Anomalous contribution to galactic rotation curves due to stochastic spacetime

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We consider a proposed alternative to quantum gravity, in which the spacetime metric is treated as classical, even while matter fields remain quantum. Consistency of the theory necessarily requires that the metric evolve stochastically. Here, we show that this stochastic behaviour leads to a modification of general relativity at low accelerations. In the low acceleration regime, the variance in the acceleration produced by the gravitational field is high in comparison to that produced by the Newtonian potential, and acts as an entropic force, causing a deviation from Einstein's theory of general relativity. We show that in this "diffusion regime", the entropic force acts from a gravitational point of view, as if it were a contribution to the matter distribution. We compute modifications to the expectation value of the metric via the path integral formalism, and find a stochastic contribution which corresponds to a cosmological constant, anti-correlated with a contribution which has been used to fit galactic rotation curves without dark matter. We caution that a greater understanding of this effect is needed before conclusions can be drawn, most likely through numerical simulations, and provide a template for computing the deviation from general relativity which serves as an experimental signature of the Brownian motion of spacetime.

According to the standard model of cosmology, ΛCDM, visible matter makes up only 5% of its contents, with dark energy or a cosmological constant Λ and cold dark matter (CDM) making up the remaining part. There is strong evidence for this. Dark energy or a cosmological constant appears to drive the expansion of the universe, while dark matter can account for the flatness of galactic rotation curves [1]. It is observed in the CMB power-spectrum [2, 3], by gravitational lensing [4] such as that observed in the Bullet Slluster, through dispersion relations of elliptical galaxies [5], mass estimates of galaxy clusters [6], and appears to be required for the formation of galaxies in the early universe [7]. However, despite large scale efforts, neither dark energy nor dark matter have been directly detected. Their apparent existence is only felt through their gravitational field. Discoveries in physics are often indirect. The neutrino was conjectured by Pauli to exist in 1939 in order to account for energy conservation in β-decay, and only gave a signal in a particle detector 26 years later. But in the absence of any direct evidence for dark energy or dark matter it is natural to wonder whether they may be unnecessary scientific constructs like celestial spheres, ether, or the planet Vulcan, all of which were superseded by simpler explanations. Gravity has a long history of being a trickster.

Several attempts to modify gravity without dark matter have been proposed. In 1983 Milgrom [8] found that if a theory had the property that either the law of inertia were modified, or Newton's theory of gravity was modified at low acceleration such that

$$a = \begin{cases} a_N & \text{when } a \gg a_0, \\ \sqrt{a_0 a_N} & \text{when } a \ll a_0. \end{cases} \tag{1}$$

with a_N the Newtonian acceleration, and a_0 a parameter of order $10^{-10}\ m/s^2$, then the flatness of rotation curves and the Tully-Fischer relation [9] would follow. These are effects currently attributed to dark matter. He called this Modified Newtonian Dynamics (MOND) [10–12]. Here, we would like to point out that this behaviour is reminiscent of Brownian motion, with a mean of a_N, and standard deviation of $\sqrt{a_N}$, a statement which we hope becomes clearer as we progress. In 1983 Milgrom also observed [8] that the MOND acceleration (in the units of this article, $c = \hbar = 1$), is given by

$$a_0 \approx \frac{1}{2\pi}\sqrt{\frac{\Lambda}{3}}, \tag{2}$$

a coincidence that has yet to be explained. A number of theories have been proposed to reproduce MOND phenomenology [11–17], but thus far no satisfying fundamental theory which reproduces this behaviour has been found. The problem is that while it is easier to modify a theory at high energy, modifying a theory at low energy while still respecting current experimental bounds is difficult. It is also important to emphasise that MOND has yet to account for the results of gravitational lensing or the CMB power spectrum. However, it seems reasonable to wait regarding these, because MOND is not a fundamental theory.

Another approach, initiated by Mannheim, comes from fitting rotation curves to the spherically symmetric metric which is a solution to conformal gravity [18–20], an approach we will discuss in more detail after having presented the results of the path integral.

Here, we calculate the effect on rotation curves due to a recently proposed alternative to quantum gravity [21, 22]. We will, in particular, use the path integral formulation developed in [22, 23], with Zach Weller-Davies. The theory was not developed to explain dark matter or energy, but rather, to reconcile quantum theory with gravity. However, it was already noted in [21], that diffusion in the metric could result in stronger gravitational fields when one might otherwise expect none to be present, and that this raised the possibility that gravitational diffusion may explain galactic rotation curves and galaxy formation without the need for dark matter. Here, we will add weight to this intuition.

We will find that even when the bare cosmological constant is zero, one should typically expect a small one due to stochastic fluctuations. This is intriguing because the bare cosmological constant is taken to be zero when no matter is present, to preserve positivity of correlation functions [24]. This provides a potential explanation for its small but non-zero value.

Looking ahead, we further find that stochastic fluctuations act as if they are a positive contribution to the mass, and that these become relevant at an acceleration scale, which sets a scale at which gravity is modified, which is a necessary condition for MOND behaviour. This is discussed through Eq. (21). The path integral also contains another dominant fluctuation, which contributes a linear term $\gamma_1 r$ to the Schwarzschild and Newtonian solution, akin to that found in conformal gravity. γ_1 has units of acceleration. Although the theory does not precisely predict the coincidence of Eq. (2), it gives a numerically similar relationship between the fluctuation corresponding to the acceleration γ_1 of $\gamma_1 \approx \Lambda R_H$, with R_H the Hubble radius and with the relationship becoming $\gamma_1 \approx \sqrt{\Lambda}$ in a Λ-dominated universe and results in flat rotation curves in a region far from the galactic centre, but with possible deviations at larger distances depending on how parameters are chosen. Our parameters are not coupling constants, but correspond to boundary conditions which likely require a fuller understanding of the dynamics of the theory. Since quantum-classical theories of gravity are very restricted, a better understanding of this is likely to enable astrophysical tests of the quantum nature of spacetime.

The theory of [21, 22] describes spacetime classically, while the matter fields are quantised. This necessarily requires the evolution of spacetime to have a stochastic component [25]. The full path integral of the theory is given in Section A. Here, we only study the classical limit of the theory. In this limit, the quantum matter degrees of freedom will have decohered, but in the dynamics of the classical-quantum framework, the classical degrees of freedom still undergo stochastic evolution. We will also not concern ourselves with the evolution of the matter degrees of freedom, and thus only represent them by their mass density $m(x)$, neglecting the Hamiltonian term which governs their evolution. Therefore, we can represent the path integral as

$$\varrho(\Sigma_f, m_f, t_f) = \int \mathcal{D}g \, \mathcal{N} e^{\mathcal{I}[g,m,t_i,t_f]} \varrho(\Sigma_i, m_i, t_i), \tag{3}$$

where the action contains a gauge-fixing term so that the path integral over metrics g is over geometries. The path integral determines the probability density $\varrho(\Sigma_f, m, t_f)$ of a final spatial surface Σ_f given an initial spatial surface, and $\mathcal{N}$ is a normalisation factor. This is much simpler in the Newtonian limit, where we can parameterise the metric in terms of the Newtonian gravitational potential Φ. In this limit, the action of [22] was found to be [26]

$$\mathcal{I}[\Phi, m, t_i, t_f] = -\frac{D_0(1-\beta)}{G_N^2} \int_{t_i}^{t_f} dt d\vec{x} \left(\nabla^2\Phi - 4\pi G_N m(x)\right)^2. \tag{4}$$

Here β is required to be less than 1/3 by consideration of positivity of the full action, but for the correlation functions of the full theory to be positive semidefinite, it is required to be negative [24]. It is naturally assumed to be of order $\mathcal{O}(1)$, and in this Newtonian limit, $\beta < 1$ is clearly sufficient – it is what is required for the path integral to suppress paths away from Poisson's equation. D_0/G_N^2 is a dimensionless coupling constant, which determines the scale of fluctuations. $\beta = 1/3$ without the matter couplings, would correspond to the conformally invariant theory. The action is similar in form to that of the Onsager-Machlup function [27]. The action can also be derived as the weak field limit of the action for Nordstrom gravity which is diffeomorphism invariant and completely positive [28].

Crucially, we see that the action (4) is in the form of an equation of motion squared, and has a global maximum when the equations of motion are satisfied

$$\langle \nabla^2\Phi - 4\pi G_N m \rangle = 0. \tag{5}$$

As shown in [26], this action derived as the weak field limit of [21, 22] is a path integral formulation of the model of [29] when a local noise kernel is chosen. Since Eq (5) is linear in Φ, when $m(x)$ has a definite distribution rather than being a statistical mixture of distributions we have that $\nabla^2\langle\Phi\rangle$ satisfies Poisson's equation, and so on expectation, there is no difference between the expectation value of Φ and its deterministic value. Nonetheless, the action of Eq (4) is extremised not only by Φ which satisfies Poisson's equation, but also by more general field configurations that make the action variation vanish for fixed endpoints. In vacuum, when $m(x) = 0$, the Φ which extremise the action of Equation (4) is found to be the biharmonic equation

$$\nabla^4 \Phi = 0, \tag{6}$$

which have solutions away from $x = 0$ given by

$$\Phi_{MPP}(x) = -\frac{\kappa_m}{4\pi|x|} + \kappa_0 - \kappa_1 8\pi|x| + \kappa_2|x|^2. \tag{7}$$

The first two terms are the standard Newtonian potential plus an arbitrary constant term, while the last two additional terms do not satisfy the standard vacuum Poisson's equation, and are therefore local rather than global maxima. However, they still make substantial contributions to the path integral. Note that the κ_m term and the κ_1 term are Green's functions for ∇^2 and ∇^4 respectively, and for this reason we've explicitly put in the sign and factors of π. Solutions to the biharmonic equation with a source can be found for example, in [30], the difference here being that the source is $4\pi\nabla^2\Phi$.

We wish to emphasise that these are not equations of motion in the usual sense. The dominant contribution to the path integral comes from the solution to Poisson's equation, while the rest merely represent stochastic deviations from Poisson's equation which are not too suppressed in the path integral given boundary conditions. Other configurations also contribute with a probability weighed by the path integral.

Therefore, we will call the generalised configurations such as those of Eq. (7) *Most Probable Paths* (MPPs), adopting the language used in the study of diffusive dynamics [31–33]. We include a simple example of how these contribute to the path integral in the case of Brownian motion with a step function potential in Appendix D. The κ should be static, since this non-relativistic action follows from a local relativistic one. For now, it is worth foreshadowing our final result by noting that the contribution of κ_1 to the most probable path has units of acceleration, and the κ_2 contribution is a solution to general relativity if there is a constant matter density and has the same units as the cosmological constant.

If we were to substitute the most probable path of Equation (7) into the 0th order action of (4) in a vacuum region, then the κ_m, $m(x)$ and κ_0 term don't contribute to the action if the Newtonian term is used as a Green's function for the matter distribution $m(x)$. They can therefore be set by the boundary conditions, as is done in solving Poisson's equation. The κ_2 however, does contribute to the action and is therefore suppressed. We will neglect the κ_1 term in this brief discussion, because it doesn't satisfy appropriate boundary conditions if in vacuum, but it would still contribute significantly to the path integral (see comment at the end of this article). If we substitute the other terms of the MPP of Eq (7), into the 0th order action, we get, what we will call the *MPP-action*

$$I_{MPP} = -\frac{D_{0,T}(1-\beta)}{4\pi G_N^2}\int d^3x\,(6\kappa_2)^2 \quad , \tag{8}$$

where $D_{0,T}/G_N^2$ has units of time, since the paths must be static as they are inherited from the relativistic theory, and we neglect the integral over time. This should be recalled when adding powers of c back to the prefactor of $D_{0,T}/G_N^2$. When we are substituting the most probable paths, this is reminiscent of the on-shell action used in quantum field theory and captures the leading order terms to the path integral. If we include other terms in the action (such as the κ_1 term), then the action will allow us to calculate the relative probabilities of the paths we substitute into the action regardless of whether they extremise the action.

If we add a source term $J(x)O(\kappa)$ to the action with an arbitrary function $O(\kappa)$ of the parameters $\kappa = \{\kappa_m, \kappa_0, \kappa_1, \kappa_2\}$, we can construct a partition function

$$Z_{MPP}[J] = \mathcal{N}\int \mathcal{D}\kappa\, e^{\mathcal{I}[\Phi_{MPP},m,J]}, \tag{9}$$

with $\mathcal{N}$ the normalisation factor. This can be used to compute correlation functions of the parameters κ. However, for this simple example, we immediately see that this is a normal distribution in κ_2 with a standard deviation which scales like $G_N/\sqrt{D_{0,T}V}$ where V is the spatial volume of the region we are considering. We will see that κ_2 is equivalent to a small cosmological constant of arbitrary sign, and we emphasise that it appears as a necessary fluctuation even though the deterministic equations of motion don't allow for it. Since it has some variance, we would be surprised if we found it to be 0. Care should be taken with the MPP action. If we toss a 1000 coins, slightly biased towards heads, the most probably single configuration is all heads. A more natural characterisation of the outcome would be in terms of the expected number of heads vs tails, which also characterises any local sample, provided it is sufficiently large. Note that unlike the Brownian motion path integral in Appendix D, or the full relativistic one, this is a non-dynamical path integral, but if the dynamics is slow enough, we would expect it to characterise the final distribution.

Let us now move to the relativistic case. Here, the static solution is the appropriate metric for considering the effect of stochastic fluctuations over large distances. We therefore consider a spherically symmetric metric of the form

$$ds^2 = -e^{2\phi(r)}dt^2 + e^{-2\psi(r)}dr^2 + e^{-2\chi(r)}r^2\, d\Omega^2, \tag{10}$$

with Ω the 2-dimensional solid angle. In general relativity, we could perform a coordinate transformation to set $\chi(r) = 0$. In vacuum, Einstein's equation would require $\phi(r) = \psi(r) = \frac{1}{2}\log(1 - 2M/r)$ and we would arrive at the usual Schwarzschild solution. Here, we consider more general metrics of the form

$$\phi(r) = \psi(r) = \frac{1}{2}\log\left(1 - \frac{2F(r)}{r}\right), \quad \chi(r) = 0, \tag{11}$$

with $F(r)$ a function which we will take to be a power series in r, motivated in part by the most probable path of Equation (7). Note that $1 - 2F/r = 0$ is a horizon, and so F is bounded in a similar way to the example we consider in Section D of Brownian motion with a wall.

The full dynamical action of [22] is reproduced as Equation (A2) of Appendix A and in Appendix B we include a more detailed description of the sketch we are giving here. The purely gravitational part of the action, is given by

$$\mathcal{I} = -\frac{D_0}{G_N^2}\int d^4x\, \sqrt{-g}\big(\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} - \beta\,\mathcal{R}^2\big). \tag{12}$$

If we now substitute the ansatz of Eq (11) into this action, we find the simple form

$$\mathcal{I}_F(r) = -\frac{8\pi D_0}{G_N^2}\int dr\,\left((1-2\beta)F''(r)^2 + \frac{(4-8\beta)}{r^2}F'(r)^2 - \frac{8\beta}{r}F'(r)F''(r)\right), \tag{13}$$

where the angular part has been already integrated out so that the path integral is a two-dimensional Gaussian distribution in the variables F' and F''. Here, F' has dimensions of acceleration and F'' has dimensions of the cosmological constant. To see this more clearly, let us consider the power expansion of F

$$\begin{aligned}\frac{2F(r)}{r} &= \sum_{n=-\infty}^{\infty}\gamma_n r^n \\ &= \cdots + \frac{\gamma_m}{r} + \gamma_0 + \gamma_1 r + \gamma_2 r^2 + \cdots\,, \end{aligned}\tag{14}$$

where in the second line we have written the terms relevant to the length scales we are considering. The γ_m ends up dropping from the action, and we will henceforth set it to $2G_N M$, since at order r^{-1} it is the standard Schwarzschild term which can here be determined from boundary conditions. It makes no difference for the purposes of this discussion whether we include the γ_i corresponding to other higher or lower powers, or functions like $\log(r)$. The reason for this is that the series is linear in the γ_i, so that when we substitute the expansion back into the action, we find that the coefficients follow a multivariate Gaussian distribution with zero mean and non-zero correlation. We can then perform the Gaussian integrals over all other γ_i we are not interested in, and the action for the remaining $\gamma_0, \gamma_1, \gamma_2$ will not change. The reason we are most interested in these contributions, is that negative powers of r will not contribute far away from the mass distribution of a galaxy which is our zone of interest. On the other hand, higher powers of the expansion will be heavily suppressed by the action once the radial integral is performed, as can be verified by including them. We will see that they represent fluctuations larger than our Hubble volume. For this reason we focus only on the correlation between γ_0, γ_1 and γ_2, and inclusion of the other γ_i would not effect our conclusions.

If we now substitute the power series of Eq. (14) into the action and integrate from $r = 0$ to some r_{max}, we obtain

$$\mathcal{I}_\gamma = -\frac{6\pi D_{0,T}V}{G_N^2}\left(\frac{5-18\beta}{r_{\max}^2}\gamma_1^2 + 6(1-4\beta)\gamma_2^2 + \frac{9(1-4\beta)}{r_{max}}\gamma_1\gamma_2\right), \tag{15}$$

where we have dropped the constant term γ_0 for ease of presentation, as it doesn't alter the conclusions nor contribute to rotation curves. $V = \frac{4}{3}\pi r_{max}^3$, and we could absorb it into the coupling constant $D_{0,T}/G_N^2$ which renormalises it and gives it units of Planck length to the 4th power l_p^4, but we leave it in place to keep track of units. The analysis doesn't change much if we integrated from the inner horizon to r_{max}. Since we are considering large-scale fluctuations which exist over all space, they are naturally suppressed by a volume element, so it's interesting to see that γ_2 is only suppressed by this amount. We also see here, or by explicit calculation, that higher powers in the expansion of Eq. (14)

would be more suppressed, which motivated us to integrate out such higher powers. They represent fluctuations of a length scale which are not felt inside our Hubble volume. This gives the path integral over $\gamma = \{\gamma_1, \gamma_2\}$

$$Z_\gamma = \mathcal{N} \int \mathcal{D}\gamma e^{\mathcal{I}_\gamma[\gamma,m]}. \tag{16}$$

Integrating over 4-geometries is here limited to 4-geometries which have the metric.

$$ds^2 = -\left(1 - \frac{2MG_N}{r} - \gamma_0 - \gamma_1 r - \gamma_2 r^2\right) dt^2 + \left(1 - \frac{2MG_N}{r} - \gamma_0 - \gamma_1 r - \gamma_2 r^2\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{17}$$

This is the *MK metric* of [30, 34], which is used to fit galactic rotation curves [18–20, 30, 35]. Here we will not fit γ_1, but instead determine it from the path integral. While this metric is a solution to conformal gravity [34], it is not a solution to general relativity, nonetheless, it does contribute to the classical-quantum path integral, as can be seen from Eq. (15). While conformal gravity has issues with negative norm ghosts, this is not an issue here [24] (c.f. [36–42]). Moreover, criticisms of using the MK metric in fitting rotation curves akin to those presented in [43–45] are also not applicable to the classical-quantum theory of [21, 22] which is not conformally invariant, but rather scale-invariant without matter [24]. Scale invariance is broken by the matter action. Furthermore, although in conformal gravity, the Newtonian potential now depends on the mass distribution of the source rather than just the mass [46], the correct Newtonian potential is the dominant saddle of our path integral.

For now, we see that γ_2 corresponds to the cosmological constant term of Schwarzschild deSitter, while γ_1 contributes to the geodesic equation of stars far from the galactic centre. The expectation values of γ_1 and γ_2 are zero, but the two random variables are normally distributed and inversely correlated as can be seen by inspection of Eq. (15) and (16). The covariance matrix of the normal distribution determined from the path integral Eq. (15) is

$$\begin{aligned}
\Sigma_{11} &= \frac{2r_{max}^2}{3(13-36\beta)} \frac{G_N^2}{D_{0,T}V} \\
\Sigma_{22} &= \frac{(5-18\beta)}{9(13-36\beta)(1-4\beta)} \frac{G_N^2}{D_{0,T}V} \\
\Sigma_{12} &= -\frac{r_{max}}{2(13-36\beta)} \frac{G_N^2}{D_{0,T}V}
\end{aligned} \tag{18}$$

and the conditional mean of γ_1, given an observation of γ_2 is given by

$$\mu_{\gamma_1|\gamma_2,r_{max}} = -\frac{9}{2}\gamma_2\, r_{max} \left(\frac{1-4\beta}{5-18\beta}\right), \tag{19}$$

here plotted in Figure 2. We can now ask, given that we observe a value of $\gamma_2 \approx \Lambda/3$, where $\Lambda \approx 10^{-52}\ m^{-2}$ is the cosmological constant, and an r_{max} given by the Hubble radius $R_H \approx 10^{26}\ m$ over which we obesrve it, what does this tell us about the value we expect to see of γ_1? Recall that we expect β to be negative and of order 1, but from Fig 2 we see that we are insensitive to it's value. In this case we find a mean value of γ_1 to be of $\mu_{\gamma_1|\gamma_2,r_{max}} \approx -10^{-26}$ m^{-1}. Putting units of c back in, we find that γ_1 is of the order of the MOND acceleration $\gamma_1 \approx 10^{-10}\ m/s^2$.

This is the value initially used to fit galactic rotation curves for larger spiral galaxies, in the context of conformal gravity [19] . In the region where the transition between the rising $\gamma_1 r$ term is of the order of the falling $G_N M/r$ term, the rotation curve is roughly flat. γ_1 is not too large such that it runs afoul of experimental bounds on solar system evolution. However, to account for smaller dwarf galaxies, γ_1 is adjusted as $\gamma_1(M) = \gamma_1(1 + M/10^{10}M_\odot)$ with $\gamma_1 \approx 10^{-28}m^{-1}$. Here it should be said that CDM simulations also have trouble – the core-cusp problem[47–49], so dark matter models also require novel matter properties, additional assumptions or further considerations[50]. To account for more recent data, γ_2, is taken to be a $\kappa \approx 10^{-50}m^{-2}$ [20] to "flatten the curve" while giving a slight rise which is claimed to be observed [51] (see in contrast [52]).

Without further considering the dynamics of the matter distribution, we have no reason at this point, to select the constants in the most probable path, because they correspond to boundary conditions. It is reasonable to set γ_2 to be the value of the cosmological constant given its observation over the scale of the Hubble radius, and then use this to predict γ_1. It is not inconsistent with our obtained γ_1 for $\gamma_2 \approx \kappa$ over galactic distances, since $\gamma_1 \approx \mu_{\gamma_1|\gamma_2,r_{max}}$ when $r_{max} \approx 100kpc$ since the observed rotation curves in each galaxy only extends to a radius of that order (the galactic disk of larger galaxies). This would require viewing γ_2 as a fluctuation which appears as if it were a dark energy contribution akin to some form of quintessence [53–55], which is not inconsistent with observation, but without a greater understanding of galaxy formation in a cosmological setting, little can be said. Instead, the first conclusion we would like to draw, is that the order of magnitude estimates suggest the theory makes predictions broadly in line with current observations, and suggest that simulations of the theory, combined with astrophysical observations, could be used to test its anomalous behavior.

Efforts to understand the effect of these stochastic fluctuations in cosmology are initiated in [56] with Emanuele Panella and Andrew Pontzen, where we find some evidence that early time stochastic fluctuations in the gravitational degree of freedom do behave as a positive matter fluctuation, in terms of how they scale with the expansion factor in a Friedman-Robertson-Walker spacetime. We also find some evidence that in order to describe our universe, the fluctuations in both the cosmological constant and gravitational degrees of freedom have to be small, or have to have occurred at an early time. In particular, we find that our current Λ cannot only be the result of large scale fluctuations in the evolution of the scale factor. Cosmology studies using different models of stochastic fluctuations have been considered in [57, 58]. Other approaches have also tried to connect cosmology with the emergence of dark matter [59] and even attempt to explain dark energy as a fluctuation of the Newtonian gravitational constant [60].

In order to provide a template for further comparison between models and observation, we can already get an estimate of the parameter $D_{0,T}/G_N^2$, at least at large distances if we take the cosmological constant to be the result of stochastic fluctuations. Since we expect that we live in a typical universe, this tells us that the variance in γ_2 should be of the order of Λ^2, so that the value of γ_2 we witness is typical. From the covariance matrix above, we can see that this sets $D_{0,T}/G_N^2$ to be of the order of $D_{0,T}/G_N^2 \approx 1/\Lambda^2 V_H$, with V_H the Hubble volume. In units with c it is perhaps easiest to think in terms of a diffusion coefficient 4-density $\mathcal{D}_2 := G_N^2/D_{0,T}c^3 V_H \approx 10^{-104} m^{-4}$. A fuller analysis of the variances can be found in Appendix Section B. Note that this could explain both the small but non-zero value of the cosmological constant, at least in terms of $D_{0,T}$, since we find in [24] that a bare cosmological constant can't be included for reasons of complete positivity, and may lead to explainations of the coincidence that $a_0 \approx \frac{1}{2\pi}\sqrt{\frac{\Lambda}{3}}$ [8] – both coincidences which has thus far received no satisfying explanation. It also may explains why general relativity is modified at low acceleration. Crucially, the fact that γ_1 and γ_2 are anti-correlated comes out of the path integral. We have also not needed to fine-tune β.

Let us now highlight the weaknesses of the calculation. To make it tractable analytically, we have restricted ourselves to spherically symmetric and static spacetimes, with metrics of the form of Eqs. (17). Allowing ψ and ϕ to be different would be desirable. This would double the number of parameters in the action, and may give further insight into galactic rotation curves. A greater understanding of more general metric fluctuations does seem in order. For simplicity we have restricted ourselves to understanding the correlation in γ_1 and γ_2. They reflect different length scales of stochastic fluctuations, but there are correlations between them, and for example, the higher powers in the expansion. Here, the full normal distribution reflected in Eq. (13) may provide some insight into the distribution of what is currently taken to be dark matter, but one must be careful since anything can be fit to a power series, and a fuller understanding of the probability distribution is required, as we don't know what other terms may contribute. Here, the cosmological constant term serves as a reasonable candle to measure against. A fuller principle component analysis may be useful, via the Kosambi–Karhunen–Loève theorem [61].

For spherically symmetric matter distributions, it is natural for the expectation value of the metric and its variance to be spherically symmetric. However, any realisation of the stochastic noise is highly non-uniform and is not constant in time, while the metric ansatz is constant in time and uniform over the sphere at each radius r. Static spacetimes were chosen because, in the relativistic theory, we would not expect large scale fluctuations except those which are already present, and conjecture that given the scale, the γ_2 term represents fluctuations which have been baked in during inflation. The R^2 term, which dominates the covariant path integral, could allow for Starobinski inflation [62–64], which is favoured by CMB data [65]. The dynamics of other contributions to the path integral are unknown. We are also here required to renormalize $D_{0,T}$ by the volume element. While this does not effect the mean values we derive, nor the relative variances of γ_1 vs γ_2, it may make the $D_{0,T}$ we estimate here, difficult to relate to that measured at shorter distance scales. A greater understanding of the renormalisation flow will be required in order to relate bounds on $D_{0,T}$ coming from astronomy data to the scale relevant for tabletop experiments such as those proposed in [25] based on the decoherence vs diffusion trade-off [25, 66].

We have also only considered correlations in larger scale fluctuations rather than short distance fluctuations. We would like to better understand how these arise from the local time-dependent fluctuations present in (4). We would therefor like to cross-check the results given here, with what the theory predicts for local fluctuations. For this, it is worthwhile to look at the post-Newtonian expansion of the full theory. Here, we can see that local stochastic fluctuations lead to an acceleration scale, below which the laws of gravity are modified. This is most easily done in isotropic coordinates, which we do in Appendix C. In these coordinates, the action is given by Eq. (C9)

$$\mathcal{I} = -\frac{D_0 c^5}{64\pi^2 G_N^2}\int d^3x dt\, e^{\frac{2\Phi}{c^2}}\left\{\left(\nabla^2\Phi - \frac{(\nabla\Phi)^2}{2c^2} - 4e^{-\frac{2\Phi}{c^2}}\pi G m\right)^2 + \frac{3}{c^4}(\nabla\Phi)^4 - 4\beta\left(\nabla^2\Phi - \frac{(\nabla\Phi)^2}{2c^2} - 4e^{-\frac{2\Phi}{c^2}}\pi G m\right)^2\right\} \tag{20}$$

where we have put powers of c back in to highlight terms which contribute only at higher order. In each term, the acceleration squared $(\nabla\Phi)^2/c^2$ plays an important role. Let us take $\beta = 0$ for simplicity (the argument doesn't change much if it's non-zero), and let's also drop the $\frac{3}{c^4}(\nabla\Phi)^4$ for further simplicity – its inclusion will only enhance the argument we are about to make. In this case, this action says that on expectation, the scalar gravitational potential Φ must satisfy

$$\left\langle e^{\frac{\Phi}{c^2}}\left(\nabla^2\Phi - \frac{1}{2c^2}(\nabla\Phi)^2 - 4e^{-\frac{2\Phi}{c^2}}\pi G m(x)\right)\right\rangle = 0 \tag{21}$$

Here, we immediately see, that when $\langle(\nabla\Phi)^2\rangle >> \langle\nabla\Phi\rangle^2$, we will see on average, a deviation from the Newtonian limit of general relativity. Indeed from Eq. (21), we see that the extra variance acts like a positive mass term. We call the regime when $\langle(\nabla\Phi)^2\rangle >> \langle\nabla\Phi\rangle^2$, the *diffusion regime*, since when the acceleration $|\nabla\Phi|$ is small in comparison to its standard deviation, we will see a deviation from the Newtonian law of gravity. In Appendix D, we define an entropic force to be just such a deviation from the deterministic equations. This is distinct from the entropic force used by Verlinde in the context of Holography, in which gravity itself is proposed as an entropic force which also acts as dark matter[67].

If the diffusion in the acceleration is relatively constant far from the galactic center, then this naturally picks out a universal acceleration scale as occurs in MOND phenomenology. Once the acceleration drops below the level set by the diffusion in $|\nabla\Phi|$, we have a deviation from the Newtonian law, and indeed, the post-Newtonian corrections. If $|\nabla\Phi|$ is instead above the diffusion regime, the expectation value $\langle\nabla\Phi\rangle$ obeys the post-Newtonian equations of motion, which explains why PPN tests of general relativity are unaffected by the stochastic fluctuations of [21, 22].

While this study demonstrates that galactic rotation curves can undergo modification due to stochastic fluctuations, a phenomenon attributed to dark matter, it is important to acknowledge the existence of separate, independent evidence supporting ΛCDM. In particular, in the CMB power spectrum, in gravitational lensing, in the necessity of dark matter for structure formation, and in a varied collection of other methods used to estimate the mass in galaxies. These now form an important set of tools with which to test the theory of [21, 22].

From a theoretical point of view, another caveat we wish to highlight, concerns the negative definiteness of the action. This is required in order to give finite probability distributions, and suppress paths which deviate from Einstein's equation. While the weak field limit has this property, the generalised deWitt metric, Eq. (A3), is not positive semidefinite, nonetheless the negative contributions to the path integral appear to correspond to non-dynamical degrees of freedom [24, 68]. One corresponds to the Gauss-Bonnet term, which in 4 spatial dimensions is a purely topological term and also a total divergence. Since we don't sum over topologies, its bulk contribution is benign. The total divergence is usually discarded as a boundary term at spatial and temporal infinity which does not effect local physics, but whether this can be done here is less clear, since the final condition is not determined by the initial condition. The other negative contribution, corresponds to the magnetic part of the Weyl curvature which is also non-dynamical, in the sense of being made up of only first time-derivatives in the metric. In [68], we find there are discretizations of the path integral, such that the magnetic Weyl term merely contributes to the normalisation, and thus appears benign, but the consequences of this are not yet fully understood. This is briefly previewed in [24]. This concern doesn't effect the calculation here, because the Weyl curvature term is positive definite on the metrics we consider, and comes into the action with an overall minus sign if the Gauss-Bonnet identity is used. Nonetheless, care should be taken in extending this work to dynamical spacetimes [69] until this issue is better understood.

Let us finally examine the implications of our results on tabletop experiments performed on Earth. Here, we have found that stochastic fluctuations in the acceleration with a standard deviation of the MOND acceleration, could explain both the small value of the cosmological constant and perhaps the flatness of galactic rotation curves. We have estimated the value of the diffusion 4-density $G_N^2/D_{0,T}V_Hc^3$ which corresponds to this, to be of order Λ^2. This rules out terrestrial experiments being able to detect the long-range stochastic fluctuations discussed here, although it says little about shorter range fluctuations. We can now use the decoherence vs diffusion trade-off [25] to ask what this says about terrestrial decoherence experiments. There, we found that the decoherence rate corresponding to the path integral of Eq. (A2) or its weak field limit [26] to be $\lambda = 2D_{0,T}c^3M^2/V_\lambda$, where M is the mass of the particle in the interference experiment, and V_λ is the volume of the wave-packet. Note that this decoherence rate is not the Diosi-Penrose rate [70–72], since the theory considered here is ultra-local and linear. There is also no genuine decoherence, since the quantum state stays pure, conditioned on the classical degrees of freedom [26], here taken to be spacetime. Only when integrating out the gravitational degrees of freedom does the state decohere. Since the value of $D_{0,T}c^3$ is of order $10^{-20}\ m^3s^{-1}kg^{-2}$, $M \approx 10^{-24}$ kg for fullerene molecules, $V_\lambda \approx 10^{-25}\ m^3$ for the wavepacket volume estimated in the experiment of [73] gives a decoherence rate $\lambda \approx 10^{-43}\ s^{-1}$, orders of magnitude below current bounds of about .1 s^{-1} [73]. However, this does not reflect the fact that the $D_{0,T}$ that we have estimated here, is at a volume scale set by V_H, the Hubble volume and may renormalise. This however does not suggest that decoherence experiments and tests from anomalous heating [74–88] are uneffected. Using the decoherence vs diffusion trade-off again, we can compute the stochastic gravitational fluctuations at short distances, and we found that such short range fluctuations would violate current bounds on precision tests of the gravitational field, unless gravity were modified at short distances. This does appear to be the case – in [24] we find that the pure gravity theory is asymptotically free, meaning that the stochastic fluctuations of the metric get weaker at shorter distances, at a scale which needs to be determined from experiment, as is the case with dimensional transmutation in QCD. Whatever the form of this stochastic noise, it will contribute to apparent decoherence through secondary heating[89], and thus decoherence experiments and bounds due to heating may constrain the theory further, or rule it out. To understand this better, one likely requires the nonlinear contributions as in Eq. (21) which acts like a mass on average when the variance becomes large and plausibly reduces the diffusion at short distances.

This also suggests that measurements of gravity at short distances[90] are of interest in understanding this apparent behaviour, as well as conducting experiments to test the quantum vs classical nature of spacetime [25, 91–95]. At this point, it is too early to make bold claims, and a greater understanding of the theoretical and experimental constraints is required. It is possible that the effects derived here could be the result of a fully quantum theory of gravity, which [21, 22] describes as an effective theory. This we regard as unlikely because we don't expect the stochastic fluctuations of spacetime to be so large in a quantum theory of gravity. The parameter space of such an effective theory has been found with Isaac Layton in [96]. We also do not know how to make such a theory renormalisable. Baring such a theory, it would appear that 95% of the energy in the universe is due to stochastic fluctuations of spacetime, whose origin is either due to a fundamental breakdown in predictability or an environment which does not obey the laws of classical or quantum theory [97].

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Comment: Two parts of v1 of this preprint were criticized in [98], neither of which formed part of our analysis. First, they note that if in the Newtonian case, one has an r term in the potential in the vacuum, then it cannot satisfy matching boundary conditions for a localised source if it is to satisfy the MPP equation with matter $\nabla^4\Phi(x) = 4\pi G_N \nabla^2 m(x)$. This is true. However as already emphasised in v1, the MPP is not an equation of motion but only gives *the most probable path*, and other paths including the r term in vacuum contribute. This can be seen by also inserting the κ_1 term into the action which gives in the spherically symmetric case

$$I'_{MPP} = -\frac{D_{0,T}(1-\beta)}{G_N^2}\int r^2 dr\left(6\kappa_2 + \frac{2\kappa_1}{|r|}\right)^2 \quad , \tag{22}$$

We see κ_1 anti-correlated with the cosmological constant term κ_2. While suppressed, it still contributes enough to influence galactic rotation curves. The path integral calculation demonstrates this. If you observe an r^2 term (which is an extremal path) corresponding to a fluctuation which acts as a cosmological constant then you should expect to see an r term. This is unchanged when a localised matter distribution is present. In the actual calculation, we consider a power law expansion. It is true however, that a fuller principle component analysis would be useful, via the Kosambi–Karhunen–Loève theorem [61], to determine the appropriate contributions more fully.

Next, they claim we drop a constant term in our discussion, but we did not, we explicitly keep and note it. This was in the context of a discussion, where we tried to give the reader further intuition concerning Eq. (21), which very clearly does set a natural acceleration scale. When the mean acceleration drops below its variance, the acceleration will act on expectation as a positive matter source. This point has been removed in v2 to not distract from (21). The separate discussion around the different anomalous contributions to rotation curves comes after Eq. (19) which we have expanded.

They make a third point already discussed by us in v1 – that by itself, the random variables found in Eqs. (22) or (15) are zero on expectation. This is true, but because they have variance, we would be very surprised if we observed a cosmological constant of zero. Furthermore, given that the galaxies we observe all formed under the dynamics of the same cosmological constant, we would also expect each of their linear fluctuations to be anti-correlated with the cosmological constant term, provided the conditional variance in the linear term is small enough, which indeed is the case. If we post-select on one global observation, it can provide a boundary condition for the remaining terms. This however does depend on the underlying dynamics in cosmology, a point made often in v1.

Although in the linearised theory, there is no preferred sign to the terms, this is not the case once non-linear corrections are included, as (21) makes clear, and in the example of Brownian motion with a potential given in

Appendix D. We find evidence for this in [56].

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Appendix A: The classical-quantum action in the classical limit.

Examples of consistent ways to couple classical and quantum systems via a master-equation approach have been known since the 90's [66, 99, 100]. One can derive the most general form of consistent classical-quantum (CQ) dynamics, by demanding that the dynamics preserves the split of classical and quantum degrees of freedom, and preserve the positivity and normalisation of probabilities [21, 101]. This can then be used to construct a master equation for general relativity via the Hamiltonian formulation [21, 102, 103]. Recently a path integral formulation of classical-quantum dynamics was introduced with Zach Weller-Davies [23] and used to formulate a manifestly covariant path integral for classical general relativity coupled to quantum fields [22]. A measurement and feedback approach in the case of sourcing the Newtonian potential by quantum matter has also been pursued [29, 89, 104], as well as an unravelling approach [105, 106]. These can be applied to the weak field limits of gravitational theories [26, 28] and to cosmology [56].

In the present article, we don't need the full CQ path integral, since we are interested in the limit where the matter fields behave classically. But for completeness, we present the full path integral of [22]

$$\varrho(g_f,\phi_f^+,\phi_f^-,t_f) = \int \mathcal{N}\mathcal{D}g\mathcal{D}\phi^+\mathcal{D}\phi^- \; e^{\mathcal{I}_{CQ}[g,\phi^+,\phi^-_,t_i,t_f]}\varrho(g_i,\phi_i^+,\phi_i^-,t_i), \tag{A1}$$

where $\mathcal{N}$ is a normalisation factor and the action takes the form of:

$$\begin{aligned}\mathcal{I}_{CQ}[g,\phi^+,\phi^-,t_i,t_f] = \int_{t_i}^{t_f} d^4x \Bigg[& i\big(\mathcal{L}_Q[g,\phi^+] - \mathcal{L}_Q[g,\phi^-]\big) \\ & - \frac{\mathrm{Det}[-g]}{8}\big(T^{\mu\nu}[\phi^+] - T^{\mu\nu}[\phi^-]\big)D_{0,\mu\nu\rho\sigma}[g]\big(T^{\rho\sigma}[\phi^+] - T^{\rho\sigma}[\phi^-]\big) \\ & - \frac{\mathrm{Det}[-g]c^6}{128\pi^2 G_N^2}\left(G^{\mu\nu} - \frac{8\pi G_N}{c^4}\bar{T}^{\mu\nu}[\phi^+,\phi^-]\right)D_{0,\mu\nu\rho\sigma}[g]\left(G^{\rho\sigma} - \frac{8\pi G_N}{c^4}\bar{T}^{\rho\sigma}[\phi^+,\phi^-]\right)\Bigg].\end{aligned} \tag{A2}$$

Here $\mathcal{L}_Q$ is the quantum Lagrangian density including the appropriate metric factors, the bra and ket fields $\phi^\pm$ can be any quantum fields, $\bar{T}[\phi^+,\phi^-]$ is the average of the bra and ket fields of the stress-energy tensor and we have taken D_0 to saturate the decoherence-diffusion trade-off [25] such that both the decoherence and diffusion coefficients are written in terms of D_0. The bare cosmological constant must be taken to be zero for the action to be completely positive [24]. Since we do not consider the dynamics of the matter distribution, and consider the decohered case when $T^{\mu\nu}[\phi^+] = T^{\mu\nu}[\phi^-]$, only the final line of the action is used here, and $\bar{T}^{\rho\sigma}[\phi^+,\phi^-]$ can be replaced by the classical stress-energy tensor.

The decoherence and diffusion coefficient $D_{0,\mu\nu\rho\sigma}[g]$ is then chosen to be ultra-local so that it can be written in terms of the generalised deWitt metric [107, 108]:

$$D_{0,\mu\nu\rho\sigma} = 64\pi^2\frac{D_0}{\sqrt{-g}}\big(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} - 2\beta g_{\mu\nu}g_{\rho\sigma}\big). \tag{A3}$$

where we have renormalised the constant to absorb the factor of $128\pi^2$ in order to simplify the calculations. It was found in [24] that positivity requirements impose $\beta \leq \frac{1}{3}$ and, if one requires the propagator to be positive, one will have to take $\beta < 0$. In the absence of mass, the value of $\beta = \frac{1}{3}$ would correspond to conformal gravity.

In the absence of matter and energy, and with the choice of diffusion coefficient delineated in Eq. (A3), the action of Equation (A2) is reduced to a purely diffusive term in the metric degrees of freedom. The first two lines of the action of Eq. (A2) don't contribute, and with the stress-energy being zero, we are left with

$$\mathcal{I}[g] = -\frac{D_0 c^6}{G_N^2}\int d^4x\,\sqrt{-g}\Big(G^{\mu\nu}G_{\mu\nu} - \beta\, G^2\Big), \tag{A4}$$

where G is the trace of the Einstein tensor. Next, we used the identities connecting the Einstein and the Ricci tensor in dimension 4.

$$\begin{aligned}&G^{\mu\nu}G_{\mu\nu} = \mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}, \\ &G^2 = \mathcal{R}^2,\end{aligned} \tag{A5}$$

to write the action as

$$\mathcal{I}[g] = -\frac{4\pi D_0 c^6}{G_N^2}\int dr\,\sqrt{-g}\Big(\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} - \beta\,\mathcal{R}^2\Big). \tag{A6}$$

This action is related to that of quadratic gravity [39, 109–111], but doesn't suffer from negative norm ghosts [24]. As far as we know, it is unrelated to other forms of stochastic gravity [112] (c.f. [113]) whose purpose is to approximate the quantum stress energy tensor beyond the semi-classical regime (c.f. [106]).

Appendix B: A stochastic modification to the Schwarzschild metric

We start with the most general spherically symmetric metric

$$ds^2 = -e^{2\phi}dt^2 + e^{-2\psi}dr^2 + e^{-2\chi}r^2(d\theta^2 + sin^2(\theta)d\phi^2). \tag{B1}$$

When deriving Schwarzschild, one usually redefines r to reduce the metric to two free parameters before using Einstein's equation. We will do that here for simplicity, but it's important to note that this sort of coordinate system is not completely sensible here, since the metric is undergoing stochastic changes which would require one to constantly redefine r to obtain a strictly static object. However, we should be able to redefine r to remove $e^{2\chi}$ on expectation.

Let us consider metrics of the generalised Schwarzschild form

$$\phi(r) = \psi(r) = \frac{1}{2}\log\left(1 - \frac{2F(r)}{r}\right). \tag{B2}$$

Given this choice of metric, the curvature terms appearing in the diffusion action are:

$$G^{\mu\nu}G_{\mu\nu} = \mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} = \frac{2}{r^2}F''(r)^2 + \frac{8}{r^4}F'(r)^2, \tag{B3}$$

$$G^2 = \mathcal{R}^2 = \frac{4}{r^2}\left(\nabla^2 F(r)\right)^2, \tag{B4}$$

which when inserted in Eq. (A4) leads to the action

$$\mathcal{I} = -\frac{16\pi D_{0,T}}{G^2}\int dr\left(\frac{1}{2}F(r)''^2 + \frac{2}{r^2}F(r)'^2 - \beta(\nabla^2 F(r))^2\right), \tag{B5}$$

where we have already integrated over the angular part given that the action is spherically symmetric.

Whe now consider the power expansion of $F(r)$ of Eq. (14) and substitute it back into the action to obtain

$$\mathcal{I}_\gamma = -\frac{8\pi D_{0,T}}{G_N^2}\int_0^{r_{max}} dr\left((5-18\beta)\gamma_1^2 + 18(1-4\beta)\gamma_2^2 r^2 + 18(1-4\beta)\gamma_1\gamma_2 r\right), \tag{B6}$$

which, when integrated, gives

$$\mathcal{I}_\gamma = -\frac{6\pi D_{0,T}V}{G_N^2}\left((5-18\beta)\frac{\gamma_1^2}{r_{max}^2} + 6(1-4\beta)\gamma_2^2 + 9(1-4\beta)\frac{\gamma_1\gamma_2}{r_{max}}\right). \tag{B7}$$

where $V = \frac{4}{3}\pi r_{max}^3$. We can now see that the exponentiated action now represents a bivariate normal distribution for the parameters γ_1 and γ_2. This represents the values of the stochastic contributions to the path integral.

$$\Phi = -\frac{G_N M}{r} - \frac{\gamma_0}{2} - \frac{\gamma_1}{2}r - \frac{\gamma_2}{2}r^2. \tag{B8}$$

When inserted into the path integral of Eq. (16), it computes the normalised probability distribution to:

$$\begin{aligned} f(\gamma) &= \frac{1}{\mathcal{N}}\exp\left(-\frac{6\pi D_{0,T}V}{G_N^2}\left((5-18\beta)\frac{\gamma_1^2}{r_{max}^2} + 6(1-4\beta)\gamma_2^2 + 9(1-4\beta)\frac{\gamma_1\gamma_2}{r_{max}}\right)\right), \\ \mathcal{N} &= \frac{r_{max}}{3\sqrt{3(1-4\beta)(13-36\beta)}}\frac{G_N^2}{D_{0,T}V}. \end{aligned} \tag{B9}$$

which can be seen in the contour plot:

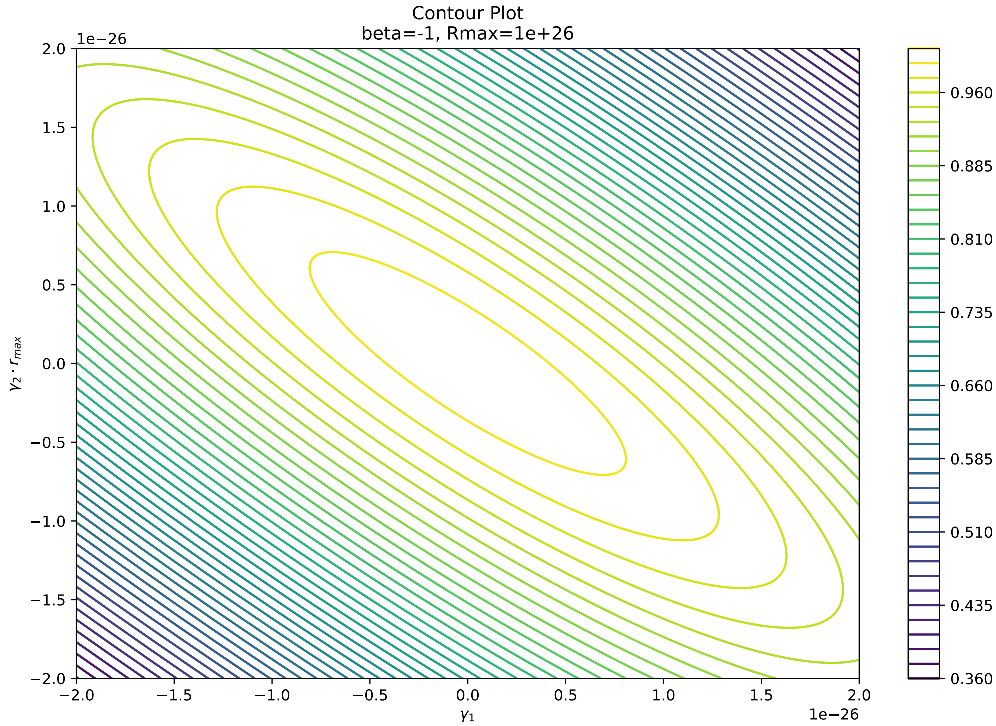

FIG. 1. Contour plot of the probability distribution of γ_1 and γ_2 defined in Eq. (B9). The negative correlation of the two variables is easily seen. To enhance the visibility of the plot, we have plotted γ_1 against $\gamma_2 * r_{max}$, and we chose a value of $r_{max} = R_H \approx 10^{26}\, m$, which represents the order of magnitude of the Hubble radius. We picked an indicative value of $\beta = -1$, but different beta will only tune the correlation as long as $\beta < \frac{1}{4}$.

We see that γ_1 and γ_2 are correlated variables with symmetric covariance matrix $\text{Cov}(\gamma_1, \gamma_2) = \Sigma_{ij}$ given by:

$$\begin{aligned}
\Sigma_{11} &= \frac{2r_{max}^2}{3(13-36\beta)} \frac{G_N^2}{D_{0,T}V}, \\
\Sigma_{22} &= \frac{(5-18\beta)}{9(13-36\beta)(1-4\beta)} \frac{G_N^2}{D_{0,T}V}, \\
\Sigma_{12} &= -\frac{r_{max}}{2(13-36\beta)} \frac{G_N^2}{D_{0,T}V},
\end{aligned} \tag{B10}$$

where Σ_{11} and Σ_{22} are the variances of γ_1 and γ_2 and the two variables have correlation coefficients given by

$$\rho_{12} = -\frac{3\sqrt{3}}{2\sqrt{2}} \sqrt{\frac{1-4\beta}{5-18\beta}}, \tag{B11}$$

with the negative correlation easily seen from the plot in Figure 1. We see that the two variables are negatively correlated with each other and have zero expected value. In other words, if we did not know the value of any of the two constants through other means, we would take their expectation value to be zero. Moreover, we expect one to decrease as the other increases and vice versa. However, through observational cosmology, we know that our universe presents a positive cosmological constant Λ, which has to be manually inserted in Einstein's equations by hand and contributes to the Weak field Newtonian limit as

$$\Phi = -\frac{GM}{r} - \frac{\Lambda}{3} r^2, \tag{B12}$$

where the r^2 dependence represents a global contribution. At this point, we can make the connection with the γ_2 factor. Given that we observe a value of $\gamma_2 = \frac{\Lambda}{3}$, what is the value of γ_1 that we expect? We can compute this by finding the conditional expectation

$$\begin{aligned}\mu_{\gamma_1|\gamma_2,r_{max}} &= \mu_{\gamma_1} + \rho_{12}\frac{\sigma_{\gamma_1}}{\sigma_{\gamma_2}}(\gamma_2 - \mu_{\gamma_2}) \\ &= -\frac{9}{2}\gamma_2 r_{max}\left(\frac{1-4\beta}{5-18\beta}\right),\end{aligned} \tag{B13}$$

where we know that $\mu_{\gamma_1} = \mu_{\gamma_2} = 0$. We now substitute the $\gamma_2 = \frac{\Lambda}{3}$ value that we observe and choose $r_{max} = R_H = 1.37 * 10^{26}\,m$ to be the Hubble radius. The Hubble radius gives a scale of the distance beyond which galaxies are receding from us faster than the speed of light due to the expansion of the Universe. Therefore, we arrive at:

$$\begin{aligned}\mu_{\gamma_1|\gamma_2,r_{max}} &= -\frac{3}{2}\Lambda R_H\left(\frac{1-4\beta}{5-18\beta}\right) \\ &= -2.28 * 10^{-26}\left(\frac{1-4\beta}{5-18\beta}\right).\end{aligned} \tag{B14}$$

Restoring the units of c means multiplying the above expression by c^2, obtaining

$$\mu_{\gamma_1|\gamma_2,r_{max}} = -2.06 * 10^{-9}\left(\frac{1-4\beta}{5-18\beta}\right), \tag{B15}$$

which, as $\beta \to 0$ tends to

$$\mu_{\gamma_1|\gamma_2,r_{max}} = -4.11 * 10^{-10}\ m/s^2, \tag{B16}$$

as plotted in Figure 2.

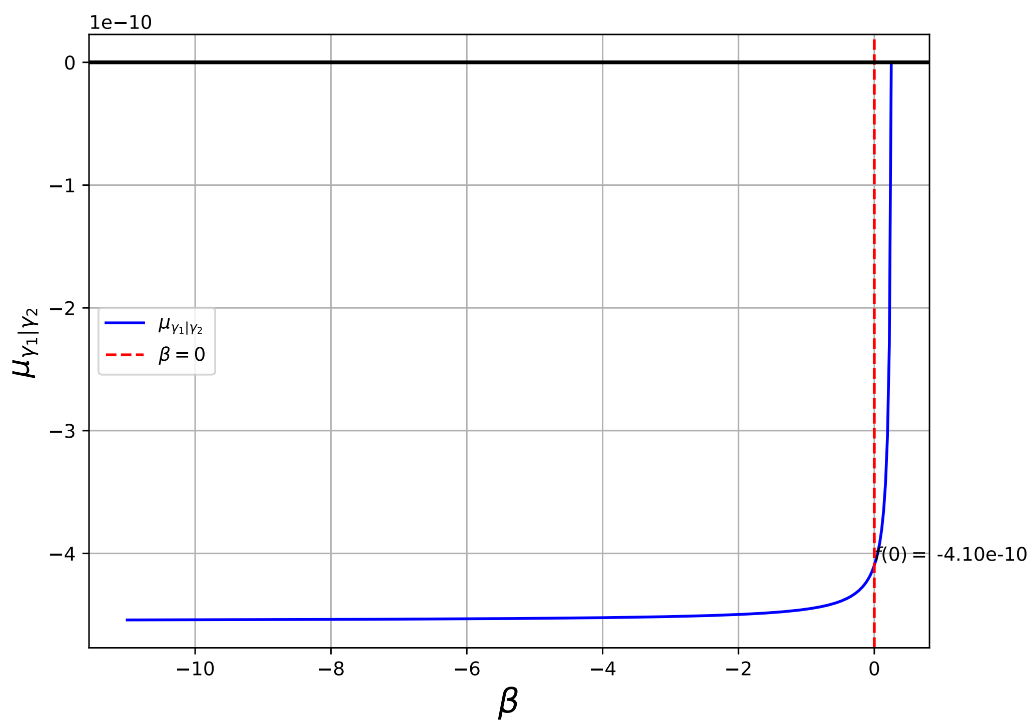

FIG. 2. The plot represents the conditional expectation of the value of γ_1 given the observed value of γ_2 as a function of β. For this plot, we have chosen $\gamma_2 = \frac{\Lambda}{3}$.

In a dark energy-dominated universe, the value of the Hubble radius can be expressed in terms of the cosmological constant as $R_H = \sqrt{\frac{3}{\Lambda}}$, meaning that the expected value of γ_1 is

$$\mu_{\gamma_1|\gamma_2,r_{max}} = -\frac{3}{2}\sqrt{3\Lambda}\left(\frac{1-4\beta}{5-18\beta}\right). \tag{B17}$$

Statistical analysis of the results

Given the observed value of Λ, we want to test two things. Firstly, how many standard deviations the observed value of γ_2 is from the predicted mean of 0, and secondly how many standard deviations the observed value of γ_1 needed for fitting is. Given that we know the two values follow a bivariate gaussian distribution, we can perform a Z-test. Given that we have a free parameter D_0, this would allow us to understand the range of possible values of the decoherence constant required for thee results to sit within 1 standard deviation of their expectation.

To perform the Z-test of γ_2 we recall that the observed value is $\gamma_2 = \frac{\Lambda}{3}$ and compute

$$Z_{\gamma_2} = \frac{\Lambda/3}{\sqrt{\Sigma_{22}}}, \tag{B18}$$

where Σ_{22} is the variance of γ_2 and we use the value of the maximal radius as the Hubble radius. We obtain

$$Z_{\gamma_2} = \frac{\Lambda\sqrt{D_{0,T}V_H}}{G_N}\sqrt{\frac{(1-4\beta)(13-36\beta)}{5-18\beta}}. \tag{B19}$$

We want the Z score of γ_2 to be less than 1, such that the observed value of γ_2 lies withing one standard deviation from the mean. Given that $V_H \approx 10^{79}$, plugging the values in the formula, we obtain

$$D_{0,T} \leq \frac{Z_{\gamma_2}^2 G_N^2}{V_H\Lambda^2} f(\beta), \tag{B20}$$

with $f(\beta) = \sqrt{\frac{(1-4\beta)(13-36\beta)}{5-18\beta}}$. When substituting the back the units of c, the formula gives

$$D_{0,T}c^3 \leq 1.34 \cdot 10^{-21} f(\beta)\, \frac{m^3}{s \cdot kg^2}. \tag{B21}$$

Therefore, we see that we are within one standard deviation for any value of D_0 with an order of magnitude less than $10^{-21} f(\beta)\, \frac{m^4}{s \cdot kg^2}$.

We can now perform the same computation for the observed value of γ_1 This could be the MOND value $\frac{a_0}{c^2} \approx 1.33 \cdot 10^{-27}$ or the value of γ found in [20] The conditional variance of the result is given by

$$\begin{aligned}\sigma^2_{\gamma_1|\gamma_2,r_{max}} &= \Sigma_{11}(1-\rho_{12}^2)\\ &= \frac{G^2 r_{max}^2}{4D_{0,T}V(15-54\beta)},\end{aligned} \tag{B22}$$

which we can now use to perform the Z-test using the Hubble parameters

$$\begin{aligned}Z_{\gamma_1} &= \frac{\gamma_{obs} - \mu_{\gamma_1|\gamma_2,r_{max}}}{\sigma_{\gamma_1|\gamma_2,r_{max}}}\\ &= \frac{2\sqrt{D_{0,T}V_H(15-54\beta)}}{G_N R_H}\left(\gamma_{obs} + \frac{3(1-4\beta)\Lambda R_H}{2(5-18\beta)}\right),\end{aligned} \tag{B23}$$

which we can rearrange to obtain

$$D_{0,T} \leq \frac{G_N^2 R_H^2 Z_{\gamma_1}^2}{V_H}\frac{(5-18\beta)}{3(2(5-18\beta)\gamma_{obs} + 3(1-4\beta)\Lambda R_H)^2}. \tag{B24}$$

When substituting numbers, restoring units of c and setting $Z_{\gamma_1} = 1$, we obtain (for $\beta = -1$)

$$D_{0,T} c^3 \leq 1.09 \cdot 10^{-22} \, \frac{m^3}{s \cdot kg^2}, \tag{B25}$$

which means that if D_0 is such that the observed MOND acceleration is withing one standard deviation of the conditional expected value, it will automatically be such that the observed value of Λ is withing one standard deviation of the model.

Appendix C: The stochastic action for the isotropic metric, and the Newtonian limit

For the purpose of this section, we only consider a static matter distribution with negligible contributions from matter pressure, frame velocity and specific energy density. In other words, we are only interested in higher-order corrections coming from the gravitational potential Φ itself. We implicitly choose a homogeneous isotropic universe in which resides an isolated Post-Newtonian system with coordinates such that the outer region far from the isolated system is in freefall with respect to the surrounding cosmological model but at rest with respect to a frame in which the universe appears isotropic. It is then possible to show that one can construct a local quasi-Cartesian system in which metric and matter degrees of freedom can all be evaluated consistently with the Post-Newtonian approximation. Lastly, one might need to take into account the extent of preferred frame effects including frame dragging and the coordinate velocity of the frame relative to the mean rest frame of the universe. All the aforementioned effects can be summarised through what is known as the Parametrised Post-Newtonian formalism (PPN), whose first formulation dates back to Eddington in 1922. When formulated in a coordinate frame moving along with the physical system of interest, post-Newtonian effects can be summarised through the metric (with units of c):

$$\begin{aligned} g_{00} &\approx -c^2 \left(1 + \frac{2\Phi}{c^2} + \frac{2\beta\Phi^2}{c^4} + f(\alpha_i, \beta, \gamma, \zeta_i, V_i, W_i)\right) + \mathcal{O}(c^6), \\ g_{ij} &\approx \left(1 - \frac{2\gamma\Phi}{c^2}\right) \delta_{ij} + \mathcal{O}(c^4), \\ g_{0i} &\approx h(\alpha_i, \gamma, \zeta_i, V_i, W_i) + \mathcal{O}(c^5), \end{aligned} \tag{C1}$$

where α_i, ζ_i with $i = \{1, 2, 3\}$ represents respectively the extent of preferred frame effects and the extent of failure in the conservation of energy, β measures the amount of nonlinearity in the superposition law for gravity, γ the amount of curvature produced by a unit rest mass and V_i, W_i effects related to the frame velocity [114, 115]. The strength of the Parametrised Post Newtonian formalism is that it can be applied to theories of gravity outside of general relativity. However, to describe the post-Newtonian limit of general relativity one takes $\alpha_i = \zeta_i = 0$ and $\beta = \gamma = 1$, which is what we will do in this paper.

Given these premises, we write the isotropic metric as

$$ds^2 = -c^2 e^{\frac{2\Phi}{c^2}} dt^2 + e^{-\frac{2\Phi}{c^2}} \delta_{ij} dx^i dx^j. \tag{C2}$$

One may worry that the exponential form of this metric may not be consistent at higher orders in the expansion, for example, not all terms in the expansion may be physically relevant. However, for all effects and purposes, in this paper, we will never exceed order $\mathcal{O}(c^4)$, such that the metric matches perfectly with the PPN formalism. For the matter distribution, we will take the Stress-Energy tensor to be that of pressureless dust, being given by

$$T_{00} = m \, e^{2\Phi}, \quad T^{ij} = 0, \quad T^{0i} = 0. \tag{C3}$$

Using the isotropic metric (with $c = 1$), the components of the action of Eq. (A4)

$$G^{\mu\nu} G_{\mu\nu} = e^{4\Phi} \left(3 \left(\nabla\Phi\right)^4 + \left(\left(\nabla\Phi\right)^2 - 2\left(\nabla^2\Phi\right)\right)^2\right), \tag{C4}$$

$$G^2 = 4e^{4\Phi} \left(\left(\nabla\Phi\right)^2 - \nabla^2\Phi\right)^2, \tag{C5}$$

The coupling to matter can be deduced from the final line of Eq (A2) since in the classical limit the system decoheres and for a decohered system, there is no distinction between $\bar{T}_{\mu\nu}$ and $T_{\mu\nu}$. The components are given by

$$G^{\mu\nu} T_{\mu\nu} = -m \, e^{2\Phi} \left(\left(\nabla\Phi\right)^2 - 2\left(\nabla^2\Phi\right)\right), \tag{C6}$$

$$T^{\mu\nu} T_{\mu\nu} = m^2, \tag{C7}$$

$$T^\mu_\mu = -m. \tag{C8}$$

where T is the trace of the stress energy tensor.

The full action for the isotropic metric is thus

$$\mathcal{I} = -\frac{D_0 c^5}{64\pi^2 G_N^2}\int d^3\vec{x}dt\, e^{\frac{2\Phi}{c^2}}\left[\left(\nabla^2\Phi - \frac{(\nabla\Phi)^2}{2c^2} - 4e^{-\frac{2\Phi}{c^2}}\pi G m\right)^2 + \frac{3}{c^4}(\nabla\Phi)^4 - 4\beta\left(\nabla^2\Phi - \frac{(\nabla\Phi)^2}{2c^2} - 4e^{-\frac{2\Phi}{c^2}}\pi G m\right)^2\right] \tag{C9}$$

where we have put in powers of c as one can use it to perform an expansion in powers of $1/c^2$. One immediately sees that at $0'th$ order in $1/c^2$, we recover the Newtonian action of (4).

Appendix D: Entropic forces

A canonical example of an entropic force is that due to a polymer which is initially curled up in a low entropy state, but will unfurl or diffuse into a higher entropy state, with its ends exerting a force [116, 117]. Another is a gas in a box fitted with a piston on one side, which is slowly pushed out as the gas diffuses. Note that in the main body of this article, we do not consider deriving gravity as an entropic force [118–122], but rather consider the entropic force that gravity exerts. The purpose of this section is to define entropic forces as applicable out of equilibrium and based only on the equations of motion. It will also give an example that can be solved in a similar manner to the gravitational case and has similar features.

Consider Newton's law $F(q) = m\ddot{q}$. This is a deterministic equation, but we can consider the case where the system is in a probability distribution over q, in which case we still expect Newton's law to be satisfied on expectation

$$\langle m\ddot{q} - F(q)\rangle = 0. \tag{D1}$$

The important ingredient is that the mean value of the force felt by the particle depends on the second and higher moment of its position, and so it generally doesn't follow its deterministic trajectory because the average of the position equation of motion is not the same as the equation of motion of the average position. We therefore define the entropic force F_S to be

$$F_S(q) = F(\langle q\rangle) - \langle F(q)\rangle, \tag{D2}$$

since it captures the extra force due to diffusion. A simple example is given by $F = -\alpha q^2$ for the cubic potential corresponding to $V(q) = \alpha q^3/3$. The time derivative of the particle's mean momentum obeys $\langle\dot{p}\rangle = \alpha\langle q^2\rangle$ which can be significantly larger than $\langle q\rangle^2$. Another example is Brownian motion of a particle in a box with a piston. The presence of a wall on the other side suffices to ensure that the mean value of the particle's position q will change with time as the piston is pushed out. If there were no diffusion or wall, the particle's average position does not change. The wall placed at $q = 0$ makes it impossible for $\langle q_f\rangle = q_0$ when $\langle q^2\rangle$ is non-zero. After all, given enough time, the reflecting boundary at the origin will skew the average final position in the direction opposite to the wall.

Indeed, as $\langle q^2\rangle$ becomes greater and greater than $\langle q\rangle^2$, (possibly due to elapsed time or a temperature increase of the heath bath) the presence of the wall makes it so that the average final position will be further and further away from the mean. We will call this the *diffusion regime*, since the second moment of the observable is comparable to its variance and is influencing the observable equations of motion, in comparison to the case where the mean value of the observable is given by its deterministic value, which for a free Brownian particle corresponds to the final position being identical to its initial position $\langle q_f\rangle \approx q_0$.

We will now explicitly show the example of a Brownian particle with a wall, and show that it has very similar features to the gravitational example discussed in the main body of the paper, and see that it can be solved in a similar way.

Brownian motion with a wall

Consider the path integral for a free particle undergoing Brownian motion with no drift. The probability of finding a particle at $q(t_f) = q_f$ given that at $t = 0$ it was at $q(0) = q_0$ and had velocity v_0 and acceleration a_0 is given by the Onsager-Machlup path integral

$$P(q_f|q_0,\dot{q}_0,\ddot{q}_0) = \frac{1}{\mathcal{N}}\int_{q_0}^{q_f}\mathcal{D}q\, e^{-\frac{1}{2D_2}\int_0^{t_f}(\ddot{q})^2 dt}. \tag{D3}$$

Note that the path integral acts to suppress the probability of paths which do not satisfy $\ddot{q} = 0$, by an amount controlled by the diffusion constant D_2. The larger D_2 is, the more stochasticity we are likely to find in the paths which are realised. This is an equivalent description of the dynamics often described by the Langevin equation, $\ddot{q} = F(q)/m + j(t)$, with F/m the drift produced by a deterministic force F (here set to 0), and $j(t)$ a stochastic white noise process. The dynamics can also be described via the Fokker-Planck equation [123] or Ito calculus [124] and we refer the interested reader to [123] for a derivation of the Onsager-Machlup path integral, or [31] for a discussion of Brownian motion in the context of path integrals of similar form to Eq. (D3).

We now imagine that there is a step function $V\Theta(-q)$ potential (we could take $V \to \infty$). This prevents the particle from going to the negative values. we can express this by modifying the OM Lagrangian to be

$$\mathcal{L}_{OM}(\ddot{q}) = -\frac{1}{2D_2}\left(\frac{d^2}{dt^2}|q|\right)^2, \tag{D4}$$

the variation of the Lagrangian provides the fourth-order Euler-Lagrange equation for the most probable paths:

$$\frac{d^4}{dt^4}|q| = 0, \tag{D5}$$

with general solution

$$q_{MPP}(t) = \alpha_0 + \alpha_1 t + \frac{1}{2}\alpha_2 t^2 + \frac{1}{6}\alpha_3 t^3. \tag{D6}$$

This is remarkably similar to (15). When substituting back into the action, we see that the terms corresponding to the deterministic solution α_0 and α_1 (which is the global minimum) drop out due to the second-order time derivative. Therefore, we are allowed to fix them through initial conditions on $q(0)$ and $\dot{q}(0)$. The action then takes the form of a bivariate Gaussian distribution, which when integrated from the initial time $t_0 = 0$ to the final time t_f becomes:

$$e^{\mathcal{S}_{OM}} = \exp\left(-\frac{t_f}{3D_2}\left(3\alpha_2^2 + 3\alpha_3\alpha_2 t_f + \alpha_3^2 t_f^2\right)\right). \tag{D7}$$

At this point, we can relate α_2 and α_3 to other known initial conditions or final conditions, and the action will act as the probability weight of the most probable path given the specified conditions. However, we could also use it to find the average final position. For the sake of simplicity and to obtain an analytical expression, we assume that the particle starts with no acceleration and that at the final time is at positon Q. In particular, we fix $q(0) = q_0 > 0$ and $\dot{q}(0) = \ddot{q}(0) = 0$, such that the particle begins on the right-hand side of the wall with zero initial velocity and acceleration. This fixes $\alpha_1 = \alpha_2 = 0$. The last condition is fixed by setting $q(t_f) = Q \geq 0$, the final position of the particle, arriving at

$$q(t) = q_0 + \frac{(Q - q_0)t^3}{t_f^3}. \tag{D8}$$

We can now substitute the solution into the Lagrangian to perform a saddle point approximation and integrate it up to the final time. We arrive at the action which determines the probability weighting of the most probable path given initial and final conditions:

$$\mathcal{S}_{OM}(Q) = -\frac{6(Q - q_0)^2}{D_2 t_f^3}, \tag{D9}$$

we can now integrate over all possible final positions to normalise the integral

$$\int_{-\infty}^{\infty} dQ\, P(Q|q_0, \dot{q}_0, \ddot{q}_0) = \frac{1}{\mathcal{N}}\int_{-\infty}^{\infty} dQ\, e^{-\frac{6(Q-q_0)^2}{D_2 t_f^3}} = 1, \tag{D10}$$

to find

$$\mathcal{N} = \sqrt{\frac{D_2 \pi\, t_f^3}{6}}. \tag{D11}$$

At this point, we can compute the average final position by keeping in mind that there is a wall at $q = 0$ such that

$$\begin{aligned}\langle Q\rangle &= \sqrt{\frac{6}{D_2 \pi\, t_f^3}} \int_{-\infty}^{\infty} dQ\, |Q| e^{-\frac{6(Q-q_0)^2}{D_2 t_f^3}} \\ &= \frac{1}{12}\left[6q_0 + \left(1+\Gamma_{\mathcal{R}}\left(-\frac{1}{2}, 0, \frac{6q_0^2}{D_2 t_f^3}\right)\right) + \sqrt{\frac{6D_2}{\pi}} t_f^{3/2} e^{-\frac{6q_0^2}{D_2 t_f^3}} - 6q_0 \mathrm{Erf}\left(\sqrt{\frac{6}{D_2 t_f^3}} q_0\right)\right],\end{aligned} \tag{D12}$$

where Erf is the error function and $\Gamma_{\mathcal{R}}$ is the regularised Gamma function.

This solution is very insightful. As the diffusion vanishes $D_2 \to 0$ or the final time goes to zero $t_f \to 0$, the argument of the error function, the exponential and the regularised gamma function go to infinity. The error function and the exponential vanish while the gamma function becomes 1, leaving $\langle Q\rangle = q_0$. As one would expect for a situation where there is either no diffusion or no time has elapsed, the final average position is the same as the initial one, which is also the deterministic behaviour. Even more interesting, the same happens when q_0 is very large; indeed, if the particle is very far from the wall, it will not feel its effect until enough time has passed, as it can be seen in Figure 3, such that there is an opposite effect between the growth of q_0 and that of t_f.

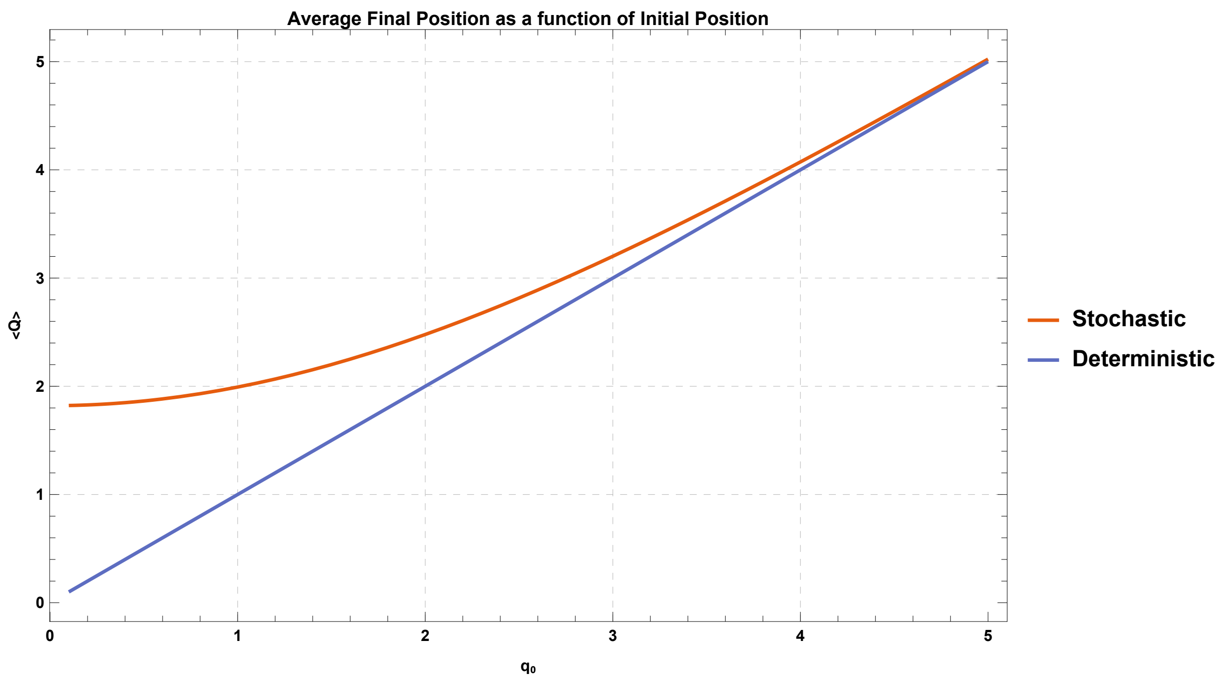

FIG. 3. Average final position as a function of initial position according to Equation (D12) for fixed final time $t_f = 5$ and diffusion coefficient $D_2 = \frac{1}{2}$. In the presence of a wall, the closer the Brownian particle starts to the reflective wall at $q = 0$, the more its average final position will diverge from its deterministic value. The particle is assumed to start with zero velocity and acceleration.

Lastly, one could assume the particle is not too far from the wall and perform a short time expansion to arrive at

$$\langle Q\rangle = q_0 + \frac{1}{2}\sqrt{\frac{D_2}{6\pi} t_f^3}, \tag{D13}$$

such that one sees that the average final position increases as $t^{3/2}$. In Figure 4 we show the probability density function of the final positions obtained from a Monte Carlo simulation.

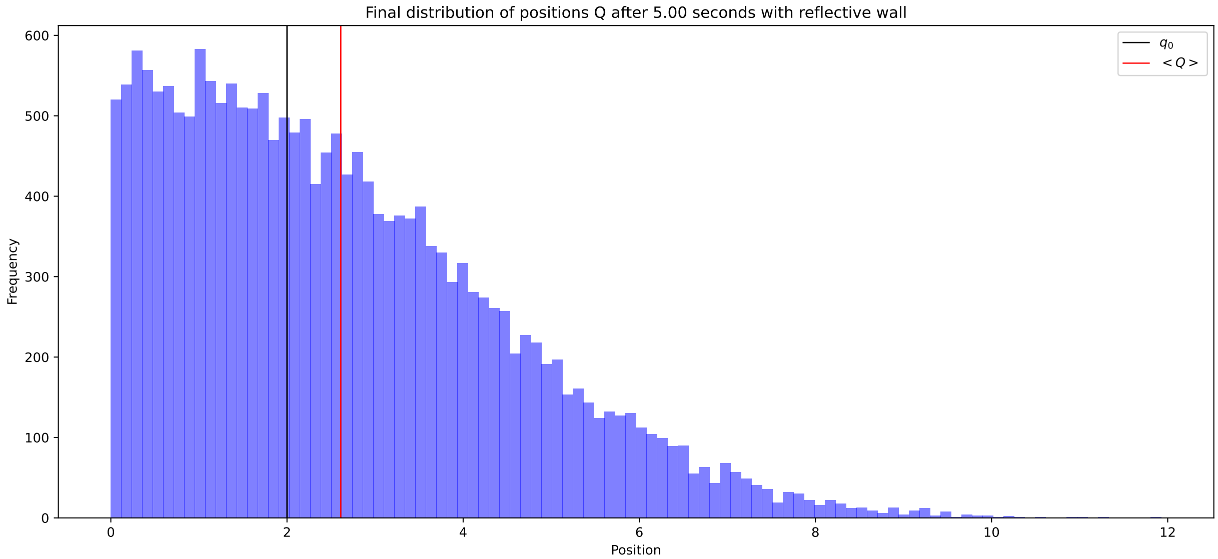

FIG. 4. Monte Carlo simulation of the probability density function of the final position for the Brownian particle with a wall with $t_f = 5$. As one can see, the initial position is $q_0 = 2$. In the deterministic case, this should correspond to the final position in the absence of initial velocity and acceleration. However, the average final position is skewed to the right, as the presence of the wall produces an entropic force which creates a deviation.