \boldsymbol{K_0}^{I_0}.\sigma \in \Xi$,
– $\mathsf{op} : \forall\boldsymbol{\beta}^J : \boldsymbol{K''}^J.A' \Rightarrow B' \in \sigma[\boldsymbol{S_0}^{I_0}/\boldsymbol{\alpha_0}^{I_0}]$,
– $\vdash \Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2$,
– $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash \boldsymbol{S_0}^{I_0} : \boldsymbol{K_0}^{I_0}$, and
– $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash \boldsymbol{T}^J : \boldsymbol{K''}^J$.

By cases (1) and (2), we have

– $\vdash \Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$,

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– $\Gamma_1, \Gamma_2[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] \vdash \boldsymbol{S_0}[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]^{I_0} : \boldsymbol{K_0}^{I_0}$, and

– $\Gamma_1, \Gamma_2[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] \vdash \boldsymbol{T}[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]^{J} : \boldsymbol{K''}^{J}$.

Since

– $((l\,\boldsymbol{S_0}^{I_0})^\uparrow)[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] = (l\,\boldsymbol{S_0}[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]^{I_0})^\uparrow$ and

– $\mathbb{0}[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] = \mathbb{0}$,

T_OP derives

$$\Gamma_1, \Gamma_2[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] \vdash \mathsf{op}_{l\,\boldsymbol{S_0}[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]^{I_0}}\,\boldsymbol{T}[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]^{J} : A'_0[\boldsymbol{T}[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]^{J}/\boldsymbol{\beta^J}] \to_{(l\,\boldsymbol{S_0}[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]^{I_0})^\uparrow} B'_0[\boldsymbol{T}[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]^{J}/\boldsymbol{\beta^J}] \mid \mathbb{0}$$

where

$$\mathsf{op} : \forall \boldsymbol{\beta^J} : \boldsymbol{K''}^{J}.A'_0 \Rightarrow B'_0 \in \sigma[\boldsymbol{S_0}[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]^{I_0}/\boldsymbol{\alpha_0}^{I_0}].$$

Without loss of generality, we can assume that, for any $i \in I$, $\alpha_i \notin \mathrm{FTV}(A') \cup \mathrm{FTV}(B')$, and $(\{\alpha_i\} \cup \mathrm{FTV}(S_i)) \cap (\{\boldsymbol{\alpha_0}^{I_0}\} \cup \{\boldsymbol{\beta^J}\}) = \emptyset$. Then,

– $A'[\boldsymbol{T^J}/\boldsymbol{\beta^J}][\boldsymbol{S^I}/\boldsymbol{\alpha^I}] = A'_0[\boldsymbol{T}[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]^{J}/\boldsymbol{\beta^J}]$ and

– $B'[\boldsymbol{T^J}/\boldsymbol{\beta^J}][\boldsymbol{S^I}/\boldsymbol{\alpha^I}] = B'_0[\boldsymbol{T}[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]^{J}/\boldsymbol{\beta^J}]$

hold. Therefore, the required result is achieved.

Case T_HANDLING: For some N, e', A', ε', l, $\boldsymbol{S_0}^{N}$, $\boldsymbol{\alpha_0}^{N}$, $\boldsymbol{K_0}^{N}$, h, and σ, the following are given:

– $e = \mathbf{handle}_{l\,\boldsymbol{S_0}^{N}}\,e'\,\mathbf{with}\,h$,

– $\Gamma_1, \boldsymbol{\alpha^I} : \boldsymbol{K^I}, \Gamma_2 \vdash e' : A' \mid \varepsilon'$,

– $l :: \forall \boldsymbol{\alpha_0}^{N} : \boldsymbol{K_0}^{N}.\sigma \in \Xi$,

– $\Gamma_1, \boldsymbol{\alpha^I} : \boldsymbol{K^I}, \Gamma_2 \vdash \boldsymbol{S_0}^{N} : \boldsymbol{K_0}^{N}$,

– $\Gamma_1, \boldsymbol{\alpha^I} : \boldsymbol{K^I}, \Gamma_2 \vdash_{\sigma[\boldsymbol{S_0}^{N}/\boldsymbol{\alpha_0}^{N}]} h : A' \Rightarrow^\varepsilon A$, and

– $(l\,\boldsymbol{S_0}^{N})^\uparrow \odot \varepsilon \sim \varepsilon'$.

By the induction hypothesis, case (2), and the fact that a typelike substitution is homomorphism for $\odot$ and $\sim$, we have

– $\Gamma_1, \Gamma_2[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] \vdash \boldsymbol{S_0}[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]^{N} : \boldsymbol{K_0}^{N}$,

– $\Gamma_1, \Gamma_2[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] \vdash e'[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] : A'[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] \mid \varepsilon'[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]$,

– $\Gamma_1, \Gamma_2[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] \vdash_{\sigma[\boldsymbol{S_0}^{N}/\boldsymbol{\alpha_0}^{N}][\boldsymbol{S^I}/\boldsymbol{\alpha^I}]} h[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] : A'[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] \Rightarrow^{\varepsilon[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]} A[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]$, and

– $(l\,\boldsymbol{S_0}[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]^{N})^\uparrow \odot \varepsilon[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] \sim \varepsilon'[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]$.

Now, because we can assume that

– $\{\boldsymbol{\alpha^I}\} \cap \{\boldsymbol{\alpha_0}^{N}\} = \emptyset$ and

– $\{\boldsymbol{\alpha_0}^{N}\} \cap \mathrm{FTV}(\boldsymbol{S^I}) = \emptyset$

without loss of generality, we have

$$\Gamma_1, \Gamma_2[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] \vdash_{\sigma[\boldsymbol{S_0}[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]^{N}/\boldsymbol{\alpha_0}^{N}]} h[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] : A'[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] \Rightarrow^{\varepsilon[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]} A[\boldsymbol{S^I}/\boldsymbol{\alpha^I}].$$

Thus, T_HANDLING derives

$$\Gamma_1, \Gamma_2[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] \vdash \mathbf{handle}_{l\,\boldsymbol{S_0}[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]^{N}}\,e[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]\,\mathbf{with}\,h[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] : B[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] \mid \varepsilon[\boldsymbol{S^I}/\boldsymbol{\alpha^I}].$$

Case H_RETURN: For some x and e_r, the following are given:

– $h = \{\,\mathbf{return}\,y \mapsto e_r\,\}$,

– $\sigma = \{\}$, and

– $\Gamma_1, \boldsymbol{\alpha^I} : \boldsymbol{K^I}, \Gamma_2, x : A \vdash e_r : B \mid \varepsilon$.

By the induction hypothesis, we have

– $\Gamma_1, \Gamma_2[\boldsymbol{S^I}/\boldsymbol{\alpha^I}], x : A[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] \vdash e_r[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] : B[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] \mid \varepsilon[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]$.

Thus, H_RETURN derives

$$\Gamma_1, \Gamma_2 \vdash_{\{\}} \{\,\mathbf{return}\,x \mapsto e_r[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]\,\} : A[\boldsymbol{S^I}/\boldsymbol{\alpha^I}] \Rightarrow^{\varepsilon[\boldsymbol{S^I}/\boldsymbol{\alpha^I}]} B[\boldsymbol{S^I}/\boldsymbol{\alpha^I}].$$

Case H_OP: Without loss of generality, we can choose $\boldsymbol{\beta^J}$ such that:

– $\{\boldsymbol{\beta^J}\} \cap \{\boldsymbol{\alpha^I}\} = \emptyset$ and

  – $\{\boldsymbol{\beta}^J\} \cap \mathrm{FTV}(\boldsymbol{S}^I) = \emptyset$.

For some h', σ', op, A', B', and e, the following are given:

  – $h = h' \uplus \{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e\}$,

  – $\sigma = \sigma' \uplus \{\mathsf{op} : \forall\boldsymbol{\beta}^J : \boldsymbol{K}^J.A' \Rightarrow B'\}$,

  – $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash_{\sigma'} h' : A \Rightarrow^\varepsilon B$, and

  – $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2, \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A', k : B' \to_\varepsilon B \vdash e : B \mid \varepsilon$.

By the induction hypothesis and Definition 1.10, we have

  – $\sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = \sigma'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \uplus \{\mathsf{op} : \forall\boldsymbol{\beta}^J : \boldsymbol{K}^J.A'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \Rightarrow B'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]\}$,

  – $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash_{\sigma'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} h'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \Rightarrow^{\varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$, and

  – $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I], \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I], k : B'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \to_{\varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash e[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid$
   $\varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

Thus, H_OP derives

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash_{\sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} h'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \uplus \{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]\} : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \Rightarrow^{\varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I].$$

 ■

Lemma 3.11 (Well-kinded of Subtyping).

- If $\Gamma \vdash A <: B$, then $\Gamma \vdash A : \mathbf{Typ}$ and $\Gamma \vdash B : \mathbf{Typ}$.

- If $\Gamma \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon$, then $\Gamma \vdash A_i : \mathbf{Typ}$ and $\Gamma \vdash \varepsilon_i : \mathbf{Eff}$ for $i \in \{1, 2\}$.

Proof. Straightforward by mutual induction on the subtyping derivations with Lemma 3.6. ■

Lemma 3.12 (Well-kinded of Typing).

(1) If $\Gamma \vdash e : A \mid \varepsilon$, then $\Gamma \vdash A : \mathbf{Typ}$ and $\Gamma \vdash \varepsilon : \mathbf{Eff}$.

(2) If $\Gamma \vdash_\sigma h : A \Rightarrow^\varepsilon B$, then $\Gamma \vdash A : \mathbf{Typ}$ and $\Gamma \vdash B : \mathbf{Typ}$ and $\Gamma \vdash \varepsilon : \mathbf{Eff}$.

Proof. By mutual induction on derivations of the judgments. We proceed by cases on the typing rule applied lastly to the derivation.

Case T_VAR: We are given $\varepsilon = \mathbb{0}$ and $\vdash \Gamma$ and $\Gamma = \Gamma_1, x : A, \Gamma_2$ for some x, Γ_1, and Γ_2. Because $\vdash \Gamma$, it is easy to prove that $\Gamma_1 \vdash A : \mathbf{Typ}$ using Lemma 3.1. Then, by Lemma 3.5, $\Gamma_1, x : A, \Gamma_2 \vdash A : \mathbf{Typ}$. We also have $\Gamma \vdash \mathbb{0} : \mathbf{Eff}$ because $\mathbb{0}$ is well-formedness-preserving.

Case T_ABS: For some f, x, e', B, C, and ε', the following are given:

- $e = \mathbf{fun}\,(f, x, e')$,

- $A = B \to_{\varepsilon'} C$,

- $\varepsilon = \mathbb{0}$, and

- $\Gamma, f : B \to_{\varepsilon'} C, x : B \vdash e' : C \mid \varepsilon'$.

Since $\mathbb{0}$ is well-formedness-preserving, we have $\Gamma \vdash \mathbb{0} : \mathbf{Eff}$. By the induction hypothesis, we have $\Gamma, f : B \to_{\varepsilon'} C, x : B \vdash C : \mathbf{Typ}$. By Lemma 3.1, we have $\vdash \Gamma, f : B \to_{\varepsilon'} C, x : B$. Since only C_VAR can derive $\vdash \Gamma, f : B \to_{\varepsilon'} C, x : B$, we have $\Gamma, f : B \to_{\varepsilon'} C \vdash B : \mathbf{Typ}$. By Lemma 3.1, we have $\vdash \Gamma, f : B \to_{\varepsilon'} C$. Since only C_VAR can derive $\vdash \Gamma, f : B \to_{\varepsilon'} C$, we have $\Gamma \vdash B \to_{\varepsilon'} C : \mathbf{Typ}$.

Case T_APP: For some v_1, v_2, and B, the following are given:

- $e = v_1\,v_2$,

- $\Gamma \vdash v_1 : B \to_\varepsilon A \mid \mathbb{0}$, and

- $\Gamma \vdash v_2 : B \mid \mathbb{0}$.

By the induction hypothesis, we have $\Gamma \vdash B \to_\varepsilon A : \mathbf{Typ}$ and $\Gamma \vdash \mathbb{0} : \mathbf{Eff}$. Since only K_FUN can derive $\Gamma \vdash B \to_\varepsilon A : \mathbf{Typ}$, we have $\Gamma \vdash A : \mathbf{Typ}$ and $\Gamma \vdash \varepsilon : \mathbf{Eff}$ as required.

Case T_TABS: For some α, K, e', B, and ε', the following are given:

- $e = \Lambda\alpha : K.e'$,

- $A = \forall\alpha : K.B^{\varepsilon'}$,

- $\varepsilon = \mathbb{0}$, and

- $\Gamma, \alpha : K \vdash e' : B \mid \varepsilon'$.

Since $\mathbb{0}$ is well-formedness-preserving, we have $\Gamma \vdash \mathbb{0} : \mathbf{Eff}$. By the induction hypothesis, we have $\Gamma, \alpha : K \vdash B : \mathbf{Typ}$ and $\Gamma, \alpha : K \vdash \varepsilon' : \mathbf{Eff}$. Thus, K_POLY derives $\Gamma \vdash \forall \alpha : K.B^{\varepsilon'} : \mathbf{Typ}$.

Case T_TAPP: For some v, S, A', ε', α, and K, the following are given:

- $e = v\,S$,
- $A = A'[S/\alpha]$,
- $\varepsilon = \varepsilon'[S/\alpha]$,
- $\Gamma \vdash v : \forall \alpha : K.A'^{\varepsilon'} \mid \mathbb{0}$, and
- $\Gamma \vdash S : K$.

By the induction hypothesis, we have $\Gamma \vdash \forall \alpha : K.A'^{\varepsilon'} : \mathbf{Typ}$. Since only K_POLY can derive $\Gamma \vdash \forall \alpha : K.A'^{\varepsilon'} : \mathbf{Typ}$, we have $\Gamma, \alpha : K \vdash A' : \mathbf{Typ}$ and $\Gamma, \alpha : K \vdash \varepsilon' : \mathbf{Eff}$. By Lemma 3.10(2), we have $\Gamma \vdash A'[S/\alpha] : \mathbf{Typ}$ and $\Gamma \vdash \varepsilon'[S/\alpha] : \mathbf{Eff}$ as required.

Case T_LET: For some x, e_1, e_2, and B, the following are given:

- $e = (\mathbf{let}\ x = e_1\ \mathbf{in}\ e_2)$,
- $\Gamma \vdash e_1 : B \mid \varepsilon$, and
- $\Gamma, x : B \vdash e_2 : A \mid \varepsilon$.

By the induction hypothesis, we have $\Gamma, x : B \vdash A : \mathbf{Typ}$ and $\Gamma \vdash \varepsilon : \mathbf{Eff}$. By $\Delta(\Gamma, x : B) = \Delta(\Gamma)$ and Lemma 3.2(2) and Lemma 3.6, we have $\Gamma \vdash A : \mathbf{Typ}$ as required.

Case T_SUB: For some A' and ε', the following are given:

- $\Gamma \vdash e : A' \mid \varepsilon'$ and
- $\Gamma \vdash A' \mid \varepsilon' <: A \mid \varepsilon$.

By Lemma 3.11, we have $\Gamma \vdash A : \mathbf{Typ}$ and $\Gamma \vdash \varepsilon : \mathbf{Typ}$.

Case T_OP: For some op, l, $\boldsymbol{S}^I$, $\boldsymbol{T}^J$, σ, $\boldsymbol{\alpha}^I$, $\boldsymbol{K}^I$, $\boldsymbol{\beta}^J$, $\boldsymbol{K'}^J$, A', B', the following are given:

- $e = \mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J$,
- $A = (A'[\boldsymbol{T}^J/\boldsymbol{\beta}^J]) \to_{(l\,\boldsymbol{S}^I)^\uparrow} (B'[\boldsymbol{T}^J/\boldsymbol{\beta}^J])$,
- $l :: \forall \boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi$,
- $\mathsf{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K'}^J.A' \Rightarrow B' \in \sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$,
- $\vdash \Gamma$,
- $\Gamma \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$, and
- $\Gamma \vdash \boldsymbol{T}^J : \boldsymbol{K'}^J$.

Since $\mathbb{0}$ is well-formedness-preserving, we have $\Gamma \vdash \mathbb{0} : \mathbf{Eff}$. Without loss of generality, we can assume that $\boldsymbol{\alpha}^I$ and $\boldsymbol{\beta}^J$ do not occur in Γ. Then, because there exist some A'' and B'' such that

- $\boldsymbol{\alpha}^I : \boldsymbol{K}^I, \boldsymbol{\beta}^J : \boldsymbol{K'}^J \vdash A'' : \mathbf{Typ}$,
- $\boldsymbol{\alpha}^I : \boldsymbol{K}^I, \boldsymbol{\beta}^J : \boldsymbol{K'}^J \vdash B'' : \mathbf{Typ}$,
- $A''[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = A'$, and
- $B''[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = B'$,

Lemma 3.5 and 3.10(2) imply $\Gamma \vdash A'[\boldsymbol{T}^J/\boldsymbol{\beta}^J] : \mathbf{Typ}$ and $\Gamma \vdash B'[\boldsymbol{T}^J/\boldsymbol{\beta}^J] : \mathbf{Typ}$. Thus, K_FUN derives $\Gamma \vdash (A'[\boldsymbol{T}^J/\boldsymbol{\beta}^J]) \to_{(l\,\boldsymbol{S}^I)^\uparrow} (B'[\boldsymbol{T}^J/\boldsymbol{\beta}^J]) : \mathbf{Typ}$.

Case T_HANDLING: For some A', σ, N, $\boldsymbol{\alpha}^N$, and $\boldsymbol{S}^N$, we have

$$\Gamma \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : A' \Rightarrow^\varepsilon A.$$

By the induction hypothesis, we have $\Gamma \vdash A : \mathbf{Typ}$ and $\Gamma \vdash \varepsilon : \mathbf{Eff}$.

Case H_RETURN: For some x and e_r, we have

$$\Gamma, x : A \vdash e_r : B \mid \varepsilon.$$

By the induction hypothesis, we have

- $\Gamma, x : A \vdash B : \textbf{Typ}$ and
- $\Gamma, x : A \vdash \varepsilon : \textbf{Eff}$.

By Lemma 3.2(2), we have

- $\Delta(\Gamma) \vdash B : \textbf{Typ}$ and
- $\Delta(\Gamma) \vdash \varepsilon : \textbf{Eff}$.

By Lemma 3.6, we have

- $\Gamma \vdash B : \textbf{Typ}$ and
- $\Gamma \vdash \varepsilon : \textbf{Eff}$.

Now, we have $\vdash \Gamma, x : A$ by Lemma 3.9. Since only C_Var can derive $\vdash \Gamma, x : A$, we have $\Gamma \vdash A : \textbf{Typ}$.

Case H_Op: For some h' and σ', we have $\Gamma \vdash_{\sigma'} h' : A \Rightarrow^{\varepsilon} B$. By the induction hypothesis, we have $\Gamma \vdash A : \textbf{Typ}$ and $\Gamma \vdash B : \textbf{Typ}$ and $\Gamma \vdash \varepsilon : \textbf{Eff}$.

∎

Lemma 3.13 (Inversion of Subtyping).
(1) If $\Gamma \vdash C <: A_1 \to_{\varepsilon_1} B_1$ and $\Gamma \vdash \mathbb{0} : \textbf{Eff}$, then $C = A_2 \to_{\varepsilon_2} B_2$ such that $\Gamma \vdash A_1 <: A_2$, $\Gamma \vdash B_2 <: B_1$, and $\Gamma \vdash \varepsilon_2 \oslash \varepsilon_1$.

(2) If $\Gamma \vdash C <: \forall \alpha : K.A_1{}^{\varepsilon_1}$ and $\Gamma \vdash \mathbb{0} : \textbf{Eff}$, then $C = \forall \alpha : K.A_2{}^{\varepsilon_2}$ such that $\Gamma, \alpha : K \vdash A_2 <: A_1$ and $\Gamma, \alpha : K \vdash \varepsilon_2 \oslash \varepsilon_1$.

Proof.
(1) By induction on a derivation of $\Gamma \vdash C <: A_1 \to_{\varepsilon_1} B_1$. We proceed by case analysis on the subtyping rule applied lastly to this derivation.

    Case ST_Refl: $\Gamma \vdash A_1 \to_{\varepsilon_1} B_1 : \textbf{Typ}$ and $C = A_1 \to_{\varepsilon_1} B_1$ are given. Because only K_Fun can derive $\Gamma \vdash A_1 \to_{\varepsilon_1} B_1 : \textbf{Typ}$, we have $\Gamma \vdash A_1 : \textbf{Typ}$, $\Gamma \vdash \varepsilon_1 : \textbf{Eff}$, and $\Gamma \vdash B_1 : \textbf{Typ}$. By ST_Refl, $\Gamma \vdash A_1 <: A_1$ and $\Gamma \vdash B_1 <: B_1$ hold. By Lemma 3.3(1), $\Gamma \vdash \varepsilon_1 \oslash \varepsilon_1$ holds.

    Case ST_Fun: Clearly.

    Case others: Cannot happen.

(2) By induction on a derivation of $\Gamma \vdash C <: \forall \alpha : K.A_1{}^{\varepsilon_1}$. We proceed by case analysis on the subtyping rule applied lastly to this derivation.

    Case ST_Refl: $\Gamma \vdash \forall \alpha : K.A_1{}^{\varepsilon_1} : \textbf{Typ}$ and $C = \forall \alpha : K.A_1{}^{\varepsilon_1}$ are given. Because only K_Poly can derive $\Gamma \vdash \forall \alpha : K.A_1{}^{\varepsilon_1} : \textbf{Typ}$, we have $\Gamma, \alpha : K \vdash A_1 : \textbf{Typ}$ and $\Gamma, \alpha : K \vdash \varepsilon_1 : \textbf{Eff}$. By ST_Refl, $\Gamma, \alpha : K \vdash A_1 <: A_1$ holds. By Lemma 3.3(1), $\Gamma, \alpha : K \vdash \varepsilon_1 \oslash \varepsilon_1$.

    Case ST_Poly: Clearly.

    Case others: Cannot happen.

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Lemma 3.14 (Inversion).
(1) If $\Gamma \vdash v : A \mid \varepsilon$, then $\Gamma \vdash v : A \mid \mathbb{0}$.

(2) If $\Gamma \vdash \textbf{fun}\,(f, x, e) : A_1 \to_{\varepsilon_1} B_1 \mid \varepsilon$, then $\Gamma, f : A_2 \to_{\varepsilon_2} B_2, x : A_2 \vdash e : B_2 \mid \varepsilon_2$ for some A_2, ε_2, and B_2 such that $\Gamma \vdash A_2 \to_{\varepsilon_2} B_2 <: A_1 \to_{\varepsilon_1} B_1$.

(3) If $\Gamma \vdash \Lambda \alpha : K.e : \forall \alpha : K.A_1{}^{\varepsilon_1} \mid \varepsilon$, then $\Gamma, \alpha : K \vdash e : A_1 \mid \varepsilon_1$.

(4) If $\Gamma \vdash \mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J : A_1 \to_{\varepsilon_1} B_1 \mid \varepsilon$, then the following hold:
- $l :: \forall \boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi$,
- $\mathsf{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K'}^J.A \Rightarrow B \in \sigma[\boldsymbol{S}^I / \boldsymbol{\alpha}^I]$,
- $\vdash \Gamma$,
- $\Gamma \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$,
- $\Gamma \vdash \boldsymbol{T}^J : \boldsymbol{K'}^J$,
- $\Gamma \vdash A_1 <: A[\boldsymbol{T}^J / \boldsymbol{\beta}^J]$,
- $\Gamma \vdash B[\boldsymbol{T}^J / \boldsymbol{\beta}^J] <: B_1$, and

- $\Gamma \vdash (l\,\boldsymbol{S}^I)^{\uparrow} \otimes \varepsilon_1$

for some $\boldsymbol{\alpha}^I$, $\boldsymbol{K}^I$, σ, $\boldsymbol{\beta}^J$, $\boldsymbol{K'}^J$, A, and B.

(5) If $\Gamma \vdash v_1\,v_2 : B \mid \varepsilon$, then there exists some type A such that $\Gamma \vdash v_1 : A \to_\varepsilon B \mid \mathbb{0}$ and $\Gamma \vdash v_2 : A \mid \mathbb{0}$.

Proof.

(1) By induction on a derivation of $\Gamma \vdash v : A \mid \varepsilon$. We proceed by cases on the typing rule applied lastly to this derivation.

Case T_VAR**:** Clearly because of $\varepsilon = \mathbb{0}$.

Case T_ABS**:** Clearly because of $\varepsilon = \mathbb{0}$.

Case T_TABS**:** Clearly because of $\varepsilon = \mathbb{0}$.

Case T_OP**:** Clearly because of $\varepsilon = \mathbb{0}$.

Case T_SUB**:** For some A' and ε', the following are given:
- $\Gamma \vdash v : A' \mid \varepsilon'$ and
- $\Gamma \vdash A' \mid \varepsilon' <: A \mid \varepsilon$.

By the induction hypothesis, $\Gamma \vdash v : A' \mid \mathbb{0}$. Since only ST_COMP derives $\Gamma \vdash A' \mid \varepsilon' <: A \mid \varepsilon$, we have $\Gamma \vdash A' <: A$ and $\Gamma \vdash \varepsilon' \otimes \varepsilon$. Because of Lemma 3.3(1), $\Gamma \vdash \mathbb{0} \otimes \mathbb{0}$ holds. By T_SUB, we have $\Gamma \vdash v : A \mid \mathbb{0}$ as required.

Case others**:** Cannot happen.

(2) By induction on a derivation of $\Gamma \vdash \mathbf{fun}\,(f, x, e) : A_1 \to_{\varepsilon_1} B_1 \mid \varepsilon$. We proceed by cases on the typing rule applied lastly to this derivation.

Case T_ABS**:** $\Gamma, f : A_1 \to_{\varepsilon_1} B_1, x : A_1 \vdash e : B_1 \mid \varepsilon_1$ is given. By Lemma 3.12, we have $\Gamma \vdash A_1 \to_{\varepsilon_1} B_1 :$ **Typ**. Thus, ST_REFL derives $\Gamma \vdash A_1 \to_{\varepsilon_1} B_1 <: A_1 \to_{\varepsilon_1} B_1$.

Case T_SUB**:** For some C and ε', the following are given:
- $\Gamma \vdash \mathbf{fun}\,(f, x, e) : C \mid \varepsilon'$ and
- $\Gamma \vdash C \mid \varepsilon' <: A_1 \to_{\varepsilon_1} B_1 \mid \varepsilon$.

Since only ST_COMP derives $\Gamma \vdash C \mid \varepsilon' <: A_1 \to_{\varepsilon_1} B_1 \mid \varepsilon$, we have $\Gamma \vdash C <: A_1 \to_{\varepsilon_1} B_1$. By Lemma 3.13(1), $C = A_2 \to_{\varepsilon_2} B_2$ for some A_2, ε_2, and B_2. By the induction hypothesis and Lemma 3.4, the required results are achieved.

Case others**:** Cannot happen.

(3) By induction on a derivation of $\Gamma \vdash \Lambda\alpha : K.e : \forall \alpha : K.A_1{}^{\varepsilon_1} \mid \varepsilon$. We proceed by cases on the typing rule applied lastly to this derivation.

Case T_TABS**:** Clearly.

Case T_SUB**:** For some B and ε', the following are given:
- $\Gamma \vdash \Lambda\alpha : K.e : B \mid \varepsilon'$ and
- $\Gamma \vdash B \mid \varepsilon' <: \forall \alpha : K.A_1{}^{\varepsilon_1} \mid \varepsilon$.

Since only ST_COMP derives $\Gamma \vdash B \mid \varepsilon' <: \forall \alpha : K.A_1{}^{\varepsilon_1} \mid \varepsilon$, we have $\Gamma \vdash B <: \forall \alpha : K.A_1{}^{\varepsilon_1}$. By Lemma 3.13(2), we have $B = \forall \alpha : K.A_2{}^{\varepsilon_2}$ for some A_2 and ε_2 such that
- $\Gamma, \alpha : K \vdash A_2 <: A_1$ and
- $\Gamma, \alpha : K \vdash \varepsilon_2 \otimes \varepsilon_1$.

By the induction hypothesis, we have $\Gamma, \alpha : K \vdash e : A_2 \mid \varepsilon_2$. Thus, T_SUB derives $\Gamma, \alpha : K \vdash e : A_1 \mid \varepsilon_1$, because ST_COMP derives $\Gamma, \alpha : K \vdash A_2 \mid \varepsilon_2 <: A_1 \mid \varepsilon_1$.

Case others**:** Cannot happen.

(4) By induction on a derivation of $\Gamma \vdash \mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J : A_1 \to_{\varepsilon_1} B_1 \mid \varepsilon$. We proceed by cases on the typing rule applied lastly to this derivation.

Case T_OP**:** For some $\boldsymbol{\alpha}^I$, $\boldsymbol{K}^I$, σ, $\boldsymbol{\beta}^J$, $\boldsymbol{K'}^J$, A, and B, the following are given:
- $l :: \forall \boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi$,
- $\mathsf{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K'}^J.A \Rightarrow B \in \sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$,
- $\vdash \Gamma$,
- $\Gamma \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$,

- $\Gamma \vdash \boldsymbol{T}^J : \boldsymbol{K'}^J$,
- $A_1 = A[\boldsymbol{T}^J/\boldsymbol{\beta}^J]$,
- $B_1 = B[\boldsymbol{T}^J/\boldsymbol{\beta}^J]$, and
- $\varepsilon_1 = (l\,\boldsymbol{S}^I)^\uparrow$.

By Lemma 3.12, we have $\Gamma \vdash A_1 \to_{\varepsilon_1} B_1 : \mathbf{Typ}$. Since only K_FUN can derive $\Gamma \vdash A_1 \to_{\varepsilon_1} B_1 : \mathbf{Typ}$, we have

- $\Gamma \vdash A[\boldsymbol{T}^J/\boldsymbol{\beta}^J] : \mathbf{Typ}$,
- $\Gamma \vdash (l\,\boldsymbol{S}^I)^\uparrow : \mathbf{Eff}$, and
- $\Gamma \vdash B[\boldsymbol{T}^J/\boldsymbol{\beta}^J] : \mathbf{Typ}$.

Thus, the required results are achieved by ST_REFL and Lemma 3.3(1).

Case T_SUB: For some C and ε', the following are given:
- $\Gamma \vdash \mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J : C \mid \varepsilon'$ and
- $\Gamma \vdash C \mid \varepsilon' <: A_1 \to_{\varepsilon_1} B_1 \mid \varepsilon$.

Since only ST_COMP can derive $\Gamma \vdash C \mid \varepsilon' <: A_1 \to_{\varepsilon_1} B_1 \mid \varepsilon$, we have $\Gamma \vdash C <: A_1 \to_{\varepsilon_1} B_1$. By Lemma 3.13(1), we have $C = A_2 \to_{\varepsilon_2} B_2$ such that
- $\Gamma \vdash A_1 <: A_2$,
- $\Gamma \vdash B_2 <: B_1$, and
- $\Gamma \vdash \varepsilon_2 \oslash \varepsilon_1$.

By the induction hypothesis,
- $l :: \forall \boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi$,
- $\mathsf{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K'}^J.A \Rightarrow B \in \sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$,
- $\vdash \Gamma$,
- $\Gamma \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$,
- $\Gamma \vdash \boldsymbol{T}^J : \boldsymbol{K'}^J$,
- $\Gamma \vdash A_2 <: A[\boldsymbol{T}^J/\boldsymbol{\beta}^J]$,
- $\Gamma \vdash B[\boldsymbol{T}^J/\boldsymbol{\beta}^J] <: B_2$, and
- $\Gamma \vdash (l\,\boldsymbol{S}^I)^\uparrow \oslash \varepsilon_2$.

By Lemma 3.4 and Lemma 3.3(2), the required result is achieved.

Case others: Cannot happen.

(5) By induction on a derivation of $\Gamma \vdash v_1\,v_2 : B \mid \varepsilon$. We proceed by cases on the typing rule applied lastly to this derivation.

Case T_APP: Clearly.

Case T_SUB: For some B' and ε', the following are given:
- $\Gamma \vdash v_1\,v_2 : B' \mid \varepsilon'$ and
- $\Gamma \vdash B' \mid \varepsilon' <: B \mid \varepsilon$.

By the induction hypothesis, we have
- $\Gamma \vdash v_1 : A \to_{\varepsilon'} B' \mid \mathbb{0}$ and
- $\Gamma \vdash v_2 : A \mid \mathbb{0}$

for some A. By Lemma 3.12, we have $\Gamma \vdash A : \mathbf{Typ}$ and $\Gamma \vdash \mathbb{0} : \mathbf{Eff}$. Thus, ST_REFL derives $\Gamma \vdash A <: A$ and Lemma 3.3(1) derives $\Gamma \vdash \mathbb{0} \oslash \mathbb{0}$. Therefore, by ST_FUN and ST_COMP, $\Gamma \vdash A \to_{\varepsilon'} B' \mid \mathbb{0} <: A \to_\varepsilon B \mid \mathbb{0}$. Then, by T_SUB, $\Gamma \vdash v_1 : A \to_\varepsilon B \mid \mathbb{0}$.

Case others: Cannot happen.

■

Lemma 3.15 (Canonical Form).
(1) If $\emptyset \vdash v : A \to_\varepsilon B \mid \varepsilon'$, then either of the following holds:

- $v = \mathbf{fun}\,(f, x, e)$ for some f, x, and e, or
- $v = \mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J$ for some op, l, $\boldsymbol{S}^I$, and $\boldsymbol{T}^J$.

(2) If $\emptyset \vdash v : \forall \alpha : K.A^\varepsilon \mid \varepsilon'$, then $v = \Lambda \alpha : K.e$ for some e.

Proof.

(1) By induction on a derivation of $\Gamma \vdash v : A \to_\varepsilon B \mid \varepsilon'$. We proceed by cases on the typing rule applied lastly to this derivation.

    Case T_VAR: Cannot happen.

    Case T_ABS: Clearly.

    Case T_SUB: For some C, the following are given:
- $\Gamma \vdash v : C \mid \varepsilon''$ and
- $\Gamma \vdash C \mid \varepsilon'' <: A \to_\varepsilon B \mid \varepsilon'$.

    By Lemma 3.13(1), we have $C = A_1 \to_{\varepsilon_1} B_1$ for some A_1, ε_1, and B_1. By the induction hypothesis, the required result is achieved.

    Case T_OP: Clearly.

    Case others: Cannot happen.

(2) By induction on a derivation of $\Gamma \vdash v : \forall \alpha : K.A^\varepsilon \mid \varepsilon'$. We proceed by cases on the typing rule applied lastly to this derivation.

    Case T_VAR: Cannot happen.

    Case T_TABS: Clearly.

    Case T_SUB: For some B, the following are given:
- $\Gamma \vdash v : B \mid \varepsilon''$ and
- $\Gamma \vdash B \mid \varepsilon'' <: \forall \alpha : K.A^\varepsilon \mid \varepsilon'$.

    By Lemma 3.13(2), we have $B = \forall \alpha : K.{A_1}^{\varepsilon_1}$ for some A_1 and ε_1. By the induction hypothesis, the required result is achieved.

    Case others: Cannot happen.

                                                    ■

Lemma 3.16 (Inversion of Handler Typing).

  (1) If $\Gamma \vdash_\sigma h : A \Rightarrow^\varepsilon B$, then there exist some x and e_r such that $\mathbf{return}\, x \mapsto e_r \in h$ and $\Gamma, x : A \vdash e_r : B \mid \varepsilon$.

  (2) If $\Gamma \vdash_\sigma h : A \Rightarrow^\varepsilon B$ and $\mathsf{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K}^J.A' \Rightarrow B' \in \sigma$, then
- $\mathsf{op}\, \boldsymbol{\beta}^J : \boldsymbol{K}^J\, p\, k \mapsto e \in h$ *and*
- $\Gamma, \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A', k : B' \to_\varepsilon B \vdash e : B \mid \varepsilon$

  for some p, k, and e.

Proof. (1) By induction on a derivation of $\Gamma \vdash_\sigma h : A \Rightarrow^\varepsilon B$. We proceed by cases on the typing rule applied lastly to this derivation.

    Case H_RETURN: Clearly.

    Case H_OP: Clearly by the induction hypothesis.

(2) By induction on a derivation of $\Gamma \vdash_\sigma h : A \Rightarrow^\varepsilon B$. We proceed by cases on the typing rule applied lastly to this derivation.

    Case H_RETURN: Clearly because there is no operation belonging to $\{\}$.

    Case H_OP: For some h', σ', op', $\boldsymbol{\beta'}^{J'}$, $\boldsymbol{K'}^{J'}$, A'', B'', p', k', e'', the following are given:
- $h = h' \uplus \{\mathsf{op}'\, \boldsymbol{\beta'}^{J'} : \boldsymbol{K'}^{J'}\, p'\, k' \mapsto e'\}$,
- $\sigma = \sigma' \uplus \{\mathsf{op}' : \forall \boldsymbol{\beta'}^{J'} : \boldsymbol{K'}^{J'}.A'' \Rightarrow B''\}$,
- $\Gamma \vdash_{\sigma'} h' : A \Rightarrow^\varepsilon B$, and
- $\Gamma, \boldsymbol{\beta'}^{J'} : \boldsymbol{K'}^{J'}, p' : A'', k' : B'' \to_\varepsilon B \vdash e : B \mid \varepsilon$.

    If $\mathsf{op} = \mathsf{op}'$, then clearly.

    If $\mathsf{op} \neq \mathsf{op}'$, then clearly by the induction hypothesis.

                                                     ■

Lemma 3.17 (Independence of Evaluation Contexts). *If $\Gamma \vdash E[e] : A \mid \varepsilon$, then there exist some A' and ε' such that*
- $\Gamma \vdash e : A' \mid \varepsilon'$, *and*

- $\Gamma, \Gamma' \vdash E[e'] : A \mid \varepsilon$ *holds for any* e' *and* Γ' *such that* $\Gamma, \Gamma' \vdash e' : A' \mid \varepsilon'$.

Proof. By induction on a derivation of $\Gamma \vdash E[e] : A \mid \varepsilon$. We proceed by cases on the typing rule applied lastly to this derivation.

Case T_LET: If $E = \Box$, then the required result is achieved immediately.

If $E \neq \Box$, then we have

- $E = (\textbf{let } x = E' \textbf{ in } e_2)$,
- $\Gamma \vdash E'[e] : B \mid \varepsilon$, and
- $\Gamma, x : B \vdash e_2 : A \mid \varepsilon$,

for some x, E', e_2, and B. By the induction hypothesis, there exist some A' and ε' such that

- $\Gamma \vdash e : A' \mid \varepsilon'$, and
- for any e' and Γ' such that $\Gamma, \Gamma' \vdash e' : A' \mid \varepsilon'$, typing judgment $\Gamma, \Gamma' \vdash E'[e'] : B \mid \varepsilon$ is derivable.

Let e' be an expression and Γ' be a typing context such that $\Gamma, \Gamma' \vdash e' : A' \mid \varepsilon'$. Without loss of generality, we can assume $x \notin \text{dom}(\Gamma')$. The induction hypothesis result implies $\Gamma, \Gamma' \vdash E'[e'] : B \mid \varepsilon$. By Lemma 3.5 and T_LET, it suffices to show that $\vdash \Gamma, \Gamma'$, which is implied by Lemma 3.9.

Case T_SUB: For some A' and ε', given are the following:

- $\Gamma \vdash E[e] : A' \mid \varepsilon'$ and
- $\Gamma \vdash A' \mid \varepsilon' <: A \mid \varepsilon$.

By the induction hypothesis, there exist some A'' and ε'' such that

- $\Gamma \vdash e : A'' \mid \varepsilon''$, and
- for any e' and Γ' such that $\Gamma, \Gamma' \vdash e' : A'' \mid \varepsilon''$, typing judgment $\Gamma, \Gamma' \vdash E[e'] : A' \mid \varepsilon'$ is derivable.

Let e' be an expression and Γ' be a typing context such that $\Gamma, \Gamma' \vdash e' : A'' \mid \varepsilon''$. Since only ST_COMP can derive $\Gamma \vdash A' \mid \varepsilon' <: A \mid \varepsilon$, we have $\Gamma \vdash A' <: A$ and $\Gamma \vdash \varepsilon' \oslash \varepsilon$. We have $\Gamma, \Gamma' \vdash A' <: A$ and $\Gamma, \Gamma' \vdash \varepsilon' \oslash \varepsilon$ by Lemma 3.9, Lemma 3.5(2), and Lemma 3.5(3). Thus, because $\Gamma, \Gamma' \vdash E[e'] : A' \mid \varepsilon'$ by the induction hypothesis result, ST_COMP and T_SUB derive $\Gamma, \Gamma' \vdash E[e'] : A \mid \varepsilon$.

Case T_HANDLING: If $E = \Box$, then the required result is achieved immediately.

If $E \neq \Box$, then we have

- $E = \textbf{handle}_{l\,\boldsymbol{S}^N}\, E' \,\textbf{with}\, h$,
- $\Gamma \vdash E'[e] : A' \mid \varepsilon'$, and
- $(l\,\boldsymbol{S}^N)^{\uparrow} \odot \varepsilon \sim \varepsilon'$,

for some l, $\boldsymbol{S}^N$, E', h, A', and ε'. By the induction hypothesis, there exist some A'' and ε'' such that

- $\Gamma \vdash e : A'' \mid \varepsilon''$, and
- for any e' and Γ' such that $\Gamma, \Gamma' \vdash e' : A'' \mid \varepsilon''$, typing judgment $\Gamma, \Gamma' \vdash E'[e'] : A' \mid \varepsilon'$ is derivable.

Because the premises of T_HANDLING other than the typing of handled expressions are independent of the handled expressions, the required result is achieved by Lemma 3.9, Lemma 3.5, and Lemma 3.2(2).

Case others: Clearly because $E = \Box$.

$\blacksquare$

Lemma 3.18 (Progress). *If* $\emptyset \vdash e : A \mid \varepsilon$*, then one of the following holds:*

- e *is a value;*

- *There exists some expression* e' *such that* $e \longrightarrow e'$*; or*

- *There exist some* $\textsf{op}$*,* l*,* $\boldsymbol{S}^I$*,* $\boldsymbol{T}^J$*,* v*,* E*, and* n *such that* $e = E[\textsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v]$ *and* $n-\text{free}(l\,\boldsymbol{S}^I, E)$.

Proof. By induction on a derivation of $\emptyset \vdash e : A \mid \varepsilon$. We proceed by cases on the typing rule applied lastly to this derivation.

Case T_VAR: Cannot happen.

Case T_ABS: e is a value because of $e = \textbf{fun}\,(f_1, x_1, e_1)$ for some f_1, x_1, and e_1.

Case T_APP: For some v_1, v_2, and B, the following are given:

- $e = v_1\, v_2$,
- $\emptyset \vdash v_1 : B \to_\varepsilon A \mid \mathbb{0}$, and
- $\emptyset \vdash v_2 : B \mid \mathbb{0}$.

By case analysis on the result of Lemma 3.15(1) on $\emptyset \vdash v_1 : B \to_\varepsilon A \mid \mathbb{0}$.

If $v_1 = \mathbf{fun}\,(f_1, x_1, e_1)$ for some f_1, x_1, and e_1, then R_APP derives $e \longmapsto e_1[\mathbf{fun}\,(f_1, x_1, e_1)/f_1][v_2/x]$.

If $v_1 = \mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J$ for some op, l, $\boldsymbol{S}^I$, $\boldsymbol{T}^J$, then the required result is implied by Lemma 3.14(4) and the fact that $e = \square[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v_2]$.

Case T_TABS: e is a value because of $e = \Lambda\alpha : K.e_1$ for some α, K, and e_1.

Case T_TAPP: For some v, α, S, K, A_1, and ε_1, the following are given:

- $e = v\,S$,
- $A = A_1[S/\alpha]$,
- $\varepsilon = \varepsilon_1[S/\alpha]$,
- $\emptyset \vdash v : \forall\alpha : K.A_1{}^{\varepsilon_1} \mid \mathbb{0}$, and
- $\emptyset \vdash S : K$.

By Lemma 3.15(2), we have $v = \Lambda\alpha : K.e_1$ for some e_1. Thus, R_TAPP derives $e \longmapsto e_1[S/\alpha]$.

Case T_LET: For some x, e_1, e_2, and B, given are the following:

- $e = (\mathbf{let}\,x = e_1\,\mathbf{in}\,e_2)$,
- $\emptyset \vdash e_1 : B \mid \varepsilon$, and
- $x : B \vdash e_2 : A \mid \varepsilon$.

By the induction hypothesis, we proceed by cases on the following conditions:

(1) e_1 is a value,

(2) There exists some e_1' such that $e_1 \longrightarrow e_1'$,

(3) There exist some op, l, $\boldsymbol{S}^I$, $\boldsymbol{T}^J$, v, E, and n such that $e_1 = E[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v]$ and $n-\mathrm{free}(l\,\boldsymbol{S}^I, E)$.

> ***Case* (1):** R_LET derives $e \longmapsto e_2[v_1/x]$ because e_1 is a value v_1.

> ***Case* (2):** Since only E_EVAL can derive $e_1 \longrightarrow e_1'$, we have
> - $e_1 = E_1[e_{11}]$,
> - $e_1' = E_1[e_{12}]$, and
> - $e_{11} \longmapsto e_{12}$,
>
> for some E_1, e_{11}, and e_{12}. Let $E = (\mathbf{let}\,x = E_1\,\mathbf{in}\,e_2)$. E_EVAL derives $e \longrightarrow E[e_{12}]$ because of $e = E[e_{11}]$.

> ***Case* (3):** Clearly because $e = (\mathbf{let}\,x = E[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v]\,\mathbf{in}\,e_2)$ and $n-\mathrm{free}(l\,\boldsymbol{S}^I, \mathbf{let}\,x = E\,\mathbf{in}\,e_2)$.

Case T_SUB: Clearly by the induction hypothesis.

Case T_OP: e is a value because of $e = \mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J$ for some op, l, $\boldsymbol{S}^I$, and $\boldsymbol{T}^J$.

Case T_HANDLING: For some e_1, h, l, $\boldsymbol{S}^N$, $\boldsymbol{\alpha}^N$, $\boldsymbol{K}^N$, A_1, and ε_1, given are the following:

- $e = \mathbf{handle}_{l\,\boldsymbol{S}^N}\,e_1\,\mathbf{with}\,h$,
- $\emptyset \vdash e_1 : A_1 \mid \varepsilon_1$,
- $l :: \forall\boldsymbol{\alpha}^N : \boldsymbol{K}^N.\sigma \in \Xi$,
- $\emptyset \vdash \boldsymbol{S}^N : \boldsymbol{K}^N$,
- $\emptyset \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : A_1 \Rightarrow^\varepsilon A$, and
- $(l\,\boldsymbol{S}^N)^\uparrow \odot \varepsilon \sim \varepsilon_1$.

By the induction hypothesis, we proceed by cases on the following conditions:

(1) e_1 is a value,

(2) There exists some e_1' such that $e_1 \longrightarrow e_1'$,

(3) There exist some op', l', $\boldsymbol{S'}^{N'}$, $\boldsymbol{T}^J$, v, E, and n such that $e_1 = E[\mathsf{op}'_{l'\,\boldsymbol{S'}^{N'}}\,\boldsymbol{T}^J\,v]$ and $n-\mathrm{free}(l'\,\boldsymbol{S'}^{N'}, E)$.

***Case* (1):** By Lemma 3.16(1), there exists some x and e_r such that **return** $x \mapsto e_r \in h$. Thus, R_HANDLE1 derives $e \longmapsto e_r[v_1/x]$ because e_1 is a value v_1.

***Case* (2):** Since only E_EVAL can derive $e_1 \longrightarrow e_1'$, we have

- $e_1 = E_1[e_{11}]$,
- $e_1' = E_1[e_{12}]$, and
- $e_{11} \longmapsto e_{12}$,

for some E, e_{11}, and e_{12}. Let $E = \mathbf{handle}_{l\,\boldsymbol{S}^N}\,E_1\,\mathbf{with}\,h$. E_EVAL derives $e \longrightarrow E[e_{12}]$ because of $e = E[e_{11}]$.

***Case* (3):** If $l\,\boldsymbol{S}^N \neq l'\,\boldsymbol{S'}^{N'}$, then $e = (\mathbf{handle}_{l\,\boldsymbol{S}^N}\,E\,\mathbf{with}\,h)[\mathsf{op}_{l'\,\boldsymbol{S'}^{N'}}\,\boldsymbol{T}^J\,v]$ and $n{-}\mathrm{free}(l'\,\boldsymbol{S'}^{N'}, \mathbf{handle}_{l\,\boldsymbol{S}^N}\,E\,\mathbf{with}\,h)$.

If $l\,\boldsymbol{S}^N = l'\,\boldsymbol{S'}^{N'}$, then by Lemma 3.17 and 3.14(4), we have

- $l' :: \forall\boldsymbol{\alpha'}^{N'} : \boldsymbol{K'}^{N'}.\sigma' \in \Xi$ and
- $\mathsf{op}' : \forall\boldsymbol{\beta'}^J : \boldsymbol{K_0'}^J.A' \Rightarrow B' \in \sigma'[\boldsymbol{S'}^{N'}/\boldsymbol{\alpha'}^{N'}]$,

for some $\boldsymbol{\alpha'}^{N'}$, $\boldsymbol{K'}^{N'}$, σ', $\boldsymbol{\beta'}^J$, A', and B'. Therefore, since $l\,\boldsymbol{S}^N = l'\,\boldsymbol{S'}^{N'}$, we have

- $\sigma = \sigma'$,
- $\boldsymbol{\alpha}^N = \boldsymbol{\alpha'}^{N'}$, and
- $\boldsymbol{K}^N = \boldsymbol{K_0}^{N'}$.

By $\emptyset \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : A_1 \Rightarrow^\varepsilon A$ and $\mathsf{op}' : \forall\boldsymbol{\beta'}^J : \boldsymbol{K_0'}^J.A' \Rightarrow B' \in \sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]$ and Lemma 3.16(2), we have

$$\mathsf{op}'\,\boldsymbol{\beta'}^J : \boldsymbol{K_0'}^J\,p\,k \mapsto e' \in h$$

for some p, k, and e'. If $n = 0$, the evaluation of e proceeds by R_HANDLE2. Otherwise, there exists some m such that $n = m + 1$ and $m{-}\mathrm{free}(l\,\boldsymbol{S}^N, \mathbf{handle}_{l\,\boldsymbol{S}^N}\,E\,\mathbf{with}\,h)$.

∎

Lemma 3.19 (Preservation in Reduction)**.** *If $\emptyset \vdash e : A \mid \varepsilon$ and $e \longmapsto e'$, then $\emptyset \vdash e' : A \mid \varepsilon$.*

Proof. By induction on a derivation of $\Gamma \vdash e : A \mid \varepsilon$. We proceed by cases on the typing rule applied lastly to this derivation.

***Case* T_VAR:** There is no e' such that $e \longmapsto e'$.

***Case* T_ABS:** There is no e' such that $e \longmapsto e'$.

***Case* T_APP:** Since only R_APP can derive $e \longmapsto e'$, we have

- $e = (\mathbf{fun}\,(f_1, x_1, e_1))\,v_2$,
- $\emptyset \vdash \mathbf{fun}\,(f_1, x_1, e_1) : A_1 \to_\varepsilon A \mid \mathbb{0}$,
- $\emptyset \vdash v_2 : A_1 \mid \mathbb{0}$, and
- $e' = e_1[\mathbf{fun}\,(f_1, x_1, e_1)/f_1][v_2/x_1]$

for some f_1, x_1, e_1, v_2, and A_1. By Lemma 3.14(2), we have

- $f_1 : A_2 \to_{\varepsilon_2} B_2, x_1 : A_2 \vdash e_1 : B_2 \mid \varepsilon_2$ and
- $\emptyset \vdash A_2 \to_{\varepsilon_2} B_2 <: A_1 \to_\varepsilon A$.

for some A_2, ε_2, and B_2. Thus, T_ABS derives $\emptyset \vdash \mathbf{fun}\,(f_1, x_1, e_1) : A_2 \to_{\varepsilon_2} B_2 \mid \mathbb{0}$. By Lemma 3.13(1), we have

- $\emptyset \vdash A_1 <: A_2$,
- $\emptyset \vdash B_2 <: A$, and
- $\emptyset \vdash \varepsilon_2 \leqslant \varepsilon$.

By Lemma 3.9 and Lemma 3.5(3), we have $f_1 : A_2 \to_{\varepsilon_2} B_2, x_1 : A_2 \vdash B_2 <: A$. Because Lemma 3.5(2), ST_COMP derives $f_1 : A_2 \to_{\varepsilon_2} B_2, x_1 : A_2 \vdash B_2 \mid \varepsilon_2 <: A \mid \varepsilon$. Therefore, T_SUB derives $f_1 : A_2 \to_{\varepsilon_2} B_2, x_1 : A_2 \vdash e_1 : A \mid \varepsilon$. Since T_SUB derives $\emptyset \vdash v_2 : A_2 \mid \mathbb{0}$, Lemma 3.7(5) makes $\emptyset \vdash e_1[\mathbf{fun}\,(f_1, x_1, e_1)/f_1][v_2/x_1] : A \mid \varepsilon$ hold as required.

***Case* T_TABS:** There is no e' such that $e \longmapsto e'$.

Case T_TAPP**:** Since only R_TAPP derives $e \longmapsto e'$, we have

- $e = (\Lambda \alpha : K.e_1) \, S$,
- $A = A_1[S/\alpha]$,
- $\varepsilon = \varepsilon_1[S/\alpha]$,
- $\emptyset \vdash \Lambda \alpha : K.e_1 : \forall \alpha : K.A_1^{\varepsilon_1} \mid \mathbb{0}$,
- $\emptyset \vdash S : K$, and
- $e' = e_1[S/\alpha]$

for some α, K, e_1, S, A_1, and ε_1. By Lemma 3.14(3), we have $\alpha : K \vdash e_1 : A_1 \mid \varepsilon_1$. Thus, Lemma 3.10(5) makes $\emptyset \vdash e_1[S/\alpha] : A_1[S/\alpha] \mid \varepsilon_1[S/\alpha]$ hold as required.

Case T_LET**:** Since only R_LET derives $e \longmapsto e'$, we have

- $e = (\mathbf{let} \, x = v \, \mathbf{in} \, e_1)$,
- $\emptyset \vdash v : B \mid \varepsilon$,
- $x : B \vdash e_1 : A \mid \varepsilon$, and
- $e' = e_1[v/x]$

for some x, v, e_1, and B. By Lemma 3.14(1) and Lemma 3.7(5), we have $\emptyset \vdash e_1[v/x] : A \mid \varepsilon$ as required.

Case T_SUB**:** For some A' and ε', we have

- $\emptyset \vdash e : A' \mid \varepsilon'$ and
- $\emptyset \vdash A' \mid \varepsilon' <: A \mid \varepsilon$.

By the induction hypothesis, we have $\emptyset \vdash e' : A' \mid \varepsilon'$. Thus, T_SUB derives $\emptyset \vdash e' : A \mid \varepsilon$ as required.

Case T_OP**:** There is no e' such that $e \longmapsto e'$.

Case T_HANDLING**:** We proceed by cases on the derivation rule which derives $e \longmapsto e'$.

    Case R_HANDLE1**:** We have

- $e = \mathbf{handle}_{l \, \boldsymbol{S}^I} \, v \, \mathbf{with} \, h$,
- $\mathbf{return} \, x \mapsto e_r \in h$,
- $\emptyset \vdash v : B \mid \varepsilon'$,
- $l :: \forall \boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi$,
- $\Gamma \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$,
- $\emptyset \vdash_{\sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} h : B \Rightarrow^\varepsilon A$,
- $(l \, \boldsymbol{S}^I)^\uparrow \odot \varepsilon \sim \varepsilon'$, and
- $e' = e_r[v/x]$

for some l, $\boldsymbol{S}^I$, $\boldsymbol{\alpha}^I$, $\boldsymbol{K}^I$, σ, v, h, B, and ε'. By $\emptyset \vdash_{\sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} h : B \Rightarrow^\varepsilon A$ and $\mathbf{return} \, x \mapsto e_r \in h$ and Lemma 3.16(1), we have

$$x : B \vdash e_r : A \mid \varepsilon.$$

By Lemma 3.14(1), we have $\emptyset \vdash v : B \mid \mathbb{0}$. Thus, Lemma 3.7(5) makes $\emptyset \vdash e_r[v/x] : A \mid \varepsilon$ hold as required.

    Case R_HANDLE2**:** We have

- $e = \mathbf{handle}_{l \, \boldsymbol{S}^N} \, E[\mathsf{op}_0 {}_{l \, \boldsymbol{S}^N} \, \boldsymbol{T}^J \, v] \, \mathbf{with} \, h$,
- $l :: \forall \boldsymbol{\alpha}^N : \boldsymbol{K}^N.\sigma \in \Xi$,
- $\emptyset \vdash \boldsymbol{S}^N : \boldsymbol{K}^N$,
- $\mathsf{op}_0 \, \boldsymbol{\beta_0}^J : \boldsymbol{K_0}^J \, p_0 \, k_0 \mapsto e_0 \in h$,
- $0{-}\mathrm{free}(l \, \boldsymbol{S}^N, E)$,
- $\emptyset \vdash E[\mathsf{op}_0 {}_{l \, \boldsymbol{S}^N} \, \boldsymbol{T}^J \, v] : B \mid \varepsilon'$,
- $\emptyset \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : B \Rightarrow^\varepsilon A$,
- $(l \, \boldsymbol{S}^N)^\uparrow \odot \varepsilon \sim \varepsilon'$, and
- $e' = e_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J][v/p_0][\lambda z.\mathbf{handle}_{l \, \boldsymbol{S}^N} \, E[z] \, \mathbf{with} \, h/k_0]$

for some l, $\boldsymbol{S}^N$, E, op_0, $\boldsymbol{T}^J$, v, h, $\boldsymbol{\alpha}^N$, $\boldsymbol{K}^N$, σ, $\boldsymbol{\beta_0}^J$, $\boldsymbol{K_0}^J$, p_0, k_0, e_0, B, and ε'. By Lemma 3.17, there exist some B_1 and ε_1 such that

- $\emptyset \vdash \mathsf{op}_{0\,l\,\boldsymbol{S}^N}\,\boldsymbol{T}^J\,v : B_1 \mid \varepsilon_1$, and
- for any e'' and Γ'', if $\Gamma'' \vdash e'' : B_1 \mid \varepsilon_1$, then $\Gamma'' \vdash E[e''] : B \mid \varepsilon'$.

By Lemma 3.14(5), we have $\emptyset \vdash \mathsf{op}_{0\,l\,\boldsymbol{S}^N}\,\boldsymbol{T}^J : A_1 \rightarrow_{\varepsilon_1} B_1 \mid \mathbb{0}$ and $\emptyset \vdash v : A_1 \mid \mathbb{0}$ for some A_1. By Lemma 3.14(4) and 3.16(2), we have

- $\mathsf{op}_0 : \forall \boldsymbol{\beta_0}^J : \boldsymbol{K_0}^J.A_0 \Rightarrow B_0 \in \sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]$,
- $\emptyset \vdash \boldsymbol{S}^N : \boldsymbol{K}^N$,
- $\emptyset \vdash \boldsymbol{T}^J : \boldsymbol{K_0}^J$,
- $\emptyset \vdash A_1 <: A_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J]$,
- $\emptyset \vdash B_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J] <: B_1$, and
- $\emptyset \vdash (l\,\boldsymbol{S}^N)^\uparrow \circledS \varepsilon_1$.

for some A_0 and B_0. Thus, T_SUB with $\emptyset \vdash \mathbb{0} \circledS \mathbb{0}$ implied by Lemma 3.3 derives

$$\emptyset \vdash v : A_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J] \mid \mathbb{0}.$$

By Lemma 3.11, we have $\emptyset \vdash B_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J] : \mathbf{Typ}$. Thus, C_VAR derives $\vdash z : B_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J]$. By $\emptyset \vdash \mathbb{0} : \mathbf{Eff}$, $\emptyset \vdash \varepsilon_1 : \mathbf{Eff}$ implied by Lemma 3.12, and $\mathbb{0} \odot \varepsilon_1 \sim \varepsilon_1$, we have $\emptyset \vdash \mathbb{0} \circledS \varepsilon_1$. Since T_VAR and T_SUB derives $z : B_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J] \vdash z : B_1 \mid \varepsilon_1$, we have

$$z : B_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J] \vdash \mathbf{handle}_{l\,\boldsymbol{S}^N}\,E[z]\,\mathbf{with}\,h : A \mid \varepsilon$$

by the result of Lemma 3.17, Lemma 3.5, and T_HANDLING. Thus, T_ABS derives

$$\emptyset \vdash \lambda z.\mathbf{handle}_{l\,\boldsymbol{S}^N}\,E[z]\,\mathbf{with}\,h : B_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J] \rightarrow_\varepsilon A \mid \mathbb{0}.$$

Since

$$\boldsymbol{\beta_0}^J : \boldsymbol{K_0}^J, p_0 : A_0, k_0 : B_0 \rightarrow_\varepsilon A \vdash e_0 : A \mid \varepsilon$$

by $\emptyset \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : B \Rightarrow^\varepsilon A$ and $\mathsf{op}_0 : \forall \boldsymbol{\beta_0}^J : \boldsymbol{K_0}^J.A_0 \Rightarrow B_0 \in \sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]$ and Lemma 3.16(2), Lemma 3.10(5) and Lemma 3.7(5) imply

$$\emptyset \vdash e_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J][v/p_0][\lambda z.\mathbf{handle}_{l\,\boldsymbol{S}^N}\,E[z]\,\mathbf{with}\,h/k_0] : A \mid \varepsilon$$

as required. ∎

Lemma 3.20 (Preservation). *If $\emptyset \vdash e : A \mid \varepsilon$ and $e \longrightarrow e'$, then $\emptyset \vdash e' : A \mid \varepsilon$.*

Proof. Since only E_EVAL derives $e \longrightarrow e'$, we have

- $e = E[e_1]$,

- $e' = E[e_2]$, and

- $e_1 \longmapsto e_2$.

By Lemma 3.17, there exist some A' and ε' such that

- $\emptyset \vdash e_1 : A' \mid \varepsilon'$, and

- for any e_1' and Γ', if $\Gamma' \vdash e_1' : A' \mid \varepsilon'$, then $\Gamma' \vdash E[e_1'] : A \mid \varepsilon$.

By Lemma 3.19, we have $\emptyset \vdash e_2 : A' \mid \varepsilon'$. Thus, $\emptyset \vdash E[e_2] : A \mid \varepsilon$ holds as required. ∎

Lemma 3.21. *If $n-\mathrm{free}(L, E)$, then $n = \mathsf{0}$.*

Proof. Straightforward by the induction on the derivation of $n-\mathrm{free}(L, E)$. ∎

Lemma 3.22. *If $\Gamma \vdash E[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v] : A \mid \varepsilon$ and $n-\mathrm{free}(l\,\boldsymbol{S}^I, E)$, then $(l\,\boldsymbol{S}^I)^\uparrow \circledS \varepsilon$.*

Proof. By induction on a derivation of $\Gamma \vdash E[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v] : A \mid \varepsilon$. We proceed by case analysis on the typing rule applied lastly to this derivation.

Case T_APP: For some B, we have

- $E = \square$,
- $\Gamma \vdash \mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J : B \rightarrow_\varepsilon A \mid \mathbb{0}$, and

- $\Gamma \vdash v : B \mid \mathbb{0}$.

By Lemma 3.14(4), we have $\Gamma \vdash (l \, \boldsymbol{S}^I)^{\uparrow} \oslash \varepsilon$. Thus, the required result is achieved.

Case T_LET: For some x, E_1, e, and B, we have

- $E = (\mathbf{let}\; x = E_1 \;\mathbf{in}\; e)$,
- $\Gamma \vdash E_1[\mathsf{op}_{l \, \boldsymbol{S}^I} \boldsymbol{T}^J \, v] : B \mid \varepsilon$, and
- $\Gamma, x : B \vdash e : A \mid \varepsilon$.

By $n-\mathrm{free}(l \, \boldsymbol{S}^I, E_1)$ and the induction hypothesis, we have $(l \, \boldsymbol{S}^I)^{\uparrow} \oslash \varepsilon$ as required.

Case T_SUB: For some A' and ε', we have

- $\Gamma \vdash E[\mathsf{op}_{l \, \boldsymbol{S}^I} \boldsymbol{T}^J \, v] : A' \mid \varepsilon'$ and
- $\Gamma \vdash A' \mid \varepsilon' <: A \mid \varepsilon$.

Since only ST_COMP can derive $\Gamma \vdash A' \mid \varepsilon' <: A \mid \varepsilon$, we have $\Gamma \vdash \varepsilon' \oslash \varepsilon$. By the induction hypothesis, we have $(l \, \boldsymbol{S}^I)^{\uparrow} \oslash \varepsilon'$. By the associativity of $\odot$, we have $(l \, \boldsymbol{S}^I)^{\uparrow} \oslash \varepsilon$ as required.

Case T_HANDLING: For some l', $\boldsymbol{S'}^{I'}$, E_1, h, B, and ε', we have

- $E = \mathbf{handle}_{l' \, \boldsymbol{S'}^{I'}} E_1 \;\mathbf{with}\; h$,
- $\Gamma \vdash E_1[\mathsf{op}_{l \, \boldsymbol{S}^I} \boldsymbol{T}^J \, v] : B \mid \varepsilon'$, and
- $(l' \, \boldsymbol{S'}^{I'})^{\uparrow} \odot \varepsilon \sim \varepsilon'$.

By Lemma 3.21, we have $l \, \boldsymbol{S}^I \neq l' \, \boldsymbol{S'}^{I'}$ and $0-\mathrm{free}(l \, \boldsymbol{S}^I, E_1)$. By the induction hypothesis, we have $(l \, \boldsymbol{S}^I)^{\uparrow} \oslash \varepsilon'$. Thus, safety condition (2) makes $(l \, \boldsymbol{S}^I)^{\uparrow} \oslash \varepsilon$ hold as required.

Case others: Cannot happen.

∎

Lemma 3.23 (Effect Safety). *If* $\Gamma \vdash E[\mathsf{op}_{l \, \boldsymbol{S}^I} \boldsymbol{T}^J \, v] : A \mid \varepsilon$ *and* $n-\mathrm{free}(l \, \boldsymbol{S}^I, E)$, *then* $\varepsilon \not\sim \mathbb{0}$.

Proof. Assume that $\varepsilon \sim \mathbb{0}$. By Lemma 3.22, we have $(l \, \boldsymbol{S}^I)^{\uparrow} \oslash \varepsilon$. Therefore, we have $(l \, \boldsymbol{S}^I)^{\uparrow} \odot \varepsilon' \sim \mathbb{0}$ for some ε'. However, this is contradictory with safety condition (1). ∎

Theorem 3.24 (Type and Effect Safety). *If* $\emptyset \vdash e : A \mid \mathbb{0}$ *and* $e \longrightarrow^* e'$ *and* $e' \not\longrightarrow$, *then* e' *is a value.*

Proof. By Lemma 3.20, $\emptyset \vdash e' : A \mid \mathbb{0}$ (it is easy to extend Lemma 3.20 to multi-step evaluation). By Lemma 3.23, $e' \neq E[\mathsf{op}_{l \, \boldsymbol{S}^N} \boldsymbol{T}^J \, v]$ for any E, l, $\boldsymbol{S}^N$, op, $\boldsymbol{T}^J$, and v such that $n-\mathrm{free}(l \, \boldsymbol{S}^I, E)$ for some n. Thus, by Lemma 3.18, we have the fact that e' is a value. ∎

3.2 Properties with Shallow Handlers

This section assumes that the safety conditions in Definition 1.45 hold.

Lemma 3.25 (Weakening). *Suppose that* $\vdash \Gamma_1, \Gamma_2$ *and* $\mathrm{dom}(\Gamma_2) \cap \mathrm{dom}(\Gamma_3) = \emptyset$.

(1) If $\vdash \Gamma_1, \Gamma_3$*, then* $\vdash \Gamma_1, \Gamma_2, \Gamma_3$.

(2) If $\Gamma_1, \Gamma_3 \vdash S : K$*, then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash S : K$.

(3) If $\Gamma_1, \Gamma_3 \vdash A <: B$*, then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash A <: B$.

(4) If $\Gamma_1, \Gamma_3 \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2$*, then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2$.

(5) If $\Gamma_1, \Gamma_3 \vdash e : A \mid \varepsilon$*, then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash e : A \mid \varepsilon$.

(6) If $\Gamma_1, \Gamma_3 \vdash_\sigma h : A^{\varepsilon'} \Rightarrow^\varepsilon B$*, then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash_\sigma h : A^{\varepsilon'} \Rightarrow^\varepsilon B$.

Proof.

(1)(2) Similarly to Lemma 3.5(1) and (2).

(3)(4) Similarly to Lemma 3.5(3) and (4).

(5)(6) By mutual induction on derivations of the judgments. We proceed by case analysis on the rule applied lastly to the derivation.

Case T_SHANDLING**:** For some N, e', A', ε', l, $\boldsymbol{S}^N$, $\boldsymbol{K}^N$, h, and σ, the following are given:

- $e = \mathbf{handle}_{l\,\boldsymbol{S}^N}\, e' \,\mathbf{with}\, h$,
- $\Gamma_1, \Gamma_3 \vdash e' : A' \mid \varepsilon'$,
- $l :: \forall \boldsymbol{\alpha}^N : \boldsymbol{K}^N.\sigma \in \Xi$,
- $\Gamma_1, \Gamma_3 \vdash \boldsymbol{S}^N : \boldsymbol{K}^N$,
- $\Gamma_1, \Gamma_3 \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : A'^{\varepsilon'} \Rightarrow^\varepsilon A$, and
- $(l\,\boldsymbol{S}^N)^\uparrow \odot \varepsilon \sim \varepsilon'$.

By the induction hypothesis and case (2), we have

- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash e' : A' \mid \varepsilon'$,
- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \boldsymbol{S}^N : \boldsymbol{K}^N$, and
- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : A'^{\varepsilon'} \Rightarrow^\varepsilon A$.

Thus, T_SHANDLING derives

$$\Gamma_1, \Gamma_2, \Gamma_3 \vdash \mathbf{handle}_{l\,\boldsymbol{S}^N}\, e \,\mathbf{with}\, h : A \mid \varepsilon.$$

Case SH_RETURN**:** Without loss of generality, we can choose x such that $x \notin \mathrm{dom}(\Gamma_2)$. For some e_r, the following are given:

- $h = \{\,\mathbf{return}\, x \mapsto e_r\,\}$,
- $\sigma = \{\}$,
- $\Gamma_1, \Gamma_3, x : A \vdash e_r : B \mid \varepsilon$, and
- $\Gamma_1, \Gamma_3 \vdash \varepsilon' : \mathbf{Eff}$.

By the induction hypothesis, we have $\Gamma_1, \Gamma_2, \Gamma_3, x : A \vdash e_r : B \mid \varepsilon$. By Lemma 3.25(2), we have $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \varepsilon' : \mathbf{Eff}$. Thus, SH_RETURN derives $\Gamma_1, \Gamma_2, \Gamma_3 \vdash_{\{\}} \{\,\mathbf{return}\, x \mapsto e_r\,\} : A^{\varepsilon'} \Rightarrow^\varepsilon B$.

Case SH_OP**:** Without loss of generality, we can choose $\boldsymbol{\beta}^J$ and p and k such that:

- $\{\boldsymbol{\beta}^J\} \cap \mathrm{dom}(\Gamma_2) = \emptyset$,
- $p \notin \mathrm{dom}(\Gamma_2)$, and
- $k \notin \mathrm{dom}(\Gamma_2)$.

For some h', σ', op, A', B', and e, the following are given:

- $h = h' \uplus \{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\, p\, k \mapsto e\}$,
- $\sigma = \sigma' \uplus \{\mathsf{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K}^J.A' \Rightarrow B'\}$,
- $\Gamma_1, \Gamma_3 \vdash_{\sigma'} h' : A^{\varepsilon'} \Rightarrow^\varepsilon B$, and
- $\Gamma_1, \Gamma_3, \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A', k : B' \rightarrow_{\varepsilon'} B \vdash e : B \mid \varepsilon$.

By the induction hypothesis, we have

- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash_{\sigma'} h' : A^{\varepsilon'} \Rightarrow^\varepsilon B$ and
- $\Gamma_1, \Gamma_2, \Gamma_3, \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A', k : B' \rightarrow_{\varepsilon'} B \vdash e : B \mid \varepsilon$.

Thus, SH_OP derives $\Gamma_1, \Gamma_2, \Gamma_3 \vdash_\sigma h' \uplus \{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\, p\, k \mapsto e\} : A^{\varepsilon'} \Rightarrow^\varepsilon B$.

Case others**:** Similarly to Lemma 3.5(5) and (6).

$\blacksquare$

Lemma 3.26 (Substitution of values). *Suppose that* $\Gamma_1 \vdash v : A \mid \mathbb{0}$.

(1) If $\vdash \Gamma_1, x : A, \Gamma_2$, *then* $\vdash \Gamma_1, \Gamma_2$.

(2) If $\Gamma_1, x : A, \Gamma_2 \vdash S : K$, *then* $\Gamma_1, \Gamma_2 \vdash S : K$.

(3) If $\Gamma_1, x : A, \Gamma_2 \vdash B <: C$, *then* $\Gamma_1, \Gamma_2 \vdash B <: C$.

(4) If $\Gamma_1, x : A, \Gamma_2 \vdash B_1 \mid \varepsilon_1 <: B_2 \mid \varepsilon_2$, *then* $\Gamma_1, \Gamma_2 \vdash B_1 \mid \varepsilon_1 <: B_2 \mid \varepsilon_2$.

(5) If $\Gamma_1, x : A, \Gamma_2 \vdash e : B \mid \varepsilon$, *then* $\Gamma_1, \Gamma_2 \vdash e[v/x] : B \mid \varepsilon$.

(6) If $\Gamma_1, x : A, \Gamma_2 \vdash_\sigma h : B^{\varepsilon'} \Rightarrow^\varepsilon C$, *then* $\Gamma_1, \Gamma_2 \vdash_\sigma h[v/x] : B^{\varepsilon'} \Rightarrow^\varepsilon C$.

Proof.(1)(2) Similarly to Lemma 3.7(1) and (2).

(3)(4) Similarly to Lemma 3.7(3) and (4).

(5)(6) By mutual induction on derivations of the judgments. We proceed by case analysis on the rule applied lastly to the derivation.

Case T_SHANDLING: For some N, e', A', ε', l, $\boldsymbol{S}^N$, $\boldsymbol{\alpha}^N$, $\boldsymbol{K}^N$, h, and σ, the following are given:
- $e = \mathbf{handle}_{l\,\boldsymbol{S}^N}\, e'\,\mathbf{with}\, h$,
- $\Gamma_1, x : A, \Gamma_2 \vdash e' : A' \mid \varepsilon'$,
- $l :: \forall \boldsymbol{\alpha}^N : \boldsymbol{K}^N.\sigma \in \Xi$,
- $\Gamma_1, x : A, \Gamma_2 \vdash \boldsymbol{S}^N : \boldsymbol{K}^N$,
- $\Gamma_1, x : A, \Gamma_2 \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : A'^{\varepsilon'} \Rightarrow^\varepsilon B$, and
- $(l\,\boldsymbol{S}^N)^\uparrow \odot \varepsilon \sim \varepsilon'$.

By the induction hypothesis and case (2), we have
- $\Gamma_1, \Gamma_2 \vdash e'[v/x] : A' \mid \varepsilon'$,
- $\Gamma_1, \Gamma_2 \vdash \boldsymbol{S}^N : \boldsymbol{K}^N$, and
- $\Gamma_1, \Gamma_2 \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h[v/x] : A'^{\varepsilon'} \Rightarrow^\varepsilon A$.

Thus, T_SHANDLING derives

$$\Gamma_1, \Gamma_2 \vdash \mathbf{handle}_{l\,\boldsymbol{S}^N}\, e'[v/x]\,\mathbf{with}\, h[v/x] : B \mid \varepsilon.$$

Case SH_RETURN: Without loss of generality, we can choose y such that $y \neq x$ and $y \notin \mathrm{FV}(v)$. For some e_r, the following are given:
- $h = \{\mathbf{return}\, y \mapsto e_r\}$,
- $\sigma = \{\}$,
- $\Gamma_1, x : A, \Gamma_2, y : B \vdash e_r : C \mid \varepsilon$, and
- $\Gamma_1, x : A, \Gamma_2 \vdash \varepsilon' : \mathbf{Eff}$.

By the induction hypothesis, we have $\Gamma_1, \Gamma_2, y : B \vdash e_r[v/x] : C \mid \varepsilon$. By Lemma 3.26(2), we have $\Gamma_1, \Gamma_2 \vdash \varepsilon' : \mathbf{Eff}$. Thus, SH_RETURN derives

$$\Gamma_1, \Gamma_2 \vdash_{\{\}} \{\mathbf{return}\, y \mapsto e_r[v/x]\} : B^{\varepsilon'} \Rightarrow^\varepsilon C.$$

Case SH_OP: Without loss of generality, we can choose $\boldsymbol{\beta}^J$ and p and k such that:
- $p \neq x$,
- $k \neq x$,
- $p \notin \mathrm{FV}(v)$,
- $k \notin \mathrm{FV}(k)$, and
- $\{\boldsymbol{\beta}^J\} \cap \mathrm{FTV}(v) = \emptyset$.

For some h', σ', op, A', B', and e, the following are given:
- $h = h' \uplus \{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\, p\, k \mapsto e\}$,
- $\sigma = \sigma' \uplus \{\mathsf{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K}^J.A' \Rightarrow B'\}$,
- $\Gamma_1, x : A, \Gamma_2 \vdash_{\sigma'} h' : B^{\varepsilon'} \Rightarrow^\varepsilon C$, and
- $\Gamma_1, x : A, \Gamma_2, \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A', k : B' \to_{\varepsilon'} C \vdash e : C \mid \varepsilon$.

By the induction hypothesis, we have
- $\Gamma_1, \Gamma_2 \vdash_{\sigma'} h'[v/x] : A^{\varepsilon'} \Rightarrow^\varepsilon B$ and
- $\Gamma_1, \Gamma_2, \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A', k : B' \to_{\varepsilon'} B \vdash e[v/x] : B \mid \varepsilon$.

Thus, SH_OP derives

$$\Gamma_1, \Gamma_2 \vdash_\sigma h'[v/x] \uplus \{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\, p\, k \mapsto e[v/x]\} : B^{\varepsilon'} \Rightarrow^\varepsilon C$$

.

Case others: Similarly to Lemma 3.7(5) and (6).

∎

Lemma 3.27 (Substitution of Typelikes). *Suppose that* $\Gamma_1 \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$.
(1) If $\vdash \Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2$, *then* $\vdash \Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

(2) If $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash T : K$, *then* $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash T[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : K$.

(3) If $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash A <: B$, then $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] <: B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

(4) If $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2$, then $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash A_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] <: A_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

(5) If $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash e : A \mid \varepsilon$, then $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash e[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

(6) If $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash_\sigma h : A^{\varepsilon'} \Rightarrow^\varepsilon B$, then $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash_{\sigma[\boldsymbol{S}/\boldsymbol{\alpha}]} h[\boldsymbol{S}/\boldsymbol{\alpha}] : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^{\varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} \Rightarrow^{\varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

Proof.(1)(2) Similarly to Lemma 3.10(1) and (2).

(3)(4) Similarly to Lemma 3.10(3) and (4).

(5)(6) By mutual induction on derivations of the judgments. We proceed by case analysis on the rule applied lastly to the derivation.

Case T_SHANDLING: For some N, e', A', ε', l, $\boldsymbol{S_0}^N$, $\boldsymbol{\alpha_0}^N$, $\boldsymbol{K_0}^N$, h, and σ, the following are given:

- $e = \mathbf{handle}_{l\, \boldsymbol{S_0}^N}\, e'\, \mathbf{with}\, h$,
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash e' : A' \mid \varepsilon'$,
- $l :: \forall \boldsymbol{\alpha_0}^N : \boldsymbol{K_0}^N.\sigma \in \Xi$,
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash \boldsymbol{S_0}^N : \boldsymbol{K_0}^N$,
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash_{\sigma[\boldsymbol{S_0}^N/\boldsymbol{\alpha_0}^N]} h : A'^{\varepsilon'} \Rightarrow^\varepsilon A$, and
- $(l\,\boldsymbol{S_0}^N)^\uparrow \odot \varepsilon \sim \varepsilon'$.

By the induction hypothesis, case (2), and that a typelike substitution is homomorphism for $\odot$ and $\sim$, we have

- $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash e'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : A'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$,
- $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash \boldsymbol{S_0}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^N : \boldsymbol{K_0}^N$,
- $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash_{\sigma[\boldsymbol{S_0}^N/\boldsymbol{\alpha_0}^N][\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} h[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : A'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^{\varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} \Rightarrow^{\varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$, and
- $(l\,\boldsymbol{S_0}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^N)^\uparrow \odot \varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \sim \varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

Now, because we can assume that

- $\{\boldsymbol{\alpha}^I\} \cap \{\boldsymbol{\alpha_0}^N\} = \emptyset$ and
- $\{\boldsymbol{\alpha_0}^N\} \cap \mathrm{FTV}(\boldsymbol{S}^I) = \emptyset$

without loss of generality, we have

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash_{\sigma[\boldsymbol{S_0}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^N/\boldsymbol{\alpha_0}^N]} h[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : A'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^{\varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} \Rightarrow^{\varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I].$$

Thus, T_SHANDLING derives

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash \mathbf{handle}_{l\, \boldsymbol{S_0}[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^N}\, e[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]\, \mathbf{with}\, h[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I].$$

Case SH_RETURN: For some x and e_r, the following are given:

- $h = \{\mathbf{return}\, y \mapsto e_r\}$,
- $\sigma = \{\}$,
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2, x : A \vdash e_r : B \mid \varepsilon$, and
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash \varepsilon' : \mathbf{Eff}$.

By the induction hypothesis, we have

- $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I], x : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash e_r[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

By Lemma 3.27(2), we have

- $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash \varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : \mathbf{Eff}$.

Thus, SH_RETURN derives

$$\Gamma_1, \Gamma_2 \vdash_{\{\}} \{\mathbf{return}\, x \mapsto e_r[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]\} : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^{\varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} \Rightarrow^{\varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I].$$

Case SH_OP: Without loss of generality, we can choose $\boldsymbol{\beta}^J$ such that:

- $\{\boldsymbol{\beta}^J\} \cap \{\boldsymbol{\alpha}^I\} = \emptyset$ and

– $\{\boldsymbol{\beta}^J\} \cap \mathrm{FTV}(\boldsymbol{S}^I) = \emptyset$.

For some h', σ', op, A', B', and e, the following are given:

– $h = h' \uplus \{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e\}$,
– $\sigma = \sigma' \uplus \{\mathsf{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K}^J.A' \Rightarrow B'\}$,
– $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash_{\sigma'} h' : A'^{\varepsilon'} \Rightarrow^\varepsilon B$, and
– $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2, \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A', k : B' \to_{\varepsilon'} B \vdash e : B \mid \varepsilon$.

By the induction hypothesis and Definition 1.10, we have

– $\sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] = \sigma'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \uplus \{\mathsf{op} : \forall \boldsymbol{\beta}^J : \boldsymbol{K}^J.A'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \Rightarrow B'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]\}$,
– $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash_{\sigma'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} h'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^{\varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} \Rightarrow^{\varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$, and
– $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I], \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I], k : B'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \to_{\varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash e[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

Thus, SH_OP derives

$$\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash_{\sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} h'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \uplus \{\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]\} : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]^{\varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} \Rightarrow^{\varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I].$$

***Case* others:** Similarly to Lemma 3.10(5) and (6). ∎

Lemma 3.28 (Well-formedness of contexts in typing judgments)**.**

- *If $\Gamma \vdash e : A \mid \varepsilon$, then $\vdash \Gamma$.*

- *If $\Gamma \vdash_\sigma h : A^{\varepsilon'} \Rightarrow^\varepsilon B$, then $\vdash \Gamma$.*

Proof. Straightforward by mutual induction on the derivations. ∎

Lemma 3.29 (Well-kinded of Typing)**.**
- *If $\Gamma \vdash e : A \mid \varepsilon$, then $\Gamma \vdash A : \mathbf{Typ}$ and $\Gamma \vdash \varepsilon : \mathbf{Eff}$.*

- *If $\Gamma \vdash_\sigma h : A^{\varepsilon'} \Rightarrow^\varepsilon B$, then $\Gamma \vdash A : \mathbf{Typ}$ and $\Gamma \vdash \varepsilon' : \mathbf{Eff}$ and $\Gamma \vdash B : \mathbf{Typ}$ and $\Gamma \vdash \varepsilon : \mathbf{Eff}$.*

Proof. By mutual induction on derivations of the judgments. We proceed by cases on the typing rule applied lastly to the derivation.

***Case* T_SHANDLING:** For some A', ε', σ, N, $\boldsymbol{\alpha}^N$, and $\boldsymbol{S}^N$, we have

$$\Gamma \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : A'^{\varepsilon'} \Rightarrow^\varepsilon A.$$

By the induction hypothesis, we have $\Gamma \vdash A : \mathbf{Typ}$ and $\Gamma \vdash \varepsilon : \mathbf{Eff}$.

***Case* SH_RETURN:** For some x and e_r, we have

- $\Gamma, x : A \vdash e_r : B \mid \varepsilon$ and
- $\Gamma \vdash \varepsilon' : \mathbf{Eff}$.

By the induction hypothesis, we have

- $\Gamma, x : A \vdash B : \mathbf{Typ}$ and
- $\Gamma, x : A \vdash \varepsilon : \mathbf{Eff}$.

By Lemma 3.2(2), we have

- $\Delta(\Gamma) \vdash B : \mathbf{Typ}$ and
- $\Delta(\Gamma) \vdash \varepsilon : \mathbf{Eff}$.

By Lemma 3.6, we have

- $\Gamma \vdash B : \mathbf{Typ}$ and
- $\Gamma \vdash \varepsilon : \mathbf{Eff}$.

Now, we have $\vdash \Gamma, x : A$ by Lemma 3.28. Since only C_VAR can derive $\vdash \Gamma, x : A$, we have $\Gamma \vdash A : \mathbf{Typ}$.

***Case* SH_OP:** For some h' and σ', we have $\Gamma \vdash_{\sigma'} h' : A^{\varepsilon'} \Rightarrow^\varepsilon B$. By the induction hypothesis, we have $\Gamma \vdash A : \mathbf{Typ}$ and $\Gamma \vdash \varepsilon' : \mathbf{Eff}$ and $\Gamma \vdash B : \mathbf{Typ}$ and $\Gamma \vdash \varepsilon : \mathbf{Eff}$.

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***Case* others:** Similarly to Lemma 3.12(1) and (2).

∎

Lemma 3.30 (Inversion).

(1) *If $\Gamma \vdash v : A \mid \varepsilon$, then $\Gamma \vdash v : A \mid \mathbb{0}$.*

(2) *If $\Gamma \vdash \mathbf{fun}\,(f, x, e) : A_1 \to_{\varepsilon_1} B_1 \mid \varepsilon$, then $\Gamma, f : A_2 \to_{\varepsilon_2} B_2, x : A_2 \vdash e : B_2 \mid \varepsilon_2$ for some A_2, ε_2, and B_2 such that $\Gamma \vdash A_2 \to_{\varepsilon_2} B_2 <: A_1 \to_{\varepsilon_1} B_1$.*

(3) *If $\Gamma \vdash \Lambda\alpha : K.e : \forall\alpha : K.A_1{}^{\varepsilon_1} \mid \varepsilon$, then $\Gamma, \alpha : K \vdash e : A_1 \mid \varepsilon_1$.*

(4) *If $\Gamma \vdash \mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J : A_1 \to_{\varepsilon_1} B_1 \mid \varepsilon$, then the following hold:*

- $l :: \forall\boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi$,
- $\mathsf{op} : \forall\boldsymbol{\beta}^J : \boldsymbol{K'}^J.A \Rightarrow B \in \sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$,
- $\vdash \Gamma$,
- $\Gamma \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$,
- $\Gamma \vdash \boldsymbol{T}^J : \boldsymbol{K'}^J$,
- $\Gamma \vdash A_1 <: A[\boldsymbol{T}^J/\boldsymbol{\beta}^J]$,
- $\Gamma \vdash B[\boldsymbol{T}^J/\boldsymbol{\beta}^J] <: B_1$, *and*
- $\Gamma \vdash (l\,\boldsymbol{S}^I)^\uparrow \oslash \varepsilon_1$

for some $\boldsymbol{\alpha}^I$, $\boldsymbol{K}^I$, σ, $\boldsymbol{\beta}^J$, $\boldsymbol{K'}^J$, A, and B.

(5) *If $\Gamma \vdash v_1\,v_2 : B \mid \varepsilon$, then there exists some type A such that $\Gamma \vdash v_1 : A \to_\varepsilon B \mid \mathbb{0}$ and $\Gamma \vdash v_2 : A \mid \mathbb{0}$.*

Proof. Similarly to Lemma 3.14; Lemmas 3.28 and 3.29 are used instead of Lemmas 3.9 and 3.12, respectively.

∎

Lemma 3.31 (Canonical Form).

(1) *If $\emptyset \vdash v : A \to_\varepsilon B \mid \varepsilon'$, then either of the following holds:*

- $v = \mathbf{fun}\,(f, x, e)$ *for some f, x, and e, or*
- $v = \mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J$ *for some op, l, $\boldsymbol{S}^I$, and $\boldsymbol{T}^J$.*

(2) *If $\emptyset \vdash v : \forall\alpha : K.A^\varepsilon \mid \varepsilon'$, then $v = \Lambda\alpha : K.e$ for some e.*

Proof. Similarly to Lemma 3.15.

∎

Lemma 3.32 (Inversion of Handler Typing).

(1) *If $\Gamma \vdash_\sigma h : A^{\varepsilon'} \Rightarrow^\varepsilon B$, then there exist some x and e_r such that $\mathbf{return}\,x \mapsto e_r \in h$ and $\Gamma, x : A \vdash e_r : B \mid \varepsilon$.*

(2) *If $\Gamma \vdash_\sigma h : A^{\varepsilon'} \Rightarrow^\varepsilon B$ and $\mathsf{op} : \forall\boldsymbol{\beta}^J : \boldsymbol{K}^J.A' \Rightarrow B' \in \sigma$, then*

- $\mathsf{op}\,\boldsymbol{\beta}^J : \boldsymbol{K}^J\,p\,k \mapsto e \in h$ *and*
- $\Gamma, \boldsymbol{\beta}^J : \boldsymbol{K}^J, p : A', k : B' \to_{\varepsilon'} B \vdash e : B \mid \varepsilon$

for some p, k, and e.

Proof. (1) By induction on a derivation of $\Gamma \vdash_\sigma h : A^{\varepsilon'} \Rightarrow^\varepsilon B$. We proceed by cases on the typing rule applied lastly to this derivation.

***Case* H_RETURN:** Clearly.

***Case* H_OP:** Clearly by the induction hypothesis.

(2) By induction on a derivation of $\Gamma \vdash_\sigma h : A^{\varepsilon'} \Rightarrow^\varepsilon B$. We proceed by cases on the typing rule applied lastly to this derivation.

***Case* H_RETURN:** Clearly because there is no operation belonging to $\{\}$.

***Case* H_OP:** For some h', σ', op', $\boldsymbol{\beta'}^{J'}$, $\boldsymbol{K'}^{J'}$, A'', B'', p', k', e'', and ε', the following are given:

- $h = h' \uplus \{\mathsf{op}'\,\boldsymbol{\beta'}^{J'} : \boldsymbol{K'}^{J'}\,p'\,k' \mapsto e'\}$,
- $\sigma = \sigma' \uplus \{\mathsf{op}' : \forall\boldsymbol{\beta'}^{J'} : \boldsymbol{K'}^{J'}.A'' \Rightarrow B''\}$,
- $\Gamma \vdash_{\sigma'} h' : A^{\varepsilon'} \Rightarrow^\varepsilon B$, and

- $\Gamma, \boldsymbol{\beta'}^{J'} : \boldsymbol{K'}^{J'}, p' : A'', k' : B'' \to_{\varepsilon'} B \vdash e : B \mid \varepsilon$.

If $\mathsf{op} = \mathsf{op}'$, then clearly.

If $\mathsf{op} \neq \mathsf{op}'$, then clearly by the induction hypothesis. ∎

Lemma 3.33 (Independence of Evaluation Contexts). *If $\Gamma \vdash E[e] : A \mid \varepsilon$, then there exist some A' and ε' such that*

- $\Gamma \vdash e : A' \mid \varepsilon'$, *and*

- $\Gamma, \Gamma' \vdash E[e'] : A \mid \varepsilon$ *holds for any e' and Γ' such that $\Gamma, \Gamma' \vdash e' : A' \mid \varepsilon'$.*

Proof. Similarly to Lemma 3.17; Lemmas 3.25 and 3.28 are used instead of Lemmas 3.5 and 3.9, respectively. ∎

Lemma 3.34 (Progress). *If $\emptyset \vdash e : A \mid \varepsilon$, then one of the following holds:*

- *e is a value;*

- *There exists some expression e' such that $e \longrightarrow e'$; or*

- *There exist some op, l, $\boldsymbol{S}^I$, $\boldsymbol{T}^J$, v, E, and n such that $e = E[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v]$ and $n-\mathrm{free}(l\,\boldsymbol{S}^I, E)$.*

Proof. Similarly to Lemma 3.18; Lemmas 3.30, 3.31, and 3.32 are used instead of Lemmas 3.14, 3.15, and 3.16, respectively. ∎

Lemma 3.35 (Preservation in Reduction). *If $\emptyset \vdash e : A \mid \varepsilon$ and $e \longmapsto e'$, then $\emptyset \vdash e' : A \mid \varepsilon$.*

Proof. By induction on a derivation of $\Gamma \vdash e : A \mid \varepsilon$. We proceed by cases on the typing rule applied lastly to this derivation.

Case T_SHANDLING: We proceed by cases on the derivation rule which derives $e \longmapsto e'$.

Case R_HANDLE1: We have

- $e = \mathbf{handle}_{l\,\boldsymbol{S}^I}\,v\,\mathbf{with}\,h$,
- $\mathbf{return}\,x \mapsto e_r \in h$,
- $\emptyset \vdash v : B \mid \varepsilon'$,
- $l :: \forall \boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi$,
- $\emptyset \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$,
- $\emptyset \vdash_{\sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} h : B^{\varepsilon'} \Rightarrow^\varepsilon A$,
- $(l\,\boldsymbol{S}^I)^\uparrow \odot \varepsilon \sim \varepsilon'$, and
- $e' = e_r[v/x]$

for some l, $\boldsymbol{S}^I$, $\boldsymbol{\alpha}^I$, $\boldsymbol{K}^I$, σ, v, h, B, and ε'. By $\emptyset \vdash_{\sigma[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} h : B^{\varepsilon'} \Rightarrow^\varepsilon A$ and $\mathbf{return}\,x \mapsto e_r \in h$ and Lemma 3.32(1), we have

$$x : B \vdash e_r : A \mid \varepsilon.$$

By Lemma 3.30(1), we have $\emptyset \vdash v : B \mid \mathbb{0}$. Thus, Lemma 3.26(5) makes $\emptyset \vdash e_r[v/x] : A \mid \varepsilon$ hold as required.

Case R_HANDLE2: We have

- $e = \mathbf{handle}_{l\,\boldsymbol{S}^N}\,E[\mathsf{op}_{0\,l\,\boldsymbol{S}^N}\,\boldsymbol{T}^J\,v]\,\mathbf{with}\,h$,
- $l :: \forall \boldsymbol{\alpha}^N : \boldsymbol{K}^N.\sigma \in \Xi$,
- $\emptyset \vdash \boldsymbol{S}^N : \boldsymbol{K}^N$,
- $\mathsf{op}_0\,\boldsymbol{\beta_0}^J : \boldsymbol{K_0}^J\,p_0\,k_0 \mapsto e_0 \in h$,
- $0-\mathrm{free}(l\,\boldsymbol{S}^N, E)$,
- $\emptyset \vdash E[\mathsf{op}_{0\,l\,\boldsymbol{S}^N}\,\boldsymbol{T}^J\,v] : B \mid \varepsilon'$,
- $\emptyset \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : B^{\varepsilon'} \Rightarrow^\varepsilon A$,
- $(l\,\boldsymbol{S}^N)^\uparrow \odot \varepsilon \sim \varepsilon'$, and
- $e' = e_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J][v/p_0][\lambda z.E[z]/k_0]$

for some l, $\boldsymbol{S}^N$, E, op_0, $\boldsymbol{T}^J$, v, h, $\boldsymbol{\alpha}^N$, $\boldsymbol{K}^N$, σ, $\boldsymbol{\beta_0}^J$, $\boldsymbol{K_0}^J$, p_0, k_0, e_0, B, and ε'. By Lemma 3.33, there exist some B_1 and ε_1 such that

- $\emptyset \vdash \mathsf{op}_{0\,l\,\boldsymbol{S}^N}\,\boldsymbol{T}^J\,v : B_1 \mid \varepsilon_1$, and

- for any e'' and Γ'', if $\Gamma'' \vdash e'' : B_1 \mid \varepsilon_1$, then $\Gamma'' \vdash E[e''] : B \mid \varepsilon'$.

By Lemma 3.30(5), we have $\emptyset \vdash \mathsf{op}_{0\,l\,\boldsymbol{S}^N}\,\boldsymbol{T}^J : A_1 \to_{\varepsilon_1} B_1 \mid \mathbb{0}$ and $\emptyset \vdash v : A_1 \mid \mathbb{0}$ for some A_1. By Lemma 3.30(4) and 3.32(2), we have

- $\mathsf{op}_0 : \forall \boldsymbol{\beta_0}^J : \boldsymbol{K_0}^J . A_0 \Rightarrow B_0 \in \sigma[\boldsymbol{S}^N / \boldsymbol{\alpha}^N]$,
- $\emptyset \vdash \boldsymbol{S}^N : \boldsymbol{K}^N$,
- $\emptyset \vdash \boldsymbol{T}^J : \boldsymbol{K_0}^J$,
- $\emptyset \vdash A_1 <: A_0[\boldsymbol{T}^J / \boldsymbol{\beta_0}^J]$,
- $\emptyset \vdash B_0[\boldsymbol{T}^J / \boldsymbol{\beta_0}^J] <: B_1$, and
- $\emptyset \vdash (l\,\boldsymbol{S}^N)^\uparrow \oslash \varepsilon_1$.

for some A_0 and B_0. Thus, T_SUB with $\emptyset \vdash \mathbb{0} \oslash \mathbb{0}$ implied by Lemma 3.3 derives

$$\emptyset \vdash v : A_0[\boldsymbol{T}^J / \boldsymbol{\beta_0}^J] \mid \mathbb{0}.$$

By Lemma 3.11, we have $\emptyset \vdash B_0[\boldsymbol{T}^J / \boldsymbol{\beta_0}^J] : \mathbf{Typ}$. Thus, C_VAR derives $\vdash z : B_0[\boldsymbol{T}^J / \boldsymbol{\beta_0}^J]$. By $\emptyset \vdash \mathbb{0} : \mathbf{Eff}$, $\emptyset \vdash \varepsilon_1 : \mathbf{Eff}$ implied by Lemma 3.12, and $\mathbb{0} \odot \varepsilon_1 \sim \varepsilon_1$, we have $\emptyset \vdash \mathbb{0} \oslash \varepsilon_1$. Since T_VAR and T_SUB derives $z : B_0[\boldsymbol{T}^J / \boldsymbol{\beta_0}^J] \vdash z : B_1 \mid \varepsilon_1$, we have

$$z : B_0[\boldsymbol{T}^J / \boldsymbol{\beta_0}^J] \vdash E[z] : B \mid \varepsilon'$$

by the result of Lemma 3.33. Thus, T_ABS derives

$$\emptyset \vdash \lambda z. E[z] : B_0[\boldsymbol{T}^J / \boldsymbol{\beta_0}^J] \to_{\varepsilon'} B \mid \mathbb{0}.$$

Since

$$\boldsymbol{\beta_0}^J : \boldsymbol{K_0}^J, p_0 : A_0, k_0 : B_0 \to_{\varepsilon'} B \vdash e_0 : A \mid \varepsilon$$

by $\emptyset \vdash_{\sigma[\boldsymbol{S}^N / \boldsymbol{\alpha}^N]} h : B^{\varepsilon'} \Rightarrow^\varepsilon A$ and $\mathsf{op}_0 : \forall \boldsymbol{\beta_0}^J : \boldsymbol{K_0}^J . A_0 \Rightarrow B_0 \in \sigma$ and Lemma 3.32(2), Lemma 3.27(5) and Lemma 3.26(5) imply

$$\emptyset \vdash e_0[\boldsymbol{T}^J / \boldsymbol{\beta_0}^J][v / p_0][\lambda z. E[z] / k_0] : A \mid \varepsilon$$

as required.

Case others: Similarly to Lemma 3.19; Lemmas 3.30, 3.25, 3.28, 3.29, 3.26, and 3.27 are used instead of Lemmas 3.14, 3.5, 3.9, 3.12, 3.7, and 3.10 respectively. ∎

Lemma 3.36 (Preservation). *If $\emptyset \vdash e : A \mid \varepsilon$ and $e \longrightarrow e'$, then $\emptyset \vdash e' : A \mid \varepsilon$.*

Proof. Similarly to Lemma 3.20; Lemmas 3.33 and 3.35 are used instead of Lemmas 3.17 and 3.19. ∎

Lemma 3.37. *If $\Gamma \vdash E[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v] : A \mid \varepsilon$ and $n-\mathrm{free}(l\,\boldsymbol{S}^I, E)$, then $\Gamma \vdash (l\,\boldsymbol{S}^I)^\uparrow \oslash \varepsilon$.*

Proof. Similarly to Lemma 3.22; Lemma 3.30 is used instead of Lemma 3.14. ∎

Lemma 3.38 (Effect Safety). *If $\Gamma \vdash E[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v] : A \mid \varepsilon$ and $n-\mathrm{free}(l\,\boldsymbol{S}^I, E)$, then $\varepsilon \not\sim \mathbb{0}$.*

Proof. Similarly to Lemma 3.23; Lemma 3.37 is used instead of Lemma 3.22. ∎

Theorem 3.39 (Type and Effect Safety). *If $\emptyset \vdash e : A \mid \mathbb{0}$ and $e \longrightarrow^* e'$ and $e' \not\longrightarrow$, then e' is a value.*

Proof. Similarly to Theorem 3.24; Lemmas 3.36, 3.38, and 3.34 are used instead of Lemmas 3.20, 3.23, and 3.18, respectively. ∎

3.3 Properties with Lift Coercions

This section assumes that the safety conditions in Definition 1.45 and the safety condition for lift coercions in Definition 1.46 hold.

Lemma 3.40 (Weakening). *Suppose that $\vdash \Gamma_1, \Gamma_2$ and $\mathrm{dom}(\Gamma_2) \cap \mathrm{dom}(\Gamma_3) = \emptyset$.*

(1) If $\vdash \Gamma_1, \Gamma_3$, then $\vdash \Gamma_1, \Gamma_2, \Gamma_3$.

(2) If $\Gamma_1, \Gamma_3 \vdash S : K$, then $\Gamma_1, \Gamma_2, \Gamma_3 \vdash S : K$.

(3) If $\Gamma_1, \Gamma_3 \vdash A <: B$, then $\Gamma_1, \Gamma_2, \Gamma_3 \vdash A <: B$.

(4) If $\Gamma_1, \Gamma_3 \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2$, then $\Gamma_1, \Gamma_2, \Gamma_3 \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2$.

(5) If $\Gamma_1, \Gamma_3 \vdash e : A \mid \varepsilon$, then $\Gamma_1, \Gamma_2, \Gamma_3 \vdash e : A \mid \varepsilon$.

(6) If $\Gamma_1, \Gamma_3 \vdash_\sigma h : A \Rightarrow^\varepsilon B$, then $\Gamma_1, \Gamma_2, \Gamma_3 \vdash_\sigma h : A \Rightarrow^\varepsilon B$.

Proof.(1)(2) Similarly to Lemma 3.5(1) and (2).

(3)(4) Similarly to Lemma 3.5(3) and (4).

(5)(6) By mutual induction on derivations of the judgments. We proceed by case analysis on the rule applied lastly to the derivation.

Case T_LIFT: For some e', L, and ε', the following are given:
- $e = [e']_L$,
- $\Gamma_1, \Gamma_3 \vdash e' : A \mid \varepsilon'$,
- $\Gamma_1, \Gamma_3 \vdash L : \mathbf{Lab}$, and
- $(L)^\uparrow \odot \varepsilon' \sim \varepsilon$.

By the induction hypothesis and case (2), we have
- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash e' : A \mid \varepsilon'$ and
- $\Gamma_1, \Gamma_2, \Gamma_3 \vdash L : \mathbf{Lab}$.

Thus, T_LIFT derives $\Gamma_1, \Gamma_2, \Gamma_3 \vdash [e']_L : A \mid \varepsilon$.

Case others: Similarly to Lemma 3.5(5) and (6). ∎

Lemma 3.41 (Substitution of values). *Suppose that $\Gamma_1 \vdash v : A \mid \mathbb{0}$.*

(1) If $\vdash \Gamma_1, x : A, \Gamma_2$, then $\vdash \Gamma_1, \Gamma_2$.

(2) If $\Gamma_1, x : A, \Gamma_2 \vdash S : K$, then $\Gamma_1, \Gamma_2 \vdash S : K$.

(3) If $\Gamma_1, x : A, \Gamma_2 \vdash B <: C$, then $\Gamma_1, \Gamma_2 \vdash B <: C$.

(4) If $\Gamma_1, x : A, \Gamma_2 \vdash B_1 \mid \varepsilon_1 <: B_2 \mid \varepsilon_2$, then $\Gamma_1, \Gamma_2 \vdash B_1 \mid \varepsilon_1 <: B_2 \mid \varepsilon_2$.

(5) If $\Gamma_1, x : A, \Gamma_2 \vdash e : B \mid \varepsilon$, then $\Gamma_1, \Gamma_2 \vdash e[v/x] : B \mid \varepsilon$.

(6) If $\Gamma_1, x : A, \Gamma_2 \vdash_\sigma h : B^{\varepsilon'} \Rightarrow^\varepsilon C$, then $\Gamma_1, \Gamma_2 \vdash_\sigma h[v/x] : B^{\varepsilon'} \Rightarrow^\varepsilon C$.

Proof.(1)(2) Similarly to Lemma 3.7(1) and (2).

(3)(4) Similarly to Lemma 3.7(3) and 3.7(4).

(5)(6) By mutual induction on derivations of the judgments. We proceed by case analysis on the rule applied lastly to the derivation.

Case T_LIFT: For some e', ε', and L, the following are given:
- $e = [e']_L$,
- $\Gamma_1, x : A, \Gamma_2 \vdash e' : B \mid \varepsilon'$,
- $\Gamma_1, x : A, \Gamma_2 \vdash L : \mathbf{Lab}$, and
- $(L)^\uparrow \odot \varepsilon' \sim \varepsilon$.

By the induction hypothesis and case 3.41(2), we have
- $\Gamma_1, \Gamma_2 \vdash e'[v/x] : B \mid \varepsilon'$ and
- $\Gamma_1, \Gamma_2 \vdash L : \mathbf{Lab}$.

Thus, T_LIFT derives $\Gamma_1, \Gamma_2 \vdash [e'[v/x]]_L : A \mid \varepsilon$.

Case others: Similarly to Lemma 3.7(5) and (6). ∎

Lemma 3.42 (Substitution of Typelikes). *Suppose that $\Gamma_1 \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$.*

(1) If $\vdash \Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2$, then $\vdash \Gamma_1, \Gamma_2[\boldsymbol{S}^I / \boldsymbol{\alpha}^I]$.

(2) If $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash T : K$, then $\Gamma_1, \Gamma_2[\boldsymbol{S}^I / \boldsymbol{\alpha}^I] \vdash T[\boldsymbol{S}^I / \boldsymbol{\alpha}^I] : K$.

(3) If $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash A <: B$, then $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] <: B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

(4) If $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash A_1 \mid \varepsilon_1 <: A_2 \mid \varepsilon_2$, then $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash A_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon_1[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] <: A_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

(5) If $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash e : A \mid \varepsilon$, then $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash e[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

(6) If $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash_\sigma h : A \Rightarrow^\varepsilon B$, then $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash_{\sigma[\boldsymbol{S}/\boldsymbol{\alpha}]} h[\boldsymbol{S}/\boldsymbol{\alpha}] : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \Rightarrow^{\varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

Proof.(1)(2) Similarly to Lemma 3.10(1) and (2).

(3)(4) Similarly to Lemma 3.10(3) and 3.10(3).

(5)(6) By mutual induction on derivations of the judgments. We proceed by case analysis on the rule applied lastly to the derivation.

    Case T_LIFT: For some e', ε', and L, the following are given:
- $e = [e']_L$,
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash e' : A \mid \varepsilon'$,
- $\Gamma_1, \boldsymbol{\alpha}^I : \boldsymbol{K}^I, \Gamma_2 \vdash L : \textbf{Lab}$, and
- $(L)^\uparrow \odot \varepsilon' \sim \varepsilon$.

    By the induction hypothesis, case 3.42(2), and the fact that a typelike substitution is homomorphism for $\odot$ and $\sim$, we have
- $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash e'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$,
- $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash L[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] : \textbf{Lab}$, and
- $(L)^\uparrow[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \odot \varepsilon'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \sim \varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

    Thus, T_LIFT derives $\Gamma_1, \Gamma_2[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \vdash [e'[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]]_{L[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]} : A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I] \mid \varepsilon[\boldsymbol{S}^I/\boldsymbol{\alpha}^I]$.

    Case others: Similarly to Lemma 3.10(5) and (6). ∎

Lemma 3.43 (Well-formedness of contexts in typing judgments)**.**

- *If $\Gamma \vdash e : A \mid \varepsilon$, then $\vdash \Gamma$.*

- *If $\Gamma \vdash_\sigma h : A \Rightarrow^\varepsilon B$, then $\vdash \Gamma$.*

Proof. Straightforward by mutual induction on the derivations. ∎

Lemma 3.44 (Well-kinded of Typing)**.**
- *If $\Gamma \vdash e : A \mid \varepsilon$, then $\Gamma \vdash A : \textbf{Typ}$ and $\Gamma \vdash \varepsilon : \textbf{Eff}$.*

- *If $\Gamma \vdash_\sigma h : A^{\varepsilon'} \Rightarrow^\varepsilon B$, then $\Gamma \vdash A : \textbf{Typ}$ and $\Gamma \vdash \varepsilon' : \textbf{Eff}$ and $\Gamma \vdash B : \textbf{Typ}$ and $\Gamma \vdash \varepsilon : \textbf{Eff}$.*

Proof. By mutual induction on derivations of the judgments. We proceed by cases on the typing rule applied lastly to the derivation.
Case T_LIFT: For some e', ε', and L, the following are given:

- $e = [e']_L$,
- $\Gamma \vdash e' : A \mid \varepsilon'$,
- $\Gamma \vdash L : \textbf{Lab}$, and
- $(L)^\uparrow \odot \varepsilon' \sim \varepsilon$.

    By the induction hypothesis, we have $\Gamma \vdash A : \textbf{Typ}$ and $\Gamma \vdash \varepsilon' : \textbf{Eff}$. $(-)^\uparrow$, $\odot$, and $\sim$ preserve well-formedness, we have $\Gamma \vdash \varepsilon : \textbf{Eff}$.

Case others: Similarly to Lemma 3.12(1) and (2). ∎

Lemma 3.45 (Inversion)**.**
(1) If $\Gamma \vdash v : A \mid \varepsilon$, then $\Gamma \vdash v : A \mid \mathbb{0}$.

(2) If $\Gamma \vdash \textbf{fun}\,(g, x, e) : A_1 \to_{\varepsilon_1} B_1 \mid \varepsilon$, then $\Gamma, g : A_2 \to_{\varepsilon_2} B_2, x : A_2 \vdash e : B_2 \mid \varepsilon_2$ for some A_2, ε_2, and B_2 such that $\Gamma \vdash A_2 \to_{\varepsilon_2} B_2 <: A_1 \to_{\varepsilon_1} B_1$.

(3) If $\Gamma \vdash \Lambda\alpha : K.e : \forall\alpha : K.A_1{}^{\varepsilon_1} \mid \varepsilon$, then $\Gamma, \alpha : K \vdash e : A_1 \mid \varepsilon_1$.

(4) If $\Gamma \vdash \mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J : A_1 \to_{\varepsilon_1} B_1 \mid \varepsilon$, then the following hold:

- $l :: \forall\boldsymbol{\alpha}^I : \boldsymbol{K}^I.\sigma \in \Xi$,
- $\mathsf{op} : \forall\boldsymbol{\beta}^J : \boldsymbol{K'}^J.A \Rightarrow B \in \sigma$,
- $\vdash \Gamma$,
- $\Gamma \vdash \boldsymbol{S}^I : \boldsymbol{K}^I$,
- $\Gamma \vdash \boldsymbol{T}^J : \boldsymbol{K'}^J$,
- $\Gamma \vdash A_1 <: A[\boldsymbol{S}^I/\boldsymbol{\alpha}^I][\boldsymbol{T}^J/\boldsymbol{\beta}^J]$,
- $\Gamma \vdash B[\boldsymbol{S}^I/\boldsymbol{\alpha}^I][\boldsymbol{T}^J/\boldsymbol{\beta}^J] <: B_1$, and
- $\Gamma \vdash (l\,\boldsymbol{S}^I)^{\uparrow} \otimes \varepsilon_1$

for some $\boldsymbol{\alpha}^I$, $\boldsymbol{K}^I$, σ, $\boldsymbol{\beta}^J$, $\boldsymbol{K'}^J$, A, and B.

(5) If $\Gamma \vdash v_1\,v_2 : B \mid \varepsilon$, then there exists some type A such that $\Gamma \vdash v_1 : A \to_{\varepsilon} B \mid \mathbb{0}$ and $\Gamma \vdash v_2 : A \mid \mathbb{0}$.

Proof. Similarly to Lemma 3.14; Lemmas 3.43 and 3.44 are used instead of Lemmas 3.9 and 3.12, respectively. ∎

Lemma 3.46 (Canonical Form)**.**
(1) If $\emptyset \vdash v : A \to_{\varepsilon} B \mid \varepsilon'$, then either of the following holds:

- $v = \mathbf{fun}\,(g, x, e)$ for some g, x, and e, or
- $v = \mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J$ for some op, l, $\boldsymbol{S}^I$, and $\boldsymbol{T}^J$.

(2) If $\emptyset \vdash v : \forall\alpha : K.A^{\varepsilon} \mid \varepsilon'$, then $v = \Lambda\alpha : K.e$ for some e.

Proof. Similarly to Lemma 3.15. ∎

Lemma 3.47 (Independence of Evaluation Contexts)**.** *If $\Gamma \vdash E[e] : A \mid \varepsilon$, then there exist some A' and ε' such that*

- $\Gamma \vdash e : A' \mid \varepsilon'$, *and*
- $\Gamma, \Gamma' \vdash E[e'] : A \mid \varepsilon$ *holds for any e' and Γ' such that $\Gamma, \Gamma' \vdash e' : A' \mid \varepsilon'$.*

Proof. By induction on a derivation of $\Gamma \vdash E[e] : A \mid \varepsilon$. We proceed by cases on the typing rule applied lastly to this derivation.

Case T_Lift**:** If $E = \square$, then the required result is achieved immediately.

If $E \neq \square$, then we have

- $E = [E']_L$,
- $\Gamma \vdash E'[e] : A \mid \varepsilon'$,
- $\Gamma \vdash L : \mathbf{Lab}$, and
- $(L)^{\uparrow} \odot \varepsilon' \sim \varepsilon$,

for some E', L, and ε'. By the induction hypothesis, there exist some A' and ε'' such that

- $\Gamma \vdash e : A' \mid \varepsilon''$ and
- for any e' and Γ' such that $\Gamma, \Gamma' \vdash e' : A' \mid \varepsilon''$, typing judgment $\Gamma, \Gamma' \vdash E'[e'] : A \mid \varepsilon'$ is derivable.

Let e' be an expression and Γ' be a typing context such that $\Gamma, \Gamma' \vdash e' : A' \mid \varepsilon'$. The induction hypothesis gives us $\Gamma, \Gamma' \vdash E'[e'] : A \mid \varepsilon'$. By Lemma 3.40(2), we have $\Gamma, \Gamma' \vdash L : \mathbf{Lab}$. Thus, T_Lift derives $\Gamma, \Gamma' \vdash [E'[e']]_L : A \mid \varepsilon$ as required.

Case others**:** Similarly to Lemma 3.17; Lemmas 3.40 and 3.43 are used instead of Lemmas 3.5 and 3.9, respectively. ∎

Lemma 3.48 (Progress)**.** *If $\emptyset \vdash e : A \mid \varepsilon$, then one of the following holds:*

- *e is a value;*
- *There exists some expression e' such that $e \longrightarrow e'$; or*

- *There exist some* op, l, $\boldsymbol{S}^I$, $\boldsymbol{T}^J$, v, E, *and* n *such that* $e = E[\text{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v]$ *and* $n-\text{free}(l\,\boldsymbol{S}^I, E)$.

Proof. By induction on a derivation of $\emptyset \vdash e : A \mid \varepsilon$. We proceed by cases on the typing rule applied lastly to this derivation.

Case T_LIFT: For some e_1, L and ε_1, the following are given:

- $e = [e_1]_L$,
- $\emptyset \vdash e_1 : A \mid \varepsilon_1$, and
- $\emptyset \vdash (L)^\uparrow \odot \varepsilon_1 \sim \varepsilon$.

By the induction hypothesis, we proceed by cases on the following conditions:

(1) e_1 is a value,

(2) There exists some e_1' such that $e_1 \longrightarrow e_1'$,

(3) There exist some op, l, $\boldsymbol{S}^I$, $\boldsymbol{T}^J$, v, E, and n such that $e_1 = E[\text{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v]$ and $n-\text{free}(l\,\boldsymbol{S}^I, E)$.

 Case (1): R_LIFT derives $[e_1]_L \longmapsto e_1$ because e_1 is a value.

 Case (2): Since only E_EVAL can derive $e_1 \longrightarrow e_1'$, we have

- $e_1 = E_1[e_{11}]$,
- $e_1' = E_1[e_{12}]$, and
- $e_{11} \longmapsto e_{12}$,

for some E_1, e_{11}, and e_{12}. Let $E = ([E_1]_L)$. E_EVAL derives $e \longrightarrow E[e_{12}]$ because of $e = E[e_{11}]$.

 Case (3): If $L \neq l\,\boldsymbol{S}^I$, then we have $n-\text{free}(l\,\boldsymbol{S}^I, [E]_L)$.

  If $L = l\,\boldsymbol{S}^I$, then we have $n+1-\text{free}(l\,\boldsymbol{S}^I, [E]_L)$.

Case others: Similarly to Lemma 3.18; Lemmas 3.45 and 3.46 are used instead of Lemmas 3.14 and 3.15, respectively.

∎

Lemma 3.49 (Preservation in Reduction). *If* $\emptyset \vdash e : A \mid \varepsilon$ *and* $e \longmapsto e'$, *then* $\emptyset \vdash e' : A \mid \varepsilon$.

Proof. By induction on a derivation of $\Gamma \vdash e : A \mid \varepsilon$. We proceed by cases on the typing rule applied lastly to this derivation.

Case T_LIFT: Since only R_LIFT derives $e \longmapsto e'$, we have

- $e = [v]_L$,
- $\emptyset \vdash v : A \mid \varepsilon_1$,
- $\emptyset \vdash L : \textbf{Lab}$,
- $(L)^\uparrow \odot \varepsilon_1 \sim \varepsilon$, and
- $e' = v$.

for some v, L, and ε_1. By Lemma 3.45(1), we have $\emptyset \vdash v : A \mid \mathbb{0}$. By $\mathbb{0} \odot \varepsilon \sim \varepsilon$, we have $\emptyset \vdash v : A \mid \varepsilon$ as required.

Case others: Similarly to Lemma 3.19; Lemmas 3.45, 3.40, 3.43, 3.44, 3.41, and 3.42 are used instead of Lemmas 3.14, 3.5, 3.9, 3.12, 3.7, and 3.10 respectively.

∎

Lemma 3.50 (Preservation). *If* $\emptyset \vdash e : A \mid \varepsilon$ *and* $e \longrightarrow e'$, *then* $\emptyset \vdash e' : A \mid \varepsilon$.

Proof. Similarly to Lemma 3.20; Lemmas 3.47 and 3.49 are used instead of Lemmas 3.17 and 3.19. ∎

Definition 3.51 (Label Inclusion).

Label Inclusion $\boxed{L \oslash^n \varepsilon}$

$$\frac{}{L \oslash^0 \varepsilon}\ \text{LI\_EMPTY} \qquad \frac{L \oslash^n \varepsilon_1 \quad (L)^\uparrow \odot \varepsilon_1 \sim \varepsilon_2}{L \oslash^{n+1} \varepsilon_2}\ \text{LI\_HANDLING}$$

$$\frac{L \oslash^n \varepsilon_1 \quad (L')^\uparrow \odot \varepsilon_1 \sim \varepsilon_2 \quad L \neq L'}{L \oslash^n \varepsilon_2}\ \text{LI\_NOHANDLING}$$

Lemma 3.52. *If $L \oslash^n \varepsilon_1$ and $\varepsilon_1 \odot \varepsilon_2 \sim \varepsilon_3$, then $L \oslash^n \varepsilon_3$.*

Proof. By induction on a derivation of $L \oslash^n \varepsilon_1$. We proceed by case analysis on the rule applied lastly to this derivation.

Case LI_EMPTY: We have $n = 0$. LI_EMPTY derives $L \oslash^0 \varepsilon_3$ as required.

Case LI_HANDLING: We have

- $n = n' + 1$,
- $L \oslash^{n'} \varepsilon_4$, and
- $(L)^\uparrow \odot \varepsilon_4 \sim \varepsilon_1$,

for some n' and ε_4. By the induction hypothesis, we have $L \oslash^{n'} \varepsilon_5$ such that $\varepsilon_4 \odot \varepsilon_2 \sim \varepsilon_5$. Thus, LI_HANDLING derives $L \oslash^{n'+1} \varepsilon_3$ as required.

Case LI_NOHANDLING: We have

- $L \oslash^n \varepsilon_4$,
- $(L')^\uparrow \odot \varepsilon_4 \sim \varepsilon_1$, and
- $L \neq L'$,

for some L' and ε_4. By the induction hypothesis, we have $L \oslash^n \varepsilon_5$ such that $\varepsilon_4 \odot \varepsilon_2 \sim \varepsilon_5$. Thus, LI_NOHANDLING derives $L \oslash^n \varepsilon_3$ as required. ∎

Lemma 3.53. *If $L \oslash^{n+1} \varepsilon_2$ and $(L)^\uparrow \odot \varepsilon_1 \sim \varepsilon_2$, then $L \oslash^n \varepsilon_1$.*

Proof. By induction on a derivation of $L \oslash^{n+1} \varepsilon_2$. We proceed by case analysis on the rule lastly applied to this derivation.

Case LI_EMPTY: Cannot happen.

Case LI_HANDLING: We have

- $L \oslash^n \varepsilon_1'$ and
- $(L)^\uparrow \odot \varepsilon_1' \sim \varepsilon_2$

for some ε_1'. By safety condition (3), we have $\varepsilon_1 \sim \varepsilon_1'$. By Lemma 3.52 and $\varepsilon_1' \odot \mathbb{0} \sim \varepsilon_1$, we have $L \oslash^n \varepsilon_1$ as required.

Case LI_NOHANDLING: We have

- $L \oslash^{n+1} \varepsilon_3$,
- $(L')^\uparrow \odot \varepsilon_3 \sim \varepsilon_2$, and
- $L \neq L'$,

for some L' and ε_3. By safety condition (2) and $L \neq L'$, we have $(L)^\uparrow \odot \varepsilon_4 \sim \varepsilon_3$ for some ε_4. By safety condition (3), we have $\varepsilon_1 \sim (L')^\uparrow \odot \varepsilon_4$. By the induction hypothesis, we have $L \oslash^n \varepsilon_4$. Thus, LI_NOHANDLING derives $L \oslash^n \varepsilon_1$ as required. ∎

Lemma 3.54. *If $L \oslash^n \varepsilon_2$ and $(L')^\uparrow \odot \varepsilon_1 \sim \varepsilon_2$ and $L \neq L'$, then $L \oslash^n \varepsilon_1$.*

Proof. By induction on a derivation of $L \oslash^n \varepsilon_2$. We proceed by case analysis on the rule lastly applied to this derivation.

Case LI_EMPTY: We have $n = 0$. LI_EMPTY derives $L \oslash^0 \varepsilon_1$ as required.

Case LI_HANDLING: We have

- $n = n' + 1$,
- $L \oslash^{n'} \varepsilon_3$, and
- $(L)^\uparrow \odot \varepsilon_3 \sim \varepsilon_2$,

for some n' and ε_3. By safety condition (2) and $L \neq L'$, we have $(L')^\uparrow \odot \varepsilon_4 \sim \varepsilon_3$ for some ε_4. By safety condition (3), we have $\varepsilon_1 \sim (L)^\uparrow \odot \varepsilon_4$. By the induction hypothesis, we have $L \oslash^{n'} \varepsilon_4$. Thus, LI_HANDLING derives $L \oslash^{n'+1} \varepsilon_1$ as required.

Case LI_NoHandling**:** We have

- $L \oslash^n \varepsilon_3$,
- $(L'')^{\uparrow} \odot \varepsilon_3 \sim \varepsilon_2$, and
- $L \neq L''$,

for some L'' and ε_3.

If $L' = L''$, then we have $\varepsilon_1 \sim \varepsilon_3$ by safety condition (3). Thus, Lemma 3.52 gives us $L \oslash^n \varepsilon_1$ as required.

If $L' \neq L''$, then we have $(L')^{\uparrow} \odot \varepsilon_4 \sim \varepsilon_3$ for some ε_4 by safety condition (2) and $L' \neq L''$. By safety condition (3), we have $\varepsilon_1 \sim (L'')^{\uparrow} \odot \varepsilon_4$. By the induction hypothesis, we have $L \oslash^n \varepsilon_4$. Thus, LI_NoHandling derives $L \oslash^n \varepsilon_1$ as required.

∎

Lemma 3.55. *If* $\emptyset \vdash E[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v] : A \mid \varepsilon$ *and* $n{-}\mathrm{free}(l\,\boldsymbol{S}^I, E)$, *then* $l\,\boldsymbol{S}^I \oslash^{n+1} \varepsilon$.

Proof. By induction on a derivation of $\emptyset \vdash E[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v] : A \mid \varepsilon$. We proceed by case analysis on the typing rule applied lastly to this derivation.

Case T_App**:** For some B, we have

- $E = \square$,
- $\emptyset \vdash \mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J : B \to_\varepsilon A \mid \mathbb{0}$, and
- $\emptyset \vdash v : B \mid \mathbb{0}$.

By Lemma 3.45(4), we have $\emptyset \vdash (l\,\boldsymbol{S}^I)^{\uparrow} \oslash \varepsilon$. Thus, the required result is achieved.

Case T_Let**:** For some x, E_1, e, and B, we have

- $E = (\mathbf{let}\,x = E_1\,\mathbf{in}\,e)$,
- $\emptyset \vdash E_1[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v] : B \mid \varepsilon$,
- $n{-}\mathrm{free}(l\,\boldsymbol{S}^I, E_1)$, and
- $x : B \vdash e : A \mid \varepsilon$.

By the induction hypothesis, we have $l\,\boldsymbol{S}^I \oslash^{n+1} \varepsilon$ as required.

Case T_Sub**:** For some A'' and ε'', we have

- $\emptyset \vdash E[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v] : A' \mid \varepsilon'$ and
- $\emptyset \vdash A' \mid \varepsilon' <: A \mid \varepsilon$.

By the induction hypothesis, we have $l\,\boldsymbol{S}^I \oslash^{n+1} \varepsilon'$. Since only ST_Comp can derive $\emptyset \vdash A' \mid \varepsilon' <: A \mid \varepsilon$, we have $\emptyset \vdash \varepsilon' \oslash \varepsilon$. Thus, Lemma 3.52 derives $l\,\boldsymbol{S}^I \oslash^{n+1} \varepsilon$ as required.

Case T_Lift**:** For some L, ε', and E', we have

- $E = [E']_L$,
- $\emptyset \vdash E'[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v] : A \mid \varepsilon'$,
- $\emptyset \vdash L : \mathbf{Lab}$, and
- $(L)^{\uparrow} \odot \varepsilon' \sim \varepsilon$.

If $l\,\boldsymbol{S}^I \neq L$, then $n{-}\mathrm{free}(l\,\boldsymbol{S}^I, E')$. By the induction hypothesis, we have $l\,\boldsymbol{S}^I \oslash^{n+1} \varepsilon'$. LI_NoHandling derives $l\,\boldsymbol{S}^I \oslash^{n+1} \varepsilon$ as required.

If $l\,\boldsymbol{S}^I = L$, then there exists some m such that $n = m + 1$ and $m{-}\mathrm{free}(l\,\boldsymbol{S}^I, E')$. By the induction hypothesis, we have $l\,\boldsymbol{S}^I \oslash^{m+1} \varepsilon'$. LI_Handling derives $l\,\boldsymbol{S}^I \oslash^{m+2} \varepsilon$ as required.

Case T_Handling**:** For some l', $\boldsymbol{S'}^{I'}$, E_1, h, B, and ε', we have

- $E = \mathbf{handle}_{l'\,\boldsymbol{S'}^{I'}}\,E_1\,\mathbf{with}\,h$,
- $\emptyset \vdash E_1[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v] : B \mid \varepsilon'$, and
- $(l'\,\boldsymbol{S'}^{I'})^{\uparrow} \odot \varepsilon \sim \varepsilon'$.

If $l\,\boldsymbol{S}^I \neq l'\,\boldsymbol{S'}^{I'}$, then $n{-}\mathrm{free}(l\,\boldsymbol{S}^I, E_1)$. By the induction hypothesis, we have $l\,\boldsymbol{S}^I \oslash^{n+1} \varepsilon'$. By Lemma 3.54, we have $l\,\boldsymbol{S}^I \oslash^{n+1} \varepsilon$.

If $l\,\boldsymbol{S}^I = l'\,\boldsymbol{S'}^{I'}$, then $n+1{-}\mathrm{free}(l\,\boldsymbol{S}^I, E_1)$. By the induction hypothesis, we have $l\,\boldsymbol{S}^I \oslash^{n+2} \varepsilon'$. By Lemma 3.53, we have $l\,\boldsymbol{S}^I \oslash^{n+1} \varepsilon$.

***Case* others:** Cannot happen.

■

Lemma 3.56 (No Inclusion by Empty Effect). *If $L \oslash^n \varepsilon$ and $\varepsilon \sim \mathbb{0}$, then $n = 0$.*

Proof. By induction on the derivation of $L \oslash^n \varepsilon$. We proceed by case analysis on the rule applied lastly to this derivation.

***Case* LI_EMPTY:** Clearly.

***Case* LI_HANDLING:** This case cannot happen. If this case happens, we have $(L)^\uparrow \odot \varepsilon' \sim \varepsilon$ for some m and ε'. Thus, we have $(L)^\uparrow \odot \varepsilon' \sim \mathbb{0}$ by $\varepsilon \sim \mathbb{0}$. However, it is contradictory with safety condition (1).

***Case* LI_NoHANDLING:** This case cannot happen. If this case happens, we have $(L')^\uparrow \odot \varepsilon' \sim \varepsilon$ for some L' and ε'. Thus, we have $(L')^\uparrow \odot \varepsilon' \sim \mathbb{0}$ by $\varepsilon \sim \mathbb{0}$. However, it is contradictory with safety condition (1).

■

Lemma 3.57 (Effect Safety). *If $\emptyset \vdash E[\mathsf{op}_{l \, \boldsymbol{S}^I} \, \boldsymbol{T}^J \, v] : A \mid \varepsilon$ and $n-\mathrm{free}(l \, \boldsymbol{S}^I, E)$, then $\varepsilon \nsim \mathbb{0}$.*

Proof. Assume that $\varepsilon \sim \mathbb{0}$. By Lemma 3.55 and Lemma 3.52, we have $l \, \boldsymbol{S}^I \oslash^{n+1} \mathbb{0}$. However, it is contradictory with Lemma 3.56.

■

Theorem 3.58 (Type and Effect Safety). *If $\emptyset \vdash e : A \mid \mathbb{0}$ and $e \longrightarrow^* e'$ and $e' \nrightarrow$, then e' is a value.*

Proof. Similarly to Theorem 3.24; Lemmas 3.50 , 3.57, and 3.48 are used instead of Lemmas 3.20 , 3.23, and 3.18, respectively.

■

3.4 Properties with Type-Erasure Semantics

This section assumes that the safety conditions in Definition 1.45 and the safety condition for type-erasure semantics in Definition 1.47 hold, and that the semantics adapts R_HANDLE2'instead of R_HANDLE2.

Remark 3.59. *The change of semantics only affects Lemma 3.18, Lemma 3.19, Lemma 3.20, Lemma 3.22, Lemma 3.23, and Theorem 3.24. Therefore, we can use other lemmas in this type-erasure setting.*

Lemma 3.60 (Progress). *If $\emptyset \vdash e : A \mid \varepsilon$, then one of the following holds:*

- *e is a value;*

- *There exists some e' such that $e \longrightarrow e'$; or*

- *There exist some op, l, $\boldsymbol{S}^I$, $\boldsymbol{T}^J$, v, E, and n such that $eE[\mathsf{op}_{l \, \boldsymbol{S}^I} \, \boldsymbol{T}^J \, v]$ and $n-\mathrm{free}(l, E)$.*

Proof. By induction on a derivation of $\emptyset \vdash e : A \mid \varepsilon$. We proceed by case analysis on the typing rule applied lastly to this derivation.

***Case* T_HANDLING:** For some l, $\boldsymbol{S}^N$, h, e_1, A_1, ε_1, $\boldsymbol{\alpha}^N$, $\boldsymbol{K}^N$, σ, given are the following:

- $e = \mathbf{handle}_{l \, \boldsymbol{S}^N} \, e_1 \, \mathbf{with} \, h$,

- $\emptyset \vdash e_1 : A_1 \mid \varepsilon_1$,

- $l :: \forall \boldsymbol{\alpha}^N : \boldsymbol{K}^N.\sigma \in \Xi$,

- $\emptyset \vdash \boldsymbol{S}^N : \boldsymbol{K}^N$,

- $\emptyset \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : A_1 \Rightarrow^\varepsilon A$, and

- $(l \, \boldsymbol{S}^N)^\uparrow \odot \varepsilon \sim \varepsilon_1$.

By the induction hypothesis, we proceed by case analysis on the following conditions:

(1) e_1 is a value,

(2) There exists some e_1' such that $e_1 \longrightarrow e_1'$, and

(3) There exist some op', l', $\boldsymbol{S'}^{N'}$, $\boldsymbol{T}^J$, v, E, and n such that $e_1 = E[\mathsf{op}'_{l' \, \boldsymbol{S'}^{N'}} \, \boldsymbol{T}^J \, v]$ and $n-\mathrm{free}(l', E)$.

***Case* (1):** By Lemma 3.16(1), there exists some x and e_r such that $\mathbf{return} \, x \mapsto e_r \in h$. Thus, R_HANDLE1 derives $e \longmapsto e_r[v_1/x]$ because e_1 is a value v_1.

***Case* (2):** Since only E_EVAL can derive $e_1 \longrightarrow e_1'$, we have

- $e_1 = E_1[e_{11}]$,
- $e_1' = E_1[e_{12}]$, and

- $e_{11} \longmapsto e_{12}$,

for some E_1, e_{11}, and e_{12}. Let $E = \mathbf{handle}_{l\,\boldsymbol{S}^N} E_1 \mathbf{with}\, h$, E_EVAL derives $e \longrightarrow E[e_{12}]$ because $e = E[e_{11}]$.

Case **(3):** If $l \neq l'$, then $e = (\mathbf{handle}_{l\,\boldsymbol{S}^N} E \mathbf{with}\, h)[\mathsf{op}'_{l'\,\boldsymbol{S}'^{N'}} \boldsymbol{T}^J v]$ and $n-\mathrm{free}(l', \mathbf{handle}_{l\,\boldsymbol{S}^N} E \mathbf{with}\, h)$. If $l = l'$, then by Lemma 3.17 and 3.14(4), we have

- $l' :: \forall \boldsymbol{\alpha}'^{N'} : \boldsymbol{K}'^{N'}.\sigma' \in \Xi$ and
- $\mathsf{op}' : \forall \boldsymbol{\beta}'^J : \boldsymbol{K_0}'^J.A' \Rightarrow B' \in \sigma'[\boldsymbol{S}'^{N'}/\boldsymbol{\alpha}'^{N'}]$,

for some $\boldsymbol{\alpha}'^{N'}$, $\boldsymbol{K}'^{N'}$, σ', $\boldsymbol{\beta}'^J$, A', and B'. Therefore, since $l = l'$, we have

- $\boldsymbol{\alpha}^N = \boldsymbol{\alpha}'^{N'}$,
- $\boldsymbol{K}^N = \boldsymbol{K}'^{N'}$, and
- $\sigma = \sigma'$.

By $\emptyset \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : A_1 \Rightarrow^\varepsilon A$, $\mathsf{op}' : \forall \boldsymbol{\beta}'^J : \boldsymbol{K_0}'^J.A'' \Rightarrow B'' \in \sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]$ for some A'' and B'', and Lemma 3.16(2), we have

$$\mathsf{op}'\, \boldsymbol{\beta}'^J : \boldsymbol{K_0}'^J\, p\, k \mapsto e' \in h$$

for some p, k, and e'. If $n = 0$, the evaluation of e proceeds by R_HANDLE2'. Otherwise, there exists some m such that $n = m + 1$ and $m-\mathrm{free}(l, \mathbf{handle}_{l\,\boldsymbol{S}^N} E \mathbf{with}\, h)$.

Case **others:** Similarly to Lemma 3.18.   ∎

Lemma 3.61. *If $n-\mathrm{free}(l, E)$, then $n = 0$.*

Proof. Straightforward by the induction on the derivation of $n-\mathrm{free}(l, E)$.   ∎

Lemma 3.62. *If $\Gamma \vdash E[\mathsf{op}_{l\,\boldsymbol{S}^I} \boldsymbol{T}^J v] : A \mid \varepsilon$ and $n-\mathrm{free}(l, E)$, then $(l\,\boldsymbol{S}^I)^\uparrow \oslash \varepsilon$.*

Proof. By induction on a derivation of $\Gamma \vdash E[\mathsf{op}_{l\,\boldsymbol{S}^I} \boldsymbol{T}^J v] : A \mid \varepsilon$. We proceed by case analysis on the typing rule applied lastly to this derivation.

Case T_APP: For some B, we have

- $E = \square$,
- $\Gamma \vdash \mathsf{op}_{l\,\boldsymbol{S}^I} \boldsymbol{T}^J : B \to_\varepsilon A \mid \mathbb{0}$, and
- $\Gamma \vdash v : B \mid \mathbb{0}$.

By Lemma 3.14(4), we have $\Gamma \vdash (l\,\boldsymbol{S}^I)^\uparrow \oslash \varepsilon$. Thus, the required result is achieved.

Case T_LET: For some x, E_1, e, and B, we have

- $E = (\mathbf{let}\, x = E_1 \mathbf{in}\, e)$,
- $\Gamma \vdash E_1[\mathsf{op}_{l\,\boldsymbol{S}^I} \boldsymbol{T}^J v] : B \mid \varepsilon$, and
- $\Gamma, x : B \vdash e : A \mid \varepsilon$.

By the induction hypothesis, we have $(l\,\boldsymbol{S}^I)^\uparrow \oslash \varepsilon$ as required.

Case T_SUB: For some A' and ε', we have

- $\Gamma \vdash E[\mathsf{op}_{l\,\boldsymbol{S}^I} \boldsymbol{T}^J v] : A' \mid \varepsilon'$ and
- $\Gamma \vdash A' \mid \varepsilon' <: A \mid \varepsilon$.

Since only ST_COMP can derive $\Gamma \vdash A' \mid \varepsilon' <: A \mid \varepsilon$, we have $\Gamma \vdash \varepsilon' \oslash \varepsilon$. By the induction hypothesis, we have $(l\,\boldsymbol{S}^I)^\uparrow \oslash \varepsilon'$. By the associativity of $\odot$, we have $(l\,\boldsymbol{S}^I)^\uparrow \oslash \varepsilon$ as required.

Case T_HANDLING: For some l', $\boldsymbol{S}'^{I'}$, E_1, h, B, and ε', we have

- $E = \mathbf{handle}_{l'\,\boldsymbol{S}'^{I'}} E_1 \mathbf{with}\, h$,
- $\Gamma \vdash E_1[\mathsf{op}_{l\,\boldsymbol{S}^I} \boldsymbol{T}^J v] : B \mid \varepsilon'$, and
- $(l'\,\boldsymbol{S}'^{I'})^\uparrow \odot \varepsilon \sim \varepsilon'$.

By Lemma 3.61, we have $l \neq l'$ and $0-\mathrm{free}(l, E_1)$. By the induction hypothesis, we have $(l\,\boldsymbol{S}^I)^\uparrow \oslash \varepsilon'$. Thus, safety condition (2) makes $(l\,\boldsymbol{S}^I)^\uparrow \oslash \varepsilon$ hold as required.

***Case* others:** Cannot happen.

∎

Lemma 3.63 (Preservation in Reduction)**.** *If $\emptyset \vdash e : A \mid \varepsilon$ and $e \longmapsto e'$, then $\emptyset \vdash e' : A \mid \varepsilon$.*

Proof. By induction on a derivation of $\Gamma \vdash e : A \mid \varepsilon$. We proceed by case analysis on the typing rule applied lastly to this derivation.

***Case* T_HANDLING:** We proceed by case analysis on the derivation rule that derives $e \longmapsto e'$.

    ***Case* R_HANDLE1:** Similarly to Lemma 3.19.

    ***Case* R_HANDLE2':** For some l, $\boldsymbol{S}^N$, E, op_0, $\boldsymbol{S'}^N$, $\boldsymbol{T}^J$, v, h, $\boldsymbol{\alpha}^N$, $\boldsymbol{K}^N$, σ, $\boldsymbol{\beta_0}^J$, $\boldsymbol{K_0}^J$, A_0, B_0, p_0, k_0, e_0, B, and ε', we have

- $e = \mathbf{handle}_{l\,\boldsymbol{S}^N}\, E[\mathsf{op}_{0\,l\,\boldsymbol{S'}^N}\, \boldsymbol{T}^J\, v]\,\mathbf{with}\, h$,
- $l :: \forall \boldsymbol{\alpha}^N : \boldsymbol{K}^N.\sigma \in \Xi$,
- $\emptyset \vdash \boldsymbol{S}^N : \boldsymbol{K}^N$,
- $\mathsf{op}_0\, \boldsymbol{\beta_0}^J : \boldsymbol{K_0}^J\, p_0\, k_0 \mapsto e_0 \in h$,
- $0-\mathrm{free}(l, E)$,
- $\emptyset \vdash E[\mathsf{op}_{0\,l\,\boldsymbol{S'}^N}\, \boldsymbol{T}^J\, v] : B \mid \varepsilon'$,
- $\emptyset \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : B \Rightarrow^\varepsilon A$,
- $(l\,\boldsymbol{S}^N)^\uparrow \odot \varepsilon \sim \varepsilon'$, and
- $e' = e_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J][v/p_0][\lambda z.\mathbf{handle}_{l\,\boldsymbol{S}^N}\, E[z]\,\mathbf{with}\, h/k_0]$.

By Lemma 3.62, we have $(l\,\boldsymbol{S'}^N)^\uparrow \oslash \varepsilon'$. Thus, we get $\boldsymbol{S'}^N = \boldsymbol{S}^N$ by $(l\,\boldsymbol{S}^N)^\uparrow \odot \varepsilon \sim \varepsilon'$ and safety condition (4). By Lemma 3.17, there exist some B_1 and ε_1 such that

- $\emptyset \vdash \mathsf{op}_{0\,l\,\boldsymbol{S}^N}\, \boldsymbol{T}^J\, v : B_1 \mid \varepsilon_1$, and
- for any e' and Γ', if $\Gamma' \vdash e' : B_1 \mid \varepsilon_1$, then $\Gamma' \vdash E[e'] : B \mid \varepsilon'$.

By Lemma 3.14(5), we have $\emptyset \vdash \mathsf{op}_{0\,l\,\boldsymbol{S}^N}\, \boldsymbol{T}^J : A_1 \rightarrow_{\varepsilon_1} B_1 \mid \mathbb{0}$ and $\emptyset \vdash v : A_1 \mid \mathbb{0}$ for some A_1. By Lemma 3.14(4) and 3.16(2), we have

- $\mathsf{op}_0 : \forall \boldsymbol{\beta_0}^J : \boldsymbol{K_0}^J.A_0 \Rightarrow B_0 \in \sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]$,
- $\emptyset \vdash \boldsymbol{S}^N : \boldsymbol{K}^N$,
- $\emptyset \vdash \boldsymbol{T}^J : \boldsymbol{K_0}^J$,
- $\emptyset \vdash A_1 <: A_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J]$,
- $\emptyset \vdash B_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J] <: B_1$, and
- $\emptyset \vdash (l\,\boldsymbol{S}^N)^\uparrow \oslash \varepsilon_1$,

for some A_0 and B_0. Thus, T_SUB with $\emptyset \vdash \mathbb{0} \oslash \mathbb{0}$ implied by Lemma 3.3 derives

$$\emptyset \vdash v : A_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J] \mid \mathbb{0}.$$

By Lemma 3.11, we have $\emptyset \vdash B_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J] : \mathbf{Typ}$. Thus, C_VAR derives $\vdash z : B_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J]$. By $\emptyset \vdash \mathbb{0} : \mathbf{Eff}$, $\emptyset \vdash \varepsilon_1 : \mathbf{Eff}$ implied by Lemma 3.12, and $\mathbb{0} \odot \varepsilon_1 \sim \varepsilon_1$, we have $\emptyset \vdash \mathbb{0} \oslash \varepsilon_1$. Since T_VAR and T_SUB derives $z : B_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J] \vdash z : B_1 \mid \varepsilon_1$, we have

$$z : B_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J] \vdash \mathbf{handle}_{l\,\boldsymbol{S}^N}\, E[z]\,\mathbf{with}\, h : A \mid \varepsilon$$

by the result of Lemma 3.17, Lemma 3.5, and T_HANDLING. Thus, T_ABS derives

$$\emptyset \vdash \lambda z.\mathbf{handle}_{l\,\boldsymbol{S}^N}\, E[z]\,\mathbf{with}\, h : B_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J] \rightarrow_\varepsilon A \mid \mathbb{0}.$$

Since

$$\boldsymbol{\beta_0}^J : \boldsymbol{K_0}^J, p_0 : A_0, k_0 : B_0 \rightarrow_\varepsilon A \vdash e_0 : A \mid \varepsilon$$

by $\emptyset \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : B \Rightarrow^\varepsilon A$ and $\mathsf{op}_0 : \forall \boldsymbol{\beta_0}^J : \boldsymbol{K_0}^J.A_0 \Rightarrow B_0 \in \sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]$ and Lemma 3.16(2), Lemma 3.10(5) and Lemma 3.7(5) imply

$$\emptyset \vdash e_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J][v/p_0][\lambda z.\mathbf{handle}_{l\,\boldsymbol{S}^N}\, E[z]\,\mathbf{with}\, h/k_0] : A \mid \varepsilon$$

as required.

***Case* others:** Similarly to Lemma 3.19.

∎

Lemma 3.64 (Preservation). *If $\emptyset \vdash e : A \mid \varepsilon$ and $e \longrightarrow e'$, then $\emptyset \vdash e' : A \mid \varepsilon$.*

Proof. Similarly to Lemma 3.20; Lemma 3.63 is used instead of Lemma 3.19. ∎

Lemma 3.65 (Effect Safety). *If $\Gamma \vdash E[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v] : A \mid \varepsilon$ and $n-\mathrm{free}(l, E)$, then $\varepsilon \approx \mathbb{0}$.*

Proof. Similarly to Lemma 3.23; Lemma 3.62 is used instead of Lemma 3.22. ∎

Theorem 3.66 (Type and Effect Safety). *If $\emptyset \vdash e : A \mid \mathbb{0}$ and $e \longrightarrow^* e'$ and $e' \not\longrightarrow$, then e' is a value.*

Proof. Similarly to Theorem 3.24; Lemmas 3.64 , 3.65, and 3.60 are used instead of Lemmas 3.20 , 3.23, and 3.18, respectively. ∎

3.5 Properties with Lift Coercions and Type-Erasure Semantics

This section assumes that the safety conditions in Definition 1.45 and the safety conditions for type-erasure semantics and lift coercions in Definition 1.47 and 1.46 hold, and that the semantics adapts R_HANDLE2' instead of R_HANDLE2.

Lemma 3.67 (Progress). *If $\emptyset \vdash e : A \mid \varepsilon$, then one of the following holds:*
- *e is a value;*
- *There exists some expression e' such that $e \longrightarrow e'$; or*
- *There exist some op, l, $\boldsymbol{S}^I$, $\boldsymbol{T}^J$, v, E, and n such that $e = E[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v]$ and $n-\mathrm{free}(l, E)$.*

Proof. Similarly to Lemma 3.48. ∎

Definition 3.68 (Label Inclusion with Type-Erasure).

Label Inclusion with Type-Erasure $\boxed{l \oslash^{\mathcal{P}} \varepsilon}$ *where* $\mathcal{P} ::= \bullet \mid \boldsymbol{S}^I \blacktriangleright \mathcal{P}$

$$\frac{}{l \oslash^{\bullet} \varepsilon} \text{ LITE\_EMPTY} \qquad \frac{l \oslash^{\mathcal{P}} \varepsilon_1 \quad (l\,\boldsymbol{S_0}^{I_0})^{\uparrow} \odot \varepsilon_1 \sim \varepsilon_2}{l \oslash^{\boldsymbol{S_0}^{I_0} \blacktriangleright \mathcal{P}} \varepsilon_2} \text{ LITE\_HANDLING}$$

$$\frac{l \oslash^{\mathcal{P}} \varepsilon_1 \quad (L)^{\uparrow} \odot \varepsilon_1 \sim \varepsilon_2 \quad \forall \boldsymbol{S_0}^{I_0}.(L \neq l\,\boldsymbol{S_0}^{I_0})}{l \oslash^{\mathcal{P}} \varepsilon_2} \text{ LITE\_NOHANDLING}$$

If $n = 0$, then $\boldsymbol{S_1}^{I_1} \blacktriangleright \cdots \blacktriangleright \boldsymbol{S_n}^{I_n} \blacktriangleright \mathcal{P}$ means $\mathcal{P}$.

Lemma 3.69. *If $l \oslash^{\mathcal{P}} \varepsilon_1$ and $\varepsilon_1 \odot \varepsilon_2 \sim \varepsilon_3$, then $l \oslash^{\mathcal{P}} \varepsilon_3$.*

Proof. By induction on a derivation of $l \oslash^{\mathcal{P}} \varepsilon_1$. We proceed by case analysis on the rule applied lastly to this derivation.

Case LITE_EMPTY: We have $\mathcal{P} = \bullet$. LITE_EMPTY derives $l \oslash^{\bullet} \varepsilon_2$ as required.

Case LITE_HANDLING: We have
- $\mathcal{P} = \boldsymbol{S}^I \blacktriangleright \mathcal{P}'$,
- $l \oslash^{\mathcal{P}'} \varepsilon_4$, and
- $(l\,\boldsymbol{S}^I)^{\uparrow} \odot \varepsilon_4 \sim \varepsilon_1$,

for some $\mathcal{P}'$, ε_4, and $\boldsymbol{S}^I$. By the induction hypothesis, we have $l \oslash^{\mathcal{P}'} \varepsilon_5$ such that $\varepsilon_4 \odot \varepsilon_2 \sim \varepsilon_5$. Thus, LITE_HANDLING derives $l \oslash^{\boldsymbol{S}^I \blacktriangleright \mathcal{P}'} \varepsilon_2$ as required.

Case LITE_NOHANDLING: We have
- $l \oslash^{\mathcal{P}} \varepsilon_4$,
- $(L)^{\uparrow} \odot \varepsilon_4 \sim \varepsilon_1$, and
- $\forall \boldsymbol{S}^I.(L \neq l\,\boldsymbol{S}^I)$,

for some L and ε_4. By the induction hypothesis, we have $l \oslash^{\mathcal{P}} \varepsilon_5$ such that $\varepsilon_4 \odot \varepsilon_2 \sim \varepsilon_5$. Thus, LITE_NOHANDLING derives $l \oslash^{\mathcal{P}} \varepsilon_3$ as required. ∎

Lemma 3.70. *If $l \oslash^{\boldsymbol{S}^I \blacktriangleright \mathcal{P}} \varepsilon_2$ and $(l\,\boldsymbol{S}^I)^{\uparrow} \odot \varepsilon_1 \sim \varepsilon_2$, then $l \oslash^{\mathcal{P}} \varepsilon_1$.*

Proof. By induction on a derivation of $l \oslash^{\boldsymbol{S}^I \blacktriangleright \mathcal{P}} \varepsilon_2$. We proceed by case analysis on the rule lastly applied to this derivation.

Case LITE_EMPTY: Cannot happen.

Case LITE_HANDLING: We have

- $l \oslash^{\mathcal{P}} \varepsilon_1'$ and
- $(l \, \boldsymbol{S}^I)^{\uparrow} \odot \varepsilon_1' \sim \varepsilon_2$

for some ε_1'. By safety condition (3), we have $\varepsilon_1 \sim \varepsilon_1'$. By Lemma 3.69 and $\varepsilon_1' \odot \mathbb{0} \sim \varepsilon_1$, we have $l \oslash^{\mathcal{P}} \varepsilon_1$ as required.

Case LITE_NOHANDLING: We have

- $l \oslash^{\boldsymbol{S}^I \blacktriangleright \mathcal{P}} \varepsilon_3$,
- $(L)^{\uparrow} \odot \varepsilon_3 \sim \varepsilon_2$, and
- $\forall \boldsymbol{S_0}^{I_0}.(L \neq l \, \boldsymbol{S_0}^{I_0})$,

for some L and ε_3. By safety condition (2) and $L \neq l \, \boldsymbol{S}^I$, we have $(l \, \boldsymbol{S}^I)^{\uparrow} \odot \varepsilon_4 \sim \varepsilon_3$ for some ε_4. By safety condition (3), we have $\varepsilon_1 \sim (L)^{\uparrow} \odot \varepsilon_4$. By the induction hypothesis, we have $l \oslash^{\mathcal{P}} \varepsilon_4$. Thus, LITE_NOHANDLING derives $l \oslash^{\mathcal{P}} \varepsilon_1$ as required. ∎

Lemma 3.71. *If $l \oslash^{\mathcal{P}} \varepsilon_2$ and $(L)^{\uparrow} \odot \varepsilon_1 \sim \varepsilon_2$ and $\forall \boldsymbol{S}^I.(L \neq l \, \boldsymbol{S}^I)$, then $l \oslash^{\mathcal{P}} \varepsilon_1$.*

Proof. By induction on a derivation of $l \oslash^{\mathcal{P}} \varepsilon_2$. We proceed by case analysis on the rule lastly applied to this derivation.

Case LITE_EMPTY: We have $\mathcal{P} = \bullet$. LITE_EMPTY derives $l \oslash^{\bullet} \varepsilon_1$ as required.

Case LITE_HANDLING: We have

- $\mathcal{P} = \boldsymbol{S}^I \blacktriangleright \mathcal{P}'$,
- $l \oslash^{\mathcal{P}'} \varepsilon_3$, and
- $(l \, \boldsymbol{S}^I)^{\uparrow} \odot \varepsilon_3 \sim \varepsilon_2$,

for some $\mathcal{P}'$, ε_3, and $\boldsymbol{S}^I$. By safety condition (2) and $L \neq l \, \boldsymbol{S}^I$, we have $(L)^{\uparrow} \odot \varepsilon_4 \sim \varepsilon_3$ for some ε_4. By safety condition (3), we have $\varepsilon_1 \sim (l \, \boldsymbol{S}^I)^{\uparrow} \odot \varepsilon_4$. By the induction hypothesis, we have $l \oslash^{\mathcal{P}'} \varepsilon_4$. Thus, LITE_HANDLING derives $l \oslash^{\boldsymbol{S}^I \blacktriangleright \mathcal{P}'} \varepsilon_1$ as required.

Case LITE_NOHANDLING: We have

- $l \oslash^{\mathcal{P}} \varepsilon_3$,
- $(L')^{\uparrow} \odot \varepsilon_3 \sim \varepsilon_2$, and
- $\forall \boldsymbol{S}^I.(L' \neq l \, \boldsymbol{S}^I)$,

for some L' and ε_3.

If $L = L'$, then we have $\varepsilon_1 \sim \varepsilon_3$ by safety condition (3). Thus, Lemma 3.69 gives us $l \oslash^{\mathcal{P}} \varepsilon_1$ as required.

If $L \neq L'$, then we have $(L)^{\uparrow} \odot \varepsilon_4 \sim \varepsilon_3$ for some ε_4 by safety condition (2) and $L \neq L'$. By safety condition (3), we have $\varepsilon_1 \sim (L')^{\uparrow} \odot \varepsilon_4$. By the induction hypothesis, we have $l \oslash^{\mathcal{P}'} \varepsilon_4$. Thus, LITE_NOHANDLING derives $l \oslash^{\mathcal{P}} \varepsilon_1$ as required. ∎

Lemma 3.72. *If $l \oslash^{\boldsymbol{S_0}^{I_0} \blacktriangleright \mathcal{P}} \varepsilon$ and $(l \, \boldsymbol{S}^I)^{\uparrow} \oslash \varepsilon$, then $\boldsymbol{S}^I = \boldsymbol{S_0}^{I_0}$.*

Proof. By induction on a derivation of $l \oslash^{\boldsymbol{S_0}^{I_0} \blacktriangleright \mathcal{P}} \varepsilon$. We proceed by case analysis on the rule lastly applied to this derivation.

Case LITE_EMPTY: Cannot happen.

Case LITE_HANDLING: We have

- $l \oslash^{\mathcal{P}} \varepsilon_1$ and
- $(l \, \boldsymbol{S_0}^{I_0})^{\uparrow} \odot \varepsilon_1 \sim \varepsilon$

for some ε_1. By safety condition (4), we have $\boldsymbol{S}^I = \boldsymbol{S_0}^{I_0}$ as required.

Case LITE_NoHandling**:** We have

- $l \otimes^{\boldsymbol{S_0}^{I_0} \blacktriangleright \mathcal{P}} \varepsilon_1$,
- $(L)^{\uparrow} \odot \varepsilon_1 \sim \varepsilon$, and
- $\forall \boldsymbol{S'}^{I'}.(L \neq \boldsymbol{S'}^{I'})$

for some L and ε_1. By safety condition (2) and $L \neq l\,\boldsymbol{S}^I$, we have $(l\,\boldsymbol{S}^I)^{\uparrow} \otimes \varepsilon_1$. Thus, by the induction hypothesis, we have $\boldsymbol{S}^I = \boldsymbol{S_0}^{I_0}$ as required. ∎

Lemma 3.73. *If $\emptyset \vdash E[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v] : A \mid \varepsilon$ and $n-\mathrm{free}(l, E)$, then $l \otimes^{\boldsymbol{S_1}^{I_1} \blacktriangleright \cdots \blacktriangleright \boldsymbol{S_n}^{I_n}, \boldsymbol{S}^I \blacktriangleright \bullet} \varepsilon$.*

Proof. By induction on a derivation of $\emptyset \vdash E[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v] : A \mid \varepsilon$. We proceed by case analysis on the typing rule applied lastly to this derivation.

Case T_App**:** For some B, we have

- $E = \square$,
- $\emptyset \vdash \mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J : B \rightarrow_{\varepsilon} A \mid \mathbb{0}$, and
- $\emptyset \vdash v : B \mid \mathbb{0}$.

By Lemma 3.45(4), we have $\emptyset \vdash (l\,\boldsymbol{S}^I)^{\uparrow} \otimes \varepsilon$. Thus, LITE_Empty and LITE_Handling derive $l \otimes^{\boldsymbol{S}^I \blacktriangleright \bullet} \varepsilon$.

Case T_Let**:** For some x, E_1, e, and B, we have

- $E = (\mathbf{let}\ x = E_1\ \mathbf{in}\ e)$,
- $\emptyset \vdash E_1[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v] : B \mid \varepsilon$,
- $n-\mathrm{free}(l, E_1)$, and
- $x : B \vdash e : A \mid \varepsilon$.

By the induction hypothesis, we have $l \otimes^{\boldsymbol{S_1}^{I_1} \blacktriangleright \cdots \blacktriangleright \boldsymbol{S_n}^{I_n}, \boldsymbol{S}^I \blacktriangleright \bullet} \varepsilon$ as required.

Case T_Sub**:** For some A' and ε', we have

- $\emptyset \vdash E[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v] : A' \mid \varepsilon'$ and
- $\emptyset \vdash A' \mid \varepsilon' <: A \mid \varepsilon$.

By the induction hypothesis, we have $l \otimes^{\boldsymbol{S_1}^{I_1} \blacktriangleright \cdots \blacktriangleright \boldsymbol{S_n}^{I_n}, \boldsymbol{S}^I \blacktriangleright \bullet} \varepsilon'$. Since only ST_Comp can derives $\emptyset \vdash A' \mid \varepsilon' <: A \mid \varepsilon$, we have $\emptyset \vdash \varepsilon' \otimes \varepsilon$. Thus, Lemma 3.69 derives $l \otimes^{\boldsymbol{S_1}^{I_1} \blacktriangleright \cdots \blacktriangleright \boldsymbol{S_n}^{I_n}, \boldsymbol{S}^I \blacktriangleright \bullet} \varepsilon$ as required.

Case T_Lift**:** For some L, ε', and E', we have

- $E = [E']_L$,
- $\emptyset \vdash E'[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v] : A \mid \varepsilon'$,
- $\emptyset \vdash L : \mathbf{Lab}$, and
- $(L)^{\uparrow} \odot \varepsilon' \sim \varepsilon$.

If $L \neq l\,\boldsymbol{S'}^{I'}$ for any $\boldsymbol{S'}^{I'}$, then we have $n-\mathrm{free}(l, E')$. By the induction hypothesis, we have $l \otimes^{\boldsymbol{S_1}^{I_1} \blacktriangleright \cdots \blacktriangleright \boldsymbol{S_n}^{I_n}, \boldsymbol{S}^I \blacktriangleright \bullet} \varepsilon'$. Thus, LITE_NoHandling derives $l \otimes^{\boldsymbol{S_1}^{I_1} \blacktriangleright \cdots \blacktriangleright \boldsymbol{S_n}^{I_n}, \boldsymbol{S}^I \blacktriangleright \bullet} \varepsilon$ as required.

If $L = l\,\boldsymbol{S'}^{I'}$ for some $\boldsymbol{S'}^{I'}$, then there exists some m such that $n = m+1$ and $m-\mathrm{free}(l, E')$. By the induction hypothesis, we have $l \otimes^{\boldsymbol{S_1}^{I_1} \blacktriangleright \cdots \blacktriangleright \boldsymbol{S_m}^{I_m}, \boldsymbol{S}^I \blacktriangleright \bullet} \varepsilon'$. Thus, LITE_Handling derives $l \otimes^{\boldsymbol{S'}^{I'}, \boldsymbol{S_1}^{I_1} \blacktriangleright \cdots \blacktriangleright \boldsymbol{S_m}^{I_m}, \boldsymbol{S}^I \blacktriangleright \bullet} \varepsilon$ as required.

Case T_Handling**:** For some l', $\boldsymbol{S'}^{I'}$, E_1, h, B, and ε', we have

- $E = \mathbf{handle}_{l'\,\boldsymbol{S'}^{I'}}\,E_1\,\mathbf{with}\,h$,
- $\emptyset \vdash E_1[\mathsf{op}_{l\,\boldsymbol{S}^I}\,\boldsymbol{T}^J\,v] : B \mid \varepsilon'$, and
- $(l'\,\boldsymbol{S'}^{I'})^{\uparrow} \odot \varepsilon \sim \varepsilon'$.

If $l \neq l'$, then $n-\mathrm{free}(l, E_1)$. By the induction hypothesis, we have $l \otimes^{\boldsymbol{S_1}^{I_1} \blacktriangleright \cdots \blacktriangleright \boldsymbol{S_n}^{I_n}, \boldsymbol{S}^I \blacktriangleright \bullet} \varepsilon'$. By Lemma 3.71, we have $l \otimes^{\boldsymbol{S_1}^{I_1} \blacktriangleright \cdots \blacktriangleright \boldsymbol{S_n}^{I_n}, \boldsymbol{S}^I \blacktriangleright \bullet} \varepsilon$ as required.

If $l = l'$, then $n + 1-\mathrm{free}(l, E_1)$. By the induction hypothesis, we have $l \otimes^{\boldsymbol{S_0}^{I_0}, \boldsymbol{S_1}^{I_1} \blacktriangleright \cdots \blacktriangleright \boldsymbol{S_n}^{I_n}, \boldsymbol{S}^I \blacktriangleright \bullet} \varepsilon'$. By Lemma 3.72, we have $\boldsymbol{S_0}^{I_0} = \boldsymbol{S'}^{I'}$. By Lemma 3.70, we have $l \otimes^{\boldsymbol{S_1}^{I_1} \blacktriangleright \cdots \blacktriangleright \boldsymbol{S_n}^{I_n}, \boldsymbol{S}^I \blacktriangleright \bullet} \varepsilon$ as required.

Case **others:** Cannot happen.

∎

Lemma 3.74 (Preservation in Reduction). *If $\emptyset \vdash e : A \mid \varepsilon$ and $e \longmapsto e'$, then $\emptyset \vdash e' : A \mid \varepsilon$.*

Proof. By induction on a derivation of $\Gamma \vdash e : A \mid \varepsilon$. We proceed by cases on the typing rule applied lastly to this derivation.

Case T_HANDLING**:** We proceed by cases on the derivation rule which derives $e \longmapsto e'$.

    Case R_HANDLE1**:** Similarly to Lemma 3.49.

    Case R_HANDLE2**:** We have

- $e = \mathbf{handle}_{l\,\boldsymbol{S}^N}\, E[\mathsf{op}_{0\,l\,\boldsymbol{S'}^N}\,\boldsymbol{T}^J\,v]\,\mathbf{with}\,h$,
- $l :: \forall \boldsymbol{\alpha}^N : \boldsymbol{K}^N.\sigma \in \Xi$,
- $\emptyset \vdash \boldsymbol{S}^N : \boldsymbol{K}^N$,
- $\mathsf{op}_0\,\boldsymbol{\beta_0}^J : \boldsymbol{K_0}^J\,p_0\,k_0 \mapsto e_0 \in h$,
- $\emptyset \vdash E[\mathsf{op}_{0\,l\,\boldsymbol{S'}^N}\,\boldsymbol{T}^J\,v] : B \mid \varepsilon'$,
- $\emptyset \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : B \Rightarrow^\varepsilon A$,
- $(l\,\boldsymbol{S}^N)^\uparrow \odot \varepsilon \sim \varepsilon'$,
- $0-\mathrm{free}(l, E)$, and
- $e' = e_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J][v/p_0][\lambda z.\mathbf{handle}_{l\,\boldsymbol{S}^N}\,E[z]\,\mathbf{with}\,h/k_0]$

for some l, $\boldsymbol{S}^N$, E, op_0, $\boldsymbol{S'}^N$, $\boldsymbol{T}^J$, v, h, $\boldsymbol{\alpha}^N$, $\boldsymbol{K}^N$, σ, $\boldsymbol{\beta_0}^J$, $\boldsymbol{K_0}^J$, p_0, k_0, e_0, B, and ε'. By Lemma 3.73, we have $l \oslash^{\boldsymbol{S'}^N} \blacktriangleright\bullet \varepsilon'$. By Lemma 3.72 and $(l\,\boldsymbol{S}^N)^\uparrow \odot \varepsilon \sim \varepsilon'$, we have $\boldsymbol{S}^N = \boldsymbol{S'}^N$. By Lemma 3.47, there exist some B_1 and ε_1 such that

- $\emptyset \vdash \mathsf{op}_{0\,l\,\boldsymbol{S}^N}\,\boldsymbol{T}^J\,v : B_1 \mid \varepsilon_1$, and
- for any e'' and Γ'', if $\Gamma'' \vdash e'' : B_1 \mid \varepsilon_1$, then $\Gamma'' \vdash E[e''] : B \mid \varepsilon'$.

By Lemma 3.45(5), we have $\emptyset \vdash \mathsf{op}_{0\,l\,\boldsymbol{S}^N}\,\boldsymbol{T}^J : A_1 \rightarrow_{\varepsilon_1} B_1 \mid \mathbb{0}$ and $\emptyset \vdash v : A_1 \mid \mathbb{0}$ for some A_1. By Lemma 3.45(4) and 3.16(2), we have

- $\mathsf{op}_0 : \forall\boldsymbol{\beta_0}^J : \boldsymbol{K_0}^J.A_0 \Rightarrow B_0 \in \sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]$,
- $\emptyset \vdash \boldsymbol{S}^N : \boldsymbol{K}^N$,
- $\emptyset \vdash \boldsymbol{T}^J : \boldsymbol{K_0}^J$,
- $\emptyset \vdash A_1 <: A_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J]$,
- $\emptyset \vdash B_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J] <: B_1$, and
- $\emptyset \vdash (l\,\boldsymbol{S}^N)^\uparrow \oslash \varepsilon_1$,

for some A_0 and B_0. Thus, T_SUB with $\emptyset \vdash \mathbb{0} \oslash \mathbb{0}$ implied by Lemma 3.3 derives

$$\emptyset \vdash v : A_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J] \mid \mathbb{0}.$$

By Lemma 3.11, we have $\emptyset \vdash B_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J] : \mathbf{Typ}$. Thus, C_VAR derives $\vdash z : B_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J]$. By $\emptyset \vdash \mathbb{0} : \mathbf{Eff}$, $\emptyset \vdash \varepsilon_1 : \mathbf{Eff}$ implied by Lemma 3.12, and $\mathbb{0} \odot \varepsilon_1 \sim \varepsilon_1$, we have $\emptyset \vdash \mathbb{0} \oslash \varepsilon_1$. Since T_VAR and T_SUB derives $z : B_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J] \vdash z : B_1 \mid \varepsilon_1$, we have

$$z : B_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J] \vdash \mathbf{handle}_{l\,\boldsymbol{S}^N}\,E[z]\,\mathbf{with}\,h : A \mid \varepsilon$$

by the result of Lemma 3.17, Lemma 3.5, and T_HANDLING. Thus, T_ABS derives

$$\emptyset \vdash \lambda z.\mathbf{handle}_{l\,\boldsymbol{S}^N}\,E[z]\,\mathbf{with}\,h : B_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J] \rightarrow_\varepsilon A \mid \mathbb{0}.$$

Since

$$\boldsymbol{\beta_0}^J : \boldsymbol{K_0}^J, p_0 : A_0, k_0 : B_0 \rightarrow_\varepsilon A \vdash e_0 : A \mid \varepsilon$$

by $\emptyset \vdash_{\sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]} h : B \Rightarrow^\varepsilon A$ and $\mathsf{op}_0 : \forall\boldsymbol{\beta_0}^J : \boldsymbol{K_0}^J.A_0 \Rightarrow B_0 \in \sigma[\boldsymbol{S}^N/\boldsymbol{\alpha}^N]$ and Lemma 3.16(2), Lemma 3.10(5) and Lemma 3.7(5) imply

$$\emptyset \vdash e_0[\boldsymbol{T}^J/\boldsymbol{\beta_0}^J][v/p_0][\lambda z.\mathbf{handle}_{l\,\boldsymbol{S}^N}\,E[z]\,\mathbf{with}\,h/k_0] : A \mid \varepsilon$$

as required.

Case **others:** Similarly to Lemma 3.49.

Lemma 3.75 (Preservation). *If $\emptyset \vdash e : A \mid \varepsilon$ and $e \longrightarrow e'$, then $\emptyset \vdash e' : A \mid \varepsilon$.*

Proof. Similarly to Lemma 3.50; Lemma 3.74 is used instead of Lemma 3.49. ■

Lemma 3.76 (No Inclusion by Empty Effect). *If $l \oslash^{\mathcal{P}} \varepsilon$ and $\varepsilon \sim \mathbb{0}$, then $\mathcal{P} = \bullet$.*

Proof. By induction on the derivation of $l \oslash^{\mathcal{P}} \varepsilon$. We proceed by case analysis on the rule applied lastly to this derivation.

Case LITE_EMPTY: Clearly.

Case LITE_HANDLING: This case cannot happen. If this case happens, we have $(l \, \boldsymbol{S_0}^{I_0})^{\uparrow} \odot \varepsilon' \sim \varepsilon$ for some ε' and $\boldsymbol{S_0}^{I_0}$. Thus, we have $(l \, \boldsymbol{S_0}^{I_0})^{\uparrow} \odot \varepsilon' \sim \mathbb{0}$ by $\varepsilon \sim \mathbb{0}$. However, it is contradictory with safety condition (1).

Case LITE_NoHANDLING: This case cannot happen. If this case happens, we have $(L)^{\uparrow} \odot \varepsilon' \sim \varepsilon$ for some L and ε'. Thus, we have $(L)^{\uparrow} \odot \varepsilon' \sim \mathbb{0}$ by $\varepsilon \sim \mathbb{0}$. However, it is contradictory with safety condition (1). ■

Lemma 3.77 (Effect Safety). *If $\emptyset \vdash E[\mathsf{op}_{l \, \boldsymbol{S}^I} \, \boldsymbol{T}^J \, v] : A \mid \varepsilon$ and $n-\mathrm{free}(l \, \boldsymbol{S}^I, E)$, then $\varepsilon \nsim \mathbb{0}$.*

Proof. Assume that $\varepsilon \sim \mathbb{0}$. By Lemma 3.73 and Lemma 3.69, we have $l \oslash^{\boldsymbol{S_1}^{I_1} \blacktriangleright \cdots \blacktriangleright \boldsymbol{S_n}^{I_n}, \boldsymbol{S}^I \blacktriangleright \bullet} \varepsilon$. However, it is contradictory with Lemma 3.76. ■

Theorem 3.78 (Type and Effect Safety). *If $\emptyset \vdash e : A \mid \mathbb{0}$ and $e \longrightarrow^* e'$ and $e' \not\longrightarrow$, then e' is a value.*

Proof. Similarly to Theorem 3.58; Lemmas 3.75 , 3.77, and 3.67 are used instead of Lemmas 3.50 , 3.57, and 3.48, respectively. ■

3.6 Safety Conditions about Instances

Lemma 3.79. *In Example 1.23, we write a and b to denote $\{\}$ or ρ or $\{L\}$. If $a_1 \underline{\cup} \cdots \underline{\cup} a_m \sim_{\mathrm{Set}} b_1 \underline{\cup} \cdots \underline{\cup} b_n$, then*

- *for any $i \in \{1, \ldots, m\}$, $a_i = \{\}$ or there exists some j such that $a_i = b_j$, and*

- *for any $j \in \{1, \ldots, n\}$, $b_j = \{\}$ or there exists some i such that $a_i = b_j$.*

Proof. By induction on the derivation of $a_1 \underline{\cup} \cdots \underline{\cup} a_m \sim_{\mathrm{Set}} b_1 \underline{\cup} \cdots \underline{\cup} b_n$. ■

Theorem 3.80. *Example 1.23 meets safety conditions.*

Proof.
(1) Clearly by Lemma 3.79.

(2) Clearly by Lemma 3.79.
■

Lemma 3.81. *In Example 1.24, we write a and b to denote $\{\}$ or ρ or $\{L\}$. If $a_1 \underline{\sqcup} \cdots \underline{\sqcup} a_m \sim_{\mathrm{MSet}} b_1 \underline{\sqcup} \cdots \underline{\sqcup} b_n$, then*

- *for any a such that $a \neq \{\}$, the number of a_i such that $a_i = a$ is equal to the number of b_j such that $b_i = a$.*

Proof. By induction on the derivation of $a_1 \underline{\sqcup} \cdots \underline{\sqcup} a_m \sim_{\mathrm{MSet}} b_1 \underline{\sqcup} \cdots \underline{\sqcup} b_n$. ■

Theorem 3.82. *Example 1.24 meets safety conditions (for lift coercions).*

Proof.
(1) Clearly by Lemma 3.81.

(2) Clearly by Lemma 3.81.

(3) Clearly by Lemma 3.81.
■

Lemma 3.83. *In Example 1.25, we write a and b to denote $\langle \rangle$ or ρ. If $\langle L_1 \mid \langle \cdots \langle L_m \mid a \rangle \cdots \rangle \rangle \sim_{\mathrm{SimpR}} \langle L_1' \mid \langle \cdots \langle L_m' \mid b \rangle \cdots \rangle \rangle$, then*

- $a = b$,

- *for any $i \in \{1, \ldots, m\}$, there exists some j such that $L_i = L'_j$, and*

- *for any $j \in \{1, \ldots, n\}$, there exists some i such that $L_i = L'_j$.*

Proof. By induction on the derivation of $\langle L_1 \mid \langle \cdots \langle L_m \mid a \rangle \cdots \rangle\rangle \sim_{\mathrm{SimpR}} \langle L'_1 \mid \langle \cdots \langle L'_m \mid b \rangle \cdots \rangle\rangle$. ∎

Theorem 3.84. *Example 1.25 meets safety conditions.*

Proof.
   (1) Clearly by Lemma 3.83.

   (2) Clearly by Lemma 3.83.

∎

Lemma 3.85. *In Example 1.26, we write a and b to denote $\langle\rangle$ or ρ. If $\langle L_1 \mid \langle \cdots \langle L_m \mid a \rangle \cdots \rangle\rangle \sim_{\mathrm{ScpR}} \langle L'_1 \mid \langle \cdots \langle L'_m \mid b \rangle \cdots \rangle\rangle$, then*
   - *$a = b$ and*

   - *for any L, the number of L_i such that $L_i = L$ is equal to the number of L'_j such that $L'_j = L$.*

Proof. By induction on the derivation of $\langle L_1 \mid \langle \cdots \langle L_m \mid a \rangle \cdots \rangle\rangle \sim_{\mathrm{ScpR}} \langle L'_1 \mid \langle \cdots \langle L'_m \mid b \rangle \cdots \rangle\rangle$. ∎

Theorem 3.86. *Example 1.26 meets safety conditions (for lift coercions).*

Proof.
(1) Clearly by Lemma 3.85.
(2) Clearly by Lemma 3.85.
(3) Clearly by Lemma 3.85. ∎

Lemma 3.87. *In Example 1.27, we write a and b to denote $\{\}$ or ρ or $\{L\}$. If $a_1 \underline{\cup} \cdots \underline{\cup} a_m \sim_{\mathrm{ESet}} b_1 \underline{\cup} \cdots \underline{\cup} b_n$, then*
   - *for any $i \in \{1, \ldots, m\}$, $a_i = \{\}$ or there exists some j such that $a_i = b_j$ or label names of them are the same, and*

   - *for any $j \in \{1, \ldots, n\}$, $b_j = \{\}$ or there exists some i such that $a_i = b_j$ or label names of them are the same.*

Proof. By induction on the derivation of $a_1 \underline{\cup} \cdots \underline{\cup} a_m \sim_{\mathrm{ESet}} b_1 \underline{\cup} \cdots \underline{\cup} b_n$. ∎

Lemma 3.88. *In Example 1.27, we define the function FO as follows:*

$$FO(l, \{\}) = \bot \quad FO(l, \{\iota\}) = \bot \quad FO(l, \rho) = \bot \quad FO(l, \{l\, \boldsymbol{S}^I\}) = \boldsymbol{S}^I \quad FO(l, \{l'\, \boldsymbol{S}^I\}) = \bot \quad (\text{where } l \neq l')$$

$$FO(l, \varepsilon_1 \underline{\cup} \varepsilon_2) = \begin{cases} FO(l, \varepsilon_2) & (\text{if } FO(l, \varepsilon_1) = \bot) \\ FO(l, \varepsilon_1) & (\text{otherwise}) \end{cases}$$

If $\varepsilon_1 \sim_{\mathrm{ESet}} \varepsilon_2$, then for any l, $FO(l, \varepsilon_1) = FO(l, \varepsilon_2)$.

Proof. By induction on the derivation of $\varepsilon_1 \sim_{\mathrm{ESet}} \varepsilon_2$. ∎

Theorem 3.89. *Example 1.27 meets safety conditions.*

Proof.
   (1) Clearly by Lemma 3.87.

   (2) Clearly by Lemma 3.87 and 3.88.

   (4) Clearly by Lemma 3.88.

∎

Lemma 3.90. *In Example 1.28, we write a and b to denote $\{\}$ or ρ or $\{L\}$. If $a_1 \underline{\sqcup} \cdots \underline{\sqcup} a_m \sim_{\mathrm{EMSet}} b_1 \underline{\sqcup} \cdots \underline{\sqcup} b_n$, then*
   - *for any a such that $a \neq \{\}$, the number of a_i such that $a_i = a$ is equal to the number of b_j such that $b_i = a$.*

Proof. By induction on the derivation of $a_1 \sqcup \cdots \sqcup a_m \sim_{\text{EMSet}} b_1 \sqcup \cdots \sqcup b_n$. ∎

Lemma 3.91. *In Example 1.28, we define the function FO as follows:*

$$FO(l, \{\}) = \bot \quad FO(l, \{\iota\}) = \bot \quad FO(l, \rho) = \bot \quad FO(l, \{l\, \boldsymbol{S}^I\}) = \boldsymbol{S}^I \quad FO(l, \{l'\, \boldsymbol{S}^I\}) = \bot \quad (\text{where } l \neq l')$$

$$FO(l, \varepsilon_1 \sqcup \varepsilon_2) = \begin{cases} FO(l, \varepsilon_2) & (\text{if } FO(l, \varepsilon_1) = \bot) \\ FO(l, \varepsilon_1) & (\text{otherwise}) \end{cases}$$

If $\varepsilon_1 \sim_{\text{EMSet}} \varepsilon_2$, then for any l, $FO(l, \varepsilon_1) = FO(l, \varepsilon_2)$.

Proof. By induction on the derivation of $\varepsilon_1 \sim_{\text{EMSet}} \varepsilon_2$. ∎

Theorem 3.92. *Example 1.28 meets safety conditions.*

Proof.
   (1) Clearly by Lemma 3.90.

   (2) Clearly by Lemma 3.90.

   (4) Clearly by Lemma 3.91.

∎

Lemma 3.93. *In Example 1.29, we write a and b to denote $\langle\rangle$ or ρ. If $\langle L_1 \mid \langle \cdots \langle L_m \mid a \rangle \cdots \rangle\rangle \sim_{\text{ESimpR}} \langle L'_1 \mid \langle \cdots \langle L'_m \mid b \rangle \cdots \rangle\rangle$, then*
   • *$a = b$,*

   • *for any $i \in \{1, \ldots, m\}$, there exists some j such that $L_i = L'_j$ or label names of them are the same, and*

   • *for any $j \in \{1, \ldots, n\}$, there exists some i such that $L_i = L'_j$ or label names of them are the same.*

Proof. By induction on the derivation of $\langle L_1 \mid \langle \cdots \langle L_m \mid a \rangle \cdots \rangle\rangle \sim_{\text{ESimpR}} \langle L'_1 \mid \langle \cdots \langle L'_m \mid b \rangle \cdots \rangle\rangle$. ∎

Lemma 3.94. *In Example 1.29, we define the function FO as follows:*

$$FO(l, \langle\rangle) = \bot \quad FO(l, \rho) = \bot \quad FO(l, \langle l\, \boldsymbol{S}^I \mid \varepsilon\rangle) = \boldsymbol{S}^I \quad FO(l, \langle l'\, \boldsymbol{S}^I \mid \varepsilon\rangle) = FO(l, \varepsilon) \quad (\text{where } l \neq l')$$
$$FO(l, \langle \iota \mid \varepsilon\rangle) = FO(l, \varepsilon)$$

If $\varepsilon_1 \sim_{\text{ESimpR}} \varepsilon_2$, then for any l, $FO(l, \varepsilon_1) = FO(l, \varepsilon_2)$.

Proof. By induction on the derivation of $\varepsilon_1 \sim_{\text{ESimpR}} \varepsilon_2$. ∎

Theorem 3.95. *Example 1.29 meets safety conditions (for type-erasure).*

Proof.
   (1) Clearly by Lemma 3.93.

   (2) Clearly by Lemma 3.93 and 3.94.

   (4) Clearly by Lemma 3.94.

∎

Lemma 3.96. *In Example 1.30, we write a and b to denote $\langle\rangle$ or ρ. If $\langle L_1 \mid \langle \cdots \langle L_m \mid a \rangle \cdots \rangle\rangle \sim_{\text{EScpR}} \langle L'_1 \mid \langle \cdots \langle L'_m \mid b \rangle \cdots \rangle\rangle$, then*
   • *$a = b$ and*

   • *for any L, the number of L_i such that $L_i = L$ is equal to the number of L'_j such that $L'_j = L$.*

Proof. By induction on the derivation of $\langle L_1 \mid \langle \cdots \langle L_m \mid a \rangle \cdots \rangle\rangle \sim_{\text{EScpR}} \langle L'_1 \mid \langle \cdots \langle L'_m \mid b \rangle \cdots \rangle\rangle$. ∎

Lemma 3.97. *In Example 1.30, we define the function FO as follows:*

$$FO(l, \langle\rangle) = \bot \quad FO(l, \rho) = \bot \quad FO(l, \langle l\, \boldsymbol{S}^I \mid \varepsilon\rangle) = \boldsymbol{S}^I \quad FO(l, \langle l'\, \boldsymbol{S}^I \mid \varepsilon\rangle) = FO(l, \varepsilon) \quad (\text{where } l \neq l')$$
$$FO(l, \langle \iota \mid \varepsilon\rangle) = FO(l, \varepsilon)$$

If $\varepsilon_1 \sim_{\text{EScpR}} \varepsilon_2$, then for any l, $FO(l, \varepsilon_1) = FO(l, \varepsilon_2)$.

Proof. By induction on the derivation of $\varepsilon_1 \sim_{\text{EScpR}} \varepsilon_2$. ∎

Theorem 3.98. *Example 1.30 meets safety conditions (for lift coercions and type-erasure).*

Proof.
  (1) Clearly by Lemma 3.96.

  (2) Clearly by Lemma 3.96.

  (3) Clearly by Lemma 3.96.

  (4) Clearly by Lemma 3.97.

∎

Theorem 3.99 (Unsafe Effect Algebras with Lift Coercions)**.** *The effect algebras* EA_{Set} *and* EA_{SimpR} *do not meet safety condition (3). Furthermore, there exists an expression such that it is well typed under* EA_{Set} *and* EA_{SimpR}*, but its evaluation gets stuck.*

Proof. We consider only EA_{Set} here; a similar discussion can be applied to EA_{SimpR}. Recall that the operation $\odot$ in EA_{Set} is implemented by the set union, so it meets idempotence: $\{L\} \sqcup \{L\} \sim \{L\}$. Furthermore, we can use the empty set as the identity element, so $\{L\} \sqcup \{L\} \sim \{L\} \sqcup \{\}$. If safety condition (3) was met, $\{L\} \sim \{\}$ (where $\{L\}$, $\{\}$, and 0 are taken as ε_1, ε_2, and n, respectively, in Definition 1.46). However, the equivalence does not hold.

As a program that is typeable under EA_{Set}, consider $\mathbf{handle}_{\text{Exc}} \, [\mathsf{raise}_{\text{Exc}} \, \mathsf{Unit} \, ()]_{\text{Exc}} \, \mathbf{with} \, h$ where $\mathsf{Exc} :: \{\mathsf{raise} : \forall \alpha : \mathbf{Typ}.\mathsf{Unit} \Rightarrow \alpha\}$. This program can be typechecked under an appropriate assumption as illustrated by the following typing derivation:

$$\frac{\cdots \quad \{\mathsf{Exc}\} \sqcup \{\} \sim \{\mathsf{Exc}\} \quad \dfrac{\dfrac{\emptyset \vdash \mathsf{raise}_{\text{Exc}} \, \mathsf{Unit} \, () : A \mid \{\mathsf{Exc}\} \quad \{\mathsf{Exc}\} \sqcup \{\mathsf{Exc}\} \sim \{\mathsf{Exc}\}}{\emptyset \vdash [\mathsf{raise}_{\text{Exc}} \, \mathsf{Unit} \, ()]_{\text{Exc}} : A \mid \{\mathsf{Exc}\}} \text{ T\_LIFT}}{\emptyset \vdash \mathbf{handle}_{\text{Exc}} \, [\mathsf{raise}_{\text{Exc}} \, \mathsf{Unit} \, ()]_{\text{Exc}} \, \mathbf{with} \, h : B \mid \{\}} \text{ T\_HANDLING}}$$

However, the call to raise is not handled because it needs to be handled by the *second* closest effect handler. ∎

Theorem 3.100 (Unsafe Effect Algebras in Type-Erasure Semantics)**.** *The effect algebras* EA_{Set}, EA_{MSet}, EA_{SimpR}, *and* EA_{ScpR} *do not meet safety condition (4). Furthermore, there exists an expression that is well typed under these algebras and gets stuck.*

Proof. Here we focus on the effect algebra EA_{Set}, but a similar discussions can be applied to the other algebras. Recall that $\odot$ in EA_{Set} is implemented by the union operation for sets, and therefore it is commutative (i.e., it allows exchanging labels in a set no matter what label names and what type arguments are in the labels). Hence, for example, $\{l \, \mathsf{Int}\} \sqcup \{l \, \mathsf{Bool}\} \sim_{\text{Set}} \{l \, \mathsf{Bool}\} \sqcup \{l \, \mathsf{Int}\}$ for a label name l taking one type parameter. It means that EA_{Set} violates safety condition (4).

To give a program that is typeable under EA_{Set} but unsafe in the type-erasure semantics, consider the following which uses an effect label $\mathsf{Writer} :: \forall \alpha : \mathbf{Typ}.\{\mathsf{tell} : \alpha \Rightarrow \mathsf{Unit}\}$:

$\mathbf{handle}_{\mathsf{Writer} \, \mathsf{Int}} \, \mathbf{handle}_{\mathsf{Writer} \, \mathsf{Bool}}$
$\qquad \mathsf{tell}_{\mathsf{Writer} \, \mathsf{Int}} \, 42$
$\quad \mathbf{with} \, \{ \, \mathbf{return} \, x \mapsto 0 \} \uplus \{\mathsf{tell} \, p \, k \mapsto \mathbf{if} \, p \, \mathbf{then} \, 0 \, \mathbf{else} \, 42\}$
$\mathbf{with} \, \{ \, \mathbf{return} \, x \mapsto x \} \uplus \{\mathsf{tell} \, p \, k \mapsto p\}$

This program is well typed because
  - the operation call $\mathsf{tell}_{\mathsf{Writer} \, \mathsf{Int}} \, 42$ can have effect $\{\mathsf{Writer} \, \mathsf{Bool}\} \sqcup \{\mathsf{Writer} \, \mathsf{Int}\}$ via subeffecting $\{\mathsf{Writer} \, \mathsf{Int}\} \oslash \{\mathsf{Writer} \, \mathsf{Bool}\} \sqcup \{\mathsf{Writer} \, \mathsf{Int}\}$ (which holds because $\mathsf{Writer} \, \mathsf{Int}$ and $\mathsf{Writer} \, \mathsf{Bool}$ are exchangeable),

  - the inner handling expression is well typed and its effect is $\{\mathsf{Writer} \, \mathsf{Int}\}$, and

  - the outer one is well typed and its effect is $\{\}$.

Note that this typing rests on the fact that the inner handler assumes that the argument variable p of its tell clause will be replaced by Boolean values as indicated by the type argument Bool to Writer. However, this program reaches the stuck state: because the operation call is handled by the innermost handler for the label *name* Writer, the inner handler is chosen and then the Boolean parameter p of the tell clause in it will be replaced by integer 42. ∎

4 Comparison of Instances and Previous Work

4.1 Comparison to [Pretnar(2015)]

We define the targets of comparison: one is an instance of λ_{EA} (Example 1.23), and another is a minor changed language of [Pretnar(2015)].

Definition 4.1 (Minor Changed Version of [Pretnar(2015)]). *Change list:*
- *removing Boolean and if expressions,*
- *removing handlers from values and handler types from types,*
- *adding well-formedness of contexts and type, and*
- *adding well-formedness of dirt to the return rule.*

The syntax of a minor changed version of [Pretnar(2015)] is as follows.

$$
\begin{array}{rcll}
A, B & ::= & A \to \underline{C} & \textit{(value types)} \\
\underline{C}, \underline{D} & ::= & A!\Delta & \textit{(computation types)} \\
\Delta & ::= & \{\mathsf{op}_1, \ldots, \mathsf{op}_n\} & \textit{(dirt)} \\
v & ::= & x \mid \mathbf{fun}\, x \mapsto c & \textit{(values)} \\
c & ::= & \mathbf{return}\, v \mid \mathsf{op}(v; y.c) \mid \mathbf{do}\, x \leftarrow c_1 \,\mathbf{in}\, c_2 \mid v_1\, v_2 \mid & \textit{(computation)} \\
& & \mathbf{with}\, h\, \mathbf{handle}\, c & \\
h & ::= & \mathbf{handler}\, \{\mathbf{return}\, x \mapsto c_r, \mathsf{op}_1(x_1; k_1) \mapsto c_1, \ldots, \mathsf{op}_n(x_n; k_n) \mapsto c_n\} & \textit{(handlers)} \\
\Sigma & ::= & \{\mathsf{op}_1 : A_1 \to B_1, \ldots, \mathsf{op}_n : A_n \to B_n\} & \textit{(signature)} \\
\Gamma & ::= & \emptyset \mid \Gamma, x : A & \textit{(typing contexts)}
\end{array}
$$

Well-formedness rules consist of the following.

Contexts Well-formedness $\boxed{\vdash \Gamma}$

$$
\frac{}{\vdash \emptyset}\ \text{Cp\_Empty} \qquad \frac{x \notin \mathrm{dom}(\Gamma) \quad \Gamma \vdash A}{\vdash \Gamma, x : A}\ \text{Cp\_Var}
$$

Kinding $\boxed{\Gamma \vdash A}$

$$
\frac{\Gamma \vdash A \quad \Delta \subseteq \mathrm{dom}(\Sigma) \quad \Gamma \vdash B}{\Gamma \vdash A \to B!\Delta}\ \text{Kp\_Fun}
$$

Typing $\boxed{\Gamma \vdash v : A}$ $\boxed{\Gamma \vdash c : \underline{C}}$

$$
\frac{\vdash \Gamma \quad x : A \in \Gamma}{\Gamma \vdash x : A}\ \text{Tp\_Var} \qquad \frac{\Gamma, x : A \vdash c : \underline{C}}{\Gamma \vdash \mathbf{fun}\, x \mapsto c : A \to \underline{C}}\ \text{Tp\_Abs} \qquad \frac{\Gamma \vdash v : A \quad \Delta \subseteq \mathrm{dom}(\Sigma)}{\Gamma \vdash \mathbf{return}\, v : A!\Delta}\ \text{Tp\_Return}
$$

$$
\frac{\Gamma \vdash v_1 : A \to \underline{C} \quad \Gamma \vdash v_2 : A}{\Gamma \vdash v_1\, v_2 : \underline{C}}\ \text{Tp\_App}
$$

$$
\frac{\mathsf{op} : A \to B \in \Sigma \quad \Gamma \vdash v : A \quad \Gamma, y : B \vdash c : A_0!\Delta \quad \mathsf{op} \in \Delta}{\Gamma \vdash \mathsf{op}(v; y.c) : A_0!\Delta}\ \text{Tp\_OpApp}
$$

$$
\frac{\Gamma \vdash c_1 : A!\Delta \quad \Gamma, x : A \vdash c_2 : B!\Delta}{\Gamma \vdash \mathbf{do}\, x \leftarrow c_1 \,\mathbf{in}\, c_2 : B!\Delta}\ \text{Tp\_Do} \qquad \frac{\Gamma \vdash c : \underline{C} \quad \Gamma \vdash h : \underline{C} \Rightarrow \underline{D}}{\Gamma \vdash \mathbf{with}\, h\, \mathbf{handle}\, c : \underline{D}}\ \text{Tp\_Handle}
$$

Handler Typing $\boxed{\Gamma \vdash h : \underline{C} \Rightarrow \underline{D}}$

$$
\frac{\begin{array}{c} \Gamma, x : A \vdash c_r : B!\Delta' \quad \Delta \setminus \{\mathsf{op}_1, \ldots, \mathsf{op}_n\} \subseteq \Delta' \\ \forall i \in \{1, \ldots, n\}.(\mathsf{op}_i : A_i \to B_i \in \Sigma \quad \Gamma, x_i : A_i, k_i : B_i \to B!\Delta' \vdash c_i : B!\Delta') \end{array}}{\Gamma \vdash \mathbf{handler}\, \{\mathbf{return}\, x \mapsto c_r, \mathsf{op}_1(x_1; k_1) \mapsto c_1, \ldots, \mathsf{op}_n(x_n; k_n) \mapsto c_n\} : A!\Delta \Rightarrow B!\Delta'}\ \text{Hp\_Handler}
$$

Definition 4.2 (Translation from Pretnars to An Instance). *We assume that[1]:*

---

[1] These assumptions arise from our formalization of labels and operations. They are easily removed if we omit labels.

- *there exists a unique partition of Σ,*

- *any dirt is a disjoint union of the partition results of Σ, and*

- *target operations of any handlers must be one of the partition results of Σ.*

We write $\mathtt{S2s}(\Sigma)$ to denote the set of the partition results of Σ. We write $\mathtt{d2l}$ to denote the function that assigns unique label l such that $l : \mathbf{Lab} \in \Sigma_{\mathrm{lab}}$ to $s \in \mathtt{S2s}(\Sigma)$.

We define $\mathtt{d2l}(\Delta)$ as the labels whose label is $\mathtt{d2l}(s)$ where $\mathrm{dom}(s) \subseteq \Delta$ and $s \in \mathtt{S2s}(\Sigma)$. We define $\mathtt{d2l}(h)$ as $\mathtt{d2l}(s)$ where $h = \mathbf{handler}\,\{\mathbf{return}\,x \mapsto c_r, \mathsf{op}_1(x_1; k_1) \mapsto c_1, \ldots, \mathsf{op}_n(x_n; k_n) \mapsto c_n\}$ and $s = \{\mathsf{op}_1 : A_1 \rightarrow B_1, \ldots, \mathsf{op}_n : A_n \rightarrow B_n\}$.

We define $\mathtt{P2I}$ as follows.

Types

$$\mathtt{P2I}(A \rightarrow B!\Delta) = \mathtt{P2I}(A) \rightarrow_{\mathtt{P2I}(\Delta)} \mathtt{P2I}(B)$$

Dirts

$$\mathtt{P2I}(\emptyset) \;=\; \{\} \qquad \mathtt{P2I}(\Delta \uplus \mathrm{dom}(s)) \;=\; \mathtt{P2I}(\Delta) \cup \{\mathtt{d2l}(s)\} \quad (\textit{if } s \in \mathtt{S2s}(\Sigma))$$

Values

$$\mathtt{P2I}(x) \;=\; x \qquad \mathtt{P2I}(\mathbf{fun}\,x \mapsto c) \;=\; \mathbf{fun}(f, x, \mathtt{P2I}(c)) \quad (\textit{where } f \textit{ is fresh})$$

Computations

$$
\begin{aligned}
\mathtt{P2I}(\mathbf{return}\,v) &= \mathtt{P2I}(v) \\
\mathtt{P2I}(v_1\,v_2) &= \mathtt{P2I}(v_1)\,\mathtt{P2I}(v_2) \\
\mathtt{P2I}(\mathbf{do}\,x \leftarrow c_1 \,\mathbf{in}\, c_2) &= \mathbf{let}\,x = \mathtt{P2I}(c_1)\,\mathbf{in}\,\mathtt{P2I}(c_2) \\
\mathtt{P2I}(\mathsf{op}(v; y.c)) &= \mathbf{let}\,y = \mathsf{op}_{\mathtt{d2l}(s)}\,\mathtt{P2I}(v)\,\mathbf{in}\,\mathtt{P2I}(c) \quad (\textit{where } \mathsf{op} \in \mathrm{dom}(s)) \\
\mathtt{P2I}(\mathbf{with}\,h\,\mathbf{handle}\,c) &= \mathbf{handle}_{\mathtt{d2l}(h)}\,\mathtt{P2I}(c)\,\mathbf{with}\,\mathtt{P2I}(h)
\end{aligned}
$$

Handlers

$$
\begin{aligned}
\mathtt{P2I}(h) \;=\;& \{\mathbf{return}\,x \mapsto \mathtt{P2I}(c_r)\} \uplus \{\mathsf{op}_1\,x_1\,k_1 \mapsto \mathtt{P2I}(c_1)\} \uplus \cdots \uplus \{\mathsf{op}_n\,x_n\,k_n \mapsto \mathtt{P2I}(c_n)\} \\
& (\textit{where } h = \mathbf{handler}\,\{\mathbf{return}\,x \mapsto c_r, \mathsf{op}_1(x_1; k_1) \mapsto c_1, \ldots, \mathsf{op}_n(x_n; k_n) \mapsto c_n\})
\end{aligned}
$$

Effect contexts

$$
\begin{aligned}
\mathtt{P2I}(\Sigma) \;=\;& \textstyle\bigcup_{s \in \mathtt{S2s}(\Sigma)}\{\mathtt{d2l}(s) :: \{\mathsf{op}_1 : \mathtt{P2I}(A_1) \Rightarrow \mathtt{P2I}(B_1), \ldots, \mathsf{op}_n : \mathtt{P2I}(A_n) \Rightarrow \mathtt{P2I}(B_n)\}\} \\
& (\textit{where } s = \{\mathsf{op}_1 : A_1 \rightarrow B_1, \ldots, \mathsf{op}_n : A_n \rightarrow B_n\})
\end{aligned}
$$

Typing Contexts

$$\mathtt{P2I}(\emptyset) \;=\; \emptyset \qquad \mathtt{P2I}(\Gamma, x : A) \;=\; \mathtt{P2I}(\Gamma), x : \mathtt{P2I}(A)$$

Lemma 4.3. $\mathrm{dom}(\Gamma) = \mathrm{dom}(\mathtt{P2I}(\Gamma))$.

Proof. Clearly by definition of $\mathtt{P2I}$. ∎

Lemma 4.4. *If $\Delta \subseteq \mathrm{dom}(\Sigma)$, then $\Gamma \vdash \mathtt{P2I}(\Delta) : \mathbf{Eff}$ for any Γ such that $\vdash \Gamma$.*

Proof. By induction on the size of Γ.

If $\Delta = \emptyset$, then clearly because $\mathtt{P2I}(\emptyset) = \{\}$.

If $\Delta = \Delta' \uplus \mathrm{dom}(s)$ for some Δ' and $s \in \mathtt{S2s}(\Sigma)$, then $\mathtt{P2I}(\Delta) = \mathtt{P2I}(\Delta') \sqcup \{\mathtt{d2l}(s)\}$ where $\mathtt{d2l}(s) : \mathbf{Lab} \in \Sigma_{\mathrm{lab}}$. Let Γ be a typing context such that $\vdash \Gamma$. By the induction hypothesis, we have $\Gamma \vdash \mathtt{P2I}(\Delta') : \mathbf{Eff}$. Thus, K_CONS derives $\Gamma \vdash \mathtt{P2I}(\Delta') \sqcup \{\mathtt{d2l}(s)\} : \mathbf{Eff}$ because we have $\Gamma \vdash \{\mathtt{d2l}(s)\} : \mathbf{Eff}$. ∎

Lemma 4.5. *If $\vdash \Gamma$ and $x : A \in \Gamma$, then $x : \mathtt{P2I}(A) \in \mathtt{P2I}(\Gamma)$.*

Proof. By structural induction on Γ.

If $\Gamma = \emptyset$, then $x : A \in \Gamma$ cannot happen.

If $\Gamma = \Gamma', y : B$ for some y, B, and Γ', then we have $\mathtt{P2I}(\Gamma) = \mathtt{P2I}(\Gamma'), y : \mathtt{P2I}(B)$. In this case, if $x = y$, then we have $A = B$ and $y : \mathtt{P2I}(B) \in \mathtt{P2I}(\Gamma)$ as required. If $x \neq y$, then we have $x : A \in \Gamma'$. By the induction hypothesis, we have $x : \mathtt{P2I}(A) \in \mathtt{P2I}(\Gamma')$. Thus, we have $x : \mathtt{P2I}(A) \in \mathtt{P2I}(\Gamma)$ as required. ∎

Theorem 4.6.

(1) *If* $\vdash \Gamma$*, then* $\vdash \mathtt{P2I}(\Gamma)$*.*

(2) *If* $\Gamma \vdash A$*, then* $\mathtt{P2I}(\Gamma) \vdash \mathtt{P2I}(A) : \mathbf{Typ}$*.*

(3) *If* $\Gamma \vdash v : A$*, then* $\mathtt{P2I}(\Gamma) \vdash \mathtt{P2I}(v) : \mathtt{P2I}(A) \mid \{\}$*.*

(4) *If* $\Gamma \vdash c : A!\Delta$*, then* $\mathtt{P2I}(\Gamma) \vdash \mathtt{P2I}(c) : \mathtt{P2I}(A) \mid \mathtt{P2I}(\Delta)$*.*

(5) *If* $\Gamma \vdash h : A!\Delta \Rightarrow B!\Delta'$*, then* $\mathtt{P2I}(\Delta) \cup \varepsilon \sim_{\mathrm{Set}} \mathtt{d2l}(h) \cup \mathtt{P2I}(\Delta')$ *for some* ε *and there exists some* σ *such that* $\mathtt{P2I}(\Gamma) \vdash_\sigma \mathtt{P2I}(h) : \mathtt{P2I}(A) \Rightarrow^{\mathtt{P2I}(\Delta')} \mathtt{P2I}(B)$ *and* $\mathtt{d2l}(h) :: \sigma \in \mathtt{P2I}(\Sigma)$*.*

Proof.(1)(2) By mutual induction on derivations of the judgments. We proceed by case analysis on the rule applied lastly to the derivation.

Case CP_EMPTY: Clearly.

Case CP_VAR: We have
- $\Gamma = \Gamma', x : A$,
- $x \notin \mathrm{dom}(\Gamma')$, and
- $\Gamma' \vdash A$,

for some x, A, and Γ'. By the induction hypothesis and Lemma 4.3, we have $x \notin \mathrm{dom}(\mathtt{P2I}(\Gamma'))$ and $\mathtt{P2I}(\Gamma') \vdash \mathtt{P2I}(A) : \mathbf{Typ}$. Thus, C_VAR derives $\vdash \mathtt{P2I}(\Gamma'), x : \mathtt{P2I}(A)$ as required.

Case KP_FUN: We have
- $A = A_1 \to B_1!\Delta$,
- $\Gamma \vdash A_1$,
- $\Delta \subseteq \mathrm{dom}(\Sigma)$, and
- $\Gamma \vdash B_1$,

for some A_1, B_1, and Δ. By the induction hypothesis and Lemma 4.4, we have
- $\mathtt{P2I}(\Gamma) \vdash \mathtt{P2I}(A_1) : \mathbf{Typ}$,
- $\mathtt{P2I}(\Gamma) \vdash \mathtt{P2I}(\Delta) : \mathbf{Eff}$, and
- $\mathtt{P2I}(\Gamma) \vdash \mathtt{P2I}(B_1) : \mathbf{Typ}$.

Thus, K_FUN derives

$$\mathtt{P2I}(\Gamma) \vdash \mathtt{P2I}(A_1) \to_{\mathtt{P2I}(\Delta)} \mathtt{P2I}(B_1) : \mathbf{Typ}$$

as required.

(3)(4)(5) By mutual induction on derivations of the judgments. We proceed by case analysis on the rule applied lastly to the derivation.

Case TP_VAR: We have
- $v = x$,
- $\vdash \Gamma$, and
- $x : A \in \Gamma$,

for some x. By Lemma 4.5 and Theorem 4.6(1), we have
- $\vdash \mathtt{P2I}(\Gamma)$ and
- $x : \mathtt{P2I}(A) \in \mathtt{P2I}(\Gamma)$.

Thus, T_VAR derives

$$\mathtt{P2I}(\Gamma) \vdash x : \mathtt{P2I}(A) \mid \{\}$$

as required.

Case TP_ABS: We have
- $v = \mathbf{fun}\, x \mapsto c$ and
- $\Gamma, x : A \vdash c : B!\Delta$

for some x, c, A, B, and Δ. By the induction hypothesis, we have

$$\mathtt{P2I}(\Gamma), x : \mathtt{P2I}(A) \vdash \mathtt{P2I}(c) : \mathtt{P2I}(B) \mid \mathtt{P2I}(\Delta).$$

Without loss of generality, we can choose f such that

- $f \notin FV(\mathtt{P2I}(c))$,
- $f \neq x$,
- $f \notin \mathrm{dom}(\Gamma)$, and
- $\mathtt{P2I}(\mathbf{fun}\, x \mapsto c) = \mathbf{fun}(f, x, \mathtt{P2I}(c))$.

By Lemma 3.12 and Lemma 3.2(2) and Lemma 3.6, we have
- $\mathtt{P2I}(\Gamma) \vdash \mathtt{P2I}(B) : \mathbf{Typ}$ and
- $\mathtt{P2I}(\Gamma) \vdash \mathtt{P2I}(\Delta) : \mathbf{Eff}$.

By Lemma 3.9, we have $\vdash \mathtt{P2I}(\Gamma), x : \mathtt{P2I}(A)$. Since only C_Var can derive $\vdash \mathtt{P2I}(\Gamma), x : \mathtt{P2I}(A)$, we have $\mathtt{P2I}(\Gamma) \vdash \mathtt{P2I}(A) : \mathbf{Typ}$. Thus, C_Var derives

$$\vdash \mathtt{P2I}(\Gamma), f : \mathtt{P2I}(A) \rightarrow_{\mathtt{P2I}(\Delta)} \mathtt{P2I}(B).$$

Thus, Lemma 3.5 and T_Abs derives

$$\mathtt{P2I}(\Gamma) \vdash \mathbf{fun}(f, x, \mathtt{P2I}(c)) : \mathtt{P2I}(A) \rightarrow_{\mathtt{P2I}(\Delta)} \mathtt{P2I}(B) \mid \{\}$$

as required.

Case Tp_Return: We have
- $c = \mathbf{return}\, v$,
- $\Gamma \vdash v : A$, and
- $\Delta \subseteq \mathrm{dom}(\Sigma)$,

for some v. By the induction hypothesis and Lemma 4.4, we have
- $\mathtt{P2I}(\Gamma) \vdash \mathtt{P2I}(v) : \mathtt{P2I}(A) \mid \{\}$ and
- $\mathtt{P2I}(\Gamma) \vdash \mathtt{P2I}(\Delta) : \mathbf{Eff}$.

Thus, T_Sub derives
$$\mathtt{P2I}(\Gamma) \vdash \mathtt{P2I}(v) : \mathtt{P2I}(A) \mid \mathtt{P2I}(\Delta)$$

as required.

Case Tp_App: We have
- $c = v_1\, v_2$,
- $\Gamma \vdash v_1 : B \rightarrow A!\Delta$, and
- $\Gamma \vdash v_2 : B$,

for some v_1, v_2, and B. By the induction hypothesis, we have
- $\mathtt{P2I}(\Gamma) \vdash \mathtt{P2I}(v_1) : \mathtt{P2I}(B) \rightarrow_{\mathtt{P2I}(\Delta)} \mathtt{P2I}(A) \mid \{\}$ and
- $\mathtt{P2I}(\Gamma) \vdash \mathtt{P2I}(v_2) : \mathtt{P2I}(B) \mid \{\}$.

Thus, T_App derives
$$\mathtt{P2I}(\Gamma) \vdash \mathtt{P2I}(v_1)\, \mathtt{P2I}(v_2) : \mathtt{P2I}(A) \mid \mathtt{P2I}(\Delta)$$

as required.

Case Tp_OpApp: We have
- $c = \mathsf{op}(v; y.c')$,
- $\mathsf{op} : A' \rightarrow B' \in \Sigma$,
- $\Gamma \vdash v : A'$,
- $\Gamma, y : B' \vdash c' : A!\Delta$, and
- $\mathsf{op} \in \Delta$,

for some op, v, y, c', A', B', and Δ. By $\mathsf{op} \in \Delta$, there uniquely exists some s such that
- $s \in \mathtt{S2s}(\Sigma)$,
- $\mathsf{op} : A' \rightarrow B' \in s$,
- $\mathrm{dom}(s) \subseteq \Delta$.

Thus, we have
- $l :: \sigma \in \mathtt{P2I}(\Sigma)$,
- $\mathsf{op} : \mathtt{P2I}(A') \Rightarrow \mathtt{P2I}(B') \in \sigma$, and
- $\{\mathtt{d2l}(s)\} \sqcup \varepsilon \sim_{\mathrm{Set}} \mathtt{P2I}(\Delta)$,

for some σ. By the induction hypothesis, we have

- $\texttt{P2I}(\Gamma) \vdash \texttt{P2I}(v) : \texttt{P2I}(A') \mid \{\}$ and
- $\texttt{P2I}(\Gamma), y : \texttt{P2I}(B') \vdash \texttt{P2I}(c') : \texttt{P2I}(A) \mid \texttt{P2I}(\Delta)$.

Thus, T_OP and T_APP derives

$$\texttt{P2I}(\Gamma) \vdash \mathsf{op}_{\mathsf{d2l}(s)}\, \texttt{P2I}(v) : \texttt{P2I}(B') \mid \{\mathsf{d2l}(s)\}.$$

Thus, T_SUB and T_LET derives

$$\texttt{P2I}(\Gamma) \vdash \mathbf{let}\, y = \mathsf{op}_{\mathsf{d2l}(s)}\, \texttt{P2I}(v)\, \mathbf{in}\, \texttt{P2I}(c') : \texttt{P2I}(A) \mid \texttt{P2I}(\Delta)$$

as required.

Case TP_DO**:** We have
- $c = \mathbf{do}\, x \leftarrow c_1\, \mathbf{in}\, c_2$,
- $\Gamma \vdash c_1 : B!\Delta$, and
- $\Gamma, x : B \vdash c_2 : A!\Delta$,

for some x, c_1, c_2, and Δ. By the induction hypothesis, we have
- $\texttt{P2I}(\Gamma) \vdash \texttt{P2I}(c_1) : \texttt{P2I}(B) \mid \texttt{P2I}(\Delta)$ and
- $\texttt{P2I}(\Gamma), x : \texttt{P2I}(B) \vdash \texttt{P2I}(c_2) : \texttt{P2I}(A) \mid \texttt{P2I}(\Delta)$.

Thus, T_LET derives

$$\texttt{P2I}(\Gamma) \vdash \mathbf{let}\, x = \texttt{P2I}(c_1)\, \mathbf{in}\, \texttt{P2I}(c_2) : \texttt{P2I}(A) \mid \texttt{P2I}(\Delta)$$

as required.

Case TP_HANDLE**:** We have
- $c = \mathbf{with}\, h\, \mathbf{handle}\, c'$,
- $\Gamma \vdash c' : A'!\Delta'$, and
- $\Gamma \vdash h : A'!\Delta' \Rightarrow A!\Delta$.

for some c', h, A', and Δ'. By the induction hypothesis, we have
- $\texttt{P2I}(\Gamma) \vdash \texttt{P2I}(c') : \texttt{P2I}(A') \mid \texttt{P2I}(\Delta')$,
- $\mathsf{d2l}(h) :: \sigma \in \texttt{P2I}(\Sigma)$,
- $\texttt{P2I}(\Gamma) \vdash_\sigma \texttt{P2I}(h) : \texttt{P2I}(A') \Rightarrow^{\texttt{P2I}(\Delta)} \texttt{P2I}(A)$, and
- $\texttt{P2I}(\Delta') \uplus \varepsilon \sim_{\mathrm{Set}} \mathsf{d2l}(h) \uplus \texttt{P2I}(\Delta)$,

for some ε and σ. Thus, T_SUB and T_HANDLING derive

$$\texttt{P2I}(\Gamma) \vdash \mathbf{handle}_{\texttt{P2I}(h)}\, \texttt{P2I}(c')\, \mathbf{with}\, \texttt{P2I}(h) : \texttt{P2I}(A) \mid \texttt{P2I}(\Delta)$$

as required.

Case HP_HANDLER**:** We have
- $h = \mathbf{handler}\, \{\mathbf{return}\, x \mapsto c_r, \mathsf{op}_1(x_1; k_1) \mapsto c_1, \ldots, \mathsf{op}_n(x_n; k_n) \mapsto c_n\}$,
- $\Gamma, x : A \vdash c_r : B!\Delta'$,
- $\mathsf{op}_i : A_i \to B_i \in \Sigma$ for any $i \in \{1, \ldots, n\}$,
- $\Gamma, x_i : A_i, k_i : B_i \to B!\Delta' \vdash c_i : B!\Delta'$ for any $i \in \{1, \ldots, n\}$, and
- $\Delta \setminus \{\mathsf{op}_1, \ldots, \mathsf{op}_n\} \subseteq \Delta'$,

for some n, x, c_r, op_i, x_i, k_i, c_i, A_i, and B_i, where $i \in \{1, \ldots, n\}$.
By the assumptions, we have
- $s \in \texttt{S2s}(\Sigma)$ and
- $\mathsf{d2l}(h) = \mathsf{d2l}(s)$

where $s = \{\mathsf{op}_1 : A_1 \to B_1, \ldots, \mathsf{op}_n : A_n \to B_n\}$. Thus, we have $\mathsf{d2l}(h) :: \sigma \in \texttt{P2I}(\Sigma)$ where $\sigma = \{\mathsf{op}_1 : \texttt{P2I}(A_1) \Rightarrow \texttt{P2I}(B_1), \ldots, \mathsf{op}_n : \texttt{P2I}(A_n) \Rightarrow \texttt{P2I}(B_n)\}$.
By $\Delta \setminus \{\mathsf{op}_1, \ldots, \mathsf{op}_n\} \subseteq \Delta'$, we have $\Delta \subseteq \mathrm{dom}(s) \cup \Delta'$. By the assumptions, we have either $\mathrm{dom}(s) \subseteq \Delta'$ or $\mathsf{op}_i \notin \Delta'$ for any i. In any case, we have $\texttt{P2I}(\Delta) \uplus \varepsilon \sim_{\mathrm{Set}} \mathsf{d2l}(h) \uplus \texttt{P2I}(\Delta')$ for some ε.
By the induction hypothesis, we have
- $\texttt{P2I}(\Gamma), x : \texttt{P2I}(A) \vdash \texttt{P2I}(c_r) : \texttt{P2I}(B) \mid \texttt{P2I}(\Delta')$ and
- $\texttt{P2I}(\Gamma), x_i : \texttt{P2I}(A_i), k_i : \texttt{P2I}(B_i) \to_{\texttt{P2I}(\Delta')} \texttt{P2I}(B) \vdash \texttt{P2I}(c_i) : \texttt{P2I}(B) \mid \texttt{P2I}(\Delta')$ for any $i \in \{1, \ldots, n\}$.

Therefore, H_RETURN and H_OP derive $\texttt{P2I}(\Gamma) \vdash_\sigma \texttt{P2I}(h) : \texttt{P2I}(A) \Rightarrow^{\texttt{P2I}(\Delta')} \texttt{P2I}(B)$.
Thus, the required result is achieved.

$\blacksquare$

4.2 Comparison to [Hillerström et al.(2017)]

We give the targets of comparison: one is an instance of λ_{EA} (Example 1.25), and another is a minorly changed language of [Hillerström et al.(2017)].

Definition 4.7 (Minor Changed Version of [Hillerström et al.(2017)]). *Change list:*
- *removing variants and records,*
- *removing presence and handler types,*
- *removing computation kinds, and*
- *adding well-formedness rules of contexts.*

The syntax of a minor changed version of [Hillerström et al.(2017)] is as follows.

$$
\begin{array}{rcll}
V, W & ::= & x \mid \lambda x^A.M \mid \Lambda \alpha^K.M & \textit{(values)} \\
M, N & ::= & V\,W \mid V\,T \mid \textbf{return}\,M \mid \textbf{let}\,x \leftarrow M \,\textbf{in}\,N & \textit{(computations)} \\
 & & \mid (\textbf{do}\,l\,V)^E \mid \textbf{handle}\,M\,\textbf{with}\,H & \\
H & ::= & \{\textbf{return}\,x \mapsto M\} \mid H \uplus \{l\,p\,r \mapsto M\} & \textit{(handlers)} \\
A, B & ::= & A \rightarrow C \mid \forall \alpha^K.C \mid \alpha & \textit{(value types)} \\
C, D & ::= & A!E & \textit{(computations types)} \\
E & ::= & \{R\} & \textit{(effect types)} \\
R & ::= & l : P; R \mid \rho \mid \cdot & \textit{(row types)} \\
P & ::= & \mathsf{Pre}(A \rightarrow B) \mid \mathsf{Abs} & \textit{(presence types)} \\
T & ::= & A \mid C \mid E \mid R & \textit{(types)} \\
K & ::= & \mathsf{Type} \mid \mathsf{Row}_{\mathcal{L}} \mid \mathsf{Effect} & \textit{(kinds)} \\
\mathcal{L} & ::= & \emptyset \mid \{l\} \uplus \mathcal{L} & \textit{(label sets)} \\
\Gamma & ::= & \cdot \mid \Gamma, x : A & \textit{(type environments)} \\
\Delta & ::= & \cdot \mid \Delta, \alpha : K & \textit{(kind environments)}
\end{array}
$$

Well-formedness, kinding, and typing rules consist of the following.

Kinding Contexts Well-formedness $\boxed{\vdash \Delta}$

$$
\frac{}{\vdash \cdot}\ \text{KCh\_Empty} \qquad \frac{\vdash \Delta \quad \alpha \notin \mathrm{dom}(\Delta)}{\vdash \Delta, \alpha : K}\ \text{KCh\_TVar}
$$

Contexts Well-formedness $\boxed{\vdash \Delta; \Gamma}$

$$
\frac{\vdash \Delta}{\Delta \vdash \cdot}\ \text{Ch\_Empty} \qquad \frac{\Delta \vdash \Gamma \quad x \notin \mathrm{dom}(\Gamma) \quad \Delta \vdash A : \mathsf{Type}}{\Delta \vdash \Gamma, x : A}\ \text{Ch\_Var}
$$

Kinding $\boxed{\Delta \vdash T : K}$

$$
\frac{\vdash \Delta, \alpha : K}{\Delta, \alpha : K \vdash \alpha : K}\ \text{Kh\_Var} \qquad \frac{\Delta \vdash A : \mathsf{Type} \quad \Delta \vdash B : \mathsf{Type} \quad \Delta \vdash E : \mathsf{Effect}}{\Delta \vdash A \rightarrow B!E : \mathsf{Type}}\ \text{Kh\_Fun}
$$

$$
\frac{\Delta, \alpha : K \vdash A : \mathsf{Type} \quad \Delta, \alpha : K \vdash E : \mathsf{Effect}}{\Delta \vdash \forall \alpha^K.A!E : \mathsf{Type}}\ \text{Kh\_Forall} \qquad \frac{\Delta \vdash R : \mathsf{Row}_{\emptyset}}{\Delta \vdash \{R\} : \mathsf{Effect}}\ \text{Kh\_Effect}
$$

$$
\frac{\forall i \in \{1, \ldots, n\}.(P_i = \mathsf{Abs}\ \text{or}\ (P_i = \mathsf{Pre}(A_i \rightarrow B_i)\ \text{and}\ \Delta \vdash A_i : \mathsf{Type}\ \text{and}\ \Delta \vdash B_i : \mathsf{Type})) \quad \vdash \Delta}{\Delta \vdash l_1 : P_1; \cdots ; l_n : P_n; \cdot : \mathsf{Row}_{\emptyset}}\ \text{Kh\_CloseRow}
$$

$$
\frac{\begin{array}{c}\forall i \in \{1, \ldots, n\}.(P_i = \mathsf{Abs}\ \text{or}\ (P_i = \mathsf{Pre}(A_i \rightarrow B_i)\ \text{and}\ \Delta \vdash A_i : \mathsf{Type}\ \text{and}\ \Delta \vdash B_i : \mathsf{Type})) \\ \Delta \vdash \rho : \mathsf{Row}_{\mathcal{L}} \quad \mathcal{L} = \{l_1, \ldots, l_n\}\end{array}}{\Delta \vdash l_1 : P_1; \cdots ; l_n : P_n; \rho : \mathsf{Row}_{\emptyset}}\ \text{Kh\_OpenRow}
$$

Typing $\boxed{\Delta;\Gamma \vdash V : A}$ $\boxed{\Delta;\Gamma \vdash M : C}$

$$\frac{\Delta \vdash \Gamma \quad x : A \in \Gamma}{\Delta;\Gamma \vdash x : A} \ \text{Th\_Var} \qquad \frac{\Delta;\Gamma, x : A \vdash M : C}{\Delta;\Gamma \vdash \lambda x^A.M : A \to C} \ \text{Th\_Lam}$$

$$\frac{\Delta, \alpha : K;\Gamma \vdash M : C \quad \Delta \vdash \Gamma}{\Delta;\Gamma \vdash \Lambda\alpha^K.M : \forall\alpha^K.C} \ \text{Th\_PolyLam} \qquad \frac{\Delta;\Gamma \vdash V : A \to C \quad \Delta;\Gamma \vdash W : A}{\Delta;\Gamma \vdash V\,W : C} \ \text{Th\_App}$$

$$\frac{\Delta;\Gamma \vdash V : \forall\alpha^K.C \quad \Delta \vdash T : K}{\Delta;\Gamma \vdash V\,T : C[T/\alpha]} \ \text{Th\_PolyApp} \qquad \frac{\Delta;\Gamma \vdash V : A \quad \Delta \vdash E : \mathsf{Effect}}{\Delta;\Gamma \vdash \mathbf{return}\,V : A!E} \ \text{Th\_Return}$$

$$\frac{\Delta;\Gamma \vdash M : A!E \quad \Delta;\Gamma, x : A \vdash N : B!E}{\Delta;\Gamma \vdash \mathbf{let}\,x \leftarrow M \,\mathbf{in}\, N : B!E} \ \text{Th\_Let}$$

$$\frac{\Delta;\Gamma \vdash V : A \quad E = \{l : \mathsf{Pre}(A \to B); R\} \quad \Delta \vdash E : \mathsf{Effect}}{\Delta;\Gamma \vdash (\mathbf{do}\,l\,V)^E : B!E} \ \text{Th\_Do}$$

$$\frac{\Delta;\Gamma \vdash M : C \quad \Delta;\Gamma \vdash H : C \Rightarrow D}{\Delta;\Gamma \vdash \mathbf{handle}\,M\,\mathbf{with}\,H : D} \ \text{Th\_Handle}$$

Handler Typing $\boxed{\Delta;\Gamma \vdash H : C \Rightarrow D}$

$$\frac{\begin{array}{c}C = A!\{l_1 : \mathsf{Pre}(A_1 \to B_1); \cdots ; l_n : \mathsf{Pre}(A_n \to B_n); R\} \\ D = B!\{l_1 : P_1; \cdots ; l_n : P_n; R\} \quad H = \{\mathbf{return}\,x \mapsto M\} \uplus \{l_1\,y_1\,r_1 \mapsto N_1\} \uplus \cdots \uplus \{l_n\,y_n\,r_n \mapsto N_n\} \\ \Delta;\Gamma, x : A \vdash M : D \quad \forall i \in \{1, \ldots, n\}.(\Delta;\Gamma, y_i : A_i, r_i : B_i \to D \vdash N_i : D) \end{array}}{\Delta;\Gamma \vdash H : C \Rightarrow D} \ \text{Hh\_Handler}$$

Definition 4.8 (Translation from Hillerström's to An Instance). *We assume that:*

- *there exists a unique set that has any label (we call it $\mathbb{L}$),*

- *there exists a unique partition of $\mathbb{L}$,*

- *for any row, a set of presence labels in that row is a disjoint union of the partition result of $\mathbb{L}$,*

- *for any handler, target labels of that handler is one of the partition result of $\mathbb{L}$, and*

- *a unique closed type can be attached to l as presence.*

We write $\mathtt{L2S}(\mathbb{L})$ to denote the set of partition results of $\mathbb{L}$, $\mathtt{r2l}$ to denote the function that assigns a unique label l such that $l : \mathbf{Lab} \in \Sigma_{\mathrm{eff}}$ to $\mathcal{L} \in \mathtt{L2S}(\mathbb{L})$. We write $\mathtt{r2l}(H)$ to denote l such that $\mathtt{r2l}(\{l_1, \ldots, l_n\}) = l$ where $H = \{\mathbf{return}\,x \mapsto M\} \uplus \{l_1\,p_1\,r_1 \mapsto N_1\} \uplus \cdots \{l_n\,p_n\,r_n \mapsto N_n\}$. We define $\mathtt{l2T}$ as the function that takes a label l and returns the type that corresponds to the unique presence type of l. We define $\mathtt{l2Op}$ as the function that takes a label l and returns a unique operation name. We also assume that

$$\mathtt{r2l}(\{l_1, \ldots, l_n\}) :: \{\mathtt{l2Op}(l_1) : \mathtt{l2T}(l_1), \ldots, \mathtt{l2Op}(l_n) : \mathtt{l2T}(l_n)\} \in \Sigma.$$

We define $\mathtt{H2I}$ as follows.

Kinds

$$\mathtt{H2I}(\mathsf{Type}) \ = \ \mathbf{Typ} \qquad \mathtt{H2I}(\mathsf{Row}_{\mathcal{L}}) \ = \ \mathtt{H2I}(\mathsf{Effect}) = \mathbf{Eff}$$

Types

$$\mathtt{H2I}(A \to B!E) \ = \ \mathtt{H2I}(A) \to_{\mathtt{H2I}(E)} \mathtt{H2I}(B) \qquad \mathtt{H2I}(\forall\alpha^K.A!E) \ = \ \forall\alpha : \mathtt{H2I}(K).\mathtt{H2I}(A)^{\mathtt{H2I}(E)}$$

Effects

$$\begin{aligned}\mathtt{H2I}(\{R\}) \ &= \ \mathtt{H2I}(R) \\ \mathtt{H2I}(l_1 : P_1; \cdots ; l_n : P_n; \cdot) \ &= \ \langle l_1' \mid \langle \cdots \mid \langle l_m' \mid \langle\rangle\rangle\rangle\rangle \quad (\textit{where } l_i' = \mathtt{r2l}(\mathcal{L}_i) \textit{ and } \mathcal{L}_1 \uplus \cdots \mathcal{L}_m = \{l_j \mid P_j \neq \mathsf{Abs}\}) \\ \mathtt{H2I}(l_1 : P_1; \cdots ; l_n : P_n; \rho) \ &= \ \langle l_1' \mid \langle \cdots \mid \langle l_m' \mid \rho\rangle\rangle\rangle \quad (\textit{where } l_i' = \mathtt{r2l}(\mathcal{L}_i) \textit{ and } \mathcal{L}_1 \uplus \cdots \mathcal{L}_m = \{l_j \mid P_j \neq \mathsf{Abs}\})\end{aligned}$$

Values

$$\begin{aligned}\mathtt{H2I}(x) \ &= \ x \qquad \mathtt{H2I}(\Lambda\alpha^K.M) \ = \ \Lambda\alpha : \mathtt{H2I}(K).\mathtt{H2I}(M) \\ \mathtt{H2I}(\lambda x^A.M) \ &= \ \mathbf{fun}(z, x, \mathtt{H2I}(M)) \quad (\textit{where } z \textit{ is fresh})\end{aligned}$$

Computations

$$
\begin{aligned}
\texttt{H2I}(V\,W) &= \texttt{H2I}(V)\,\texttt{H2I}(W) & \texttt{H2I}(V\,T) &= \texttt{H2I}(V)\,\texttt{H2I}(T) \\
\texttt{H2I}(\mathbf{return}\,M) &= \texttt{H2I}(M) & \texttt{H2I}(\mathbf{let}\,x \leftarrow M\,\mathbf{in}\,N) &= \mathbf{let}\,x = \texttt{H2I}(M)\,\mathbf{in}\,\texttt{H2I}(N) \\
\texttt{H2I}((\mathbf{do}\,l\,V)^E) &= \texttt{l2op}(l)_{\texttt{r2l}(\mathcal{L})}\,\texttt{H2I}(V) & (\textit{where}\ l \in \mathcal{L} \in \texttt{L2S}(\mathbb{L})) \\
\texttt{H2I}(\mathbf{handle}\,M\,\mathbf{with}\,H) &= \mathbf{handle}_{\texttt{r2l}(H)}\,\texttt{H2I}(M)\,\mathbf{with}\,\texttt{H2I}(H)
\end{aligned}
$$

Handlers

$$
\texttt{H2I}(\{\mathbf{return}\,x \mapsto M\}) = \{\mathbf{return}\,x \mapsto \texttt{H2I}(M)\} \quad \texttt{H2I}(\{l\,p\,r \mapsto M\} \uplus H) = \texttt{H2I}(H) \uplus \{\texttt{l2op}(l)\,p\,r \mapsto \texttt{H2I}(M)\}
$$

Contexts

$$
\begin{aligned}
\texttt{H2I}(\cdot) &= \emptyset & \texttt{H2I}(\Gamma, x : A) &= \texttt{H2I}(\Gamma), x : \texttt{H2I}(A) \\
\texttt{H2I}(\Delta, \alpha : K) &= \texttt{H2I}(\Delta), \alpha : \texttt{H2I}(K)
\end{aligned}
$$

Lemma 4.9.

(1) *If* $\vdash \Gamma_1, \alpha : K, x : A, \Gamma_3$ *and* $\vdash \Gamma_1, x : A$, *then* $\vdash \Gamma_1, x : A, \alpha : K, \Gamma_3$.

(2) *If* $\Gamma_1, \alpha : K, x : A, \Gamma_3 \vdash S : K'$ *and* $\vdash \Gamma_1, x : A$, *then* $\Gamma_1, x : A, \alpha : K, \Gamma_3 \vdash S : K'$.

(3) *If* $\Gamma_1, \alpha : K, x : A, \Gamma_3 \vdash B <: C$ *and* $\vdash \Gamma_1, x : A$, *then* $\Gamma_1, x : A, \alpha : K, \Gamma_3 \vdash B <: C$.

(4) *If* $\Gamma_1, \alpha : K, x : A, \Gamma_3 \vdash B_1 \mid \varepsilon_1 <: B_2 \mid \varepsilon_2$ *and* $\vdash \Gamma_1, x : A$, *then* $\Gamma_1, x : A, \alpha : K, \Gamma_3 \vdash B_1 \mid \varepsilon_1 <: B_2 \mid \varepsilon_2$.

(5) *If* $\Gamma_1, \alpha : K, x : A, \Gamma_3 \vdash e : B \mid \varepsilon$ *and* $\vdash \Gamma_1, x : A$, *then* $\Gamma_1, x : A, \alpha : K, \Gamma_3 \vdash e : B \mid \varepsilon$.

(6) *If* $\Gamma_1, \alpha : K, x : A, \Gamma_3 \vdash_\sigma h : B \Rightarrow^\varepsilon C$ *and* $\vdash \Gamma_1, x : A$, *then* $\Gamma_1, x : A, \alpha : K, \Gamma_3 \vdash_\sigma h : B \Rightarrow^\varepsilon C$.

Proof. Straightforward by mutual induction on the derivations. $\blacksquare$

Theorem 4.10.

(1) *If* $\vdash \Delta$, *then* $\vdash \texttt{H2I}(\Delta)$.

(2) *If* $\Delta \vdash T : K$, *then* $\texttt{H2I}(\Delta) \vdash \texttt{H2I}(T) : \texttt{H2I}(K)$.

(3) *If* $\Delta \vdash \Gamma$, *then* $\vdash \texttt{H2I}(\Delta), \texttt{H2I}(\Gamma)$.

(4) *If* $\Delta; \Gamma \vdash V : A$, *then* $\texttt{H2I}(\Delta), \texttt{H2I}(\Gamma) \vdash \texttt{H2I}(V) : \texttt{H2I}(A) \mid \emptyset_E$.

(5) *If* $\Delta; \Gamma \vdash M : A!E$, *then* $\texttt{H2I}(\Delta), \texttt{H2I}(\Gamma) \vdash \texttt{H2I}(M) : \texttt{H2I}(A) \mid \texttt{H2I}(E)$.

(6) *If* $\Delta; \Gamma \vdash \{\mathbf{return}\,x \mapsto M\} \uplus \{l_1\,p_1\,r_1 \mapsto N_1\} \uplus \cdots \uplus \{l_n\,p_n\,r_n \mapsto N_n\} : A!E \Rightarrow B!E'$, *then*

    • $\texttt{H2I}(\Delta), \texttt{H2I}(\Gamma) \vdash_\sigma \texttt{H2I}(H) : \texttt{H2I}(A) \Rightarrow^{\texttt{H2I}(E')} \texttt{H2I}(B)$,

    • $\texttt{r2l}(H) : \sigma \in \Sigma$, *and*

    • $\langle \texttt{r2l}(H) \mid \texttt{H2I}(E') \rangle \sim_{\mathrm{SimpR}} \texttt{H2I}(E)$,

    where $\sigma = \{\texttt{l2op}(l_1) : \texttt{l2T}(l_1), \ldots, \texttt{l2op}(l_n) : \texttt{l2T}(l_n)\}$.

Proof.

(1) Straightforward by induction on the derivation.

(2) By induction on a derivation of the judgment. We proceed by case analysis on the rule applied lastly to the derivation.

    Case KH_VAR**:** We have

        – $T = \alpha$,

        – $\vdash \Delta', \alpha : K$, and

        – $\Delta = \Delta', \alpha : K$,

for some Δ'.

By definition of H2I, we have $\alpha : \mathtt{H2I}(K) \in \mathtt{H2I}(\Delta', \alpha : K)$. By case (1), K_VAR derives $\mathtt{H2I}(\Gamma) \vdash \alpha : \mathtt{H2I}(K)$

***Case* KH_FUN:** We have
- $T = A \to B!E$,
- $K = \mathsf{Type}$,
- $\Delta \vdash A : \mathsf{Type}$,
- $\Delta \vdash B : \mathsf{Type}$,
- $\Delta \vdash E : \mathsf{Effect}$,

for some A, B, and E. By the induction hypothesis, we have
- $\mathtt{H2I}(\Delta) \vdash \mathtt{H2I}(A) : \mathbf{Typ}$,
- $\mathtt{H2I}(\Delta) \vdash \mathtt{H2I}(B) : \mathbf{Typ}$, and
- $\mathtt{H2I}(\Delta) \vdash \mathtt{H2I}(E) : \mathbf{Eff}$.

Thus, K_FUN derives

$$\mathtt{H2I}(\Delta) \vdash \mathtt{H2I}(A) \to_{\mathtt{H2I}(E)} \mathtt{H2I}(B) : \mathbf{Typ}$$

as required.

***Case* KH_FORALL:** We have
- $T = \forall \alpha^{K'}.A!E$,
- $K = \mathsf{Type}$,
- $\Delta, \alpha : K' \vdash A : \mathsf{Type}$, and
- $\Delta, \alpha : K' \vdash E : \mathsf{Effect}$,

for some α, K', A, and E. By the induction hypothesis, we have
- $\mathtt{H2I}(\Delta, \alpha : K') \vdash \mathtt{H2I}(A) : \mathbf{Typ}$ and
- $\mathtt{H2I}(\Delta, \alpha : K') \vdash \mathtt{H2I}(E) : \mathbf{Eff}$.

Thus, K_POLY derives

$$\mathtt{H2I}(\Delta) \vdash \forall \alpha : \mathtt{H2I}(K').\mathtt{H2I}(A)^{\mathtt{H2I}(E)} : \mathbf{Typ}$$

as required.

***Case* KH_EFFECT:** Clearly by the induction hypothesis.

***Case* KH_CLOSEROW:** Clearly by the assumptions and K_CONS.

***Case* KH_OPENROW:** Clearly by the assumptions and K_CONS.

(3) By induction on a derivation of the judgment. We proceed by case analysis on the rule applied lastly to the derivation.

***Case* CH_EMPTY:** Clearly because case (1).

***Case* CH_VAR:** We have
- $\Gamma = \Gamma', x : A$,
- $\Delta \vdash \Gamma'$,
- $x \notin \mathrm{dom}(\Gamma')$, and
- $\Delta \vdash A : \mathsf{Type}$,

for some Γ', x, and A. By the induction hypothesis and case (2), we have
- $\vdash \mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma')$ and
- $\mathtt{H2I}(\Delta) \vdash \mathtt{H2I}(A) : \mathbf{Typ}$.

By $\vdash \mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma')$ and Lemma 3.5(2), we have $\mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma') \vdash \mathtt{H2I}(A) : \mathbf{Typ}$. By definition of H2I, we have $x \notin \mathrm{dom}(\mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma'))$. Thus, C_VAR derives

$$\vdash \mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma'), x : \mathtt{H2I}(A)$$

as required.

(4)(5)(6) By mutual induction on derivations of the judgments. We proceed by case analysis on the rule applied lastly to the derivations.

***Case* TH_VAR:** We have

- $V = x$,
- $\Delta \vdash \Gamma$, and
- $x : A \in \Gamma$,

for some x. By Theorem (3), we have $\vdash \mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma)$. By definition of $\mathtt{H2I}$, we have $x : \mathtt{H2I}(A) \in \mathtt{H2I}(\Gamma)$. Thus, T_VAR derives

$$\mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma) \vdash x : \mathtt{H2I}(A) \mid \emptyset_E$$

as required.

Case Th_LAM: We have
- $V = \lambda x^{A_0}.M$,
- $A = A_0 \to A_1!E$, and
- $\Delta; \Gamma, x : A_0 \vdash M : A_1!E$,

for some x, A_0, A_1, E, and M. By the induction hypothesis, we have

$$\mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma), x : \mathtt{H2I}(A_0) \vdash \mathtt{H2I}(M) : \mathtt{H2I}(A_1) \mid \mathtt{H2I}(E).$$

Without loss of generality, we can choose z such that
- $z \notin FV(\mathtt{H2I}(M))$,
- $z \neq x$,
- $z \notin \mathrm{dom}(\mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma))$, and
- $\mathtt{H2I}(\lambda x^{A_0}.M) = \mathbf{fun}(z, x, \mathtt{H2I}(M))$.

By Lemma 3.12 and Lemma 3.2(2) and Lemma 3.6, we have
- $\mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma) \vdash \mathtt{H2I}(A_1) : \mathbf{Typ}$ and
- $\mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma) \vdash \mathtt{H2I}(E) : \mathbf{Eff}$.

By Lemma 3.9, we have $\vdash \mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma), x : \mathtt{H2I}(A_0)$. Since only C_VAR can derive this judgment, we have $\mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma) \vdash \mathtt{H2I}(A_0) : \mathbf{Typ}$. Thus, C_VAR derives

$$\vdash \mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma), z : \mathtt{H2I}(A_0) \to_{\mathtt{H2I}(E)} \mathtt{H2I}(A_1).$$

Thus, Lemma 3.5(5) and T_ABS derives

$$\mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma) \vdash \mathbf{fun}(z, x, \mathtt{H2I}(M)) : \mathtt{H2I}(A_0) \to_{\mathtt{H2I}(E)} \mathtt{H2I}(A_1) \mid \emptyset_E$$

as required.

Case Th_POLYLAM: We have
- $V = \Lambda \alpha^K.M$,
- $A = \forall \alpha^K.B!E$,
- $\Delta, \alpha : K; \Gamma \vdash M : B!E$, and
- $\Delta \vdash \Gamma$,

for some α, K, M, B, and E. By the induction hypothesis and case (3), we have
- $\vdash \mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma)$ and
- $\mathtt{H2I}(\Delta), \alpha : \mathtt{H2I}(K), \mathtt{H2I}(\Gamma) \vdash \mathtt{H2I}(M) : \mathtt{H2I}(B) \mid \mathtt{H2I}(E)$.

By applying Lemma 4.9 repeatedly, we have

$$\mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma), \alpha : \mathtt{H2I}(K) \vdash \mathtt{H2I}(M) : \mathtt{H2I}(B) \mid \mathtt{H2I}(E).$$

Thus, T_TABS derives

$$\mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma) \vdash \Lambda \alpha : \mathtt{H2I}(K).\mathtt{H2I}(M) : \forall \alpha : \mathtt{H2I}(K).\mathtt{H2I}(B)^{\mathtt{H2I}(E)} \mid \emptyset_E$$

as required.

Case Th_APP: We have
- $M = V W$,
- $\Delta; \Gamma \vdash V : B \to A!E$, and
- $\Delta; \Gamma \vdash W : B$,

for some V, W, and B. By the induction hypothesis, we have

- $\mathrm{H2I}(\Delta), \mathrm{H2I}(\Gamma) \vdash \mathrm{H2I}(V) : \mathrm{H2I}(B) \rightarrow_{\mathrm{H2I}(E)} \mathrm{H2I}(A) \mid \emptyset_E$ and
- $\mathrm{H2I}(\Delta), \mathrm{H2I}(\Gamma) \vdash \mathrm{H2I}(W) : \mathrm{H2I}(B) \mid \emptyset_E$.

Thus, T_APP derives

$$\mathrm{H2I}(\Delta), \mathrm{H2I}(\Gamma) \vdash \mathrm{H2I}(V)\,\mathrm{H2I}(W) : \mathrm{H2I}(A) \mid \mathrm{H2I}(E)$$

as required.

Case TH_POLYAPP: We have

- $M = V\,T$,
- $A = (B!E)[T/\alpha]$,
- $\Delta; \Gamma \vdash V : \forall \alpha^K.B!E$, and
- $\Delta \vdash T : K$

for some V, T, α, K, B, and E. By the induction hypothesis and case (2), we have

- $\mathrm{H2I}(\Delta), \mathrm{H2I}(\Gamma) \vdash \mathrm{H2I}(V) : \forall \alpha : \mathrm{H2I}(K).\mathrm{H2I}(B)^{\mathrm{H2I}(E)} \mid \emptyset_E$ and
- $\mathrm{H2I}(\Delta) \vdash \mathrm{H2I}(T) : \mathrm{H2I}(K)$.

By Lemma 3.9 and Lemma 3.5(2), we have $\mathrm{H2I}(\Delta), \mathrm{H2I}(\Gamma) \vdash \mathrm{H2I}(T) : \mathrm{H2I}(K)$. Thus, T_TAPP derives

$$\mathrm{H2I}(\Delta), \mathrm{H2I}(\Gamma) \vdash \mathrm{H2I}(V)\,\mathrm{H2I}(T) : \mathrm{H2I}(B[T/\alpha]) \mid \mathrm{H2I}(E[T/\alpha])$$

as required.

Case TH_RETURN: We have

- $M = \mathbf{return}\,V$,
- $\Delta; \Gamma \vdash V : A$, and
- $\Delta \vdash E : \mathsf{Effect}$,

for some V. By the induction hypothesis and case (2), we have

- $\mathrm{H2I}(\Delta), \mathrm{H2I}(\Gamma) \vdash \mathrm{H2I}(V) : \mathrm{H2I}(A) \mid \emptyset_E$ and
- $\mathrm{H2I}(\Delta) \vdash \mathrm{H2I}(E) : \mathbf{Eff}$.

Thus, T_SUB derives

$$\mathrm{H2I}(\Delta), \mathrm{H2I}(\Gamma) \vdash \mathrm{H2I}(V) : \mathrm{H2I}(A) \mid \mathrm{H2I}(E)$$

as required.

Case TH_LET: We have

- $M = \mathbf{let}\,x \leftarrow M_0\,\mathbf{in}\,M_1$,
- $\Delta; \Gamma \vdash M_0 : B!E$, and
- $\Delta; \Gamma, x : B \vdash M_1 : A!E$,

for some x, M_0, M_1, and B. By the induction hypothesis, we have

- $\mathrm{H2I}(\Delta), \mathrm{H2I}(\Gamma) \vdash \mathrm{H2I}(M_0) : \mathrm{H2I}(B) \mid \mathrm{H2I}(E)$ and
- $\mathrm{H2I}(\Delta), \mathrm{H2I}(\Gamma), x : \mathrm{H2I}(B) \vdash \mathrm{H2I}(M_1) : \mathrm{H2I}(A) \mid \mathrm{H2I}(E)$.

Thus, T_LET derives

$$\mathrm{H2I}(\Delta), \mathrm{H2I}(\Gamma) \vdash \mathbf{let}\,x = \mathrm{H2I}(M_0)\,\mathbf{in}\,\mathrm{H2I}(M_1) : \mathrm{H2I}(A) \mid \mathrm{H2I}(E)$$

as required.

Case TH_DO: We have

- $M = (\mathbf{do}\,l\,V)^E$,
- $\Delta; \Gamma \vdash V : B$,
- $\{l : \mathsf{Pre}(B \rightarrow A); R\}$, and
- $\Delta \vdash E : \mathsf{Effect}$,

for some l, V, E, B, and R. By the induction hypothesis and case (2), we have

- $\mathrm{H2I}(\Delta), \mathrm{H2I}(\Gamma) \vdash \mathrm{H2I}(V) : \mathrm{H2I}(B) \mid \emptyset_E$ and
- $\mathrm{H2I}(\Delta) \vdash \mathrm{H2I}(E) : \mathbf{Eff}$.

There uniquely exists some $\mathcal{L}$ such that

- $\mathcal{L} \in \mathrm{L2S}(\mathbb{L})$,

    – $l \in \mathcal{L}$, and

    – $\mathcal{L} \subseteq \mathrm{dom}(E)$.

Thus, we have

    – $\mathtt{r2l}(\mathcal{L}) :: \sigma \in \Sigma$,

    – $\mathtt{l2Op}(l) : \mathtt{l2T}(l) \in \sigma$, and

    – $\langle \mathtt{r2l}(\mathcal{L}) \mid \varepsilon \rangle \sim_{\mathrm{SimpR}} \mathtt{H2I}(E)$,

for some σ and ε. Because Lemma 3.9 gives us $\vdash \mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma)$, T_Op and T_App and T_Sub derive

$$\mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma) \vdash \mathtt{l2Op}(l)_{\mathtt{r2l}(\mathcal{L})} \, \mathtt{H2I}(V) : \mathtt{H2I}(B) \mid \mathtt{H2I}(E)$$

    as required.

Case TH_HANDLE: We have

    – $M = \mathbf{handle}\, N \,\mathbf{with}\, H$,

    – $\Delta; \Gamma \vdash N : B!E'$, and

    – $\Delta; \Gamma \vdash H : B!E' \Rightarrow A!E$,

for some N, H, B, and E'. By the induction hypothesis, we have

    – $\mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma) \vdash \mathtt{H2I}(N) : \mathtt{H2I}(B) \mid \mathtt{H2I}(E')$,

    – $\mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma) \vdash_\sigma \mathtt{H2I}(H) : \mathtt{H2I}(B) \Rightarrow^{\mathtt{H2I}(E)} \mathtt{H2I}(A)$,

    – $\mathtt{r2l}(H) :: \sigma \in \Sigma$, and

    – $\langle \mathtt{r2l}(H) \mid \mathtt{H2I}(E) \rangle \sim_{\mathrm{SimpR}} \mathtt{H2I}(E')$.

for some σ. Thus, T_HANDLING derives

$$\mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma) \vdash \mathbf{handle}_{\mathtt{r2l}(H)} \, \mathtt{H2I}(N) \,\mathbf{with}\, \mathtt{H2I}(H) : \mathtt{H2I}(A) \mid \mathtt{H2I}(E)$$

    as required.

Case HH_HANDLER: We have

    – $\Delta; \Gamma, x : A \vdash M : B!E'$,

    – $\Delta; \Gamma, y_i : A_i, r_i : B_i \to B!E' \vdash N_i : B!E'$ for any $i \in \{1, \ldots, n\}$,

    – $E = \{l_1 : \mathsf{Pre}(A_1 \to B_1); \cdots ; l_n : \mathsf{Pre}(A_n \to B_n); R\}$, and

    – $E' = \{l_1 : P_1; \cdots ; l_n : P_n; R\}$,

for some A_i, B_i, and P_i, where $i \in \{1, \ldots, n\}$. By the assumptions, we have

    – $\{l_1, \ldots, l_n\} \in \mathtt{L2S}(\mathbb{L})$,

    – $\mathtt{l2T}(l_i) = \mathtt{H2I}(A_i) \Rightarrow \mathtt{H2I}(B_i)$ for any $i \in \{1, \ldots, n\}$,

    – $\mathtt{r2l}(\{l_1, \ldots, l_n\}) :: \{\mathtt{l2Op}(l_1) : \mathtt{l2T}(l_1), \ldots, \mathtt{l2Op}(l_n) : \mathtt{l2T}(l_n)\} \in \Sigma$, and

    – $\forall i \in \{1, \ldots, n\}.(P_i = \mathsf{Abs})$ or $\forall i \in \{1, \ldots, n\}.(P_i = \mathsf{Pre}(A_i \to B_i))$.

Thus, we have $\langle \mathtt{r2l}(H) \mid \mathtt{H2I}(E') \rangle \sim_{\mathrm{SimpR}} \mathtt{H2I}(E)$.

By the induction hypothesis, we have

    – $\mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma), x : \mathtt{H2I}(A) \vdash \mathtt{H2I}(M) : \mathtt{H2I}(B) \mid \mathtt{H2I}(E')$ and

    – $\mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma), y_i : \mathtt{H2I}(A_i), r_i : \mathtt{H2I}(B_i) \to_{\mathtt{H2I}(E')} \mathtt{H2I}(B) \vdash \mathtt{H2I}(N_i) : \mathtt{H2I}(B) \mid \mathtt{H2I}(E')$ for any $i \in \{1, \ldots, n\}$.

Therefore, H_RETURN and H_OP derive

$$\mathtt{H2I}(\Delta), \mathtt{H2I}(\Gamma) \vdash_{\{\mathtt{l2Op}(l_1):\mathtt{l2T}(l_1), \ldots, \mathtt{l2Op}(l_n):\mathtt{l2T}(l_n)\}} \mathtt{H2I}(H) : \mathtt{H2I}(A) \Rightarrow^{\mathtt{H2I}(E')} \mathtt{H2I}(B).$$

Thus, the required result is achieved. ∎

4.3 Comparison to [Leijen(2017)]

We give the targets of comparison: one is an instance of λ_{EA} (Example 1.26), and another is a minorly changed language of [Leijen(2017)].

Definition 4.11 (Minor Changed Version of [Leijen(2017)]). *Change list:*
- *changing implicit polymorphism to explicit polymorphism,*

- *removing constants from values,*

- *removing the assumption that the initial environment has effect declarations, and adding such declarations to Σ,*

- *adding type variables to contexts, and*

- *adding well-formedness of contexts.*
The syntax of a minor changed version of [Leijen(2017)] is as follows.

$$
\begin{array}{rcll}
e & ::= & v \mid e(e) \mid e(\tau^k) \mid \mathsf{val}\, x = e_1; e_2 \mid \mathsf{handle}\{h\}(e) & \textit{(expressions)} \\
v & ::= & x \mid op \mid \lambda x.e \mid \Lambda \alpha^k.e & \textit{(values)} \\
h & ::= & \mathsf{return}\, x \to e \mid op(x) \to e; h & \textit{(clauses)} \\
\tau^k & ::= & \alpha^k \mid c^{(k_1,\ldots,k_n)\to k}\langle \tau_1^{k_1}, \ldots, \tau_n^{k_n}\rangle & \textit{(types)} \\
k & ::= & * \mid \mathsf{e} \mid \mathsf{k} \mid (k_1,\ldots,k_n) \to k & \textit{(kinds)} \\
\sigma & ::= & \forall \alpha^k.\sigma \mid \tau^* & \textit{(type scheme)} \\
\Gamma & ::= & \emptyset \mid \Gamma, x : \sigma \mid \Gamma, \alpha^k & \textit{(typing contexts)} \\
\Sigma & ::= & \emptyset \mid \Sigma, l : \{op_1, \ldots, op_n\} & \textit{(signature environment)} \\
- \to -\, - & :: & (*, \mathsf{e}, *) \to * & \textit{(functions)} \\
\langle\rangle & :: & \mathsf{e} & \textit{(empty effect)} \\
\langle - \mid - \rangle & :: & (\mathsf{k}, \mathsf{e}) \to \mathsf{e} & \textit{(effect extension)} \\
l & ::= & c^{(k_1,\ldots,k_n)\to k}\langle \tau_1^{k_1}, \ldots, \tau_n^{k_n}\rangle & \textit{(effect labels)}
\end{array}
$$

Well-formedness rules, free type variable, and typing rules consist of the following.

Contexts Well-formedness $\boxed{\vdash \Gamma}$

$$
\frac{}{\vdash \emptyset} \;\; \text{CL\_EMPTY} \qquad
\frac{\vdash \Gamma \quad x \notin \mathrm{dom}(\Gamma) \quad \mathsf{ftv}(\tau^*)\setminus\{\overline{\alpha^{k'}}\} \subseteq \Gamma \quad \forall \overline{k}.(\{\overline{\alpha^k}\} \cap \Gamma = \emptyset)}{\vdash \Gamma, x : \forall \overline{\alpha^{k'}}.\tau^*} \;\; \text{CL\_VAR}
$$

$$
\frac{\vdash \Gamma \quad \forall k.(\alpha^k \notin \Gamma)}{\vdash \Gamma, \alpha^{k'}} \;\; \text{CL\_TVAR}
$$

Free Type Variable $\boxed{\mathsf{ftv}(\tau^k)}$ $\boxed{\mathsf{ftv}(\sigma)}$

$$
\mathsf{ftv}(\alpha^k) = \{\alpha^k\} \qquad \mathsf{ftv}(\tau_1^* \to \tau_2^{\mathsf{e}}\, \tau_3^*) = \mathsf{ftv}(\tau_1^*) \cup \mathsf{ftv}(\tau_2^{\mathsf{e}}) \cup \mathsf{ftv}(\tau_3^*) \qquad \mathsf{ftv}(\langle\rangle) = \emptyset \qquad \mathsf{ftv}(\langle\tau_1^{\mathsf{k}} \mid \tau_2^{\mathsf{e}}\rangle) = \mathsf{ftv}(\tau_1^{\mathsf{k}}) \cup \mathsf{ftv}(\tau_2^{\mathsf{e}})
$$

$$
\mathsf{ftv}(c^{(k_1,\ldots,k_n)\to \mathsf{k}}\langle \tau_1^{k_1}, \ldots, \tau_n^{k_n}\rangle) = \bigcup_{i\in\{1,\ldots,n\}} \mathsf{ftv}(\tau_i^{k_i}) \qquad \mathsf{ftv}(\forall \alpha^k.\sigma) = \mathsf{ftv}(\sigma) \setminus \{\alpha^k\}
$$

Typing $\boxed{\Gamma \vdash e : \sigma \mid \epsilon}$

$$
\frac{\vdash \Gamma \quad \Gamma(x) = \sigma \quad \mathsf{ftv}(\epsilon) \subseteq \Gamma}{\Gamma \vdash x : \sigma \mid \epsilon} \;\; \text{TL\_VAR} \qquad
\frac{\Gamma, x : \tau_1 \vdash e : \tau_2 \mid \epsilon' \quad \mathsf{ftv}(\epsilon) \subseteq \Gamma}{\Gamma \vdash \lambda x.e : \tau_1 \to \epsilon'\, \tau_2 \mid \epsilon} \;\; \text{TL\_LAM}
$$

$$
\frac{\Gamma \vdash e_1 : \sigma_1 \mid \epsilon \quad \Gamma, x : \sigma \vdash e_2 : \tau \mid \epsilon}{\Gamma \vdash \mathsf{val}\, x = e_1; e_2 : \tau \mid \epsilon} \;\; \text{TL\_LET} \qquad
\frac{\Gamma \vdash e_1 : \tau_2 \to \epsilon\, \tau \mid \epsilon \quad \Gamma \vdash e_2 : \tau_2 \mid \epsilon}{\Gamma \vdash e_1(e_2) : \tau \mid \epsilon} \;\; \text{TL\_APP}
$$

$$
\frac{\Gamma, \overline{\alpha^k} \vdash e : \tau \mid \langle\rangle \quad \mathsf{ftv}(\epsilon) \subseteq \Gamma}{\Gamma \vdash \Lambda \overline{\alpha^k}.e : \forall \overline{\alpha^k}.\tau \mid \epsilon} \;\; \text{TL\_TABS} \qquad
\frac{\Gamma \vdash e : \forall \overline{\alpha^k}.\tau \mid \epsilon \quad \mathsf{ftv}(\overline{\tau_0^k}) \subseteq \Gamma}{\Gamma \vdash e(\overline{\tau_0^k}) : \tau[\overline{\alpha^k} \mapsto \overline{\tau_0^k}] \mid \epsilon} \;\; \text{TL\_TAPP}
$$

$$
\frac{\begin{array}{c} \Gamma \vdash e : \tau \mid \langle l \mid \epsilon\rangle \quad \Gamma, x : \tau \vdash e_r : \tau_r \mid \epsilon \\ \Sigma(l) = \{op_1, \ldots, op_n\} \quad \Gamma \vdash op_i : \tau_i \to \langle l \mid \langle\rangle\rangle\, \tau_i' \mid \langle\rangle \\ \Gamma, x_i : \tau_i, resume : \tau_i' \to \epsilon\, \tau_r \vdash e_i : \tau_r \mid \epsilon \end{array}}{\Gamma \vdash \mathsf{handle}\{op_1(x_1) \to e_1; \cdots ; op_n(x_n) \to e_n; \mathsf{return}\, x \to e_r\}(e) : \tau_r \mid \epsilon} \;\; \text{TL\_HANDLE}
$$

$$
\frac{\Gamma \vdash e : \tau_1 \to \langle l_1, \ldots, l_n \mid \langle\rangle\rangle\, \tau_2 \mid \epsilon \quad \mathsf{ftv}(\epsilon') \subseteq \Gamma}{\Gamma \vdash e : \tau_1 \to \langle l_1, \ldots, l_n \mid \epsilon'\rangle\, \tau_2 \mid \epsilon} \;\; \text{TL\_OPEN}
$$

Definition 4.12 (Translation from Leijen's to An Instance)**.** *We assume that there is no constants other than* $- \to - -$, $\langle\rangle$, $\langle- \mid -\rangle$ *and* $c^{(k_1,\ldots,k_n)\to k}$. *We define* c2l *as the injective function that assigns a label name* l *such that* $l :$ *(belonging to an instance) to* $c^{(k_1,\ldots,k_n)\to k}$ *(belonging to Leijen's). We define* L2I, h2l, *and* c2l *as follows. We require* c2l *to be injective.*

Kinds

$$\text{L2I}(*) \;=\; \mathbf{Typ} \qquad \text{L2I}(\mathsf{e}) \;=\; \mathbf{Eff} \qquad \text{L2I}(\mathsf{k}) \;=\; \mathbf{Lab}$$

Types

$$
\begin{aligned}
\text{L2I}(\alpha^k) &= \alpha & \text{L2I}(\tau_1^* \to \tau_2^{\mathsf{e}} \,\tau_3^*) &= \text{L2I}(\tau_1^*) \to_{\text{L2I}(\tau_2^{\mathsf{e}})} \text{L2I}(\tau_3^*) \\
\text{L2I}(\langle\rangle) &= \langle\rangle & \text{L2I}(\langle l \mid \epsilon\rangle) &= \langle \text{L2I}(l) \mid \text{L2I}(\epsilon)\rangle \\
\text{L2I}(\forall\alpha^k.\sigma) &= \forall\alpha : \text{L2I}(k).\text{L2I}(\sigma)^{\langle\rangle} \\
\text{L2I}(c^{(k_1,\ldots,k_n)\to k}\langle\tau_1^{k_1},\ldots,\tau_n^{k_n}\rangle) &= \text{c2l}(c^{(k_1,\ldots,k_n)\to k})\,\text{L2I}(\tau_1^{k_1})\cdots\text{L2I}(\tau_n^{k_n}) \\
\text{c2l}(c^{(k_1,\ldots,k_n)\to k}) &= l \quad (\textit{where } l : \text{L2I}(k_1) \times \ldots \times \text{L2I}(k_n) \to \mathbf{Lab} \in \Sigma_{\text{eff}})
\end{aligned}
$$

Expressions

$$
\begin{aligned}
\text{L2I}(x) &= x \\
\text{L2I}(op) &= \mathsf{op}_{\text{c2l}(c)}\,\overline{\text{L2I}(\tau^k)} \quad (\textit{where } op \in \Sigma(c\langle\overline{\tau^k}\rangle)) \\
\text{L2I}(\lambda x.e) &= \mathbf{fun}(z, x, \text{L2I}(e)) \quad (\textit{where } z \textit{ is fresh}) \\
\text{L2I}(\Lambda\alpha^k.e) &= \Lambda\alpha : \text{L2I}(k).\text{L2I}(e) \\
\text{L2I}(e_1(e_2)) &= \mathbf{let}\, x = \text{L2I}(e_1)\,\mathbf{in}\,\mathbf{let}\, y = \text{L2I}(e_2)\,\mathbf{in}\, x\,y \\
\text{L2I}(e(\tau^k)) &= \mathbf{let}\, x = \text{L2I}(e)\,\mathbf{in}\, x\,\text{L2I}(\tau^k) \\
\text{L2I}(\mathsf{val}\,x = e_1; e_2) &= \mathbf{let}\, x = \text{L2I}(e_1)\,\mathbf{in}\,\text{L2I}(e_2) \\
\text{L2I}(\mathsf{handle}\{h\}(e)) &= \mathbf{handle}_{\text{h2l}(h)}\,\text{L2I}(e)\,\mathbf{with}\,\text{L2I}(h)
\end{aligned}
$$

Handlers

$$
\begin{aligned}
\text{L2I}(\mathsf{return}\,x \to e) &= \{\mathbf{return}\,x \mapsto \text{L2I}(e)\} \\
\text{L2I}(op(x) \to e; h) &= \text{L2I}(h) \uplus \{\mathsf{op}\,x\,\textit{resume} \mapsto \text{L2I}(e)\}
\end{aligned}
$$

Contexts

$$\text{L2I}(\emptyset) \;=\; \emptyset \qquad \text{L2I}(\Gamma, x : \sigma) \;=\; \text{L2I}(\Gamma), x : \text{L2I}(\sigma) \qquad \text{L2I}(\Gamma, \alpha^k) \;=\; \text{L2I}(\Gamma), \alpha : \text{L2I}(k)$$

Effect Contexts

$$
\begin{aligned}
\text{L2I}(\emptyset) &= \emptyset \\
\text{L2I}(\Sigma, c\langle\overline{\tau^k}\rangle : \{op_1,\ldots,op_n\}) &= \text{L2I}(\Sigma), \text{c2l}(c) :: \forall\overline{\alpha : \text{L2I}(k)}.\sigma \\
&\quad (\textit{where } \Gamma_0 \ni op_i : \tau_i \to \langle c\langle\overline{\tau^k}\rangle \mid \langle\rangle\rangle\,\tau_i' \\
&\quad\quad \textit{and } \sigma[\overline{\text{L2I}(\tau^k)/\overline{\alpha}}] = \{\mathsf{op}_1 : \tau_1 \Rightarrow \tau_1',\ldots,\mathsf{op}_n : \tau_n \Rightarrow \tau_n'\})
\end{aligned}
$$

Translation from Handlers to Labels

$$
\begin{aligned}
&\text{h2l}(op_1(x_1) \to e_1; \cdots; op_n(x_n) \to e_n; \mathsf{return}\,x \to e) \\
&= \begin{cases} l & (\textit{if } l = \text{c2l}(c) \textit{ and } \{op_1,\ldots,op_n\} = \Sigma(c\langle\cdots\rangle)) \\ \textit{undefined} & (\textit{otherwise}) \end{cases}
\end{aligned}
$$

Lemma 4.13. *If* $x \notin \text{dom}(\Gamma)$, *then* $x \notin \text{dom}(\text{L2I}(\Gamma))$.

Proof. Straightforward by structural induction on Γ and the definition of L2I. ∎

Lemma 4.14. $\alpha^k \in \Gamma$ *iff* $\alpha : \text{L2I}(k) \in \text{L2I}(\Gamma)$.

Proof. Straightforward by structural induction on Γ and the definition of L2I. ∎

Lemma 4.15. *If* $x : \sigma \in \Gamma$, *then* $x : \text{L2I}(\sigma) \in \text{L2I}(\Gamma)$.

Proof. Straightforward by structural induction on Γ and the definition of L2I. ∎

Lemma 4.16. *If* $\Gamma \vdash e : \sigma \mid \epsilon$, *then* $\vdash \Gamma$.

Proof. Straightforward by induction on a derivation of $\Gamma \vdash e : \sigma \mid \epsilon$. ∎

Theorem 4.17.

(1) If $\mathsf{ftv}(\tau^k) \subseteq \Gamma$ *and* $\vdash \mathtt{L2I}(\Gamma)$, *then* $\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(\tau^k) : \mathtt{L2I}(k)$.

(2) If $\vdash \Gamma$, *then* $\vdash \mathtt{L2I}(\Gamma)$.

(3) If $\Gamma \vdash e : \sigma \mid \epsilon$, *then* $\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(e) : \mathtt{L2I}(\sigma) \mid \mathtt{L2I}(\epsilon)$.

Proof.

(1) By structural induction on τ^k.

Case $\tau^k = \alpha^k$**:** We have $\alpha^k \in \Gamma$. By Lemma 4.14, we have $\alpha : \mathtt{L2I}(k) \in \mathtt{L2I}(\Gamma)$. Thus, K_VAR derives

$$\mathtt{L2I}(\Gamma) \vdash \alpha : \mathtt{L2I}(k)$$

as required.

Case $\tau^k = \tau_1^* \to \tau_2^{\mathsf{e}}\, \tau_3^*$**:** We have
- $k = *$,
- $\mathsf{ftv}(\tau_1^*) \subseteq \Gamma$,
- $\mathsf{ftv}(\tau_2^{\mathsf{e}}) \subseteq \Gamma$, and
- $\mathsf{ftv}(\tau_3^*) \subseteq \Gamma$.

By the induction hypothesis, we have
- $\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(\tau_1^*) : \mathbf{Typ}$,
- $\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(\tau_2^{\mathsf{e}}) : \mathbf{Eff}$, and
- $\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(\tau_3^*) : \mathbf{Typ}$.

Thus, K_FUN derives
$$\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(\tau_1^*) \to_{\mathtt{L2I}(\tau_2^{\mathsf{e}})} \mathtt{L2I}(\tau_3^*) : \mathbf{Typ}$$

as required.

Case $\tau^k = \langle\rangle$**:** We have $k = \mathsf{e}$. Thus, by $\vdash \mathtt{L2I}(\Gamma)$, we have

$$\mathtt{L2I}(\Gamma) \vdash \langle\rangle : \mathbf{Eff}$$

as required.

Case $\tau^k = \langle \tau_1^{\mathsf{k}} \mid \tau_2^{\mathsf{e}} \rangle$**:** We have
- $k = \mathsf{e}$,
- $\mathsf{ftv}(\tau_1^{\mathsf{k}}) \subseteq \Gamma$, and
- $\mathsf{ftv}(\tau_2^{\mathsf{e}}) \subseteq \Gamma$.

By the induction hypothesis, we have
- $\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(\tau_1^{\mathsf{k}}) : \mathbf{Lab}$ and
- $\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(\tau_2^{\mathsf{e}}) : \mathbf{Eff}$.

Thus, K_CONS derives
$$\mathtt{L2I}(\Gamma) \vdash \langle \mathtt{L2I}(\tau_1^{\mathsf{k}}) \mid \mathtt{L2I}(\tau_2^{\mathsf{e}}) \rangle : \mathbf{Eff}$$

as required.

Case $\tau^k = c^{(k_1,\ldots,k_n) \to \mathsf{k}} \langle \tau_1^{k_1}, \ldots, \tau_n^{k_n} \rangle$**:** We have $\mathsf{ftv}(\tau_i^i) \subseteq \Gamma$ for any $i \in \{1, \ldots, n\}$. By the induction hypothesis and definition of c2l, we have
- $\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(\tau_i^{k_i}) : \mathtt{L2I}(k_i)$ for any $i \in \{1, \ldots, n\}$,
- $\mathtt{c2l}(c^{(k_1,\ldots,k_n) \to \mathsf{k}}) : \mathtt{L2I}(k_1) \times \ldots \times \mathtt{L2I}(k_n) \to \mathbf{Lab} \in \Sigma_{\mathrm{eff}}$.

Thus, K_CONS derives

$$\mathtt{L2I}(\Gamma) \vdash \mathtt{c2l}(c^{(k_1,\ldots,k_n) \to \mathsf{k}}) \, \mathtt{L2I}(\tau_1^{k_1}) \cdots \mathtt{L2I}(\tau_n^{k_n}) : \mathbf{Lab}$$

as required.

(2) By induction on a derivation of the judgment. We proceed by case analysis on the rule applied lastly to the derivation.

Case CL_EMPTY**:** Clearly by C_EMPTY and the definition of L2I.

Case CL_VAR**:** We have

- $\Gamma = \Gamma', x : \forall \overline{\alpha^{k'}}.\tau^*$,
- $\vdash \Gamma'$,
- $x \notin \text{dom}(\Gamma')$,
- $\text{ftv}(\tau^*) \setminus \{\overline{\alpha^{k'}}\} \subseteq \Gamma'$, and
- $\forall \overline{k}.(\{\overline{\alpha^k}\} \cap \Gamma' = \emptyset)$,

for some Γ', x, $\overline{\alpha^{k'}}$, and τ^*. By the induction hypothesis and Lemma 4.13, we have

- $\vdash \text{L2I}(\Gamma')$ and
- $x \notin \text{dom}(\text{L2I}(\Gamma'))$.

By Lemma 4.14 and C_TVAR, we have

$$\vdash \text{L2I}(\Gamma'), \overline{\alpha} : \overline{\text{L2I}(k')}.$$

By $\text{ftv}(\tau^*) \subseteq \Gamma', \overline{\alpha^{k'}}$ and case (1), we have

$$\text{L2I}(\Gamma'), \overline{\alpha} : \overline{\text{L2I}(k')} \vdash \text{L2I}(\tau^*) : \textbf{Typ}.$$

Thus, K_POLY and C_VAR derives

$$\vdash \text{L2I}(\Gamma'), x : \text{L2I}(\forall \overline{\alpha^{k'}}.\tau^*)$$

as required.

Case CL_TVAR**:** We have

- $\Gamma = \Gamma', \alpha^k$,
- $\vdash \Gamma'$, and
- $\forall k.(\alpha^k \notin \Gamma')$,

for some Γ' and α^k. By the induction hypothesis and Lemma 4.14, we have

- $\vdash \text{L2I}(\Gamma')$ and
- $\alpha \notin \text{dom}(\text{L2I}(\Gamma'))$.

Thus, C_TVAR derives

$$\vdash \text{L2I}(\Gamma'), \alpha : \text{L2I}(k)$$

as required.

(3) By induction on a derivation of the judgement. We proceed by case analysis on the rule applied lastly to the derivation.

Case TL_VAR**:** We have

- $e = x$,
- $\Gamma(x) = \sigma$, and
- $\text{ftv}(\epsilon) \subseteq \Gamma$

for some x. By Lemma 4.16 and case (2), we have $\vdash \text{L2I}(\Gamma)$. By case (1), we have

$$\text{L2I}(\Gamma) \vdash \text{L2I}(\epsilon) : \textbf{Eff}.$$

By Lemma 4.15, we have $x : \text{L2I}(\sigma) \in \text{L2I}(\Gamma)$. Thus, T_VAR derives

$$\text{L2I}(\Gamma) \vdash x : \text{L2I}(\sigma) \mid \langle \rangle.$$

By Lemma 3.12, we have $\text{L2I}(\Gamma) \vdash \text{L2I}(\sigma) : \textbf{Typ}$. Thus, ST_REFL and Lemma 3.3(1) and T_SUB derive

$$\text{L2I}(\Gamma) \vdash x : \text{L2I}(\sigma) \mid \text{L2I}(\epsilon)$$

as required.

Case TL_LAM**:** We have

- $e = \lambda x.e'$,
- $\sigma = \tau_1 \rightarrow \epsilon' \tau_2$,

- $\Gamma, x : \tau_1 \vdash e' : \tau_2 \mid \epsilon'$, and
- $\mathsf{ftv}(\epsilon) \subseteq \Gamma$

for some x, e', τ_1, ϵ', and τ_2. By Lemma 4.16, we have $\vdash \Gamma, x : \tau_1$. Since only CL_VAR can derive $\vdash \Gamma, x : \tau_1$, we have $\vdash \Gamma$. By case (2), we have $\vdash \mathtt{L2I}(\Gamma)$. By the induction hypothesis and case (1), we have

- $\mathtt{L2I}(\Gamma, x : \tau_1) \vdash \mathtt{L2I}(e') : \mathtt{L2I}(\tau_2) \mid \mathtt{L2I}(\epsilon')$ and
- $\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(\epsilon) : \mathbf{Eff}$.

Without loss of generality, we can choose z such that

- $z \notin FV(\mathtt{L2I}(e'))$,
- $z \neq x$,
- $z \notin \mathrm{dom}(\mathtt{L2I}(\Gamma))$, and
- $\mathtt{L2I}(\lambda x.e') = \mathbf{fun}(z, x, \mathtt{L2I}(e'))$.

By Lemma 3.12 and Lemma 3.2(2) and Lemma 3.6, we have

- $\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(\tau_2) : \mathbf{Typ}$ and
- $\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(\epsilon') : \mathbf{Eff}$.

By Lemma 3.9, we have $\vdash \mathtt{L2I}(\Gamma), x : \mathtt{L2I}(\tau_1)$. Since only C_VAR can derive $\vdash \mathtt{L2I}(\Gamma), x : \mathtt{L2I}(\tau_1)$, we have $\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(\tau_1) : \mathbf{Typ}$. Thus, K_FUN derives

$$\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(\tau_1) \rightarrow_{\mathtt{L2I}(\epsilon')} \mathtt{L2I}(\tau_2) : \mathbf{Typ}.$$

Thus, C_VAR derives

$$\vdash \mathtt{L2I}(\Gamma), z : \mathtt{L2I}(\tau_1) \rightarrow_{\mathtt{L2I}(\epsilon')} \mathtt{L2I}(\tau_2).$$

Thus, Lemma 3.5 and T_ABS derives

$$\mathtt{L2I}(\Gamma) \vdash \mathbf{fun}(z, x, \mathtt{L2I}(e')) : \mathtt{L2I}(\tau_1) \rightarrow_{\mathtt{L2I}(\epsilon')} \mathtt{L2I}(\tau_2) \mid \langle \rangle.$$

Thus, ST_REFL and T_SUB derive

$$\mathtt{L2I}(\Gamma) \vdash \mathbf{fun}(z, x, \mathtt{L2I}(e')) : \mathtt{L2I}(\tau_1) \rightarrow_{\mathtt{L2I}(\epsilon')} \mathtt{L2I}(\tau_2) \mid \mathtt{L2I}(\epsilon).$$

as required.

Case TL_LET: We have

- $e = \mathsf{val}\, x = e_1; e_2$,
- $\sigma = \tau$,
- $\Gamma \vdash e_1 : \sigma' \mid \epsilon$, and
- $\Gamma, x : \sigma' \vdash e_2 : \tau \mid \epsilon$,

for some x, e_1, e_2, τ, and σ'. By the induction hypothesis, we have

- $\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(e_1) : \mathtt{L2I}(\sigma') \mid \mathtt{L2I}(\epsilon)$ and
- $\mathtt{L2I}(\Gamma, x : \sigma') \vdash \mathtt{L2I}(e_2) : \mathtt{L2I}(\tau) \mid \mathtt{L2I}(\epsilon)$.

By definition of $\mathtt{L2I}$ and T_LET, we have

$$\mathtt{L2I}(\Gamma) \vdash \mathbf{let}\, x = \mathtt{L2I}(e_1) \,\mathbf{in}\, \mathtt{L2I}(e_2) : \mathtt{L2I}(\tau) \mid \mathtt{L2I}(\epsilon)$$

as required.

Case TL_APP: We have

- $e = e_1(e_2)$,
- $\sigma = \tau$,
- $\Gamma \vdash e_1 : \tau_2 \rightarrow \epsilon\, \tau \mid \epsilon$, and
- $\Gamma \vdash e_2 : \tau_2 \mid \epsilon$,

for some e_1, e_2, τ, and τ_2. By the induction hypothesis, we have

- $\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(e_1) : \mathtt{L2I}(\tau_2) \rightarrow_{\mathtt{L2I}(\epsilon)} \mathtt{L2I}(\tau) \mid \mathtt{L2I}(\epsilon)$ and
- $\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(e_2) : \mathtt{L2I}(\tau_2) \mid \mathtt{L2I}(\epsilon)$.

Without loss of generality, we can choose x and y such that

- $x \neq y$,

- $x \notin \mathrm{dom}(\mathtt{L2I}(\Gamma))$, and
- $y \notin \mathrm{dom}(\mathtt{L2I}(\Gamma))$.

Because Lemma 3.12(1) and C_VAR give us

- $\vdash \mathtt{L2I}(\Gamma), x : \mathtt{L2I}(\tau_2) \to_{\mathtt{L2I}(\epsilon)} \mathtt{L2I}(\tau)$ and
- $\vdash \mathtt{L2I}(\Gamma), x : \mathtt{L2I}(\tau_2) \to_{\mathtt{L2I}(\epsilon)} \mathtt{L2I}(\tau), y : \mathtt{L2I}(\tau_2)$.

Thus, T_VAR and T_APP derive

$$\mathtt{L2I}(\Gamma), x : \mathtt{L2I}(\tau_2) \to_{\mathtt{L2I}(\epsilon)} \mathtt{L2I}(\tau), y : \mathtt{L2I}(\tau_2) \vdash x\,y : \mathtt{L2I}(\tau) \mid \mathtt{L2I}(\epsilon).$$

By Lemma 3.5(5), T_LET derive

$$\mathtt{L2I}(\Gamma), x : \mathtt{L2I}(\tau_2) \to_{\mathtt{L2I}(\epsilon)} \mathtt{L2I}(\tau) \vdash \mathbf{let}\ y = \mathtt{L2I}(e_2)\ \mathbf{in}\ x\,y : \mathtt{L2I}(\tau) \mid \mathtt{L2I}(\epsilon).$$

Thus, T_LET derives

$$\mathtt{L2I}(\Gamma) \vdash \mathbf{let}\ x = e_1\ \mathbf{in}\ \mathbf{let}\ y = \mathtt{L2I}(e_2)\ \mathbf{in}\ x\,y : \mathtt{L2I}(\tau) \mid \mathtt{L2I}(\epsilon)$$

as required.

Case TL_TABS: We have

- $e = \Lambda \overline{\alpha^k}.e'$,
- $\sigma = \forall \overline{\alpha^k}.\tau$,
- $\Gamma, \overline{\alpha^k} \vdash e' : \tau \mid \langle\rangle$, and
- $\mathsf{ftv}(\epsilon) \subseteq \Gamma$,

for some $\overline{\alpha^k}$, e', and τ. By Lemma 4.16, we have $\vdash \Gamma, \overline{\alpha^k}$. Since only CL_TVAR derive $\vdash \Gamma, \overline{\alpha^k}$, we have $\vdash \Gamma$. By case (2), we have $\vdash \mathtt{L2I}(\Gamma)$. By the induction hypothesis and case (1), we have

- $\mathtt{L2I}(\Gamma), \overline{\alpha : \mathtt{L2I}(k)} \vdash \mathtt{L2I}(e') : \mathtt{L2I}(\tau) \mid \langle\rangle$ and
- $\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(\epsilon) : \mathbf{Eff}$.

By applying T_TABS repeatedly, we have

$$\mathtt{L2I}(\Gamma) \vdash \Lambda \overline{\alpha : \mathtt{L2I}(k)}.\mathtt{L2I}(e') : \forall \alpha_0 : \mathtt{L2I}(k_0).(\cdots (\forall \alpha_n : \mathtt{L2I}(k_n).\mathtt{L2I}(\tau)^{\langle\rangle}) \cdots)^{\langle\rangle} \mid \langle\rangle.$$

Thus, T_SUB derives

$$\mathtt{L2I}(\Gamma) \vdash \Lambda \overline{\alpha : \mathtt{L2I}(k)}.\mathtt{L2I}(e') : \forall \alpha_0 : \mathtt{L2I}(k_0).(\cdots (\forall \alpha_n : \mathtt{L2I}(k_n).\mathtt{L2I}(\tau)^{\langle\rangle}) \cdots)^{\langle\rangle} \mid \mathtt{L2I}(\epsilon).$$

as required.

Case TL_TAPP: We have

- $e = e'(\overline{\tau_0^k})$,
- $\sigma = \tau[\overline{\alpha^k} \mapsto \overline{\tau_0^k}]$,
- $\Gamma \vdash e' : \forall \overline{\alpha^k}.\tau \mid \epsilon$, and
- $\mathsf{ftv}(\overline{\tau_0^k}) \subseteq \Gamma$,

for some e', $\overline{\tau_0^k}$, and $\overline{\alpha^k}$. By Lemma 4.16, we have $\vdash \Gamma$. By case (2), we have $\vdash \mathtt{L2I}(\Gamma)$. By the induction hypothesis and case (1), we have

- $\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(e') : \forall \alpha_0 : \mathtt{L2I}(k_0).(\cdots (\forall \alpha_n : \mathtt{L2I}(k_n).\mathtt{L2I}(\tau)^{\langle\rangle}) \cdots)^{\langle\rangle} \mid \mathtt{L2I}(\epsilon)$ and
- $\mathtt{L2I}(\Gamma) \vdash \mathtt{L2I}(\overline{\tau_0^k}) : \overline{\mathtt{L2I}(k)}$.

Without loss of generality, we can choose x such that $x \notin \mathrm{dom}(\mathtt{L2I}(\Gamma))$. By Lemma 3.12(1) and C_VAR, we have

$$\vdash \mathtt{L2I}(\Gamma), x : \forall \alpha_0 : \mathtt{L2I}(k_0).(\cdots (\forall \alpha_n : \mathtt{L2I}(k_n).\mathtt{L2I}(\tau)^{\langle\rangle}) \cdots)^{\langle\rangle}.$$

By Lemma 3.5(2), we have

$$\mathtt{L2I}(\Gamma), x : \forall \alpha_0 : \mathtt{L2I}(k_0).(\cdots (\forall \alpha_n : \mathtt{L2I}(k_n).\mathtt{L2I}(\tau)^{\langle\rangle}) \cdots)^{\langle\rangle} \vdash \overline{\tau_0^k} : \overline{\mathtt{L2I}(k)}.$$

Thus, T_VAR and applying T_TAPP repeatedly derive

$$\mathtt{L2I}(\Gamma), x : \forall \alpha_0 : \mathtt{L2I}(k_0).(\cdots (\forall \alpha_n : \mathtt{L2I}(k_n).\mathtt{L2I}(\tau)^{\langle\rangle}) \cdots)^{\langle\rangle} \vdash x\,\mathtt{L2I}(\overline{\tau_0^k}) : \mathtt{L2I}(\tau)[\overline{\tau_0^k}/\overline{\alpha^k}] \mid \langle\rangle.$$

Because Lemma 3.12(1) and Lemma 3.5(2) give us

- $\text{L2I}(\Gamma), x : \forall \alpha_0 : \text{L2I}(k_0).(\cdots (\forall \alpha_n : \text{L2I}(k_n).\text{L2I}(\tau)^{\langle\rangle}) \cdots)^{\langle\rangle} \vdash \text{L2I}(\tau)[\overline{\tau_0^k/\alpha^k}] : \mathbf{Eff}$ and
- $\text{L2I}(\Gamma), x : \forall \alpha_0 : \text{L2I}(k_0).(\cdots (\forall \alpha_n : \text{L2I}(k_n).\text{L2I}(\tau)^{\langle\rangle}) \cdots)^{\langle\rangle} \vdash \text{L2I}(\epsilon) : \mathbf{Eff}$,

ST_REFL and T_SUB derives

$$\text{L2I}(\Gamma), x : \forall \alpha_0 : \text{L2I}(k_0).(\cdots (\forall \alpha_n : \text{L2I}(k_n).\text{L2I}(\tau)^{\langle\rangle}) \cdots)^{\langle\rangle} \vdash x\,\text{L2I}(\overline{\tau_0^k}) : \text{L2I}(\tau)[\overline{\tau_0^k/\alpha^k}] \mid \text{L2I}(\epsilon).$$

Thus, T_LET derives

$$\text{L2I}(\Gamma) \vdash \mathbf{let}\, x = \text{L2I}(e') \,\mathbf{in}\, x\,\text{L2I}(\tau_0^k) : \text{L2I}(\tau)[\overline{\tau_0^k/\alpha^k}] \mid \text{L2I}(\epsilon)$$

as required.

Case TL_HANDLE: We have
- $h = op_1(x_1) \to e_1; \cdots ; op_n(x_n) \to e_n; \mathsf{return}\, x \to e_r$,
- $e = \mathsf{handle}\{h\}(e')$,
- $\sigma = \tau_r$,
- $\Gamma \vdash e' : \tau \mid \langle c\langle \overline{\tau^k}\rangle \mid \epsilon\rangle$,
- $\Gamma, x : \tau \vdash e_r : \tau_r \mid \epsilon$,
- $\Sigma(c\langle \overline{\tau^k}\rangle) = \{op_1, \ldots, op_n\}$,
- $\Gamma \vdash op_i : \tau_i \to \langle c\langle\overline{\tau^k}\rangle \mid \langle\rangle\rangle\,\tau_i' \mid \langle\rangle$, and
- $\Gamma, resume : \tau_i' \to \epsilon\,\tau_r, x_i : \tau_i \vdash e_i : \tau_r \mid \epsilon$ for any $i \in \{1, \ldots, n\}$,

for some $h, e', x, e_r, op_i, x_i, e_i, \tau_i, \tau_i'$, and $c\langle\overline{\tau^k}\rangle$ where $i \in \{1, \ldots, n\}$. By the induction hypothesis and definition of L2I, we have
- $\text{L2I}(\Gamma) \vdash \text{L2I}(e') : \text{L2I}(\tau) \mid \langle \mathsf{c2l}(c)\,\overline{\text{L2I}(\tau^k)} \mid \text{L2I}(\epsilon)\rangle$,
- $\text{L2I}(\Gamma), x : \text{L2I}(\tau) \vdash \text{L2I}(e_r) : \text{L2I}(\tau_r) \mid \text{L2I}(\epsilon)$,
- $\text{L2I}(\Gamma), x_i : \text{L2I}(\tau_i), resume : \text{L2I}(\tau_i') \to_{\text{L2I}(\epsilon)} \text{L2I}(\tau_r) \vdash \text{L2I}(e_i) : \text{L2I}(\tau_r) \mid \text{L2I}(\epsilon)$ for any $i \in \{1, \ldots, n\}$,
- $\mathsf{c2l}(c) :: \forall \overline{\alpha} : \overline{\text{L2I}(k)}.\sigma \in \text{L2I}(\Sigma)$, and
- $\sigma[\overline{\text{L2I}(\tau^k)/\overline{\alpha}}] = \{\mathsf{op}_1 : \tau_1 \Rightarrow \tau_1', \ldots, \mathsf{op}_n : \tau_n \Rightarrow \tau_n'\}$.

Because H_RETURN and H_OP derive

$$\text{L2I}(\Gamma) \vdash_{\sigma[\overline{\text{L2I}(\tau^k)/\overline{\alpha}}]} \text{L2I}(h) : \text{L2I}(\tau) \Rightarrow^{\text{L2I}(\epsilon)} \text{L2I}(\tau_r),$$

T_HANDLING derives

$$\text{L2I}(\Gamma) \vdash \mathbf{handle}_{\mathsf{h2l}(h)}\,\text{L2I}(e')\,\mathbf{with}\,\text{L2I}(h) : \text{L2I}(\tau_r) \mid \text{L2I}(\epsilon)$$

as required.

Case TL_OPEN: We have
- $\sigma = \tau_1 \to \langle l_1, \ldots, l_n \mid \epsilon'\rangle\,\tau_2$,
- $\Gamma \vdash e : \tau_1 \to \langle l_1, \ldots, l_n \mid \langle\rangle\rangle\,\tau_2 \mid \epsilon$, and
- $\mathsf{ftv}(\epsilon') \subseteq \Gamma$,

for some $\tau_1, \tau_2, l_1, \ldots, l_n$, and ϵ'. By Lemma 4.16, we have $\vdash \Gamma$. By case (2), we have $\vdash \text{L2I}(\Gamma)$. By the induction hypothesis and case (1), we have
- $\text{L2I}(\Gamma) \vdash \text{L2I}(e) : \text{L2I}(\tau_1) \to_{\langle \text{L2I}(l_1), \ldots, \text{L2I}(l_n) \mid \langle\rangle\rangle} \text{L2I}(\tau_2) \mid \text{L2I}(\epsilon)$ and
- $\text{L2I}(\Gamma) \vdash \text{L2I}(\epsilon') : \mathbf{Eff}$.

By Lemma 3.12(1), we have

$$\text{L2I}(\Gamma) \vdash \text{L2I}(\tau_1) \to_{\langle \text{L2I}(l_1), \ldots, \text{L2I}(l_n) \mid \langle\rangle\rangle} \text{L2I}(\tau_2) : \mathbf{Typ}.$$

Since only K_FUN can derive this judgment, we have
- $\text{L2I}(\Gamma) \vdash \text{L2I}(\tau_1) : \mathbf{Typ}$,
- $\text{L2I}(\Gamma) \vdash \text{L2I}(\langle \text{L2I}(l_1), \ldots, \text{L2I}(l_n) \mid \langle\rangle\rangle) : \mathbf{Eff}$, and
- $\text{L2I}(\Gamma) \vdash \text{L2I}(\tau_2) : \mathbf{Typ}$.

Thus, by ST_REFL and ST_FUN and Lemma 3.3(1) and T_SUB, we have

$$\text{L2I}(\Gamma) \vdash \text{L2I}(e) : \text{L2I}(\tau_1) \to_{\langle \text{L2I}(l_1), \ldots, \text{L2I}(l_n) \mid \epsilon'\rangle} \text{L2I}(\tau_2) \mid \text{L2I}(\epsilon)$$

as required.

$\blacksquare$

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